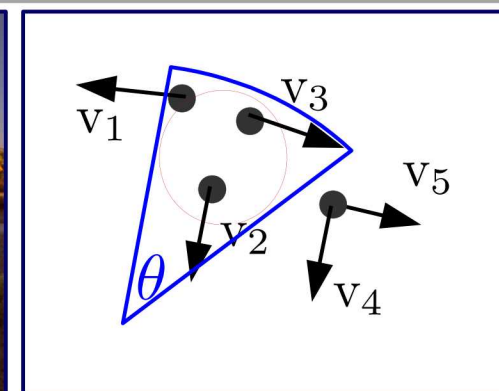
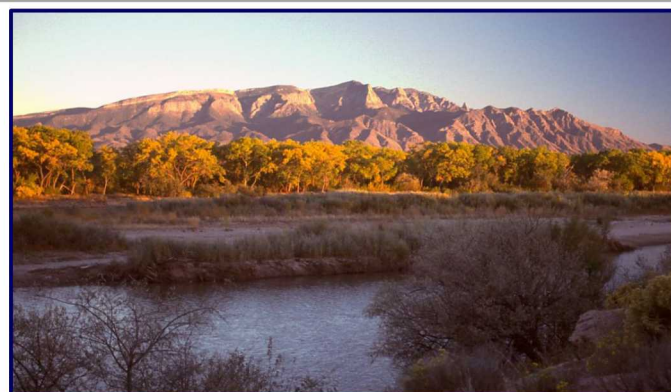
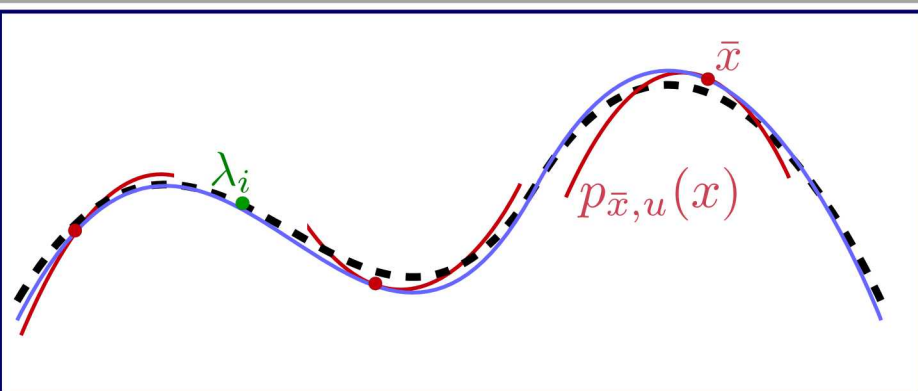


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Advances in the Approximation Theory for Generalized Moving Least Squares

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Generalized Moving Least Squares

Ingredients:

V, V^* - a *function space* (e.g. **continuous functions**) and its dual

$P = \text{span}\{p_i\}_{i=1}^Q \subset V$ - a finite dimensional *approximation space*, e.g. **polynomials**
FE analog: *local approximation space within an element*

$\Lambda^h := \{\lambda_1^h, \dots, \lambda_{N_h}^h\} \subset \Lambda \subset V^*$ - a finite set of *sampling functionals* (e.g. point evaluations)
FE analog: *degrees of freedom*

$\tau \in \mathcal{T} \subset V^*$ - a *target functional* (or a family of target functionals)

$W(\tau, \lambda_i) : (\mathcal{T} \cup \Lambda) \times (\mathcal{T} \cup \Lambda) \rightarrow \mathbb{R}$ - a *window function* correlating functionals (e.g. a radial kernel) determines the **smoothness** of reconstruction
FE analog: **1** if target and sampling functionals belong to the same element, **0** otherwise.

Example, **MLS** case:

Point cloud $X^h = \{\mathbf{x}_i^h\}_{i=1}^{N_h} \subset \Omega$, with filling distance $h = \sup_{\mathbf{x}^h \in \Omega} \min_{i=1, \dots, N_h} |\mathbf{x} - \mathbf{x}_i^h|$.

