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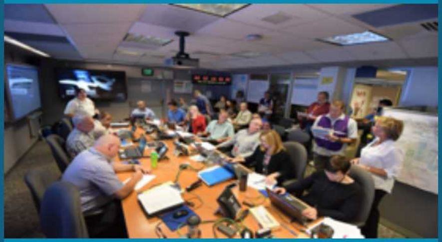
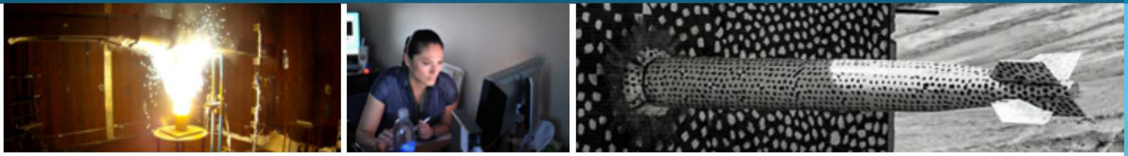
Explicit Partitioned Methods based on Monolithic Formulations of the Coupled Problem





8th International Conference on Coupled Problems, Barcelona, Spain, 3-5 June 2019

Special session on
Algorithmic and software advances in coupling methods for climate models

PRESENTED BY

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Why we like explicit partitioned methods?

- Allow reuse of existing codes: reduce development, validation & deployment time and cost
- Can improve simulation efficiency by running codes in their “sweet spots”, e.g., optimal Δt, mesh size,…
- Simplify development of large multiphysics codes, e.g., Earth System Model, CASL, etc.

In this talk we consider coupled problems with “standard” interface conditions:

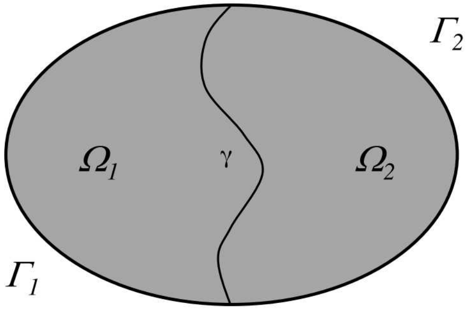
$$\dot{u}_1 + L_1 u_1 = f_1 \text{ in } \Omega_1$$
$$\dot{u}_2 + L_2 u_2 = f_2 \text{ in } \Omega_2$$

Subdomain equations

$$u_1 = u_2 \text{ on } \gamma$$
$$-F_1 = F_2 \text{ on } \gamma$$

Continuity of states

Continuity of fluxes



Next talk will focus on more interesting interface conditions relevant to ocean-atmosphere couplings.

Basic types of partitioned methods (PM)

PM having an “iterative” basis

$$\begin{tikzcd} \dot{u}_1 + L_1 u_1 = f_1 \ \text{ in } \ \Omega_1 \arrow[d, bend left=60, "u_1 \text{ or } F_1"] \\ \dot{u}_2 + L_2 u_2 = f_2 \ \text{ in } \ \Omega_2 \arrow[u, bend left=60, "u_2 \text{ or } F_2"] \end{tikzcd}$$

- Mathematically equivalent to a single step of an iterative method, e.g., fixed point, non-overlapping Schwarz, etc.
- Small number of steps ⇒ stability & accuracy issues
- Common mitigation strategies: optimized, Robin-like transmission conditions (Banks), Anderson acceleration (Pawlowski, CASL)

A. de Boer, A. van Zuijlen, H. Bijl, Review of coupling methods for non-matching meshes, CMAME 196 (2007).
Domain Decomposition Methods: recent advances and new challenges in engineering.

PM having a “monolithic” basis

$$\dot{u}_1 + L_1 u_1 = f_1 \ \text{ in } \ \Omega_1$$

$$\dot{u}_2 + L_2 u_2 = f_2 \ \text{ in } \ \Omega_2$$

$$\langle u_1 - u_2, \lambda \rangle_\gamma = 0 \quad \text{on} \quad \gamma$$

K. C. Park, C. A. Felippa, R. Ohayon, Partitioned formulation of internal fluid–structure interaction problems by localized Lagrange multipliers, CMAME 190 (2001).

- Use Lagrange multipliers to enforce continuity of states
- Lead to semidiscrete problems that are Hessenberg index-2 DAE

$$\begin{aligned} \dot{y}_1 &= f_1(t, y, z) \\ \dot{y}_2 &= f_2(t, y, z) \\ 0 &= g(t, y) \end{aligned}$$

y - differential variable
z - algebraic variable
g - algebraic constraint

Note: g does not depend on z!

- Not ”compatible” with explicit time integration: it “deletes” the constraint
- Resulting PM methods not truly explicit or resemble projection methods
- Have “hidden” constraints and are more difficult to solve

The Implicit Value Recovery (IVR) approach: an overview

Step 1: reduce the DAE index

$$\begin{aligned}\dot{y}_1 &= f_1(t,y,z)\\ \dot{y}_2 &= f_2(t,y,z)\\ 0 &= g(t,y)\end{aligned} \quad\longrightarrow\quad \begin{aligned}\dot{y}_1 &= f_1(t,y,z)\\ \dot{y}_2 &= f_2(t,y,z)\\ 0 &= g(t,y,z)\end{aligned}$$

Hessenberg index-2

Hessenberg index-1

where the Jacobian $\partial_z g$ is non-singular

Step 2: eliminate the algebraic variable

$$\begin{aligned}\dot{y}_1 &= f_1(t,y,z)\\ \dot{y}_2 &= f_2(t,y,z)\\ 0 &= g(t,y,z)\end{aligned} \quad\longrightarrow\quad \begin{aligned}\dot{y}_1 &= f_1\big(t,y,z(t,y)\big)\\ \dot{y}_2 &= f_2\big(t,y,z(t,y)\big)\end{aligned}$$

$0 = g(t,y,z)$ defines an **implicit** function $z(t,y)$

Step 3: apply explicit time integration

$$\begin{aligned}y_1^{n+1} &= f_1\big(t,y^n,z(t^n,y^n)\big)\\ y_2^{n+1} &= f_2\big(t,y^n,z(t^n,y^n)\big)\end{aligned}$$

- Subdomain equations can be solved independently!
- Explicit time integration effectively decouples the system
- Remains equivalent to the parent monolithic problem
- No splitting error!

