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Adding Magnetization to the Eddy Current Approximation of Maxwell's Equations

Christopher M. Siefert

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185
Livermore, California 94550

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ABSTRACT

The eddy current approximation to Maxwell's equation often omits terms associated with magnetization, removing permanent magnets from the domain of validity of the approximation. We show that adding these terms back into the eddy current approximation is relatively straightforward, and demonstrate this on using a simple material constitutive model.

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CONTENTS

NOTATION

Electromagnetic Quantities

Reference Frame			
Static	Moving	Units	Description
$\mathbf{E}$	$\mathcal{E}$	V/m	Electric field
$\mathbf{J}$	$\mathcal{J}$	A/m^2	Current density
$\mathbf{H}$	$\mathcal{H}$	A/m	Magnetic field strength
$\mathbf{B}$	—	T	Magnetic flux density
$\mathbf{M}$	$\mathcal{M}$	A/m	Magnetization
$\mathbf{D}$	—	C/m^2x	Electric displacement field
$\mathbf{P}$	—	C/m^2	Polarization density
q	—	C/m^3	Charge density

Kinematic Quantities

Symbol	Units	Description
ρ	kg/m^3	Density
p	Pa	Pressure
π	Pa	Pressure in absence of magnetic field
$\mathbf{v}$	m/s	Velocity

Material Parameters

Symbol	Units	Description
σ	S/m	Electrical conductivity
μ	H/m	Magnetic permeability
χ_B	—	Magnetic susceptibility

Material Constants

Symbol	Units	Description
μ_0	H/m	Permeability of free space
ε_0	F/m	Permittivity of free space

1. INTRODUCTION

Following (54.13)–(54.18) in Kovetz [2, p. 218], we begin with Maxwell's equations,

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= q, &(1)\\
\nabla\times\mathcal{H} &= \mathcal{J}+\overset{*}{D}, &(2)\\
\nabla\cdot\mathbf{B} &= 0, &(3)\\
\nabla\times\mathcal{E} &= -\overset{*}{\mathbf{B}}, &(4)\\
\mathbf{D} &= \varepsilon_0\mathbf{E}+\mathbf{P} &(5)\\
\mathcal{H} &= \mu_0^{-1}\mathbf{B}-v\times\varepsilon_0\mathbf{E}-\mathcal{M}, &(6)
\end{aligned}$$

where, following (54.19) in Kovetz [2, p. 90],

$$\overset{*}{\mathbf{C}} = \frac{\partial\mathbf{C}}{\partial t}+\mathbf{v}\nabla\cdot\mathbf{C}-\nabla\times(\mathbf{v}\times\mathbf{C}), \tag{7}$$

for any vector field **C**.

We note that the moving (script) and static (bold) frame fields are related via Galilean transformations following (6.9), (9.9) and (20.5) in Kovetz [2, pp. 25,37,77],

$$\begin{aligned}
\mathcal{M} &= \mathbf{M}+\mathbf{v}\times\mathbf{P}, &(8)\\
\mathcal{E} &= \mathbf{E}+\mathbf{v}\times\mathbf{B}, &(9)\\
\mathcal{H} &= \mathbf{H}-\mathbf{v}\times\mathbf{D}. &(10)
\end{aligned}$$

We then make the two assumptions critical to the eddy current approximation, while retaining magnetization.

Assumption 1. *Polarization, P is zero.*

Assumption 2. *The permittivity of free space,* ε_0*, is zero.*

The first assumption simply states that we do not consider polarizable materials. The second assumption alters the fundamental physics of Maxwell's equations. Combined, these two assumptions imply that free charge, q, and the electric displacement, **D** are also both zero (thus satisfying Gauss' law). This yields the reduced set of equations

$$\begin{aligned}
\nabla\times\mathcal{H} &= \mathcal{J}, &(11)\\
\nabla\cdot\mathbf{B} &= 0, &(12)\\
\nabla\times\mathcal{E} &= -\overset{*}{\mathbf{B}}, &(13)\\
\mathcal{H} &= \mu_0^{-1}\mathbf{B}-\mathcal{M}. &(14)
\end{aligned}$$

With (12), we can restate (13) as

$$\nabla\times\mathcal{E} = -\frac{\partial\mathbf{B}}{\partial t}-\nabla\times(\mathbf{v}\times\mathbf{B}). \tag{15}$$

Substituting in (9) yields the more familiar form of Faraday's Law, namely,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{16}$$

We also note that (8) holds, so in the absence of polarization, $\mathcal{M} = \mathbf{M}$. Likewise, in the absence of displacement currents, $\mathcal{H} = \mathbf{H}$. We now add a third assumption, a constitutive relation for $\mathcal{J}$,