$u \in V = C^{k+1}(\Omega)$, $P = \Pi^k(\Omega)$, $\lambda_i^h(u) = u(\mathbf{x}_i^h)$, $\tau_{\mathbf{x}} = u(\mathbf{x})$, $W = W(|\mathbf{x} - \mathbf{x}_i^h|)$

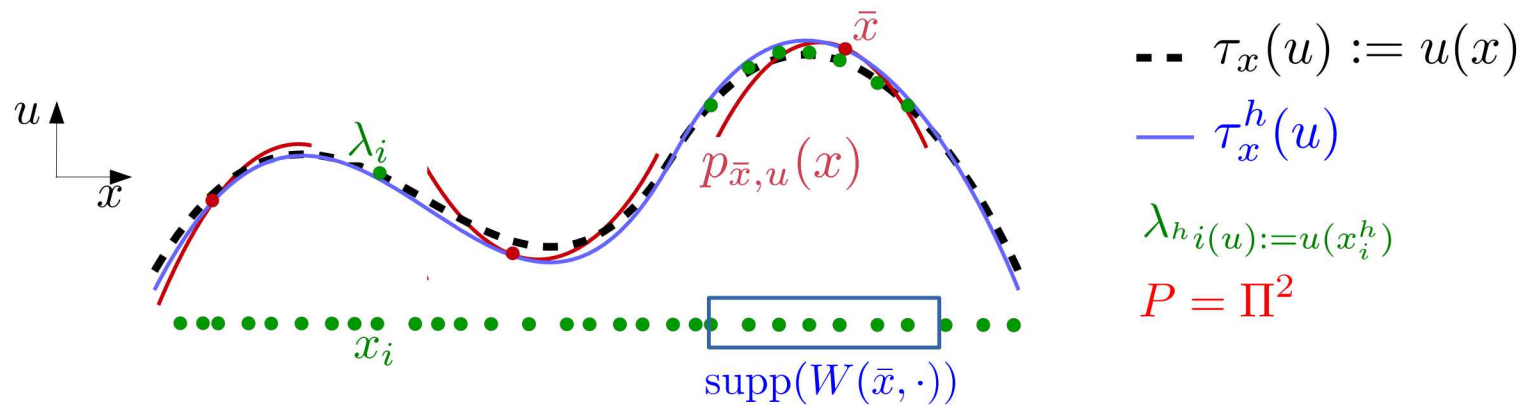
Generalized Moving Least Squares

Recipes (2 equivalent formulations):

1. Least Square formulation:

$$\tau_{\bar{\mathbf{x}}}^h(u) := \tau_{\bar{\mathbf{x}}}(p_{\bar{\mathbf{x}},u}), \quad p_{\bar{\mathbf{x}},u} = \arg \min_{p \in P} \sum_{i \in I_{\bar{\mathbf{x}}}} (\lambda_i^h(u) - \lambda_i^h(p))^2 W(\bar{\mathbf{x}}, \mathbf{x}_i^h)$$

$I_{\bar{\mathbf{x}}} := \{i : W(\bar{\mathbf{x}}, \mathbf{x}_i^h) > 0\}$



2. Constrained Optimization formulation:

$$\tau_{\bar{\mathbf{x}}}^h(u) := \sum_{i \in I_{\bar{\mathbf{x}}}} a_{\tau_{\bar{\mathbf{x}}}}^i \lambda_i^h(u), \quad \{a_{\tau_{\bar{\mathbf{x}}}}^i\} = \arg \min_{a^i} \sum_{i \in I_{\bar{\mathbf{x}}}} \frac{|a^i|^2}{W(\bar{\mathbf{x}}, \mathbf{x}_i^h)},$$

s. t. $\tau_{\bar{\mathbf{x}}}(p) = \sum_{i \in I_{\bar{\mathbf{x}}}} a^i \lambda_i^h(p), \quad \forall p \in P \quad (\tau_{\bar{\mathbf{x}}}^h(p) = \tau_{\bar{\mathbf{x}}}(p))$

Properties of Generalized Moving Least Squares

Main questions: **When is the reconstruction possible? How good is the reconstruction?**

- **Unisolvency:**

In order for $\tau_{\mathbf{x}}^h$ to exist, $\Lambda_{\mathbf{x}} := \{\lambda_i\}_{i \in I_{\mathbf{x}}}$ must be P -unisolvent, i.e. for $p \in P$

$$\forall \lambda_i \in \Lambda_{\mathbf{x}}, \lambda_i(p) = 0 \iff p = 0.$$

FE DoFs are
unisolvent by
construction

- **Approximation property:**

$$|\tau_{\mathbf{x}}^h(u) - \tau_{\mathbf{x}}(u)| \sim O(h^s), \text{ for some } s.$$

➤ Sufficient results for unisolvancy and approx. error have been provided by Wendland* for (non generalized) MLS.