Model Solid-Solid Interaction and Transmission Problems

SSI

TP

Subdomain equations

$$\begin{array}{rcll} \ddot{\boldsymbol{u}}_i - \nabla\cdot\sigma_i(\boldsymbol{u}_i) & = & \boldsymbol{f}_i & \text{in } \Omega_i\times[0,T] \\ \boldsymbol{u}_i & = & \boldsymbol{g}_i & \text{on } \Gamma_i\times[0,T] \end{array}, \quad i=1,2$$

$$\sigma_i(\boldsymbol{u}_i) = \lambda_i(\nabla\cdot\boldsymbol{u}_i)I + 2\mu_i\varepsilon(\boldsymbol{u}_i)$$

$$\begin{array}{rll} \dot{\varphi}_i - \nabla\cdot F_i(\varphi_i) = f_i & \text{in} & \Omega_i\times[0,T] \\ \varphi_i = g_i & \text{in} & \Gamma_i\times[0,T] \end{array} \quad i=1,2$$

$$F_i(\varphi_i) = \epsilon_i\nabla\varphi_i - \boldsymbol{u}\varphi_i$$

Initial conditions

$$\begin{array}{rcll} \boldsymbol{u}_i(0,\boldsymbol{x}) & = & \boldsymbol{u}_0(\boldsymbol{x})\big|_{\Omega_i} & \text{in } \Omega_i \\ \dot{\boldsymbol{u}}_i(0,\boldsymbol{x}) & = & \dot{\boldsymbol{u}}_0(\boldsymbol{x})\big|_{\Omega_i} & \text{in } \Omega_i \end{array} \quad i=1,2;$$

$$\varphi_i(\boldsymbol{x},0) = \varphi_{i,0}(\boldsymbol{x}) \quad \text{in} \quad \Omega_i \quad i=1,2;$$

Coupling conditions

$$\begin{array}{c} \boldsymbol{u}_1(\boldsymbol{x},t) = \boldsymbol{u}_2(\boldsymbol{x},t) \\ \sigma_1(\boldsymbol{x},t)\cdot\boldsymbol{n}_\gamma = \sigma_2(\boldsymbol{x},t)\cdot\boldsymbol{n}_\gamma \end{array} \quad \text{on} \quad \gamma\times[0,T].$$

$$\begin{array}{c} \varphi_1(\boldsymbol{x},t) - \varphi_2(\boldsymbol{x},t) = 0 \\ F_1(\boldsymbol{x},t)\cdot\boldsymbol{n}_\gamma = F_2(\boldsymbol{x},t)\cdot\boldsymbol{n}_\gamma \end{array} \quad \text{on} \quad \gamma\times[0,T].$$

Monolithic problem (weak form)

seek $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{t}\} \in \mathbf{H}^1_\Gamma(\Omega_1)\times\mathbf{H}^1_\Gamma(\Omega_2)\times\mathbf{H}^{-1/2}(\gamma)$ *such that*

$$\begin{array}{rcll} (\ddot{\boldsymbol{u}}_1,\boldsymbol{v}_1)_{0,\Omega_1} + \langle\boldsymbol{t},\boldsymbol{v}_1\rangle_\gamma & = & (\boldsymbol{f}_1,\boldsymbol{v}_1)_{0,\Omega_1} - (\sigma_1(\boldsymbol{u}_1),\varepsilon(\boldsymbol{v}_1))_{0,\Omega_1} & \forall\boldsymbol{v}_1\in\mathbf{H}^1_\Gamma(\Omega_1) \\ (\ddot{\boldsymbol{u}}_2,\boldsymbol{v}_2)_{0,\Omega_2} - \langle\boldsymbol{t},\boldsymbol{v}_2\rangle_\gamma & = & (\boldsymbol{f}_2,\boldsymbol{v}_2)_{0,\Omega_2} - (\sigma_2(\boldsymbol{u}_2),\varepsilon(\boldsymbol{v}_2))_{0,\Omega_2} & \forall\boldsymbol{v}_2\in\mathbf{H}^1_\Gamma(\Omega_2) \\ \langle\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{s}\rangle_\gamma & = & 0 & \forall\boldsymbol{s}\in\mathbf{H}^{-1/2}(\gamma) \end{array}$$

seek $\{\varphi_1,\varphi_2,\lambda\} \in H^1_\Gamma(\Omega_1)\times H^1_\Gamma(\Omega_2)\times H^{-1/2}(\gamma)$ *such that*

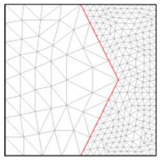
$$\begin{array}{rcll} (\dot{\varphi}_1,\psi_1)_{0,\Omega_1} + \langle\lambda,\psi_1\rangle_\gamma & = & (f_1,\psi_1)_{0,\Omega_1} - (F_1(\varphi_1),\nabla\psi_1)_{0,\Omega_1} & \forall\psi_1\in H^1_\Gamma(\Omega_1) \\ (\dot{\varphi}_2,\psi_2)_{0,\Omega_2} - \langle\lambda,\psi_2\rangle_\gamma & = & (f_2,\psi_2)_{0,\Omega_2} - (F_2(\varphi_2),\nabla\psi_2)_{0,\Omega_2} & \forall\psi_2\in H^1_\Gamma(\Omega_2) \\ \langle\varphi_1-\varphi_2,\mu\rangle_\gamma\, dS & = & 0 & \forall\mu\in H^{-1/2}(\gamma) \end{array}$$

IVR formulation for the Solid-Solid Interaction Problem

Step 1. Spatial discretization: *seek* $\{\boldsymbol{u}_1^h, \boldsymbol{u}_2^h, \boldsymbol{t}^h\} \in S_{1,\Gamma}^h \times S_{2,\Gamma}^h \times G_\gamma^h$ *such that*

$$\begin{aligned}
(\ddot{\boldsymbol{u}}_1^h, \boldsymbol{v}_1^h)_{0,\Omega_1} + \left(\boldsymbol{t}^h, \boldsymbol{v}_1^h\right)_{0,\gamma} &= (\boldsymbol{f}_1, \boldsymbol{v}_1^h)_{0,\Omega_1} - (\sigma_1(\boldsymbol{u}_1^h), \varepsilon(\boldsymbol{v}_1^h))_{0,\Omega_1} \quad \forall \boldsymbol{v}_1^h \in S_{1,\Gamma}^h \\
(\ddot{\boldsymbol{u}}_2^h, \boldsymbol{v}_2^h)_{0,\Omega_2} - \left(\boldsymbol{t}^h, \boldsymbol{v}_2^h\right)_{0,\gamma} &= (\boldsymbol{f}_2, \boldsymbol{v}_2^h)_{0,\Omega_2} - (\sigma_2(\boldsymbol{u}_2^h), \varepsilon(\boldsymbol{v}_2^h))_{0,\Omega_2} \quad \forall \boldsymbol{v}_2^h \in S_{2,\Gamma}^h \\
\left(\boldsymbol{u}_1^h - \boldsymbol{u}_2^h, \boldsymbol{s}^h\right)_{0,\gamma} &= 0 \quad \forall \boldsymbol{s}^h \in G_\gamma^h
\end{aligned}$$

$$\begin{aligned}
M_1\ddot{\mathbf{u}}_1 + G_1^T\mathbf{t} &= \mathbf{f}_1(\mathbf{u}_1) \\
M_2\ddot{\mathbf{u}}_2 - G_2^T\mathbf{t} &= \mathbf{f}_2(\mathbf{u}_2) \\
G_1\mathbf{u}_1 - G_2\mathbf{u}_2 &= 0
\end{aligned}$$



We assume spatially coincident interfaces (no gaps or overlaps) but allow non-matching grids.

