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Title: Grad-Shafranov equation for non-axisymmetric MHD equilibria

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Grad-Shafranov equation for non-axisymmetric MHD equilibria

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This talk will present a novel structural property of non-degenerate, smooth 3D MHD equilibria

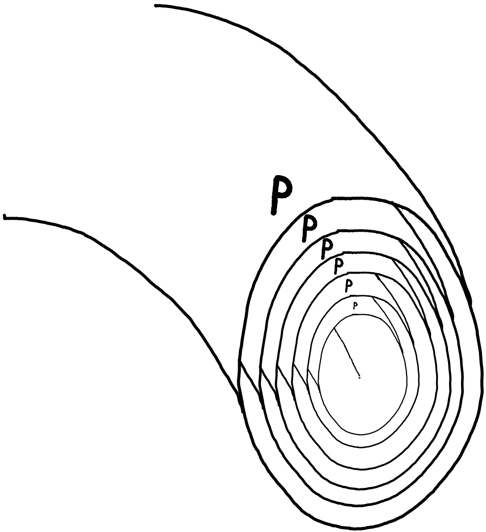
Definition (Non-degenerate equilibrium)

Let $Q \subset \mathbb{R}^3$ be compact region diffeomorphic to $D^2 \times S^1$. A *non-degenerate MHD equilibrium* is a pair $(\boldsymbol{B}, \mathrm{p})$, where $\boldsymbol{B}$ is a smooth vector field on Q that satisfies $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on ∂Q, p is a smooth function on Q,

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \nabla \mathrm{p}$$
$$\nabla \cdot \boldsymbol{B} = 0,$$

and $\nabla \mathrm{p} \neq 0$ except on a single magnetic axis $\ell_0 \subset Q$.

This talk will present a novel structural property of non-degenerate, smooth 3D MHD equilibria



Disclaimer

This talk will *not* establish existence of smooth non-degenerate 3D equilibria

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This talk will *not* establish existence of smooth non-degenerate 3D equilibria

But the results may help in such an endeavor

Outline

Circle actions:

Inputs

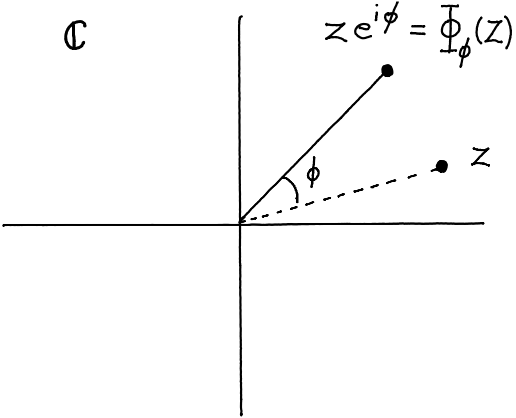
- A point $z \in Z$
- An angle $\phi \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$

Outputs

- A *rotated* point $\Phi_\phi(z) \in Z$

Circle actions are compact 1-parameter symmetries

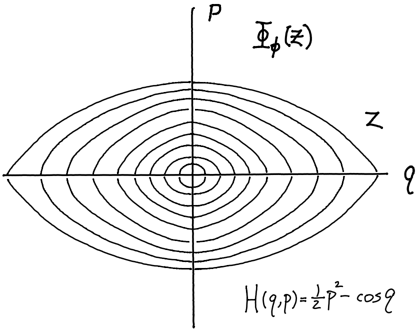
Example 1:



N.B. $\Phi_{\phi_1+\phi_2}(z) = \Phi_{\phi_1}(\Phi_{\phi_2}(z))$

Circle actions are compact 1-parameter symmetries

Example 2:



$$\Phi_\phi(z) = F_{\phi/\omega(z)}(z)$$

ω : non-linear pendulum frequency

F_t : pendulum time-advance map

Circle actions are compact 1-parameter symmetries

Definition (Circle action)

A *circle action* on a space Z is a 1-parameter family of transformations $\Phi_\phi : Z \to Z$ such that

- $\Phi_0 = \Phi_{2\pi} = \mathrm{id}_Z$ (periodicity)
- $\Phi_{\phi_1+\phi_2} = \Phi_{\phi_1} \circ \Phi_{\phi_2}$ (generalized rotation property)

for all $\phi_1, \phi_2 \in S^1$.

