NOTE: This is a preprint of a paper being submitted for publication. Contents of this paper should not be quoted or referred to without permission of the author(s).

To be published in the Proceedings of the International Workshop on Condensed Matter Theory

COMPOSITE FERMION EXCITATIONS IN FRACTIONAL QUANTUM HALL SYSTEMS

John J. Quinn

DEPARTMENT OF PHYSICS AND ASTRONOMY, UNIVERSITY OF TENNESSEE
KNOXVILLE, TN 37996

and

SOLID STATE DIVISION
OAK RIDGE NATIONAL LABORATORY
Managed by
MARTIN MARIETTA ENERGY SYSTEMS, INC.
Under
Contract No. DE-AC05-84OR21400
With the
U. S. DEPARTMENT OF ENERGY
OAK RIDGE, TENNESSEE

April 1995

COMPOSITE FERMION EXCITATIONS IN FRACTIONAL QUANTUM HALL SYSTEMS

John J. Quinn

University of Tennessee, Knoxville, TN 37996 USA
and
Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831 USA

ABSTRACT

In two dimensional systems in a strong magnetic field, electrons can be transformed into composite Fermions (CF) by attaching to each a fictitious flux tube (carrying flux Φ) and fictitious charge q, where the product $q\Phi$ is a multiple of 2π. In the mean field approximation, this transformation converts a fractionally filled electron Landau level into an integrally filled CF Landau level. This integrally filled CF Landau level corresponds to the ground state of a Laughlin incompressible fluid. Excited states are described by the n_{QE} and n_{QH}, the numbers of quasielectron and quasihole CF excitations. For N electrons on the surface of a sphere the energy and angular momentum of a quasihole (or quasielectron) are ε_{QH} and $\ell_{QH} = \frac{1}{2}(N + n_{QH} - n_{QE} - 1)$ (or ε_{QE} and $\ell_{QE} = \ell_{QH} + 1$). The lowest energy sector of the energy spectrum contains the minimum number of CF excitations consistent with the value of N and the degeneracy of the lowest Landau level, 2S+1. The first excited sector contains one additional QE-QH pair. The total angular momentum L is obtained by adding the angular momenta of QE excitations and QH excitations treated as distinguished sets of Fermions. In the absence of CF interactions, all states containing n_{QE} quasielectrons and n_{QH} quasiholes are degenerate. The interaction between CF excitations partially removes this degeneracy. The interactions between CF excitations can be determined by comparing exact numerical results for N electrons with the CF picture. This amounts to constructing a Fermi liquid theory of CF excitations, and should allow the study of low lying excitations of systems with much larger values of N than can be treated numerically.

1. INTRODUCTION

The study of the electronic properties of quasi two dimensional systems has been a very exciting area of condensed matter physics during the past two decades. [1] Among the most interesting discoveries in this area are the integral [2] and fractional [3] quantum Hall effects. For both of these effects, incompressible states of the two dimensional (2D) electron liquid are found at particular values of the two dimensional electron density for any given value of a dc magnetic field applied normal to the layer. The integral quantum Hall effect is relatively simple to understand qualitatively. The incompressibility results from an energy gap $\hbar\omega_c$ in the single particle spectrum. When all states below the gap are filled and all states above it empty, the ground state is incompressible. Excited states consist of electron-hole pair excitations and require a finite excitation energy. Both localized and extended single particle states are required in order that the experimentally observed magnetoconductivity be realized.

The fractional quantum Hall effect is more difficult to understand and more interesting from a theoretical point of view. The energy gap that gives rise to the Laughlin [4] incompressible fluid state is completely the result of the interaction between the electrons. The elementary excitations are fractionally charged Laughlin[4] quasiparticles, which satisfy fractional statistics. [5,6] The standard techniques of many body perturbation theory are incapable of treating fractional quantum Hall systems. With remarkable insight into the nature of the many body correlations, Laughlin was able to determine the form of both the ground state wave function and of the elementary excitations. Striking confirmation of Laughlin's picture was obtained by exact diagonalization of the full interacting Hamiltonian within the subspace of the lowest Landau level. [6] More recently Jain, [7] Lopez, and Fradkin [8] and Halperin, Lee and Read [9] have developed the composite Fermion picture of the two dimensional electron gas in a strong magnetic field. This picture has offered new insight into the problem of interacting 2D electron systems.

The object of this paper is to present a simple and understandable review of the composite Fermion (CF) picture as applied to fractional quantum Hall systems. Exact numerical calculations for small numbers of electrons on a spherical surface are interpreted in terms of CF excitations, and an outline of how to construct a phenomenological "Fermi liquid" interaction between CF excitations by using the numerical results for small systems as input data is presented.

The plan of the paper is as follows. Section 2 gives a very brief review of the single particle states and the integral quantum Hall effect. In Section 3 the Laughlin incompressible liquid state wave function and the fractionally charged Laughlin quasiparticles will be discussed. Section 4 gives a brief discussion of statistics in 2D systems, and a discussion of how anyon statistics arise. The realization of anyon statistics and the transformation of particle statistics in 2D systems through the introduction of fictitious flux and charge (i.e., Chern-Simons gauge fields) are discussed. In Section 5 the results of numerical diagonalization of the Hamiltonian for small numbers of interacting electrons on a spherical surface are reviewed. The behavior of the spectrum of eigenvalues E_L as a function of the total angular momentum L is discussed qualitatively. In Section 6 the composite Fermion transformation is introduced. The nature of the low lying excited states in the energy spectrum discussed in Section 5 is interpreted in terms of CF excitations above the mean field ground state. The treatment of fluctuations about the mean field by standard many body techniques is also discussed briefly. The final section contains a comparison of the CF picture with the numerical results. From this comparison a "CF interaction function" can be obtained.

---

[1] See, for example, the Proceedings of the International Conferences on Electronic Properties of Two Dimensional Systems (1975, 77, 79, 81, 83, 85, 87, 89, 91, 93) published in *Surface Science*.

[2] K. von Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* 45, (1980) 494.

[3] D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* 48 (1982) 1559.

[4] R. B. Laughlin, *Phys. Rev. Lett.* 50 (1983) 1395.

[5] B. I. Halperin, *Phys. Rev. Lett.* 52 (1984) 1583.

[6] F. D. M. Haldane, *Phys. Rev. Lett.* 51 (1983) 605.

[7] J. K. Jain, *Phys. Rev. Lett.* 63 (1989) 199.

[8] A. Lopez and E. Fradkin, *Phys. Rev. B* 44 (1991) 5246.

[9] B. I. Halperin, P. A. Lee, and N. Read, *Phys. Rev. B* 47 (1993) 7312.

This function may be useful in understanding somewhat higher energy eigenstates or in studying systems containing a larger number of electrons than one can treat by exact numerical diagonalization.