Step 2. Index reduction: Assume $\boldsymbol{u}_0(\boldsymbol{x}^-) = \boldsymbol{u}_0(\boldsymbol{x}^+)$ and $\dot{\boldsymbol{u}}_0(\boldsymbol{x}^-) = \dot{\boldsymbol{u}}_0(\boldsymbol{x}^+)$ on $\boldsymbol{\gamma}$.

$$u_1(\boldsymbol{x}, t) = u_2(\boldsymbol{x}, t) \longrightarrow \ddot{u}_1(\boldsymbol{x}, t) = \ddot{u}_2(\boldsymbol{x}, t)$$

$$\begin{aligned}
M_1\ddot{\mathbf{u}}_1 + G_1^T\mathbf{t} &= \mathbf{f}_1(\mathbf{u}_1) \\
M_2\ddot{\mathbf{u}}_2 - G_2^T\mathbf{t} &= \mathbf{f}_2(\mathbf{u}_2) \\
G_1\ddot{\mathbf{u}}_1 - G_2\ddot{\mathbf{u}}_2 &= 0
\end{aligned}$$

It is easy to check that this problem has the Hessenberg Index-1 structure: set

$y = (u_1, u_2)$;
$z = \mathbf{t}$,

and

$$f(t, y, z) = \begin{pmatrix} M_1^{-1}\left(\mathbf{f}_1(\mathbf{u}_1) - G_1^T\mathbf{t}\right) \\ M_2^{-1}\left(\mathbf{f}_2(\mathbf{u}_2) + G_2^T\mathbf{t}\right) \end{pmatrix}$$

$$g(t, y, z) = S\mathbf{t} - G_1M_1^{-1}\mathbf{f}_1(\mathbf{u}_1) + G_2M_2^{-1}\mathbf{f}_2(\mathbf{u}_2)$$

$$S = G_1M_1^{-1}G_1^T + G_2M_2^{-1}G_2^T$$

$$\begin{aligned}
\ddot{y} &= f(t, y, z) \\
0 &= g(t, y, z)
\end{aligned}$$

IVR formulation for the Solid-Solid Interaction Problem

Step 3. Reduction of the DAE to the underlying ODE: lumped mass matrix case

Monolithic problem assumes a form where

$$\left[\begin{array}{ccc:cc} M_{1,\gamma} & 0 & G_1^T & 0 & 0 \\ 0 & M_{2,\gamma} & -G_2^T & 0 & 0 \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & M_{1,0} & 0 \\ 0 & 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \mathbf{t} \\ \hdashline \ddot{\mathbf{u}}_{1,0} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) \\ \mathbf{f}_{2,\gamma}(\mathbf{u}_2) \\ 0 \\ \hdashline \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right]$$

Interface blocks

are **completely separated** from the

interior blocks

Form the Schur complement of the upper left 2x2 block in the decoupled interface system

$$\left[\begin{array}{cc:c} M_{1,\gamma} & 0 & G_1^T \\ 0 & M_{2,\gamma} & -G_2^T \\ \hdashline G_1 & -G_2 & 0 \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \hdashline \mathbf{t} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) \\ \mathbf{f}_{2,\gamma}(\mathbf{u}_2) \\ \hdashline \mathbf{0} \end{array}\right]$$

Note: solvability of the Schur complement requires G_1 and G_2 to have full column ranks

and solve the resuling equation for the interface force:

$$\left(G_1 M_{1,\gamma}^{-1} G_1^T + G_2 M_{2,\gamma}^{-1} G_2^T\right)\mathbf{t} = \left(G_1 M_{1,\gamma}^{-1}\mathbf{f}_{1,\gamma}(\mathbf{u}_1) - G_2 M_{2,\gamma}^{-1}\mathbf{f}_{2,\gamma}(\mathbf{u}_2)\right)$$

IVR formulation for the Solid-Solid Interaction Problem

Step 3. Reduction of the DAE to the underlying ODE: consistent mass matrix case

Monolithic problem assumes a form where

$$\left[\begin{array}{ccc:cc} M_{1,\gamma} & 0 & G_1^T & M_{1,\gamma 0} & 0 \\ 0 & M_{2,\gamma} & -G_2^T & 0 & M_{2,\gamma 0} \\ G_1 & -G_2 & 0 & 0 & 0 \\ \hdashline M_{1,0\gamma} & 0 & 0 & M_{1,0} & 0 \\ 0 & M_{2,0\gamma} & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \mathbf{t} \\ \hdashline \ddot{\mathbf{u}}_{1,0} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) \\ \mathbf{f}_{2,\gamma}(\mathbf{u}_2) \\ 0 \\ \hdashline \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right]$$

Interface blocks

are **not separated** from the

interior blocks

Step 3a. “Static condensation” of the interior variables

$$\left[\begin{array}{cc:c} A_1 & 0 & G_1^T \\ 0 & A_2 & -G_2^T \\ \hdashline G_1 & -G_2 & 0 \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{2,\gamma} \\ \hdashline \mathbf{t} \end{array}\right] = \left[\begin{array}{c} \widetilde{\mathbf{f}}_{1,\gamma}(\mathbf{u}_1) \\ \widetilde{\mathbf{f}}_{2,\gamma}(\mathbf{u}_2) \\ \hdashline 0 \end{array}\right] \longrightarrow$$

Step 3b. Form Schur complement and solve for **t**

$$\left(G_1 A_{1,\gamma}^{-1} G_1^T + G_2 A_{2,\gamma}^{-1}\right) \mathbf{t} = \left(G_1 A_{1,\gamma}^{-1} \widehat{\mathbf{f}}_{1,\gamma}(\mathbf{u}_1) - G_1 A_{2,\gamma}^{-1} \widehat{\mathbf{f}}_{2,\gamma}(\mathbf{u}_2)\right)$$

Modified mass and force terms

$$A_i = M_{i,\gamma} - M_{i,\gamma 0} M_{i,0}^{-1} M_{i,0\gamma} \qquad \widehat{\mathbf{f}}_{i,\gamma}(\boldsymbol{\varphi}_i) = \mathbf{f}_{i,\gamma}(\boldsymbol{\varphi}_i) - M_{i,\gamma 0} M_{i,0}^{-1} \mathbf{f}_{i,0}(\boldsymbol{\varphi}_i)$$

IVR formulation for the Solid-Solid Interaction Problem

Underlying ODE systems:

Lumped mass matrix

$$\left[\begin{array}{cc:cc} M_{1,\gamma} & 0 & 0 & 0 \\ 0 & M_{1,0} & 0 & 0 \\ \hdashline 0 & 0 & M_{2,\gamma} & 0 \\ 0 & 0 & 0 & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{1,0} \\ \hdashline \ddot{\mathbf{u}}_{2,\gamma} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) - G_1^T \mathbf{t}(\mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \hdashline \mathbf{f}_{2,\gamma}(\mathbf{u}_2) + G_2^T \mathbf{t}(\mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right] \quad \begin{array}{l} \longleftarrow \quad \Omega_1 \\ \\ \longleftarrow \quad \Omega_2 \end{array}$$

