A Multi-Parametric optimization approach for bilevel mixed-integer linear and quadratic programming problems

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Abstract

Optimization problems involving two decision makers at two different decision levels are referred to as bi-level programming problems. In this work, we present novel algorithms for the exact and global solution of two classes of bi-level programming problems, namely (i) bi-level mixed-integer linear programming problems (B-MILP) and (ii) bi-level mixed-integer convex quadratic programming problems (B-MIQP) containing both integer and bounded continuous variables at both optimization levels. Based on multi-parametric programming theory, the main idea is to recast the lower level problem as a multi-parametric programming problem, in which the optimization variables of the upper level problem are considered as bounded parameters for the lower level. The resulting exact multi-parametric mixed-integer linear or quadratic solutions are then substituted into the upper level problem, which can be solved as a set of single-level, independent, deterministic mixed-integer optimization problems. Extensions to problems including right-hand-side uncertainty on both lower and upper levels are also discussed. Finally, computational implementation and studies are presented through test problems.

Keywords: Bilevel programming, Multi parametric programming,

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1. Introduction

Optimization problems involving two decision makers at two different decision levels are referred to as bilevel programming problems: the first decision maker (upper level; leader) is solving an optimization problem which includes in its constraint set another optimization problem solved by the second decision maker (lower level; follower). This class of problems has attracted considerable attention across a broad range of research communities, including economics, sciences and engineering. It was applied to many and diverse problems that require hierarchical decision making such as transportation network planning (Boyce and Mattsson, 1999; Migdalas, 1995; Yang and Yagar, 1995), urban planning (Tam and Lam, 2004), economic planning (Gao et al., 2011; Miljkovic, 2002; Robbins and Lunday, 2016), policy decision making (Avraamidou et al., 2018), design under uncertainty (Floudas et al., 2001; Ierapetritou and Pistikopoulos, 1996; Ryu et al., 2004; Avraamidou and Pistikopoulos), design and control integration (Brengel and Seider, 1992; Luyben and Floudas, 1994a,b; Tanartkit and Biegler, 1996), hierarchical control (Avraamidou and Pistikopoulos, 2017a; Faisca et al., 2009; Katebi and Johnson, 1997), and supply chain planning (Gao and You, 2018; Calvete et al., 2010; Grossmann, 2005; Seferlis and Giannelos, 2004; Avraamidou and Pistikopoulos, 2017b; Yue and You, 2017).

Such decision making problems can involve decisions in both discrete and continuous variables. A motivating example that falls in this class is presented below.

1.1. Motivating example: Production and Distribution Planning Integration

Supply chains are systems with multiple decision levels corresponding to different activities, spanning from the procurement of raw materials to the distribution of the final products to the costumers. Even though these decisions are interlinked and can affect each other, in most cases they are considered individually (Grossmann, 2005, 2004).

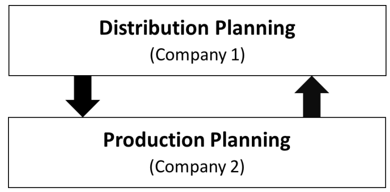

Figure 1: Schematic of the production-distribution planning problem with two companies

The significance of the integration of production and distribution decisions inside supply chains, in order to account for the interactions between them, has been recognized by different researchers (Erenguc et al., 1999; Vidal and Goetschalckx, 1997; Grossmann, 2005). Proposed integrated approaches include assuming (i) that one company controls the integrated process by owning both the processing plants and distribution centers (Gupta and Maranas, 2000; Sousa et al., 2008; Jung et al., 2004), or (ii) that the processing plants and distribution centers are owned by different companies, each trying to optimize their own objective (Calvete et al., 2010; Seferlis and Giannelos, 2004; Roghanian et al., 2007; Kuo and Han, 2011; Ivanov et al., 2013).

Considering the second case, the production-distribution planning (PD) problem can be expressed as a hierarchical decision problem, involving two different decision makers corresponding to each company. Assuming one company owns the production plants and another the distribution centers, the resulting problem is a two level hierarchical decision problem. The first level is responsible for optimizing the distribution centers overall costs and is influenced by the second level that is responsible for optimizing the production plants overall costs.

When considering the PD problem, decisions taken at both decision levels can involve both continuous (e.g. production rates, distribution rates) or discrete (e.g. choice of production plant, choice of distribution center, active routes) variables. Therefore, the integrated PD problem results into a mixed-integer bilevel programming problem with both integer and continuous variables at both optimization levels.