➤ Results has been generalized by Mirzaei**, for point derivatives as target functionals:

$$|D_{\mathbf{x}}^{\alpha, h}(u) - D_{\mathbf{x}}^{\alpha}(u)| \sim O(h^{(k+1-|\alpha|)}), \text{ where } k \text{ is the degree of the polynomials.}$$

and for weak derivatives in Sobolev spaces***.

* H. Wendland, Scattered data approximation, Vol. 17. Cambridge university press, 2004.

** D. Mirzaei et Al., IMA J. Num. Analysis, 2011.

*** D. Mirzaei, Comp. And App. Math. 2016

Generalized Moving Least Squares: our plan

Our plan is to further extend analysis of generalized moving least square, and present **general approximation results**.

- **Sampling/target functionals:** we want to consider differential forms and in particular volume integrals, fluxes over (virtual) faces and line integral over (virtual) edges. Potentially useful for *PDE discretizations and data transfer*.
- **Approximation space:** we want to consider vector polynomials and subspaces of complete polynomial spaces (e.g. div-free, curl-free spaces).
- **Go beyond smooth functions** and consider functions in Sobolev spaces (expanding on what done in Mirzaei ***)

In the following we present

- 1) **existence results** for different sampling functionals
- 2) **approximation results** for different sampling/target functionals
- 3) **generalization** of definition of “local polynomial reproduction”

*** D. Mirzaei, Comp. And App. Math. 2016

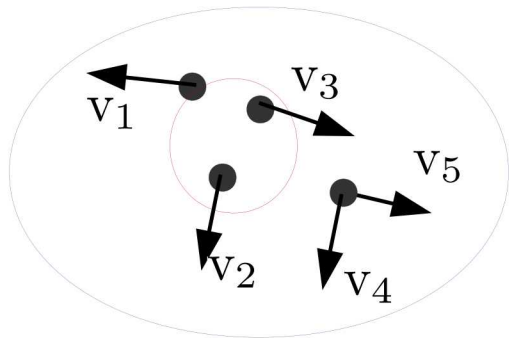
Reconstructing **vector** functions (from projections along given directions)

Associate to each particle i a position \mathbf{x}_i and a unit vector $\mathbf{v}_i \in \mathbb{R}^d$.

Sampling functional:

$$\lambda_i(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{v}_i$$

Filling distance h_ω is the radius of the smallest ball that centered at any point of Ω contains d particles whose versors $\mathbf{v}_1, \dots, \mathbf{v}_d$ contain a basis for \mathbb{R}^d with associated determinant bigger than ω ($|\det[\mathbf{v}_1, \dots, \mathbf{v}_d]| \geq \omega$).



For $h_\omega \leq C(\theta, R, m, \omega)$, $\forall p \in [\Pi^m]^d$,
 $\exists \lambda_{i_p} \in \Lambda_h$, such that $|\lambda_{i_p}(p)| \geq \rho_\omega \|p\|_{\Omega, \infty}$

Other sampling functionals we considered:

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{t}_i \quad \lambda_i^f(\mathbf{u}) = \frac{1}{|f_i|} \int_{f_i} \mathbf{u} \cdot \mathbf{n}_i \quad \lambda_i^v(u) := \frac{1}{|V_i|} \int_{V_i} u(\mathbf{y}) d\mathbf{y}$$

GMLS approximation results:

Basic technique:

$$\begin{aligned}
 |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}^h(u)| &\leq |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(p)| + |\tau_{\mathbf{x}}(p) - \tau_{\mathbf{x}}^h(u)|, \quad (\forall p \in P) \\
 &\leq |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(p)| + |\tau_{\mathbf{x}}^h(p - u)|, \quad \leftarrow \text{reconstruction property} \\
 &\leq |\tau_{\mathbf{x}}(u - p)| + \left| \sum_{i=1}^{N_p} \lambda_i(u - p) a_{\tau_{\mathbf{x}}}^i \right| \quad \leftarrow \text{GMLS definition} \\
 &\leq |\tau_{\mathbf{x}}(u - p)| + \max_{i \in I_{\mathbf{x}}} |\lambda_i(u - p)| \sum_{i \in I_{\mathbf{x}}} |a_{\tau_{\mathbf{x}}}^i|.
 \end{aligned}$$

$$\sum_{i \in I_{\mathbf{x}}} |a_{\tau_{\mathbf{x}}}^i| \leq C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\|$$