Consistent mass matrix

$$\left[\begin{array}{cc:cc} M_{1,\gamma} & M_{1,\gamma 0} & 0 & 0 \\ M_{1,0\gamma} & M_{1,0} & 0 & 0 \\ \hdashline 0 & 0 & M_{2,\gamma} & M_{2,\gamma 0} \\ 0 & 0 & M_{2,0\gamma} & M_{2,0} \end{array}\right] \left[\begin{array}{c} \ddot{\mathbf{u}}_{1,\gamma} \\ \ddot{\mathbf{u}}_{1,0} \\ \hdashline \ddot{\mathbf{u}}_{2,\gamma} \\ \ddot{\mathbf{u}}_{2,0} \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_{1,\gamma}(\mathbf{u}_1) - G_1^T \mathbf{t}(\mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{f}_{1,0}(\mathbf{u}_1) \\ \hdashline \mathbf{f}_{2,\gamma}(\mathbf{u}_2) + G_2^T \mathbf{t}(\mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{f}_{2,0}(\mathbf{u}_2) \end{array}\right] \quad \begin{array}{l} \longleftarrow \quad \Omega_1 \\ \\ \longleftarrow \quad \Omega_2 \end{array}$$

IVR formulation for the Solid-Solid Interaction Problem

Step 4. Explicit time discretization of the underlying ODE: $\ddot{D}^{n+1}(\mathbf{u}_i) \approx \ddot{\boldsymbol{u}}(t^{n+1})$

We obtain two independent sets of fully discrete equations on each subdomain:

$$\begin{bmatrix} M_{i,\gamma} & 0 \\ 0 & M_{i,0} \end{bmatrix} \begin{bmatrix} \ddot{D}^{n+1}\mathbf{u}_{i,\gamma} \\ \ddot{D}^{n+1}\mathbf{u}_{i,0} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^n_{i,\gamma} + (-1)^i G_i^T \mathbf{t}^n \\ \mathbf{f}_{i,0} \end{bmatrix}$$

Lumped mass matrix case

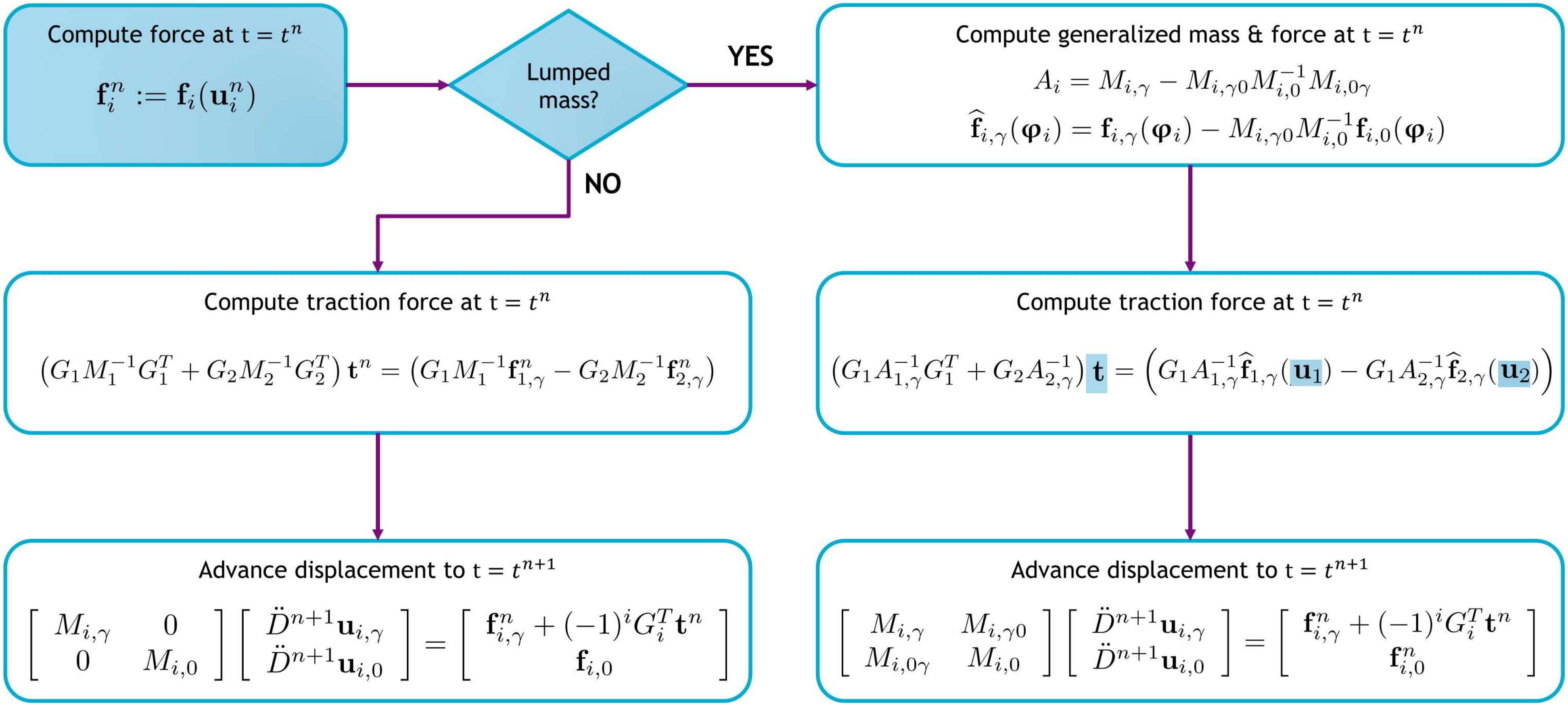
$$\begin{bmatrix} M_{i,\gamma} & M_{i,\gamma 0} \\ M_{i,0\gamma} & M_{i,0} \end{bmatrix} \begin{bmatrix} \ddot{D}^{n+1}\mathbf{u}_{i,\gamma} \\ \ddot{D}^{n+1}\mathbf{u}_{i,0} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^n_{i,\gamma} + (-1)^i G_i^T \mathbf{t}^n \\ \mathbf{f}^n_{i,0} \end{bmatrix}$$

Consistent mass matrix case

- Time discretization **both discretizes** the system in time and **partitions** the subdomain equations
- As long as time step is within the stability region of the time integrator, the partitioned scheme is stable
- Not subject to splitting errors characteristic of iterative partitioned methods
- The **only error** incurred is the **time discretization** error

Here we use the second central difference $\ddot{D}^{n+1}(\mathbf{u}_i) = (\mathbf{u}_i^{n+1} - 2\mathbf{u}_i^n + \mathbf{u}_i^{n-1})/\Delta t^2$

IVR algorithm at a glance



Compute force at t = t^n

$$\mathbf{f}_i^n := \mathbf{f}_i(\mathbf{u}_i^n)$$

Lumped mass?