1.2. *Mixed-integer bilevel programming problems (B-MIP)*

Problems such as the integrated PD problem are referred to as mixed-integer bilevel programing problems (B-MIP), and have the general form of:

$$\begin{aligned}
\min_{x_1,y_1} \quad & F_1(x,y) \\
\text{s.t.} \quad & G_1(x,y) \leq 0 \\
& H_1(x,y) = 0 \\
& x_2, y_2 \in \arg\min_{x_2,y_2} \{F_2(x,y) : G_2(x,y) \leq 0, H_2(x,y) = 0\} \\
& x = [{x_1}^T \ {x_2}^T]^T, \quad y = [{y_1}^T \ {y_2}^T]^T \\
& x \in \mathbb{R}^n, \quad y \in \mathbb{Z}^m
\end{aligned} \tag{1}$$

where x_1 is a vector of the upper level problem continuous variables, y_1 is a vector of the upper level integer variables, x_2 is a vector of the lower level problem continuous variables, y_2 is a vector of the lower level integer variables, x is a vector of all continuous variables, y is a vector of all integer variables.

The general formulation of the mixed-integer bilevel programming problem (1), corresponds to a number of different classes of problems. Table 1 classifies these problems into four categories that can be expanded to cover the linear, quadratic and non-linear sub-class of each Type, as identified by Gumus and Floudas (2005).

Table 1: Types of mixed-integer bilevel programming problems

Problem Type	**Upper level variables**	**Lower Level variables**
Type 1	Continuous and/or Integer	Continuous
Type 2	Integer	Integer
Type 3	Continuous	Integer
Type 4	Continuous and/or Integer	Continuous and Integer

1.3. *Challenges and Previous work*

Bilevel programming problems are very challenging to solve, even in the linear case (shown to be NP-hard by Hansen et al. (1992) and Deng (1998)). To strengthen these results Vicente et al. (1994) proved that even checking strict or local optimality is NP-hard.

For classes of problems where the lower level problem also involves discrete variables, the complications are further increased, typically requiring global optimization methods for their solution and often resulting to approximate solutions. The major difficulty for this class of problems arises from the fact that conventional solution methods for continuous bi-level problems are no longer applicable when integer variables exist at the lower level. One of the most widely used solution approaches for continuous bi-level problems with convex objective functions and constraints, is the transformation of the problem to a single level problem using the Karush-Kuhn-Tucker (KKT) optimality conditions. Since this method requires gradient information it is not directly applicable to bi-level problems with integer variables on the lower level, even though in some cases there is merit in using them (Gumus and Floudas, 2005; Saharidis and Ierapetritou, 2009; Mitsos, 2010). Also, the branch and bound rules used to solve mixed-integer problems cannot be directly or effectively applied to mixed-integer bi-level problems (Bard and Moore, 1990). It is worth noting here, that algorithms developed for non-convex continuous bi-level problems such as Gumus and Floudas (2001); Mitsos et al. (2008); Zhu and Guo (2017), can be extended into solution methods for mixed-integer problems.

In the literature, methods developed for the solution of mixed-integer bi-level problems have mainly addressed the linear Type 1 and 2 problems. Tables 2, 3, 4 and 5 summarize some of the most important solution methods for bi-level mixed-integer linear problems of Type 1, Type 2, Type 3 and Type 4 respectively, in the open literature. Table 6 and Table 7 summarize approaches for the solution of bi-level mixed-integer non-linear problems of Type 1, Type 2 and Type 4 [1]. Note that general strategies for the global and exact solution of Type 4 bi-level mixed-integer linear or quadratic problems are scarce.

[1] The **Notes** column in Tables 2, 3, 4, 5, 6 and 7 represents important features, limitations or advantages of the works as written in each individual manuscript.

Table 2: Indicative list of previous work on bi-level mixed-integer **linear** optimization problems of **Type 1**

Algorithm	**Reference**	**Notes**
Branch and Bound	Wen and Yang (1990)	Heuristic approach, only integer optimization variables are allowed in the upper level.
Tabu search	Wen and Huang (1996)	Only integer optimization variables in the upper level. Approximate.
Multi-Parametric Programming	Faisca et al. (2007)	Exact.
Benders decomposition	Caramia and Mari (2016) Fontaine and Minner (2014)	ϵ-optimal.

Table 3: Indicative list of previous work on bi-level mixed-integer **linear** optimization problems of **Type 2**

Algorithm	**Reference**	**Notes**
Penalty Function	Vicente et al. (1996)	Also provided theory for Type 1.
Branch and Bound	Bard and Moore (1992)	Implicit Enumeration. Assume: all binary, no upper level constraints.
Chvatal-Gomory cuts (cutting plane)	Dempe (2001)	Generates a lower bound to the problem.
Branch and Cut (cutting plane)	DeNegre and Ralphs (2009)	All binary. Based on Bard and Moore (1992)
Genetic Algorithm	Nishizaki and Sakawa (2005)	Approximate solutions.
Evolutionary Algorithm	Handoko et al. (2015)	Global optimality is not guaranteed.

Table 4: Indicative list of previous work on bi-level mixed-integer **linear** optimization problems of **Type 3**

Algorithm	**Reference**	**Notes**
Branch and Cut	Dempe and Kue (2017)	Lower level variables cannot affect the upper level constraints.
Polynomial Approximation	Dempe (2001) Dempe et al. (2000)	Cutting plane, approximate.
Parametric integer programming	Koppe et al. (2010)	Cannot guarantee optimality.