2. INTEGER QUANTUM HALL EFFECT

The Hamiltonian for a single electron confined to a two dimensional plane in the presence of a dc magnetic field $\vec{B}_o$ oriented normal to the plane is

$$H_o = \frac{1}{2m} \left(\vec{p} + \frac{e}{c}\vec{A}_o\right)^2, \tag{1}$$

where $\vec{p} = (p_x, p_y, 0)$ is the momentum operator, and $\vec{A}_o(\vec{r})$ is the vector potential associated with the magnetic field $\vec{B}_o$. If we choose the vector potential $\vec{A}_o = \frac{1}{2} B_o (-y, x, 0)$, then the single particle eigenfunctions [10] are the form $\psi_{nm}(r, \theta) = e^{im\theta} u_{nm}(r)$ and the eigenvalues are given by

$$E_{nm} = \frac{1}{2}\hbar\omega_c(2n + 1 + |m| + m). \tag{2}$$

In these expressions the principal quantum number n takes on non-negative integral values, and the angular momentum quantum number m can take on any integral value. The lowest energy states (referred to as the lowest Landau level) have $n = 0, m = 0, -1, -2, \cdots$ and have energy $E_{om} = \hbar\omega_c/2$. It is convenient to introduce a complex coordinate $Z = re^{-i\theta} = x - iy$, and to write the eigenfunctions belonging to the lowest Landau level as $\psi_{o,-m} = u_m(Z) = N_m Z^m exp(-|Z|^2/4)$, where N_m is a normalization constant. With this definition of $u_m(Z)$, m can take on any non-negative integral value. In writing these expression we are using the magnetic length $\ell = (\hbar c/eB_o)^{\frac{1}{2}}$ as the unit of length. The function $|u_m(Z)|^2$ has its maximum at a value r_m which is proportional to $m^{\frac{1}{2}}$. All of the single particle states belonging to a given Landau level are degenerate. Each Landau level is separated in energy by $\hbar\omega_c$ from neighboring levels.

If m is restricted to being less than some maximum value, N_L, chosen so that the system has "finite radial size", then the lowest Landau level contains N_L states with $m = 0, 1, \cdots N_L - 1$. The value of N_L is equal to the flux through the sample, $B_o \cdot A$ (where A is the area of the sample) divided by the flux quantum hc/e. We introduce the filling factor $\nu = NN_L^{-1}$. A decrease in the area A decreases N_L, the number of states in a Landau level. Thus an integrally filled system $(\nu = 1, 2, \cdots)$ can be compressed only at the finite cost in energy associated with promoting electrons to higher Landau levels. Even though it is non-trivial to demonstrate that the energy gap $\hbar\omega_c$ and the resulting incompressibility lead to the observed quantum Hall behavior[2] of the magnetoconductivity, this incompressibility at integral values of ν is responsible for the integral quantum Hall effect.

In this discussion we have considered only the single particle states $u_m(Z)$ in the lowest Landau level. For a system of N electrons filling the lowest Landau level, the many electron wavefunction $\Psi(Z_1, Z_2, \cdots Z_N)$ must be antisymmetric under interchange of a pair of particles. This can be accomplished by constructing an antisymmetrized product function

$$\Psi(Z_1, \cdots Z_N) = A\{u_0(Z_1)u_1(Z_2)\cdots u_{N-1}(Z_N)\}, \tag{3}$$

where A is the antisymetrizing operator. By using $u_m(Z) \propto Z^m \exp(\frac{-|Z|^2}{4})$, this can be written

[10] See, for example, S. Gasiorowicz, *"Quantum Physics"*, John Wiley and Sons, New York (1974).

$$\Psi(Z_1, Z_N) \propto \begin{vmatrix} 1 & 1 & \cdots & 1 \\ Z_1 & Z_2 & & Z_N \\ Z_1^2 & Z_2^2 & & Z_N^2 \\ \vdots & \vdots & & \vdots \\ Z_1^{N-1} & Z_2^{N-1} & \cdots & Z_N^{N-1} \end{vmatrix} \exp\left\{-\frac{1}{4}\sum_i |Z_i|^2\right\}. \quad (4)$$

The determinant in Eq. (4) is the well-known van der Monde determinant, and it can be written as $\prod_{i>j}$ $(Z_i - Z_j)$. This is easily demonstrated by subtracting column j from column i and noting $Z_{ij} = Z_i - Z_j$ is a common factor. Since this is true for every $i \neq j$, the result is apparent. The N-particle wavefunction

$$\Psi(Z_i, \cdots Z_N) \propto \left(\prod_{i>j} Z_{ij}\right) exp\left(-\tfrac{1}{4}\sum_i |Z_i|^2\right), \quad (5)$$

corresponds to a filled Landau level because the highest power of Z_i appearing in Eq. (5) is the N-1 power. This means the allowed values of $|m|$ are equal to $0, 1, \cdots N-1$, or that N_L, the number of single particle states in the Landau level, is equal to N, so that $\nu = N/N_L = 1$.

3. LAUGHLIN STATE

For $\nu = 1$ we are essentially forced to Eq. (5) by the requirement of antisymmetry imposed upon the product of single particle eigenfunctions. For fractional filling, e.g. $\nu = \frac{1}{3}$, it is not at all apparent how to construct antisymmetric product functions for N electrons in 3N states. Laughlin proposed a wave function

$$\Psi_3(Z_1, \cdots Z_N) \propto \left(\prod_{i>j} Z_{ij}^3\right) exp\left(-\frac{1}{4}\sum_i |Z_i|^2\right). \quad (6)$$

This choice was dictated by the requirement of antisymmetry, by the attempt to minimize the repulsive interaction between electrons by having the wavefunction vanish rapidly as $Z_{ij} \to 0$, and by the observation that the highest power of Z_j appearing in (6) is $N_L - 1 = 3(N-1)$ giving $\nu \to \frac{1}{3}$ as $N \to \infty$. Laughlin states at $\nu = \frac{1}{m}$, where m is any odd integer, are constructed in a similar way.

Laughlin also proposed the form of the fractionally charged elementary excitations, and it was soon established that Laughlin's fractionally charged quasiparticles were anyons satisfying fractional statistics.[5, 11]

4. STATISTICS OF IDENTICAL PARTICLES IN TWO DIMENSIONS

For two identical particles initially at positions $\vec{r}_1$ and $\vec{r}_2$ in a three dimensional space, the amplitude for the path that takes the system from the initial state $(\vec{r}_1, \vec{r}_2)$ to the same final state $(\vec{r}_1, \vec{r}_2)$ depends on whether the angle of rotation ϕ of the vector $\vec{r}_{12}$ is zero or π. The angle ϕ is only defined modulo 2π, and, what's important, are the end points $\phi = \pi$ or $\phi = 0$ representing exchange or non-exchange processes.