Outline

The metric tensor on $\mathbb{R}^3$ defines lengths and angles

The standard metric on $\mathbb{R}^3$

$$g = \delta_{ij} dx^i \, dx^j$$

Squared length of a vector $\boldsymbol{u}$

$$|\boldsymbol{u}|^2 = g(\boldsymbol{u}, \boldsymbol{u})$$

Length of a curve c

$$L(c) = \lim \sum_i \sqrt{|c(t_{i+1}) - c(t_i)|^2}$$

The metric tensor can be averaged using a circle action Φ_ϕ

The standard metric **averaged** using $\Phi_\phi = (x^1_\phi, x^2_\phi, x^3_\phi)$

$$\overline{g} = \frac{1}{2\pi}\int_0^{2\pi} \delta_{ij} dx^i_\phi \, dx^j_\phi \, d\phi$$

Squared length of a vector $\boldsymbol{u}$

$$\begin{aligned} ||\boldsymbol{u}||^2 &= \overline{g}(\boldsymbol{u}, \boldsymbol{u}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(\boldsymbol{u}_\phi, \boldsymbol{u}_\phi)\, d\phi \end{aligned}$$

Length of a curve c

$$\overline{L}(c) = \lim \sum_i \sqrt{||c(t_{i+1}) - c(t_i)||^2}$$

New squared length is mean squared length

New length is RMS length

The averaged metric is a bonafide metric

Proof.

If $\boldsymbol{u} \neq 0$:

$$\begin{aligned}\overline{g}(\boldsymbol{u},\boldsymbol{u}) &= \frac{1}{2\pi}\int_0^{2\pi} \delta_{ij} dx^i_\phi(\boldsymbol{u})\, dx^j_\phi(\boldsymbol{u})\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left([dx^1_\phi(\boldsymbol{u})]^2 + [dx^2_\phi(\boldsymbol{u})]^2 + [dx^3_\phi(\boldsymbol{u})]^2 \right) d\phi \\ &> 0.\end{aligned}$$

Moreover, if $\overline{g}(\boldsymbol{u},\boldsymbol{u}) = 0$ then $[dx^i_\phi(\boldsymbol{u})]^2 = 0$, $i = 1, 2, 3$, which implies $\boldsymbol{u} = 0$. □

Therefore $\overline{g}$ can be used to define new vector calculus/algebra operations

Averaged dot product

$$\begin{aligned} \boldsymbol{u} \bar{\cdot} \boldsymbol{v} &= \overline{g}(\boldsymbol{u}, \boldsymbol{v}) \\ &= u^i \overline{g}_{ij} v^j \\ &= [u]^T [\overline{g}][v]. \end{aligned}$$

Therefore $\overline{g}$ can be used to define new vector calculus/algebra operations

Averaged cross product

$$\boldsymbol{u}\overline{\times}\boldsymbol{v}\bar{\cdot}\boldsymbol{w} = \sqrt{\det\left[\overline{g}\right]}\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w}$$

OR

$$[\boldsymbol{u}\overline{\times}\boldsymbol{v}] = \sqrt{\det[\overline{g}]}\,[\overline{g}]^{-1}[\boldsymbol{u}\times\boldsymbol{v}].$$

Therefore $\overline{g}$ can be used to define new vector calculus/algebra operations

Averaged gradient

$$\boldsymbol{u}\bar{\cdot}\overline{\nabla}\psi = \boldsymbol{u}\cdot\nabla\psi$$

OR

$$[\overline{\nabla}\psi] = [\overline{g}]^{-1}[\nabla\psi].$$

Therefore $\overline{g}$ can be used to define new vector calculus/algebra operations

Averaged divergence

$$\overline{\nabla}\bar{\cdot}\boldsymbol{u} = \sqrt{\det\left[\overline{g}\right]}^{-1} \nabla \cdot \left(\sqrt{\det\left[\overline{g}\right]}\,\boldsymbol{u}\right)$$

Therefore $\overline{g}$ can be used to define new vector calculus/algebra operations

Averaged curl

$$\overline{\nabla}\overline{\times}\boldsymbol{u}\,\bar{\cdot}\,\boldsymbol{v}\overline{\times}\boldsymbol{w} = \boldsymbol{v}\,\bar{\cdot}\,\overline{\nabla}(\boldsymbol{w}\bar{\cdot}\boldsymbol{u}) - \boldsymbol{w}\,\bar{\cdot}\,\overline{\nabla}(\boldsymbol{v}\bar{\cdot}\boldsymbol{u}) - \boldsymbol{u}\,\bar{\cdot}\,[\boldsymbol{v},\boldsymbol{w}]$$

OR

$$[\overline{\nabla}\overline{\times}\boldsymbol{u}] = \sqrt{\det[\overline{g}]}^{-1}\nabla\times\left([\overline{g}][u]\right).$$

Averaged vector calculus satisfies the identities you would expect...