Holds for any target functional and approximation space:

$$|\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}^h(u)| \leq |\tau_{\mathbf{x}}(u - p)| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i(u - p)|, \quad p \in P$$

GMLS approximation results: **div free** spaces

$$V = \{\mathbf{u} \in [C^{k+1}(\Omega)]^d, \operatorname{div}(\mathbf{u}) = 0\}, \quad P = \{\mathbf{p} \in [\Pi^k(\Omega)]^d, \operatorname{div}(\mathbf{p}) = 0\}$$

$$\lambda_i^j(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{e}_j, \quad \tau_{\mathbf{x}}^j(\mathbf{u}) = \mathbf{u}(\mathbf{x}) \cdot \mathbf{e}_j, \quad \boldsymbol{\tau}_{\mathbf{x}} = (\tau_{\mathbf{x}}^1, \dots, \tau_{\mathbf{x}}^d)$$

The Taylor polynomial $\mathbf{p}_{\mathbf{x},u}^k$ of $\mathbf{u} \in V$, belongs to P (it's divergence free).

$$|\tau_{\mathbf{x}}^j(\mathbf{u}) - \tau_{\mathbf{x}}^{j,h}(\mathbf{u})| \leq |\tau_{\mathbf{x}}^j(\mathbf{u} - \mathbf{p}_{\mathbf{x},u}^k)| + C_W \|\tau_{\mathbf{x}}^j\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i^j(\mathbf{u} - \mathbf{p}_{\mathbf{x},u}^k)|$$

Taylor approx.
(| α | $\leq k$)
↓
0

Taylor approx.
↓
 $\mathcal{O}(h^{k+1})$

$$|\boldsymbol{\tau}_{\mathbf{x}}(\mathbf{u}) - \boldsymbol{\tau}_{\mathbf{x}}^h(\mathbf{u})| \sim \mathcal{O}(h^{k+1})$$

Note that, in general, $\operatorname{div}(\boldsymbol{\tau}_{\mathbf{x}}^h(\mathbf{u})) \neq 0$.

GMLS approximation results: integral target functionals (e.g. used for nonlocal problems)

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{t}_i, \quad V = C^{k+1}(\Omega), \quad P = [\Pi^k(\Omega)]^d$$

$$\tau_{\mathbf{x}}(u) = \text{div}(\mathbf{u})|_{\mathbf{x}} \qquad |\lambda_i^e(\mathbf{u})| \leq \|\mathbf{u}\|_{L^\infty(e_i)}$$

Take $p = p_{\mathbf{x},u}^k$, the Taylor polynomial of degree k of u at \mathbf{x} .

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau_{\mathbf{x}}^h(\mathbf{u})| \leq |\tau_{\mathbf{x}}(\mathbf{u} - p_{\mathbf{x},u}^k)| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i(\mathbf{u} - p_{\mathbf{x},u}^k)|$$

Taylor approx.
($k \geq 1$)

↓

⇓

Mirzaei**

↓

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau_{\mathbf{x}}^h(\mathbf{u})| \leq 0 + Ch^{-1} h^{k+1} \|\mathbf{u}\|_{\infty, \Omega}, \quad \text{if } |e_i| \sim h$$

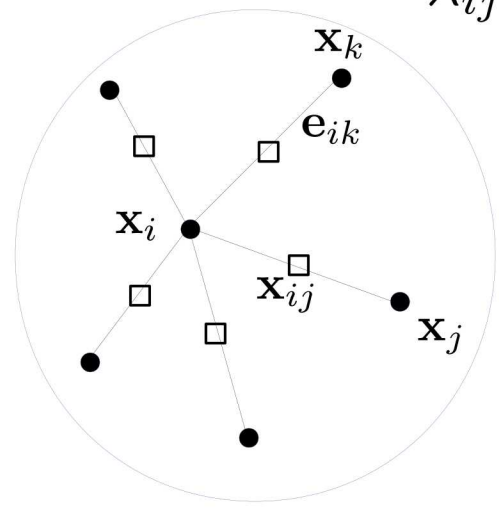
⇓

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau_{\mathbf{x}}^h(\mathbf{u})| \leq Ch^k \|\mathbf{u}\|_{\infty, \Omega}$$

** D. Mirzaei et Al., IMA J. Num. Analysis, 2011.

Application to PDE: staggered scheme for Darcy

$$\begin{cases} \operatorname{div}(\mathbf{u}) = \mathbf{0} \\ \mathbf{u} = -K \nabla \phi \end{cases}$$