YES

Compute generalized mass & force at t = t^n

$$A_i = M_{i,\gamma} - M_{i,\gamma 0} M_{i,0}^{-1} M_{i,0\gamma}$$

$$\widehat{\mathbf{f}}_{i,\gamma}(\boldsymbol{\varphi}_i) = \mathbf{f}_{i,\gamma}(\boldsymbol{\varphi}_i) - M_{i,\gamma 0} M_{i,0}^{-1} \mathbf{f}_{i,0}(\boldsymbol{\varphi}_i)$$

NO

Compute traction force at t = t^n

$$(G_1 M_1^{-1} G_1^T + G_2 M_2^{-1} G_2^T)\, \mathbf{t}^n = (G_1 M_1^{-1} \mathbf{f}_{1,\gamma}^n - G_2 M_2^{-1} \mathbf{f}_{2,\gamma}^n)$$

Compute traction force at t = t^n

$$(G_1 A_{1,\gamma}^{-1} G_1^T + G_2 A_{2,\gamma}^{-1})\, \mathbf{t} = \Big(G_1 A_{1,\gamma}^{-1} \widehat{\mathbf{f}}_{1,\gamma}(\mathbf{u}_1) - G_1 A_{2,\gamma}^{-1} \widehat{\mathbf{f}}_{2,\gamma}(\mathbf{u}_2)\Big)$$

Advance displacement to t = t^{n+1}

$$\begin{bmatrix} M_{i,\gamma} & 0 \\ 0 & M_{i,0} \end{bmatrix} \begin{bmatrix} \ddot{D}^{n+1} \mathbf{u}_{i,\gamma} \\ \ddot{D}^{n+1} \mathbf{u}_{i,0} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{i,\gamma}^n + (-1)^i G_i^T \mathbf{t}^n \\ \mathbf{f}_{i,0} \end{bmatrix}$$

Advance displacement to t = t^{n+1}

$$\begin{bmatrix} M_{i,\gamma} & M_{i,\gamma 0} \\ M_{i,0\gamma} & M_{i,0} \end{bmatrix} \begin{bmatrix} \ddot{D}^{n+1} \mathbf{u}_{i,\gamma} \\ \ddot{D}^{n+1} \mathbf{u}_{i,0} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{i,\gamma}^n + (-1)^i G_i^T \mathbf{t}^n \\ \mathbf{f}_{i,0}^n \end{bmatrix}$$

Analysis

Hessenberg Index-1 DAE requires a non-singular Jacobian $\partial_z g$. For IVR we have that

$\Longrightarrow$ $\partial_z g = S$ where $S = G_1 M_1^{-1} G_1^T + G_2 M_2^{-1} G_2^T$

$\Longrightarrow$

well-posedness of IVR requires a non-singular Schur complement S.

The monolithic equation is of the mixed type with bilinear forms given by

$$a(\boldsymbol{u}_1^h, \boldsymbol{u}_2^h; \boldsymbol{v}_1^h, \boldsymbol{v}_2^h) = (\boldsymbol{u}_1^h, \boldsymbol{v}_1^h)_{0,\Omega_1} + (\boldsymbol{u}_2^h, \boldsymbol{v}_2^h)_{0,\Omega_2} \quad \text{and} \quad b(\boldsymbol{v}_1^h, \boldsymbol{v}_2^h; \boldsymbol{t}^h) = \left(\boldsymbol{v}_1^h - \boldsymbol{v}_2^h, \boldsymbol{t}^h\right)_{0,\gamma}$$

$\Longrightarrow$ To show that the Schur complement S is non-singular we will use the Brezzi saddle-point theory:

1. $a(\cdot,\cdot)$ **is coercive on** $Z = \{(\boldsymbol{v}_1, \boldsymbol{v}_2) | b(\boldsymbol{v}_1, \boldsymbol{v}_2, \mathbf{t}) = 0\ \forall \mathbf{t}\}$.

Trivially satisfied because

$$a(\boldsymbol{v}_1^h, \boldsymbol{v}_2^h; \boldsymbol{v}_1^h, \boldsymbol{v}_2^h) = \|\boldsymbol{v}_1^h\|_{0,\Omega_1}^2 + \|\boldsymbol{v}_2^h\|_{0,\Omega_2}^2 = \left|\left|\left|\boldsymbol{v}_1^h; \boldsymbol{v}_2^h\right|\right|\right|^2$$

is a norm on $S_{1,\Gamma}^h \times S_{2,\Gamma}^h$

2. $b(\cdot,\cdot)$ **satisfies the inf-sup condition**

$$\sup_{\{\boldsymbol{v}_1^h; \boldsymbol{v}_2^h\} \in X \times X} \frac{b(\boldsymbol{v}_1^h, \boldsymbol{v}_2^h; \boldsymbol{s}^h)}{\left|\left|\left|\boldsymbol{v}_1^h; \boldsymbol{v}_2^h\right|\right|\right|} \geq \beta \|\boldsymbol{s}^h\|_Y$$

Analysis

Lemma: Assume that the Lagrange multiplier space G^h_γ satisfies the following condition: there exists an operator $Q : G^h_\gamma \mapsto S^h_{1,\Gamma} \times S^h_{2,\Gamma}$ such that

$$\|\boldsymbol{s}^h\|_{0,\gamma} \leq C_1 \left(\boldsymbol{s}^h, (Q\boldsymbol{s}^h)_1 - (Q\boldsymbol{s}^h)_2\right)_{0,\gamma} \quad \forall \boldsymbol{s}^h \in G^h_\gamma, \text{ and } |||Q(\boldsymbol{s}^h)||| \leq C_2 h^\alpha_\gamma \|\boldsymbol{s}\|_{0,\gamma}, \quad \alpha \geq 0$$

Then $b(\cdot,\cdot)$ satisfies the inf-sup condition.

Corollary: Assume that G^h_γ satisfies the above conditions. Then the matrix of the monolithic problem is uniformly bounded and its inverse is uniformly bounded in the mesh size. Furthermore, the block matrix $G^T = (G^T_1, G^T_2)$ has full column rank.

Choosing the Lagrange Multiplier space

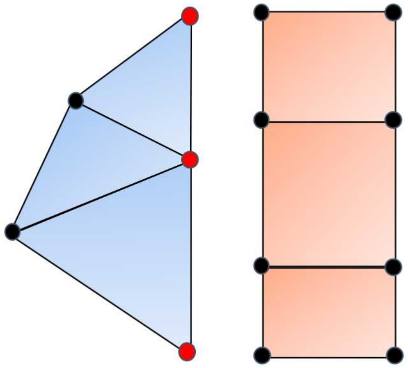
- We follow the mortar element approach and choose the Lagrange multiplier space to be one of the FE interface trace spaces:
  - IVR(1): LM using the **coarser** interface mesh
  - IVR(2): LM using the **finer** interface mesh
- With this choice an operator Q satisfying our conditions always exists!

  ⇒ Results in a monolithic formulation that satisfies the inf-sup condition
- Moreover, for this choice the condition number of *S* is bounded by a constant!