Cohomological identities

$$\overline{\nabla}\,\bar{\cdot}\,\overline{\nabla}\,\overline{\times}\,\boldsymbol{u} = 0$$

$$\overline{\nabla}\,\overline{\times}\,\overline{\nabla}\psi = 0$$

Leibniz identities

$$\overline{\nabla}\,\bar{\cdot}\,(f\boldsymbol{u}) = \boldsymbol{u}\,\bar{\cdot}\,\overline{\nabla}f + f\,\overline{\nabla}\,\bar{\cdot}\,\boldsymbol{u}$$

$$\overline{\nabla}\,\overline{\times}\,(f\boldsymbol{u}) = \overline{\nabla}f\,\overline{\times}\,\boldsymbol{u} + f\,\overline{\nabla}\,\overline{\times}\,\boldsymbol{u}$$

$$\overline{\nabla}\,\bar{\cdot}\,(\boldsymbol{u}\,\overline{\times}\,\boldsymbol{v}) = \boldsymbol{v}\,\bar{\cdot}\,\overline{\nabla}\,\overline{\times}\,\boldsymbol{u} - \boldsymbol{u}\,\bar{\cdot}\,\overline{\nabla}\,\overline{\times}\,\boldsymbol{v}$$

$$\overline{\nabla}\,\overline{\times}\,(\boldsymbol{u}\overline{\times}\boldsymbol{v}) = \boldsymbol{u}\,\overline{\nabla}\,\bar{\cdot}\,\boldsymbol{v} - \boldsymbol{v}\,\overline{\nabla}\,\bar{\cdot}\,\boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}]$$

...as well as some that are more remarkable

Lemma

Suppose Φ_ϕ is a volume-preserving circle action on Q. Let $\boldsymbol{u} = \partial_\phi \Phi_\phi|_{\phi=0}$. Then

$$\boldsymbol{u} \,\bar{\cdot}\, \overline{\nabla} ||\boldsymbol{u}||^2 = 0$$

$$\boldsymbol{u} \,\bar{\cdot}\, \overline{\nabla} \sqrt{\det [\overline{g}]} = 0$$

$$\frac{\boldsymbol{u}}{||\boldsymbol{u}||^2} \,\overline{\times}\, \overline{\nabla} \,\overline{\times}\, \frac{\boldsymbol{u}}{||\boldsymbol{u}||^2} = 0$$

The vector field $\boldsymbol{u}$ is force-free w.r.t. the averaged metric!

Outline

The classical Grad-Shafranov equation governs axisymmetric equilibria

Theorem (Grad, Rubin, Shafranov)

Suppose Q is axisymmetric and that $(\boldsymbol{B}, \mathrm{p})$ is an axisymmetric non-degenerate MHD equilibrium. There exists a smooth function $\psi : Q \to \mathbb{R}$ and single-variable functions $C(\psi), p(\psi)$ such that

$$\boldsymbol{B} = C(\psi)\frac{\boldsymbol{e}_\phi}{R} + \frac{\nabla\psi \times \boldsymbol{e}_\phi}{R}$$

and

$$-\nabla \cdot (R^{-2}\nabla\psi) = p'(\psi) + R^{-2}C(\psi)\, C'(\psi),$$

where R is the major radius.

We have shown that a similar result holds in general

Theorem (Burby, Kallinikos, MacKay)

Let $(\boldsymbol{B}, \mathrm{p})$ *be any smooth non-degenerate equilibrium in a domain* $Q \approx D^2 \times S^1$. *There exists a circle action* Φ_ϕ, *a smooth function* $\psi : Q \to \mathbb{R}$, *and smooth single-variable functions* $C(\psi), p(\psi)$ *such that*

$$\boldsymbol{B} = C(\psi)\frac{\boldsymbol{u}}{R^2} + \rho\,\frac{\overline{\nabla}\psi\,\overline{\times}\,\boldsymbol{u}}{R^2}$$

and

$$-\rho\,\overline{\nabla}\,\overline{\cdot}(R^{-2}\rho\,\overline{\nabla}\psi) + \rho\,C(\psi)(\boldsymbol{u}/R^2)\,\overline{\cdot}\,\overline{\nabla}\,\overline{\times}\,(\boldsymbol{u}/R^2)$$
$$= p'(\psi) + R^{-2}\,C(\psi)\,C'(\psi),$$

where $R^2 = ||\boldsymbol{u}||^2$ *and* $\rho = \sqrt{det\,[\overline{g}]}$.

The equation for ψ generalizes the classical GS equation in many ways

Definition

Given a circle action Φ_ϕ on Q, the nonlinear elliptic partial differential equation

$$-\rho\,\overline{\nabla}\bar{\cdot}(R^{-2}\rho\,\overline{\nabla}\psi) + \rho\,C(\psi)(\boldsymbol{u}/R^2)\,\bar{\cdot}\,\overline{\nabla}\,\overline{\times}\,(\boldsymbol{u}/R^2)$$
$$= p'(\psi) + R^{-2}\,C(\psi)\,C'(\psi),$$

is the *generalized Grad-Shafranov (GGS) equation*.