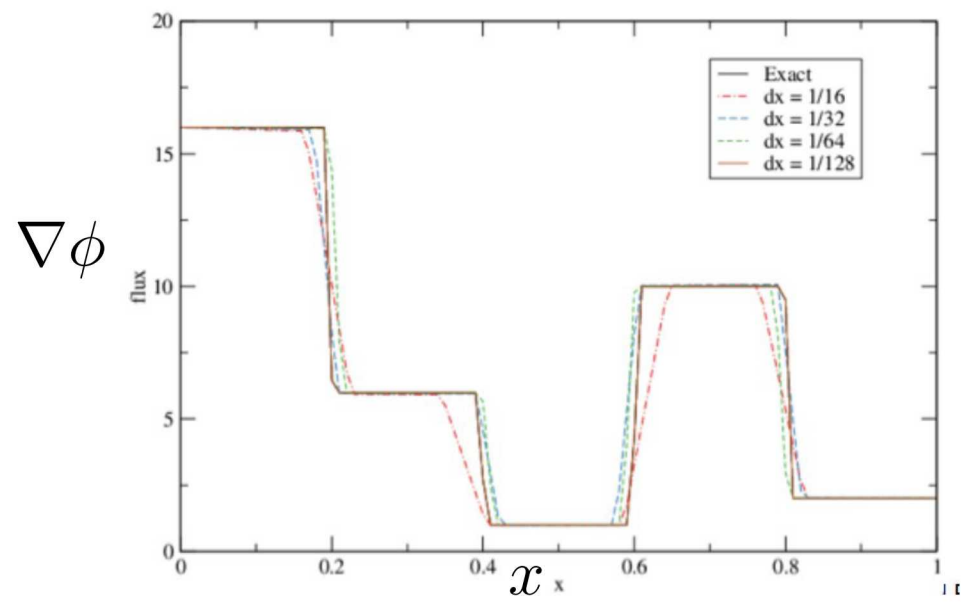
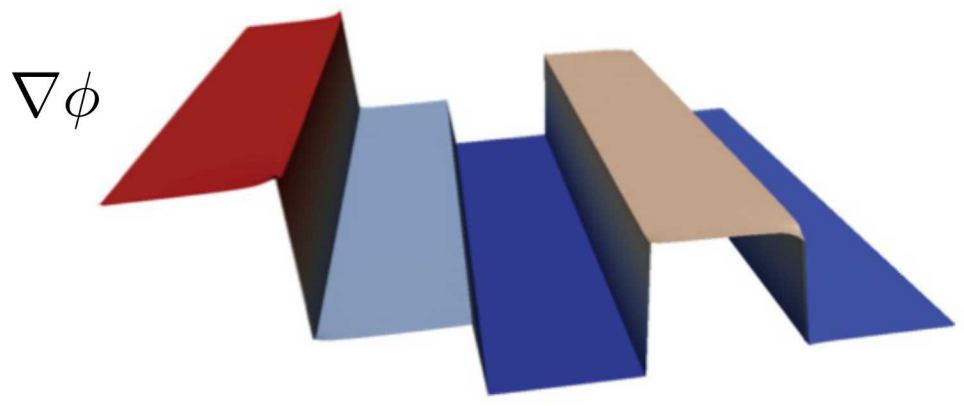
$$\lambda_{ij}(\mathbf{u}) := \int_{e_{ij}} \mathbf{u} \cdot \mathbf{t}_{ij} \approx K(\mathbf{x}_{ij})(\phi_j - \phi_i)$$

$$\tau_{\mathbf{x}_i}(\mathbf{u}) = \operatorname{div}(\mathbf{u})|_{\mathbf{x}_i}$$

$$P = [\Pi_m]^d$$

$$W(\tau_{\mathbf{x}}, \lambda_{ij}) = W(|\mathbf{x} - \mathbf{x}_{ij}|)$$

Solution of PDE w/ jumps in permeability



Result (**local polynomial reproduction**) [MLS]

If, for each $\mathbf{x} \in \Omega \subset \mathbb{R}^d$, $\Lambda^h \subset \Lambda \subset V^*$, exist $C_{\text{loc}} > 0$ and $C_a > 0$, s.t.