Assumption 3. *Ohm's law holds, namely,*

$$\mathcal{J} = \sigma \mathcal{E}. \tag{17}$$

Then, we can apply (17) to (11) to yield the moving-frame versions of Ampere's law, which results in the governing equations,

$$\nabla \times \mu_0^{-1}\mathbf{B} - \nabla \times \mathbf{M} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{18}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{19}$$

2. MATHEMATICAL IDENTITIES AND MATERIAL MODELS

2.1. Frequently Used Vector Calculus Identities

We will use certain vector calculus identities enough to note them specifically.

$$\nabla(\mathbf{A} \cdot \mathbf{C}) = (\mathbf{A} \cdot \nabla)\mathbf{C} + (\mathbf{C} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{C}) + \mathbf{C} \times (\nabla \times \mathbf{A}), \tag{20}$$

$$\frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) = \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{A} \tag{21}$$

We also note that the Kronecker product of vectors is simply an outer product,

$$\mathbf{x} \otimes \mathbf{y} = \mathbf{x}\mathbf{y}^T. \tag{22}$$

2.2. Constitutive Model for a Linearly Magnetized Material

For the case of diamagnetic or paramagnetic materials, we can use the constitutive relation

$$\mu_0 \mathbf{M} = \chi_B \mathbf{B},, \tag{23}$$

where χ_B is the magnetic susceptibility, cf. equation (4.32) in Kovetz [2, p. 154]. We call these materials linear, as the magnetization $\mathbf{M}$ is a linear function of the $\mathbf{B}$-field. Following equation (43.4) in Kovetz [2, p. 154], we can then use the magnetic susceptibility to define the permeability as

$$\mu = \frac{\mu_0}{1 - \chi_B}, \tag{24}$$

or $\chi_B = 1 - \frac{\mu_0}{\mu}$. Griffiths [1] defines magnetic susceptibility differently, $\mathbf{M} = \chi_H \mathbf{H}$[1]. This is equivalent to $\chi_H = \frac{\mu}{\mu_0} - 1$. These two versions of magnetic susceptibility are related by $(1-\chi_B)(1+\chi_H) = 1$.

Using Kovetz's definition, we can write the overall magnetization in terms of μ as,

$$\mathbf{M} = (\mu_0^{-1} - \mu^{-1})\mathbf{B}. \tag{25}$$

For linear magnetic materials, we can use (25) to rewrite (18) as

$$\nabla \times \mu^{-1}\mathbf{B} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{26}$$

In subsequent sections, especially Chapter 3, we will use the color blue to designate terms associated with linear magnetization.

2.3. Constitutive Model for a Permanently and Linearly Magnetized (PLM) Material

In exercise 70.3, Kovetz [2, p. 262] offers another constitutive model which allows for the addition of a permanently magnetized term, $\mathbf{M}_0$, which does not vary in time. We can then state this *permanent and linear magnet* (PLM) constitutive model as

$$\mathbf{M} = \mathbf{M}_0 + (\mu_0^{-1} - \mu^{-1})\mathbf{B}, \tag{27}$$

With this constitutive model, we can rewrite (18) as

$$\nabla \times \mu^{-1}\mathbf{B} - \nabla \times \mathbf{M}_0 = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{28}$$

In subsequent sections, especially Chapter 3, we will use the color red to designate terms associated with permanent magnetization and blue for those associated with linear magnetization. Thus terms associated with this model will have both terms.

3. MAGNETIZATION AND CONSERVATION OF MOMENTUM

Unlike the previous discussion, where our concerns are limited to purely electromagnetic phenomena, magnetic materials effect the magnetohydrodynamics equations through the momentum equations as well. We begin from the assumption of a perfect electromangetic fluid, (56.17) in Kovetz [2, p. 225],

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + q\mathcal{E} + (\mathcal{J} + \overset{*}{\mathbf{P}}) \times \mathbf{B} + (\mathbf{P} \cdot \nabla)\mathcal{E} + (\mathcal{M} \cdot \nabla)\mathbf{B} + \mathcal{M} \times \nabla \times \mathbf{B} + \rho \mathbf{b}, \tag{29}$$

where $\mathbf{b}$ is an arbitrary body force. Removing polarization ($\mathbf{P}$) and free charge (q) yields,

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathcal{J} \times \mathbf{B} + (\mathbf{M} \cdot \nabla)\mathbf{B} + \mathbf{M} \times \nabla \times \mathbf{B} + \rho \mathbf{b}. \tag{30}$$

[1] Griffiths calls this χ_M, but we use χ_H here instead.

Kovetz does note that when $\mathcal{M} \neq 0$ or $\mathbf{P} \neq 0$, the pressure term p can be a function of $\mathcal{E}$ and $\mathbf{B}$, which means that the pressure term is as "electromagnetic" as the other terms.