In two dimensions the angle ϕ is perfectly well-defined for a given trajectory. It's possible to keep track of how many times ϕ winds around the origin. The space of particle trajectories falls into disconnected pieces that cannot be deformed into one another (if r_{ij} is not allowed to vanish). Each piece has a definite winding number. In constructing path integrals, the weighting of trajectories can depend

[11] D. Arovas, J. R. Schrieffer, and F. Wilczek, *Phys. Rev. Lett.* 53 (1984) 722.

on a new parameter θ (defined modulo 2π) through a factor $exp(i\ \theta\ \phi\ /\ \pi)$. For $\theta = 0$ or $\theta = \pi$ we have the usual Boson or Fermion statistics. For the most general case we have

$$P_{12}\ \psi(1,2) = e^{i\theta}\ \psi\ (1,2), \tag{7}$$

and for arbitrary θ the particles are called anyons and satisfy a new form of quantum statistics. [12]

A simple way to realize anyon statistics is to add to a simple Lagrangian describing the relative motion $[\vec{r} = (r, \phi)]$ of two interacting particles a term $(q\Phi/\pi)\ \dot{\phi}$, where q and Φ are a fictitious charge and flux. For example, if

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) + \dot{\phi}(q\Phi/\pi), \tag{8}$$

the fictitious (Chern-Simons) charge-flux term has no effect on the classical equations of motions (since q and Φ are time independent). However, the canonical angular momentum is given by $p_\phi = \mu r^2\dot{\phi} + q\Phi/\pi$. Because $exp(2\ \pi\ i\ p_\phi)$ generates rotations of 2π, $\hbar^{-1}p_\phi$ must have integral eigenvalues ℓ. The kinematic angular momentum $p_\phi - (q\Phi/\pi)$ can, however, take on fractional values which will result in fractional statistics for the particles.

If we have a system of particles in 2D satisfying some particular statistics and described by a Hamiltonian

$$H = \frac{1}{2m}\ \sum_i\ [\vec{p}_i + \tfrac{e}{c}\ \vec{A}\,(r_i)]^2 + \sum_{i>j} V(r_{ij}), \tag{9}$$

then we can change the statistics by adding a fictitious charge q and flux tube carrying flux Φ to each particle. The new Hamiltonian simply has $\frac{e}{c}\vec{A}(r_i)$ replaced by $\frac{e}{c}\vec{A}(r_i) + \frac{q}{c}\vec{a}(r_i)$, where

$$\vec{a}\,(r_i) = \Phi \sum_{j \neq i} \frac{\hat{z} \times \vec{r}_{ij}}{r_{ij}^2}, \tag{10}$$

is the fictitious vector potential at $\vec{r}_i$ caused by the flux tubes on all the particles at $r_j \neq r_i$. The net effect of the added Chern-Simons term is to change the parameter θ describing the particle statistics to $\theta + (\pi q\Phi/hc)$. If Φ is equal to $p(\frac{hc}{e})$ where p is an integer, then $\theta \rightarrow \theta + p\pi\frac{q}{e}$; for $q = e, p = 1$ this will convert Bosons to Fermions $[\theta = 0 \rightarrow \theta = \pi]$ and Fermions to Bosons $[\theta = \pi \rightarrow \theta = 2\pi]$. For $p = 2$, the statistics would be unchanged by the Chern-Simons terms.

5. NUMERICAL STUDY OF SMALL NUMBERS OF ELECTRONS ON A SPHERICAL SURFACE

Haldane introduced the idea of putting a small number of electrons on a spherical surface of radius R at the center of which is a magnetic monopole of strength $2S(hc/e)$. The single particle Hamiltonian can be expressed as [13]

$$H_o = \frac{1}{2mR^2}\,(\vec{L} - \hbar S\,\hat{R})^2, \tag{11}$$

[12] A very nice introduction to fractional statistics is given in F. Wilczek, *"Fractional Statistics and Anyon Superconductivity"*, World Scientific Singapore (1990); the present discussion draws heavily on this work.

[13] G. Fano, F. Ortolani, and E. Colombo, *Phys. Rev. B* 34 (1986) 2607.

where $\vec{L}$ is the angular momentum operator and $\hat{R}$ a unit vector in the radial direction. The components of $\vec{L}$ satisfy the usual commutation rules $[L_\alpha, L_\beta] = i\hbar\, \epsilon_{\alpha\beta\gamma}\, L_\gamma$. The eigenstates of H_o can be expressed as $|l, m>$, where $|\ell, m>$ is an eigenfunction of L^2 and of L_z with eigenvalues $\hbar^2\ell(\ell+1)$ and $\hbar m$ respectively. The lowest eigenvalue occurs for $\ell = S$ and has energy $\hbar\omega_c/2$. The n^{th} excited state has $\ell = S + n$, and

$$E_n = \frac{\hbar\omega_c}{2S}\,[(S+n)(S+n+1) - S^2]. \qquad (12)$$

If we concentrate on a partially filled lowest Landau level we have only $2S+1$ degenerate single particle states (since the electron angular momentum l_e must be equal to S and its $z-$ component can take on values between $-S$ and S). From the $2S+1$ single particle states, we can construct $N_{MB} = (2S+1)![N!(2S+1-N)!]^{-1}$ antisymmetric many body states containing N electrons. For the Laughlin $\nu = 1/m$ state, $2S_{\nu=1/m} = m(N-1)$. Table 1 lists the values of $\ell_e, 2S+1, N_{MB}, L_{MAX}$ (the largest possible total angular momentum of the system), and the allowed values of L, the total angular momentum, with a superscript indicating how many times they appear. The number in parenthesis in the allowed L-value column is the total number of different L-multiplets that appear. For three electrons there are five such states, all with different L-values. For four electrons there are eighteen states; $L = 12, 10, 9, 7, 5$ and 3 each appear once, $L = 8, 2$ and 0 each appear twice, and $L = 6$ and 4 each appear three times.

One can diagonalize the interaction Hamiltonian $H_I = \sum_{i>j} e^2 r_{ij}^{-1}$ within the subspace of the N_{MB} many body states associated with the lowest Landau level. The many body states of the non-interacting system can be written $|\alpha, L, M>$, where L is the total angular momentum, M its z-component, and α is a label which distinguishes different states with the same L value (e.g. for the four electron system the three $L = 6$ states correspond to the three different values of α). Because the interaction Hamiltonian is a scalar, the Wigner-Eckart theorem tells us that

$$< \alpha'\, L'\, M' \mid H_I \mid \alpha\, LM > = \delta_{LL'}\, \delta_{MM'} < \alpha' L\, |H_I|\, \alpha L >, \qquad (13)$$

where the reduced matrix element is $< \alpha' L|H_I|\alpha L >$ independent of M. The 210 many body states of four electrons give us an 18 by 18 matrix that is block diagonal with two 3 by 3 blocks, three 2 by 2 blocks, and six 1 by 1 blocks. For small numbers of electrons these finite matrices can be easily diagonalized to obtain the many body eigenvalues and eigenfunctions.