$$\left\{ \begin{array}{l} r_{\mathbf{x}}^h(p) := \sum_i a_{\mathbf{x}}^i p(\mathbf{x}_i) = p(\mathbf{x}), \forall p \in \Pi^m \quad - \text{consistency} \\ \sum_i |a_{\mathbf{x}}^i| \leq C_a \quad - \text{uniform boundness} \\ |\mathbf{x} - \mathbf{x}_i| > C_{\text{loc}} h \Rightarrow a_{\mathbf{x}}^i = 0 \quad - \text{local support} \end{array} \right.$$

$$\implies |r_{\mathbf{x}}^h(u) - u(\mathbf{x})| \leq C_{u,m} h^{m+1}$$

Result (**generalized local reproduction**) [GMLS]

If, for each $\tau \in \mathcal{T} \subset V^*$, $\Lambda^h \subset \Lambda \subset V^*$ there exist $C_{\text{loc}}^h \rightarrow 0$, $C_{u,\mathcal{T}}^h \rightarrow 0$, $C_a^h C_{u,\Lambda}^h \rightarrow 0$, s.t.

$$\left\{ \begin{array}{l} \tau^h(p) := \sum_i a_{\tau}^i \lambda_i(p) = \tau(p), \forall p \in P \quad - \text{consistency} \\ \sum_i |a_{\tau}^i| \leq C_a^h \quad - \text{uniform boundness} \\ d(\tau, \lambda_i) > C_{\text{loc}}^h \Rightarrow a_{\tau}^i = 0, \quad - \text{local support} \\ \forall f \in V, \exists p \in P \text{ such that:} \\ \quad |\tau(u - p)| \leq C_{u,\mathcal{T}}^h \text{ and} \quad - \text{approximation} \\ \quad |\lambda_i(u - p)| \leq C_{u,\Lambda}^h, \forall \lambda_i : d(\tau, \lambda_i) \leq C_{\text{loc}}^h. \end{array} \right.$$

$$\implies |\tau^h(u) - \tau(u)| \leq C_{u,\mathcal{T}}^h + C_a^h C_{u,\Lambda}^h$$

Thank you!

$$\tau \in \mathcal{T} \subset V^*, \quad \Lambda^h = \{\lambda_i^h \in V^*\}_{i=1}^{N_h} \subset \Lambda \subset V^*$$

Pseudo-distance:

$$\mathcal{X} : (\mathcal{T} \cup \Lambda) \rightarrow \mathbb{R}^q \quad d^*(\tau, \lambda) := |\mathcal{X}(\tau) - \mathcal{X}(\lambda)|$$

Separation and filling distances:

$$h_s = \frac{1}{2} \min_{i \neq j} d^*(\lambda_i, \lambda_j), \quad h = \sup_{\tau \in \mathcal{T}} \min_i d^*(\tau, \lambda_i),$$

Quasi-uniformity:

$$h_s \leq h \leq C_{qu} h_s$$

Properties of window function $W : \mathcal{T} \times \Lambda \rightarrow \mathbb{R}$:

$$0 \leq W(\tau, \lambda) \leq M,$$

$$W(\tau, \lambda) = 0 \quad \text{if } d^*(\tau, \lambda) > C_{loc}^h$$

$$W(\tau, \lambda) \geq \rho > 0 \quad \text{if } d^*(\tau, \lambda) < \frac{1}{2} C_{loc}^h$$

Neighborhood:

$$I_\tau := \{i, | W(\tau, \lambda_i) > 0\}, \quad \bar{I}_\tau := \{i, | d^*(\tau, \lambda_i) < \frac{1}{2} C_h\}$$

Sample mapping

$$T_\tau : P \rightarrow \mathbb{R}^{\#\tilde{I}_\tau}, \quad T_\tau(v) := (\lambda_i(v))_{i \in \tilde{I}_\tau}$$

GMLS process defines a local reproduction if:

$$\forall \Lambda_h, h \leq h_0 :$$

$$\|\tau\|_{V^*} \leq C_U h^{\beta_1}, \quad \|\lambda_i\|_{V^*} \leq C_\Lambda h^{\beta_2}$$

$$\|\tilde{T}_\tau^{-1}\| \leq C_\Lambda h^{\beta_3}$$

$\forall f \in V, \exists p \in P$ such that:

$$\|f - p\|_{V \setminus \text{Ker}(\tau)} \leq C_f h^{\alpha_1} \text{ and}$$

$$\|f - p\|_{V \setminus \text{Ker}(\lambda_i)} \leq C_f h^{\alpha_2}, \quad \forall \lambda_i : d(\tau, \lambda_i) \leq C_h.$$