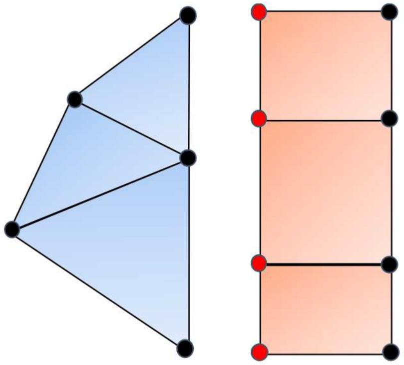
Lemma. Assume that $h_1 \leq h_2$ and let $\rho = h_2/h_1 > 1.$

1. If the Lagrange multiplier space $G^h_\gamma = G^h_1$ then $\kappa(S) \leq C\rho^{d-1}$
2. If the Lagrange multiplier space $G^h_\gamma = G^h_2$ then $\kappa(S) \leq C\rho^{d}$

IVR(1)



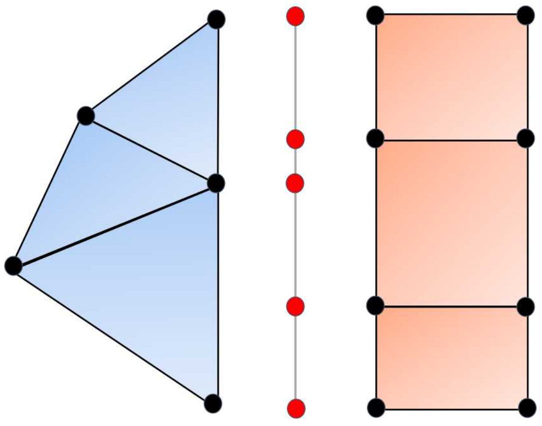
IVR(2)



An alternative choice of the Lagrange multiplier space

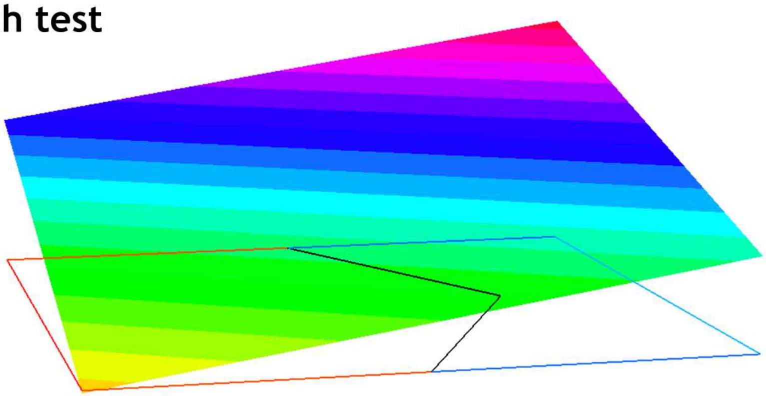
- The mortar element choice for the Lagrange multiplier is not linearly consistent
- Still converges optimally but does not recover linear fields
- As an alternative, we considered a Lagrange multiplier space defined on a common refinement grid on the interface.
- No proof of inf-sup stability for this space, but numerical results indicate that it preserves linear fields.
- In fact, easy to construct examples where inf-sup blows up – simply take two meshes with two of the nodes converging together
- Practical challenges include floating point errors and maintaining mesh quality to avoid inf-sup blow up.

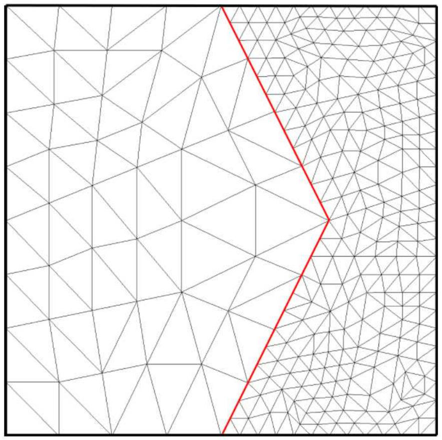
IVR(12)



Numerical examples: Solid-Solid Interaction (SSI)

Patch test





Linear Elasticity

$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i\boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

$$\lambda_i = 400 \quad \text{and} \quad \mu_i = 400$$

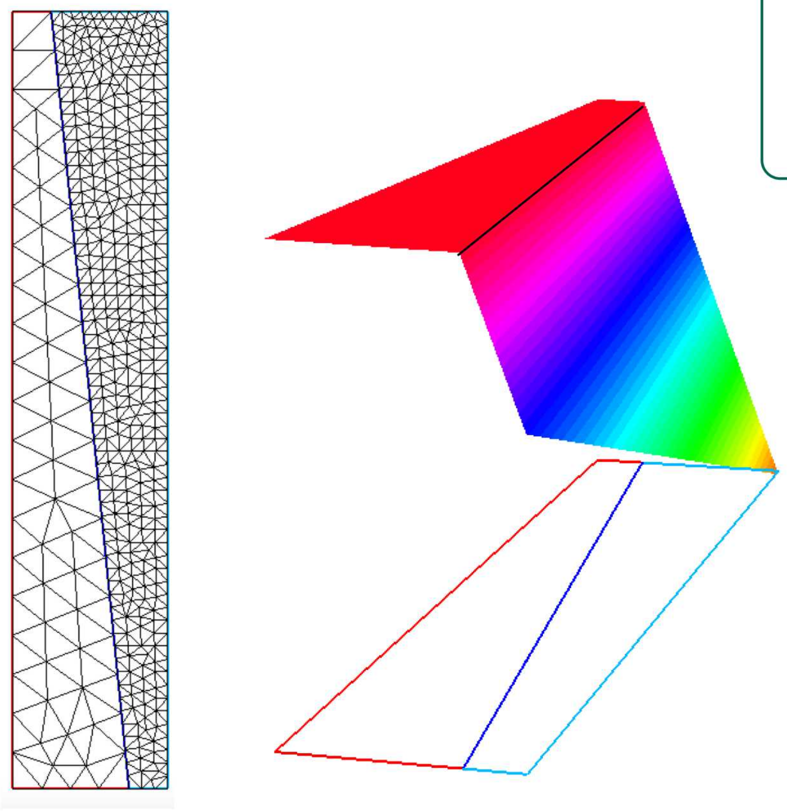
Manufactured Solution

$$\boldsymbol{u}(\boldsymbol{x}) = (3x + 5y, 8x - 4.3y)$$

Error Norm	IVR(L1)	IVR(L2)	IVR(L12)
$L^2(0,T;L^2(\Omega))$	5.166e-04	1.468e-06	2.223e-15
$L^2(0,T;H^1(\Omega))$	1.683e-02	2.679e-05	3.412e-14
LM mesh:	coarser	finer	common

Numerical examples: Solid-Solid Interaction (SSI)

Discontinuous patch test



Manufactured Solution

$$\boldsymbol{u}_1(\boldsymbol{x}) = \left(\frac{-0.9 + x + 0.1y}{0.15}, \frac{18 - 20x - 2y}{0.15} \right)$$

$$\boldsymbol{u}_2(\boldsymbol{x}) = \left(100 \left(\frac{-0.9 + x + 0.1y}{0.15} \right) - 99), 100 \left(\frac{18 - 20x - 2y}{0.15} \right) + 1980 \right)$$

$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i \boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

$$\lambda_1 = 40 \quad \text{and} \quad \mu_1 = 40$$

$$\lambda_2 = 0.4 \quad \text{and} \quad \mu_2 = 0.4$$

Error Norm	IVR(L1)	IVR(L2)	IVR(L12)
$L^2(0,T;L^2(\Omega))$	1.093e-04	5.418e-07	4.832e-13
$L^2(0,T;H^1(\Omega))$	4.591e-02	1.083e-04	6.658e-11
LM mesh:	coarser	finer	common