We should also consider the magnetic stress tensor, since the divergence of the magnetic stress tensor is how magnetic forces are actually computed in may codes. Again following the perfect electromagnetic fluid assumption, (56.15) in Kovetz [2, p. 223], we have,

$$\begin{aligned} T &= -\left[p + \frac{1}{2}\varepsilon_0 \|\mathbf{E}\|^2 + \frac{1}{2}\mu_0^{-1}\|\mathbf{B}\|^2 - \mathcal{M}\cdot\mathbf{B}\right] I \\ &\quad + \varepsilon_0 \mathbf{E}\otimes\mathbf{E} + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B} + \mathcal{E}\otimes\mathbf{P} - \mathcal{M}\otimes\mathbf{B} + \varepsilon_0 \mathbf{E}\times\mathbf{B}\times\mathbf{v}. \end{aligned} \tag{31}$$

The version of (31) with pressure, polarization, magnetization and velocity removed is often referred to as the "Maxwell stress tensor." Neglecting polarization ($\mathbf{P}$) and permittivity (ε_0), we have

$$T = -\left[p + \frac{1}{2}\mu_0^{-1}\|\mathbf{B}\|^2 - \mathbf{M}\cdot\mathbf{B}\right] I + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B} - \mathbf{M}\otimes\mathbf{B}. \tag{32}$$

When Kovetz notes that the pressure is "electromagnetic," he means something rather specific for a permeable fluid at rest, namely,

$$p = \pi + \frac{1}{2}\mathcal{M}\cdot\mathbf{B}, \tag{33}$$

where π is the pressure in the *absence* of a magnetic field [2, p. 260]. This means that the stress tensor can be rewritten as

$$T = -\left[\pi + \frac{1}{2}\mu_0^{-1}\|\mathbf{B}\|^2 - \frac{1}{2}\mathbf{M}\cdot\mathbf{B}\right] I + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B} - \mathbf{M}\otimes\mathbf{B}. \tag{34}$$

3.1. Momentum and Non-magnetized Materials

For non-magnetic materials, we can simplify (30) to

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla\pi + \mathcal{J}\times\mathbf{B} + \rho\mathbf{b}, \tag{35}$$

and (34) to

$$T_0 = -\left[\pi + \frac{1}{2}\mu_0^{-1}\|\mathbf{B}\|^2\right] I + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B}, \tag{36}$$

or

$$T_0 = \mu_0^{-1}\begin{bmatrix} Bx^2 - \frac{1}{2}\|\mathbf{B}\| & BxBy & BxBz \\ BxBy & By^2 - \frac{1}{2}\|\mathbf{B}\| & ByBz \\ BxBz & ByBz & Bz^2 - \frac{1}{2}\|\mathbf{B}\| \end{bmatrix} - \pi I. \tag{37}$$

3.2. Momentum and Linearly Magnetizable Materials

For a linearly magnetic material, (30) can be written as

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathcal{J}\times\mathbf{B} + \left((\mu_0^{-1}-\mu^{-1})\mathbf{B}\cdot\nabla\right)\mathbf{B} + (\mu_0^{-1}-\mu^{-1})\mathbf{B}\times\nabla\times\mathbf{B} + \rho\mathbf{b}, \tag{38}$$

or by (21),

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathcal{J}\times\mathbf{B} + \frac{1}{2}(\mu_0^{-1}-\mu^{-1})\nabla(\mathbf{B}\cdot\mathbf{B}) + \rho\mathbf{b}. \tag{39}$$

This yields the standard (hydrodynamic) version of Euler's first law (excepting the electromagnetic part of the pressure term), plus the Lorentz force and the additional term shown in blue. We can do the same for the magnetic stress tensor (34)

$$T_{LM} = -\left[\pi + \frac{1}{2}\|\mathbf{B}\|^2 - \frac{1}{2}(\mu_0^{-1}-\mu^{-1})\mathbf{B}\cdot\mathbf{B}\right]I + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B} - (\mu_0^{-1}-\mu^{-1})\mathbf{B}\otimes\mathbf{B}, \tag{40}$$

with the new terms again appearing in blue. We can also consider new entries of this tensor,

$$T_{LM} - T_0 = (\mu_0^{-1}-\mu^{-1})\begin{bmatrix} Bx^2 - \frac{1}{2}\|\mathbf{B}\| & BxBy & BxBz \\ BxBy & By^2 - \frac{1}{2}\|\mathbf{B}\| & ByBz \\ BxBz & ByBz & Bz^2 - \frac{1}{2}\|\mathbf{B}\| \end{bmatrix}. \tag{41}$$

Note that every entry in the tensor is changed by the addition of linear magnetization. This has the effect of making T_{LM} effectively T_0 with μ_0 replaced by μ.