In Figure 1 we display the energy spectra for an eight electron system as a function of the total angular momentum L for various values of $2S$. These result were first obtained by He et al. [14] It is clear that the states fall into a well defined low energy sector and a slightly less well defined first excited sector. For the Laughlin $\nu = \frac{1}{3}$ state $(2S = 21)$ the low energy sector consists of a singlet $L = 0$ state. For $2S$ =20 the system is one single particle state shy of having the Laughlin $\nu = \frac{1}{3}$ filling. In this case the low energy sector corresponds to having a single Laughlin quasielectron of angular momentum $L = 4$. For $2S = 19$ the low energy sector contains two Laughlin quasiparticles.

6. COMPOSITE FERMION TRANSFORMATION

The difficulty in trying to understand the fractionally filled Landau level in 2D systems results from the enormous degeneracy that is present in the non-interacting many body states. The lowest Landau level contains N_L states and $N_L = B_o A/(hc/e)$, the number of flux quanta threading the sample area A. Therefore $N_L/N = \nu^{-1}$ is equal to the number of flux quanta per electron. Let's think of the $\nu = \frac{1}{3}$ state as an example; it has three flux quanta per electron. If we attach to each electron a fictitious

[14] S. He, X. Xie, F. Zhang, *Phys. Rev. Lett.* 68 (1992) 3460.

charge q (taken equal to $-e$, the electron charge) and a fictitious flux tube (carrying flux $\Phi = 2\,p\phi_o$ directed opposite to $\vec{B}_o$, where p is an integer and ϕ_o the quantum of flux), the net effect is to give us the Hamiltonian described by Eqs. (9) and (10) and to leave the statistical parameter θ unchanged. The electrons are converted into composite Fermions which interact through the gauge field terms as well as through the Coulomb interaction.

Why would anyone want to make this transformation since it makes the Hamiltonian more complicated? The answer is simple. If the gauge field $\vec{a}(r_i)$ is replaced by its mean value, we can define an effective magnetic field $B^* = B_o + < b >$, where $< b >$ is the average magnetic field associated with the fictitious flux. Since B_o corresponds to three flux quanta per electron and $< b >$ corresponds to two flux quanta per electron directed opposite to B_o, we see that $B^* = \frac{1}{3}B_o$. The effective magnetic field acting on the composite Fermions gives a CF Landau level containing $\frac{1}{3}N_L$ states, or exactly enough states to accommodate our N particles. Thus the $\nu = \frac{1}{3}$ electron Landau level is converted by the CF transformation to a $\nu^* = 1$ CF Landau level. We know everything about the properties of a filled Landau level in 2D systems. The ground state is the antisymmetric product of single particle states containing N particles in exactly N states. Fluctuations about the mean field can be treated by standard many body perturbation theory.[8,9, 15]

The original Hamiltonian of the 2D system of electrons can be written as $H = H_o + V$, where

$$H_o = \frac{1}{2m}\int d^2r\, \psi_e^+(r)\;[\vec{p} + \frac{e}{c}\vec{A}(\vec{r})]^2\,\psi_e(r). \tag{14}$$

Here $\psi_e(r)$ is the electron annihilation operator, and V is the electron-electron interaction. The CF operator is defined by[9]

$$\psi^+(r) = \psi_e^+(r)\,exp\,[-i\,p\phi_o\int d^2r'\,arg\,(\vec{r}-\vec{r}\,')\rho(r')], \tag{15}$$

where $arg\,(\vec{r}-\vec{r}\,')$ is the angle that $\vec{r}-\vec{r}\,'$ makes with the x-axis, $\rho(r) = \psi^+(r)\,\psi(r) = \psi_e^+(r)\,\psi_e(r)$, $\phi_o = hc/e$, and p is an even integer. The Hamiltonian H_o can be expressed in terms of the CF operators as

$$H_o = \frac{1}{2m}\int d^2r\,\psi^+(r)\;[\vec{p} + \frac{e}{c}\vec{A}(\vec{r}) - \frac{e}{c}\vec{a}(\vec{r})]^2\,\psi(r), \tag{16}$$

where

$$\vec{a}(r) = p\phi_o\int d^2r'\;\frac{\hat{z}\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\;\rho(r'). \tag{17}$$

In the mean field approximation we can write $a(r) = \bar{a}(r) + \delta a(r)$, where $\bar{a}(r)$ is obtained by replacing $\rho(r')$ in Eq. (17) by its average value ρ_o. H_o can be rewritten as $H_o = H_o^* + \delta H_o$, where

$$H_o^* = \frac{1}{2m}\int d^2r\,\psi^+(r)\;[\vec{p} + \frac{e}{c}\vec{A}^*(r)]^2\,\psi(r). \tag{18}$$

Here $A^*(r) = A(r) - \bar{a}(r)$ and the effective magnetic field B^* acting on the composite Fermions is equal to $\nabla\times A^*(r)$. δH_o is simply the difference between H_o and H_o^*. It involves linear and quadratic terms in $\delta a(r)$. The quadratic term gives rise to three body interactions proportional to $\int d^2r d^2r' d^2r'' W(r-r'; r-r'')\rho(r)\rho(r')\rho(r'')$, where $\rho(r) = \psi^+(r)\psi(r)$. These terms are difficult and are usually neglected. The two body terms can be written as

[15] S. Simon and B. Halperin, *Phys. Rev. B* 48 (1993) 17386.

$$\delta H_o = \frac{1}{2}\int d^2r d^2r'\, j_\mu(\vec{r})\, V_{\mu\nu}\,(\vec{r}-\vec{r}\,')\, j_\nu(r'). \tag{19}$$

where $j_0(r) = \rho(r) - \rho_o$ is a charge density fluctuation and $j_x(r)$ and $j_y(r)$ are current density fluctuations. [16] δH_o can be added to the Coulomb interaction $V(r)$ and can be treated by standard many body perturbation theory (e.g. by generalized RPA).