Numerical examples: Solid-Solid Interaction (SSI)

Convergence

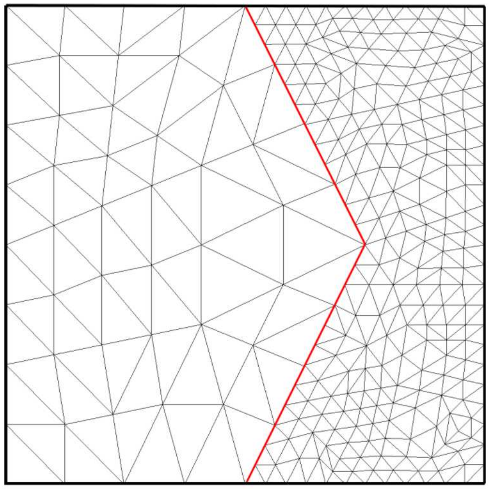
$$\ddot{\boldsymbol{u}}_i - \nabla \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \boldsymbol{f}_i$$

$$\boldsymbol{\sigma}_i(\boldsymbol{u}_i) = \lambda_i(\nabla \cdot \boldsymbol{u}_i)I + 2\mu_i\boldsymbol{\varepsilon}_i(\boldsymbol{u}_i)$$

$$\lambda = 0.864198 \quad \text{and} \quad \mu = 0.37037$$

Manufactured Solution

$$\boldsymbol{u}(\boldsymbol{x}, t) = (3\sin(x)\sin(y)\cos(t), \sin(x)\sin(y)t)$$



$L^2(\Omega)$ Error

$h_{min}(\Omega_1)$	$h_{min}(\Omega_2)$	Δt	IVR(1)	IVR(2)	IVR(12)
0.378545	0.113981	0.00371833	0.0146414	0.0146403	0.0146404
0.220723	0.0672413	0.00185917	0.00353829	0.00349268	0.00349301
0.107240	0.0359195	0.00101409	0.00095948	0.000854641	0.000854613
0.0514682	0.0196624	0.00053119	0.00033852	0.000217698	0.000217665
0.0277461	0.00957506	0.00024789	0.000141964	5.53096e-05	5.52471e-05
Rate			1.73	2.08	2.08

LM mesh: coarser finer common

$H^1(\Omega)$ Error

$h_{min}(\Omega_1)$	$h_{min}(\Omega_2)$	Δt	IVR(1)	IVR(2)	IVR(12)
0.378545	0.113981	0.00371833	0.341327	0.340643	0.340643
0.220723	0.0672413	0.00185917	0.16736	0.16385	0.163848
0.107240	0.0359195	0.00101409	0.094672	0.081204	0.0812045
0.0514682	0.0196624	0.00053119	0.0701869	0.0404745	0.0404726
0.0277461	0.00957506	0.00024789	0.0576898	0.0204939	0.0204888
Rate			0.657	1.05	1.05

Numerical examples: Transmission Problem (TP)

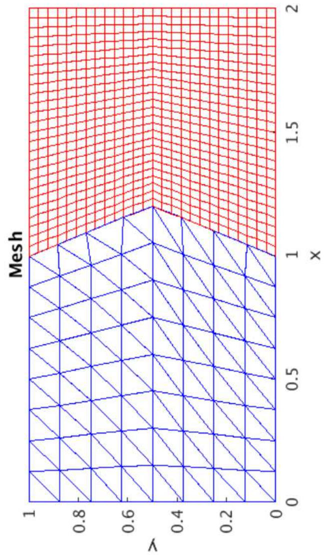
Patch test

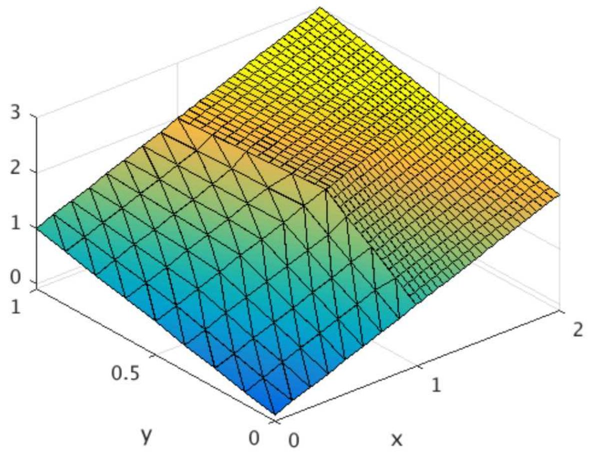
Advection Diffusion

$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

Manufactured Solution

$$\varphi_i(\mathbf{x}, t) = x + y$$





Pure Diffusion $\epsilon = 0.1$ $\mathbf{u} = 0$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	9.745e-04	1.062e-06	1.384e-13
$H^1(\Omega)$	4.089e-02	2.155e-05	3.106e-12

Moderate Advection $\epsilon = 0.1$ $\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	8.417e-03	8.477e-06	2.229e-13
$H^1(\Omega)$	3.540e-01	1.523e-04	3.176e-12

Strong Advection $\epsilon = 0.0001$ $\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	2.000e-01	3.175e-05	2.227e-13
$H^1(\Omega)$	1.136e+01	7.719e-04	4.341e-12
LM mesh:	coarser	finer	common

Numerical examples: Transmission Problem (TP)

Discontinuous patch test

Advection Diffusion

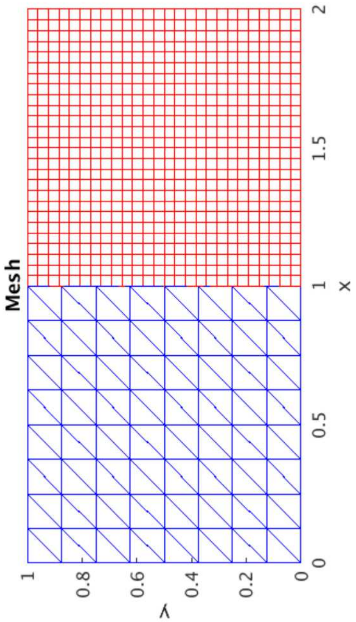
$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

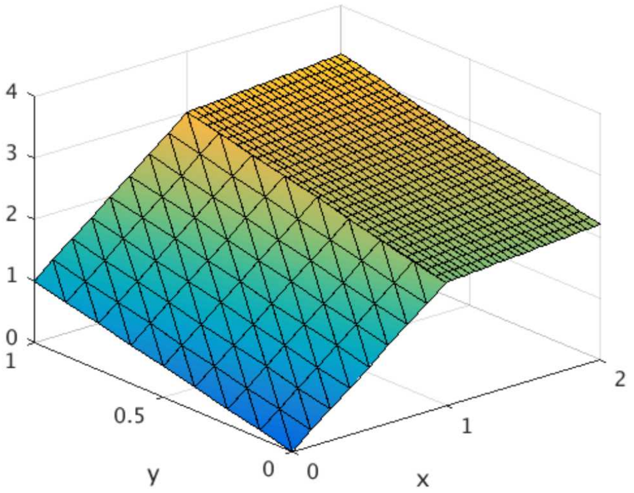
$$\epsilon_1 = 0.01 \qquad \epsilon_2 = 0.1$$