3.3. Momentum and PLM Materials

For a PLM constitutive model, (30) can be written as

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathcal{J}\times\mathbf{B} + \left(\left(\mathbf{M}_0 + (\mu_0^{-1}-\mu^{-1})\mathbf{B}\right)\cdot\nabla\right)\mathbf{B} + \left(\mathbf{M}_0 + (\mu_0^{-1}-\mu^{-1})\mathbf{B}\right)\times\nabla\times\mathbf{B} + \rho\mathbf{b}, \tag{42}$$

or by (21),

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathcal{J}\times\mathbf{B} + \frac{1}{2}(\mu_0^{-1}-\mu^{-1})\nabla(\mathbf{B}\cdot\mathbf{B}) + (\mathbf{M}_0\cdot\nabla)\mathbf{B} + \mathbf{M}_0\times\nabla\times\mathbf{B} + \rho\mathbf{b}. \tag{43}$$

Here we have a blue term very much like the linear case, but we also have two red terms associated with the permanent magnetization term $\mathbf{M}_0$.

We also note that if we assume $\mathbf{M}_0$ is constant within each element, then by (20) we have

$$\nabla(\mathbf{M}_0\cdot\mathbf{B}) = (\mathbf{M}_0\cdot\nabla)\mathbf{B} + \mathbf{M}_0\times\nabla\times\mathbf{B}, \tag{44}$$

so the momentum equation can be rewritten in terms of the so-called "magnetic loop force," $\nabla(\mathbf{M}_0\cdot\mathbf{B})$, which the force on a permanent magnetic exerted by a non-uniform magnetic field,

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathcal{J}\times\mathbf{B} + \frac{1}{2}(\mu_0^{-1}-\mu^{-1})\nabla(\mathbf{B}\cdot\mathbf{B}) + \nabla(\mathbf{M}_0\cdot\mathbf{B}) + \rho\mathbf{b}. \tag{45}$$

We can also consider the magnetic stress tensor (34) in this case we get

$$T = -\left[\pi + \frac{1}{2}\mu_0^{-1}\|\mathbf{B}\|^2 - \frac{1}{2}\left(\mathbf{M}_0 + (\mu_0^{-1} - \mu^{-1}\right)\cdot\mathbf{B}\right] I + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B} - \left(\mathbf{M}_0 + (\mu_0^{-1} - \mu^{-1}\right)\otimes\mathbf{B}, \tag{46}$$

or

$$\begin{aligned} T \;=\; & -\left[\pi + \frac{1}{2}\mu_0^{-1}\|\mathbf{B}\|^2 - \frac{1}{2}\mathbf{M}_0\cdot\mathbf{B} - \frac{1}{2}(\mu_0^{-1} - \mu^{-1})\mathbf{B}\cdot\mathbf{B}\right] I \\ & + \mu_0^{-1}\mathbf{B}\otimes\mathbf{B} - \mathbf{M}_0\otimes\mathbf{B} - (\mu_0^{-1} - \mu^{-1})\mathbf{B}\otimes\mathbf{B}, \end{aligned} \tag{47}$$

with the color coding as above. We can then consider the change in the magnetic stress tensor entry-wise,

$$\begin{aligned} T_{PLM} - T_0 \;=\; & \begin{bmatrix} \frac{1}{2}\mathbf{M}_0\cdot\mathbf{B} - M0xBx & -BxM0y & -BxM0z \\ -ByM0x & \frac{1}{2}\mathbf{M}_0\cdot\mathbf{B} - M0yBy & -ByM0z \\ -BzM0x & -BzM0y & \frac{1}{2}\mathbf{M}_0\cdot\mathbf{B} - BzM0z \end{bmatrix} \\ & + (\mu_0^{-1} - \mu^{-1}) \begin{bmatrix} Bx^2 - \frac{1}{2}\|\mathbf{B}\| & BxBy & BxBz \\ BxBy & By^2 - \frac{1}{2}\|\mathbf{B}\| & ByBz \\ BxBz & ByBz & Bz^2 - \frac{1}{2}\|\mathbf{B}\| \end{bmatrix}, \end{aligned} \tag{48}$$

Where the second term is $T_{LM} - T_0$. We note that this stress tensor *is not symmetric*.

4. CONCLUSIONS

We have derived an extension to the eddy current equations which allows for magnetization. We have shown how linearly and permanently magnetizable materials can be integrated into this framework. Moreover, we have shown how magnetization modifies the Maxwell stress tensor and have calculated the resulting magnetic forces.

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[2] A. Kovetz. *Electromagnetic Theory*. Oxford University Press, 2000.

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Name	Company Email Address	Company Name
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Email—Internal (encrypt for OUO)

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