The CF transformation can be applied to the system of N electrons on a sphere. For the $\nu = \frac{1}{3}$ state we know that $2S_{\nu=\frac{1}{3}} = 3(N-1)$. Thus for an 8 electron system, $2S_{\nu=\frac{1}{3}}$ has the value 21 as discussed earlier. For 2S close to the value $2S_{\nu=\frac{1}{3}}$, we can write $2S = 2S_{\nu=\frac{1}{3}} + n_{QH} - n_{QE}$. Thus, the state with 2S = 19 must contain at least $2QE$, and the state with $2S = 22$ must contain at least one QH. When we transform the electrons to composite Fermions, we arrive at a new effective value of 2S given by $2S^* = 2S^*_{\nu*=1} + n_{QE} - n_{QH}$. If $2S_{\nu=\frac{1}{3}} = 3(N_e-1)$, $2S^*_{\nu^*=1} = 1(N_e-1)$; however the values of n_{QE} and n_{QH} are unchanged by the CF transformation. If we concentrate on the eight electron system we have $2S^*_{\nu^*=1} = 7$ or $S^*_{\nu^*=1} = 7/2$.

For the composite Fermions S^* is the angular momentum associated with the lowest CF Landau level. If we create a $QE-QH$ pair by promoting one CF from the lowest CF Landau level to the first excited CF Landau level, we must have $l_{QH} = S^*$ and $\ell_{QE} = S^*+1$. Thus for an N electron system containing n_{QE} and n_{QH} CF excitations, we know that each QH has $\ell_{QH} = S^* = \frac{1}{2}(N + n_{QH} - n_{QE} - 1)$ and each QE has $\ell_{QE} = \ell_{QH}+1$.

The energy spectra obtained by exact numerical diagonalization can be interpreted [17] by assuming that the lowest energy sector contains the minimum number of CF excitations consistent with the values of $2S$ and N. In the absence of interactions between the quasiparticles (i.e. when $\delta H = 0$), the energy of a state containing n_{QE} quasielections and n_{QH} quasiholes is simply $n_{QE}\,\epsilon_{QE} + n_{QH}\,\epsilon_{QH}$, where ϵ_{QE} and ϵ_{QH} are the quasielectron and quasihole energies. The allowed values of the total angular moment of these states can be obtained by simple addition of the angular momenta of the quasielectrons and quasiholes treated as distinguishable sets of Fermions. Table 2[17] gives the values of $n_{QE}, n_{QH}, \ell_{QE}, \ell_{QH}$ and the allowed values of the total angular momentum L that correspond to the lowest energy sector of the energy spectrum of eight electrons at values of $2S$ going from 16 to 24. It is worth noting that a Laughlin $\nu = 2/5$ incompressible liquid state occurs at $2S$ = 16 even in the absence of any interactions between the quasielectrons. The reason for this is that as $2S$ decreases from the value $2S$ = 21 corresponding to the Laughlin state, the number of quasielectron states, $2\ell_{QE}+1$, decreases and the number of quasielectron excitations increases. At $2S$ = 16, both of these quantities have a value of five. There is only one way to put five quasielectrons into five CF states (i.e. by having the z-component of ℓ_{QE} take the values -2, -1, 0, 1, 2); this gives the state at L = 0 as the entire low energy sector. For quasiholes this does not occur. The number of available quasihole states increases as the number of quasiholes increases. Thus a Laughlin state at $\nu = 2/7$ can occur only if quasihole-quasihole interactions result in a single L = 0 state having the lowest energy in a crowded low energy sector.

The first excited sector should result from adding one additional quasielectron-quasihole pair to the low-energy sector. For example, at $2S$ = 21, the Laughlin incompressible fluid state has L = 0 and has no $QE-QH$ pairs. The first excited sector should contain a single $QE-QH$ pair. Because $\ell_{QH} = 3.5$ and $\ell_{QE} = 4.5$, this sector (in the absence of $QE-QH$ interactions) should contain eight degenerate states with energy $\epsilon_{QE}+\epsilon_{QH}$ and angular momentum L satisfying $\ell_{QE} - \ell_{QH} \le L \le \ell_{QE}+\ell_{QH}$, or going from $L = 1$ to $L = N$. Seven of these eight states appear in the first excited sector of the energy spectra obtained numerically. The $L = 1$ state is missing from this sector, and this is true for five, six, and seven electron systems too. Our conclusion is that there must be a strong $QE-QH$ repulsion when these particles are in a total angular momentum state $L = 1$. The $QE-QH$ interactions are also

[16] See, for example, Y. Chen , F. Wilczek, E. Witten, and B. I. Halperin, *Int. Journal of Mod. Phys. B*, Vol. 3 (1989) 1001.

[17] X. M. Chen and J. J. Quinn, *Solid State Commun.* 92 (1994) 865.

responsible for the magnetoroton structure of these states; the interactions remove the degeneracy and give a binding of the $QE - QH$ pair which is largest for $L \simeq 4$ and 5.

For $2S = 20$, the first excited sector should have $n_{QE} = 2$ and $n_{QH} = 1$. The angular momenta of the individual quasiparticles are $\ell_{QE} = 4$ and $\ell_{QH} = 3$. If we first add the angular momenta of the two QE, we obtain $L_{QE} = 7, 5, 3, 1$. Adding to these values $\ell_{QH} = 3$ gives total L values $L = 0 \oplus 1 \oplus 2^3 \oplus 3^3 \oplus 4^4 \oplus 5^3 \oplus 6^3 \oplus 7^2 \oplus 8^2 \oplus 9 \oplus 10$. Comparing with the numerical results, we observe that the $L = 3$ and $L = 4$ states occur twice and three times, respectively, instead of the three and four times predicted. To test whether our ideas are correct, we introduced a $QE - QH$ interaction that was strongly repulsive when the total angular momentum of a $QE - QH$ pair was equal to unity. [18] With just this interaction, the degeneracy of the first excited sector is removed, and one of the $L = 3$ states and one $L = 4$ states are pushed into the next excited sector.

To summarize our findings, we emphasize that the low lying excitations can be described in terms of the numbers n_{QE} and n_{QH}. The total angular momentum can be obtained by the addition of the individual QE and QH angular momenta, being careful to treat the QE excitations as a set of Fermions and QH excitations as a set of Fermions distinguishable from the QE excitations. The energy of the excited state would simply be the sum of the individual quasiparticle energies if interactions between quasiparticles were neglected. The interactions partially remove the degeneracy of different states having the same values of n_{QE} and n_{QH}. The CF excitation picture can be used to predict the states in the low energy sector for N larger than the values that have the studied numerically up to now. Table 3[17] gives the allowed L-value for QE states in the low energy sector for systems with N going from 9 to 12 and n_{QE} going from 1 up to 7.

7. QUASIPARTICLE-QUASIPARTICLE INTERACTIONS

In the mean field approximation the energy necessary to create a QE-QH pair is $\hbar\omega_c^*$, the CF cyclotron energy. However, the quasiparticles will interact with the Laughlin condensed state through the fluctuation Hamiltonian. The renormalized quasiparticle energy will include this "self-energy" which is difficult to calculate. We can determine the QE and QH energies phenomenologically, using the exact numerical results as input data.