All vector calculus operations defined using $\overline{g}$, as before

Basic properties of GGS equation

- Elliptic with principal symbol $\rho^2 R^{-2}||\xi||^2$
- Satisfies a variational principle
- If ψ solves GGS then $\psi_\phi = \psi \circ \Phi_\phi$ also solves GGS
- Solutions exist with $\psi = 0$ on ∂Q under mild hypotheses

The variational principle has a familiar looking Lagrangian

$$\mathcal{L}(\psi, d\psi, \boldsymbol{x}) = \frac{1}{2}\frac{\rho^2\, \overline{\nabla}\psi \,\bar{\cdot}\, \overline{\nabla}\psi}{R^2} - \frac{1}{2}\frac{C^2(\psi)}{R^2} - p(\psi)$$
$$+ \rho D(\psi)(\boldsymbol{u}/R^2) \,\bar{\cdot}\, \overline{\nabla} \,\overline{\times}\, (\boldsymbol{u}/R^2)$$
$$D(\psi) = \int^{\psi} C(\overline{\psi})\, d\overline{\psi}$$

Lagrangian = (poloidal magnetic energy) - (toroidal magnetic energy) - (pressure) + (twist)

Q: Are there surfaces perpendicular to $\boldsymbol{u}$?

A: Yes if twist vanishes

Proposition (Frobenius)

Fix $\boldsymbol{x} \in Q$. There is a neighborhood of $\boldsymbol{x}$ foliated by surfaces S perpendicular to $\boldsymbol{u}$ (using $\overline{g}$) if and only if

$$\tau_{\boldsymbol{u}} = (\boldsymbol{u}/R^2) \bar{\cdot} \overline{\nabla} \overline{\times} (\boldsymbol{u}/R^2) = 0$$

near $\boldsymbol{x}$.

A: Yes if twist vanishes

Corollary

Fix $\boldsymbol{x} \in Q$. There is a neighborhood of $\boldsymbol{x}$ foliated by surfaces S perpendicular to $\boldsymbol{u}$ (using $\overline{g}$) if and only if

$$\overline{\nabla} \overline{\times} (\boldsymbol{u}/R^2) = 0.$$

near $\boldsymbol{x}$.

Follows from $\boldsymbol{u}/R^2 =$ force-free w.r.t. $\overline{g}$

The GGS equation differs from the classical GS equation in one crucial way

Theorem (Grad, Rubin, Shafranov)

If $\psi : Q \to \mathbb{R}$ is an axisymmetric solution of the GS equation then

$$\boldsymbol{B} = C(\psi)\frac{\boldsymbol{e}_\phi}{R} + \frac{\nabla\psi \times \boldsymbol{e}_\phi}{R}$$

and $\mathrm{p} = p(\psi)$ *satisfy*

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \nabla \mathrm{p}$$
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$$\nabla \cdot \boldsymbol{B} = 0.$$

This result does *not* hold for the GGS equation!

The GGS equation differs from the classical GS equation in one crucial way

Theorem (Burby, Kallinikos, MacKay)

If $\psi : Q \to \mathbb{R}$ *is an* S^1*-invariant solution of the GGS equation then*

$$\boldsymbol{B} = C(\psi)\frac{\boldsymbol{u}}{R^2} + \rho\,\frac{\overline{\nabla}\psi\overline{\times}\boldsymbol{u}}{R^2}$$

and $\mathrm{p} = p(\psi)$ *satisfy*

$$(\overline{\nabla}\,\overline{\times}\,\boldsymbol{B})\overline{\times}\boldsymbol{B} = \overline{\nabla}\mathrm{p}$$
$$\nabla\cdot\boldsymbol{B} = 0.$$

Note that $\nabla\cdot\boldsymbol{B} = 0$ w.r.t. the standard metric

The GGS equation differs from the classical GS equation in one crucial way

This is not force balance

$$(\overline{\nabla}\,\overline{\times}\,\boldsymbol{B})\overline{\times}\boldsymbol{B} = \overline{\nabla}p$$

The GGS equation differs from the classical GS equation in one crucial way

This is force balance averaged over Φ_ϕ

$$(\overline{\nabla}\,\overline{\times}\,\boldsymbol{B})\overline{\times}\boldsymbol{B} = \overline{\nabla}\mathrm{p}$$

This implies a new procedure for constructing 3D equilibria

1. Guess a volume-preserving circle action Φ_ϕ on Q

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4. Adjust Φ_ϕ to make $\int_Q |\boldsymbol{R}|^2 \, d^3\boldsymbol{x}$ smaller
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5. Go back to first step

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Space of solutions of GGS eqn much smaller than $(\boldsymbol{B}, \mathrm{p})$-space

END