Manufactured Solution

$$\varphi_1(\mathbf{x}, t) = 2x + y$$

$$\varphi_2(\mathbf{x}, t) = 0.2x + y + 1.8$$





Pure Diffusion $\mathbf{u}_1 = \mathbf{u}_2 = 0$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	1.899e-04	3.963e-07	4.365e-14
$H^1(\Omega)$	7.510e-03	8.674e-06	1.920e-12

Moderate Advection $\mathbf{u}_1 = \mathbf{u}_2 = (-\sin(\pi/6), \cos(\pi/6))$

Error Norm	IVR(1)	IVR(2)	IVR(12)
$L^2(\Omega)$	1.269e-02	2.003e-05	1.700e-13
$H^1(\Omega)$	5.098e-01	3.573e-04	5.149e-12

LM mesh: coarser finer common

TRANSMISSION PROBLEM CONVERGENCE

Advection Diffusion
strong advection regime

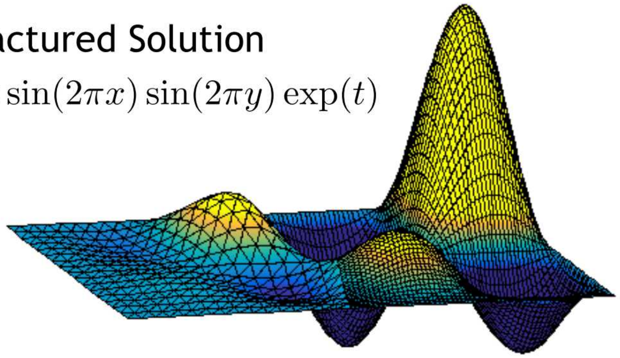
$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

$$\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$$

$$\epsilon = 0.0001$$

Manufactured Solution

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x) \sin(2\pi y) \exp(t)$$



$L^2(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	1.042e-02	1.121e-01	9.114e-02	9.114e-02
0.125	0.0357	5.102e-03	3.482e-02	3.426e-02	3.783e-02
0.0625	0.0179	2.551e-03	8.282e-03	8.279e-03	8.747e-03
0.03125	0.00893	1.272e-03	1.620e-03	1.613e-03	1.663e-03
Rate	-	-	2.04	1.95	1.94

$H^1(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	1.042e-02	3.215	2.321	2.321
0.125	0.0357	5.102e-03	1.412	1.297	1.448
0.0625	0.0179	2.551e-03	0.6381	0.6261	0.6502
0.03125	0.00893	1.272e-03	0.3012	0.2985	0.3020
Rate	-	-	1.14	0.993	0.998

TRANSMISSION PROBLEM CONVERGENCE

Advection Diffusion
moderate advection regime

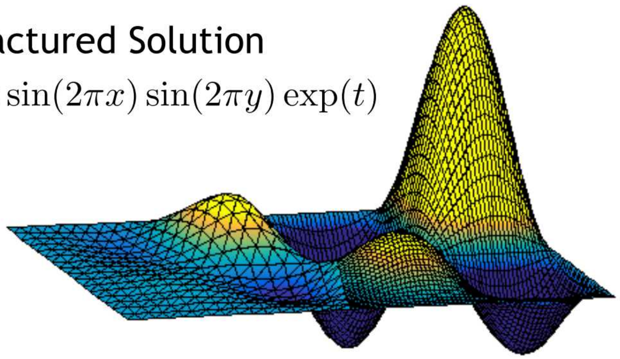
$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i - \mathbf{u}\varphi_i) = f_i$$

$$\mathbf{u} = (-\sin(\pi/6), \cos(\pi/6))$$

$$\epsilon = 0.1$$

Manufactured Solution

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x) \sin(2\pi y) \exp(t)$$



$L^2(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	9.021e-02	9.043e-02	9.043e-02
0.125	0.0357	8.681e-04	2.876e-02	2.891e-02	3.367e-02
0.0625	0.0179	2.161e-04	8.157e-03	8.181e-03	8.655e-03
0.03125	0.00893	5.391e-05	2.133e-03	2.136e-03	2.186e-03
Rate	-	-	1.80	1.80	1.81

$H^1(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	2.080	2.072	2.072
0.125	0.0357	8.681e-04	1.096	1.094	1.124
0.0625	0.0179	2.161e-04	0.5602	0.5594	0.5652
0.03125	0.00893	5.391e-05	0.2821	0.2817	0.2831
Rate	-	-	0.962	0.960	0.961

TRANSMISSION PROBLEM CONVERGENCE

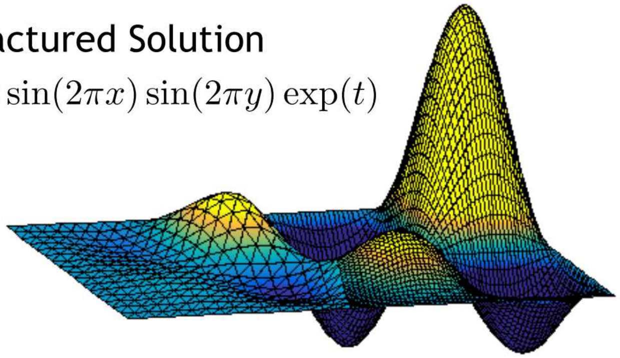
Pure Diffusion

$$\dot{\varphi}_i - \nabla \cdot (\epsilon \nabla \varphi_i) = f_i$$

$$\epsilon = 0.1$$

Manufactured Solution

$$\varphi_k(\mathbf{x}, t) = x^2 y \sin(2\pi x) \sin(2\pi y) \exp(t)$$



$L^2(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	9.573e-02	9.630e-02	9.630e-02
0.125	0.0357	8.681e-04	3.049e-02	3.061e-02	3.531e-02
0.0625	0.0179	2.161e-04	8.636e-03	8.660e-03	9.180e-03
0.03125	0.00893	5.391e-05	2.257e-03	2.261e-03	2.320e-03
Rate	-	-	1.80	1.81	1.81

$H^1(\Omega)$ Error

h(Ω_1)	h(Ω_2)	Δt	IVR(1)	IVR(2)	IVR(12)
0.25	0.0714	3.497e-03	2.049	2.049	2.049
0.125	0.0357	8.681e-04	1.092	1.091	1.116
0.0625	0.0179	2.161e-04	0.5594	0.5587	0.5643
0.03125	0.00893	5.391e-05	0.2820	0.2816	0.2830
Rate	-	-	0.955	0.956	0.955

Conclusions

Partitioned Implicit Value Recovery (IVR) Scheme

- Uses a well-posed monolithic mixed formulation to estimate boundary data
- Key idea is to consider alternative constraint, which enables explicit treatment of Lagrange multiplier
- Results in non-iterative partitioned method
- Stability and accuracy derive from the stability and accuracy of the mixed method
- Traces of subdomain finite element spaces on the interface are stable choices for the Lagrange multiplier

Next steps

- Investigate alternative coupling conditions relevant to earth system models
- See next talk