For the eight electron system the Laughlin condensed state occurs for $2S = 21$ at $L = 0$. A single QE excitation is present for $2S = 20$ and its angular momentum is $L = 4$; a single quasihole occurs for $2S = 22$ at $L = 4$. The energies ϵ_{QE} and ϵ_{QH} are simply the difference between the energy of the Laughlin condensed state $E(L = 0)$ at $2S = 21$ and the energy $E(L = 4)$ at $2S = 20(QE)$ and $2S = 22(QH)$. In considering the quasiparticle states, we keep the number of electrons constant but change the value of $2S$. The effects of the neutralizing background are subtracted from the energies obtained numerically by addition of either $-N^2/2R$ or $-(N^2 - \nu^2)/2R$. They correspond to the definitions of the quasiparticle energies used by Halperin[5] and by Fano et al.[13] respectively. In Fig. (2) we plot these two quasiparticle energies as a function of N^{-1}, to obtain ϵ_{QE} and ϵ_{QH} free of finite size effects.

The picture we are using is very reminiscent of Fermi liquid theory. The ground state is the Laughlin condensed state; it plays the role of a "vacuum" state. The elementary excitations are quasiparticles (QE or QH). We propose that the total energy can be expressed as

$$E = E_o + \sum_{QP} \epsilon_{QP} \; n_{QP} + \frac{1}{2} \sum_{QP,QP'} V_{QP-QP'} \; (L) \; n_{QP} \; n_{QP'}. \tag{20}$$

The last term represents the interactions between pairs of quasiparticles in a state of angular momentum L. We have taken the energy spectra for $5, 6, 7$, and 8 electrons, and compared the two QP states ($2QE, 2QH$, or $1QE + 1QH$) with the CF picture.

[18] X. M. Chen, S. N. Yi, and J. J. Quinn, unpublished.

In Fig. (3) the resulting $V_{QE-QH}(L)$ is plotted as a function of the total angular momentum L. The lower curves use the definition of Fano et al.[13] for the quasiparticle energies, the upper curves that of Halperin.[5] Which value is used is unimportant as long as it is used consistently.

In Fig. (4) and (5) we plot V_{QH-QH} and V_{QE-QE} versus the relative angular momentum defined by $RAM = L_{MAX} - L$ as discussed by Haldane and Rezayi [19] and by Johnson and Canright. [20] The value of V_{QP-QP} are obtained by subtracting the energies of the non-interacting quasiparticles from the numerical values of $E(L)$ for the $2QP$ states after the appropriate positive background energy correction. Here the upper curves are for the QP energies used by Fano et al.[13]

It is worth noting that the interaction energy for unlike quasiparticles depends on the total angular momentum L while for like quasiparticles it depends on the relative angular momentum $L_{MAX} - L$. This can be understood by considering the 2D plane (or sphere with $R \rightarrow \infty$). Oppositely charged quasiparticles form bound states in which both charges drift in the direction perpendicular to the line connecting them. Their spatial separation is related to the total angular momentum L. Like charges repel one another, but they actually orbit around one another due to the effect of the dc magnetic field. Their separation is related to their relative angular momentum.

So far we have not investigated carefully states with more than two quasiparticles. We believe the energy of such states can be understood on the basis of Eq. (20). We also believe that systems with large numbers of electrons can be treated using the phenomenological V_{QP-QP} to determine how the states in the low energy sectors are affected by $QP - QP$ interactions.

ACKNOWLEDGMENT

This research was supported in part by the Division of Materials Sciences of the U.S. Department of Energy under Contract No. DE-AC05-840R21400 with Martin Marietta Energy Systems, Inc. The author would like to acknowledge the collaboration of Drs. X. M. Chen and S. N. Yi on many of the still unpublished results presented in this review. He would also like to thank Prof. G. Canright and M. Johnson for helpful discussions. The assistance of a travel grant from the US Army Research Office is gratefully acknowledged.

[19] F. D. M. Haldane and E. Rezayi, *Phys. Rev. Lett.* 54 (1985) 237.
[20] M. D. Johnson and G. S. Canright, *Phys. Rev. B* 49 (1994) 2947.

TABLE AND FIGURE CAPTIONS

Table 1. For N electrons on the surface of a sphere the values of ℓ_e, the angular momentum of an electron in the lowest Landau level; $2S+1$ the degeneracy of the Landau level; $N_{MB} = (2S+1)![N!(2S+1-N)!]^{-1}$, the number of antisymmetric N electron states; L_{MAX}, the maximum value of the total angular momentum L, and the allowed L-values (the exponent gives the number of times an L-multiplet appears). The number in parenthesis is the total number of L-multiplets.

Table 2. For an eight electron system, the values of $n_{QE}, n_{QH}, \ell_{QE}, \ell_{QH}$ and the allowed values of the total angular momentum L for states in the lowest energy sector at values of $2S$ ranging from 16 to 24.

Table 3. The allowed values of the total angular momentum for systems of $N = 9$ up to $N = 12$ for states in the low energy sector as a function the number of quasielectron excitations n_{QE}. Note that Laughlin condensed states (corresponding to $\nu = 2/5$) occur at $n_{QE} = 6$ for the ten electron system and at $n_{QE} = 7$ for the twelve electron system.

Figure 1. Energy spectra for an eight-electron system on a sphere as a function of total angular momentum L. The curves, a, b, c, d, e, f correspond to 0, 1, 2, 3, 4, and 5 quasielectron ground states respectively. The energy is in units of e^2/ℓ. Low energy sectors are below the dot-dash lines.

Figure 2. ε_{QE} and ε_{QH} obtained from the numerical calculations as a function of the inverse of the number of electrons. The open symbols include the "self-energy" correction suggested in [19] and [13]. Energies are in units of e^2/ℓ.

Figure 3. V_{QE-QH} as a function of total angular momentum L for $N_e = 5, 6, 7$, and 8 electrons. The lower curves were obtained by including the "self-energy" correction in the quasiparticle energies, in the upper curves it was omitted. The correction essentially shifts the curves by a constant value. Energies are in units of e^2/ℓ.

Figure 4. V_{QH-QH} as a function of the relative angular momentum defined by $RAM = L_{MAX} - L$, where L is the total angular momentum and L_{MAX} its maximum value. Energies are in units of ϵ^2/ℓ_o is the magnetic length for $\nu = \frac{1}{3}$. Here the upper curves include the "self-energy" correction to ϵ_{QE}.

Figure 5. Same as Figure 4 but for V_{QE-QE} instead of V_{QH-QH}.

Table 1

N	ℓ_e	$2S+1$	N_{MB}	L_{MAX}	Allowed L-values
3	3	7	35	6	$6 \oplus 4 \oplus 3 \oplus 2 \oplus 0$ (5)
4	4.5	10	210	12	$12 \oplus 10 \oplus 9 \oplus 8^2 \oplus 7 \oplus 6^3 \oplus 5 \oplus 4^3 \oplus 3 \oplus 2^2 \oplus 0^2$ (18)
5	6	13	1287	20	$20 \oplus 18 \oplus 17 \oplus 16^2 \oplus 15^2 \oplus 14^3 \oplus 13^3 \oplus 12^5 \oplus 11^4 \oplus 10^6 \oplus 9^5 \oplus 8^7 \oplus 7^5 \oplus 6^7 \oplus 5^5 \oplus 4^6 \oplus 3^3 \oplus \oplus 2^4 \oplus 1 \oplus 0^2$ (73)
6	7.5	16	8008	30	$30 \oplus 28 \oplus \ldots$ (338)
7	9	19	50.382	42	$42 \oplus 40 \oplus \ldots$ (1656)
8	10.5	22	319,770	56	$56 \oplus 54 \oplus \ldots$ (8512)
9	12	25	2,042,975	72	$72 \oplus 70 \oplus \ldots$ (45.207)
10	13.5	28	13,123,110	90	$90 \oplus 88 \oplus \ldots$ (246.448)

Figure 1

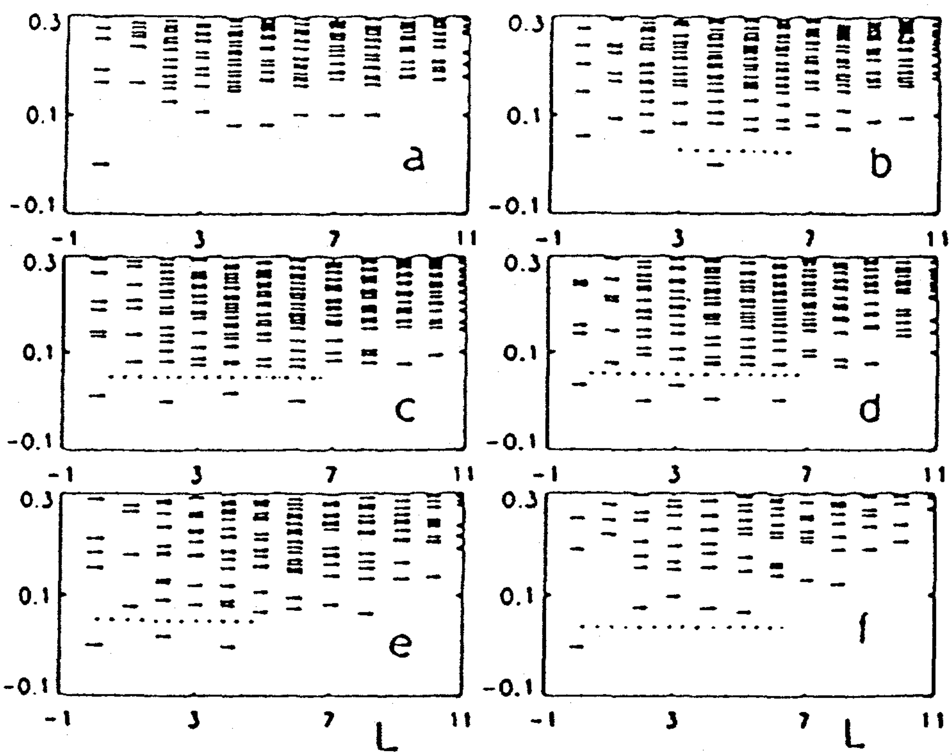

Table 2

$2S$	n_{QE}	n_{QH}	l_{QE}	l_{QH}	allowed values of L
24	0	3	6.0	5.0	$12 \oplus 10 \oplus 9 \oplus 8 \oplus 7 \oplus 6^2 \oplus 5 \oplus 4^2 \oplus 3 \oplus 2 \oplus 0$
23	0	2	5.5	4.5	$8 \oplus 6 \oplus 4 \oplus 2 \oplus 0$
22	0	1	5.0	4.0	4
21	0	0	4.5	3.5	0
20	1	0	4.0	3.0	4
19	2	0	3.5	2.5	$6 \oplus 4 \oplus 2 \oplus 0$
18	3	0	3.0	2.0	$6 \oplus 4 \oplus 3 \oplus 2 \oplus 0$
17	4	0	2.5	1.5	$4 \oplus 2 \oplus 0$
16	5	0	2.0	1.0	0

Table 3

N, n_{QE}	1	2	3	4	5	6	7
9	4.5	$7 \oplus 5 \oplus 3 \oplus 1$	$7.5 \oplus 5.5 \oplus 4.5 \oplus 3.5 \oplus 2.5 \oplus 1.5$	$6 \oplus 4 \oplus 3 \oplus 2 \oplus 0$	2.5	X	X
10	5	$8 \oplus 6 \oplus 4 \oplus 2 \oplus 0$	$9 \oplus 7 \oplus 6 \oplus 5 \oplus 4 \oplus 3^2 \oplus 1$	$8 \oplus 6 \oplus 5 \oplus 4^2 \oplus 2^2 \oplus 0$	$5 \oplus 3 \oplus 1$	0	X
11	5.5	$9 \oplus 7 \oplus 5 \oplus 3 \oplus 1$	$10.5 \oplus 8.5 \oplus 7.5 \oplus 6.5 \oplus 5.5 \oplus 4.5^2 \oplus 3.5 \oplus 2.5 \oplus 1.5$	$10 \oplus 8 \oplus 7 \oplus 6^2 \oplus 5 \oplus 4^2 \oplus 3 \oplus 2^2 \oplus 0$	$7.5 \oplus 5.5 \oplus 4.5 \oplus 3.5 \oplus 2.5 \oplus 1.5$	3	X
12	6	$10 \oplus 8 \oplus 6 \oplus 4 \oplus 2 \oplus 0$	$12 \oplus 10 \oplus 9 \oplus 8 \oplus 7 \oplus 6^2 \oplus 5 \oplus 4^2 \oplus 3 \oplus 2 \oplus 0$	$12 \oplus 10 \oplus 9 \oplus 8^2 \oplus 7 \oplus 6^3 \oplus 5 \oplus 4^3 \oplus 3 \oplus 2^2 \oplus 0^2$	$10 \oplus 8 \oplus 7 \oplus 6^2 \oplus 5 \oplus 4^2 \oplus 3 \oplus 2^2 \oplus 0$	$6 \oplus 4 \oplus 2 \oplus 0$	0

Figure2

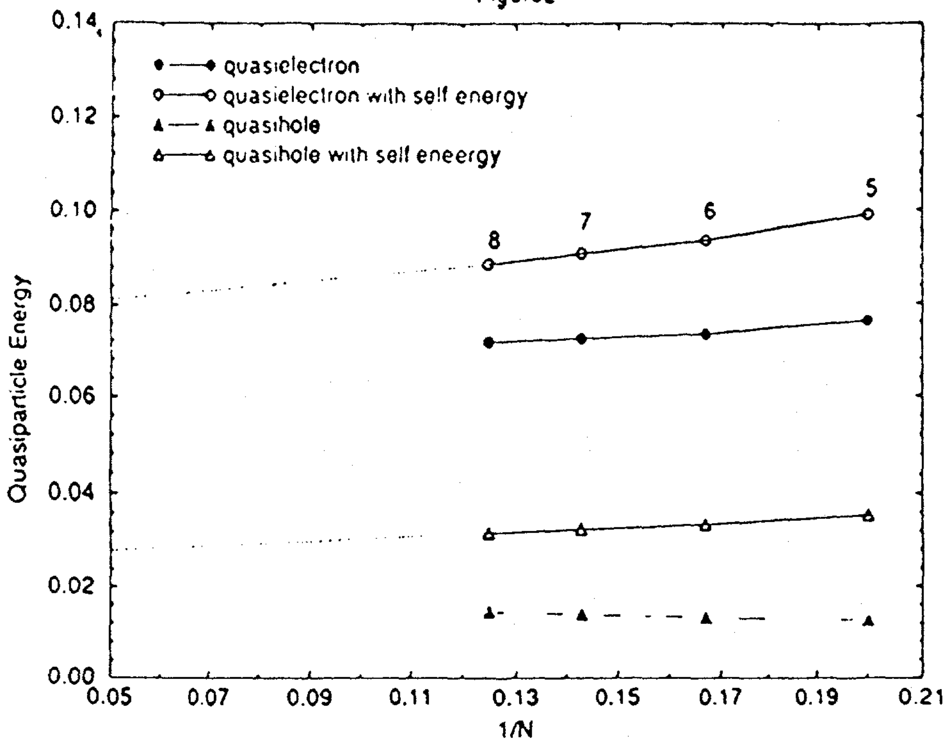

Figure 3

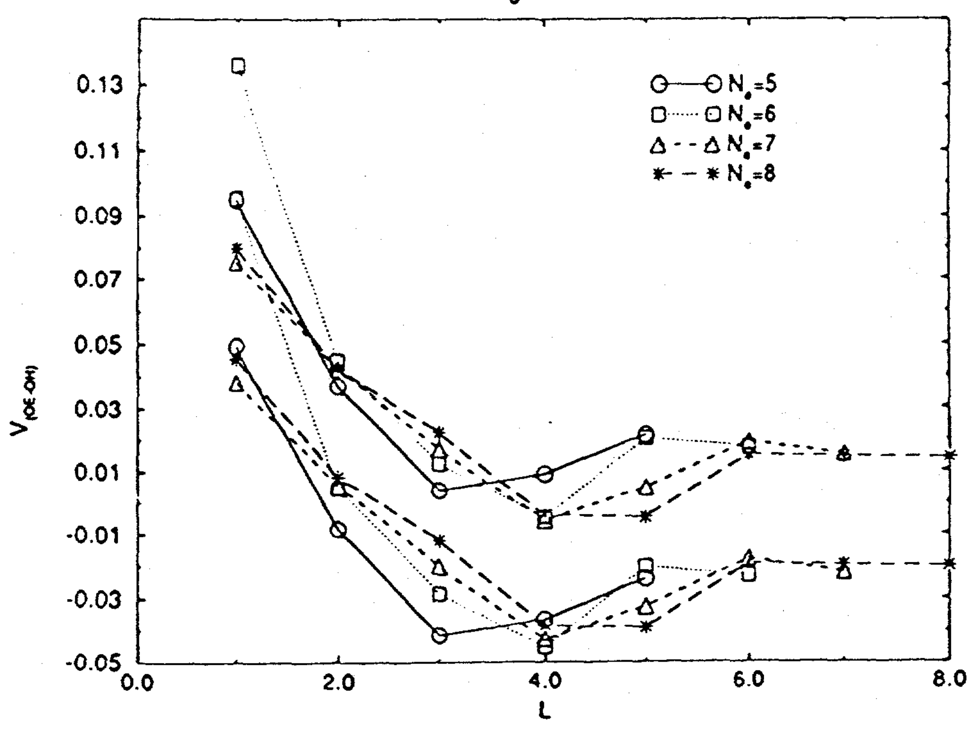

Figure 4

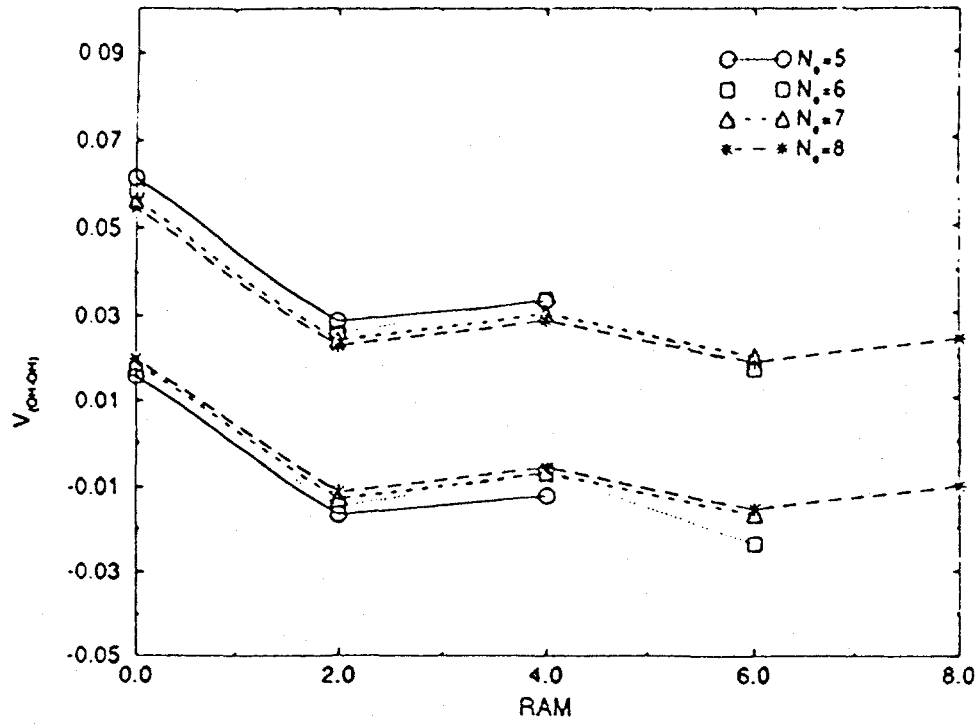

Figure 5