1.4. Key contribution

In this paper, we present global optimization algorithms for the *exact* solution of two classes of bilevel programming problems, namely:

1. Bilevel mixed-integer linear programming problems (B-MILP)
2. Bilevel mixed-integer convex quadratic programming problems (B-MIQP)

Both classes belong to sub-class Type 4 (i.e. containing both integer and continuous variables at both optimization levels), while the proposed algorithms are also applicable to problems of Types 1-3.

The proposed algorithms are a result of new developments on multi-parametric programming theory (Acevedo and Pistikopoulos, 1997; Oberdieck and Pistikopoulos, 2015; Oberdieck et al., 2016a) and our earlier results and developed algorithms on *continuous* bilevel linear and quadratic programming (Faisca et al., 2007, 2009). The main idea of this type of algorithms for the solution of mixed-integer bilevel problems is to recast the lower level problem as a multi-parametric programming problem, in which the optimization variables of the upper level problem (both continuous and integer) are considered as parameters for the lower level problem. The resulting exact parametric solutions are then substituted into the upper level problem, which can be solved as a set of single-level deterministic mixed-integer programming problems.

Table 5: Indicative list of previous work on bi-level mixed-integer **linear** optimization problems of **Type 4**

Algorithm	**Reference**	**Notes**
Branch and Bound	Moore and Bard (1990)	Implicit Enum. Cannot guarantee optimality.
Penalty Function	Dempe et al. (2005)	Approximate local solutions.
Benders decomposition	Saharidis and Ierapetritou (2009)	ϵ-optimal. Leader controls all binary variables.
Branch and Bound	Xu and Wang (2014); Xu (2012) Caramia and Mari (2016)	Only integer optimization variables in the upper level.
Lagrangean relaxation	Rahmani and MirHassani (2015)	Lower level variables cannot appear in the constraints of the upper level.
Projection-based Reformulation	Yue and You (2016) Zeng and An (2014)	ϵ-optimal.
Row-and-column generation	Poirion et al. (2015)	ϵ-optimal.
Branch-and-Cut	Fischetti et al. (2016)	Exact. Leader variables that influence the follower decisions are all integer.
	Fischetti et al. (2017)	Exact.

Table 6: Indicative list of previous work on bi-level mixed-integer **non-linear** optimization problems of **Type 1** and **Type 2**

Type	**Algorithm**	**Reference**	**Notes**
Type 1	Branch and Bound	Edmunds and Bard (1992)	Lower level is convex quadratic.
		Gumus and Floudas (2001)	Approximate.
Type 2	Parametric Analysis	Jan and Chern (1994)	Only for separable and monotone constraints and objective.
	Fuzzy Programming	Emam (2006)	Pareto optimal solution.

The proposed algorithms are implemented in a prototype toolbox, B-POP (Avraamidou and Pistikopoulos, 2018a), that uses POP's (Oberdieck et al., 2016b) mp-MILP and mp-MIQP solvers to solve B-MILP and B-MIQP problems. Computational studies are carried out to show the capabilities and scalability of B-POP. To our knowledge, B-POP is currently the only freely accessible toolbox for the solution of bi-level mixed-integer linear and quadratic programming problems.

Section 2 presents the solution algorithm for the Type 4 B-MILP, while its application is illustrated via three numerical examples. Section 3 addresses the Type 4 B-MIQP algorithm and is illustrated through two numerical examples. The theory is extended in Section 4 to cover the existence of right hand side uncertainty on both optimization levels, while Section 5 summarizes the computational implementation and computational studies of the presented algorithms.

2. Bilevel Mixed-integer Linear Programming Problems

A widely known property of the general bilevel programming problem is that the feasible set of the inner problem is parametric in terms of the decision

Table 7: Indicative list of previous work on bi-level mixed-integer **linear** optimization problems of **Type 4**

Algorithm	**Reference**	**Notes**
Simulated Annealing	Sahin and Ciric (1998)	Near global solutions.
Branch and Bound	Gumus and Floudas (2005)	The lower level must be linear in continuous variables.
Genetic Algorithms	Li and Wang (2008) Hecheng and Yuping (2008)	Lower level functions are separable or convex.
	Arroyo and Fernandez (2009)	Near optimal solutions.
Multi-Parametric Programming	Dominguez and Pistikopoulos (2010)	Reformulation via convex hull. Approximate.
Branch and Sandwich	Kleniati and Adjiman (2015)	ϵ-optimal.
Bounding	Mitsos (2010)	ϵ-optimal.
Value Function Based	Lozano and Smith (2017)	Requires all upper-level variables to be integer.

variables of the outer problem. To effectively utilize this property, Pistikopoulos and co-workers have presented a series of algorithms based on multi-parametric programming theory, which can address different classes of continuous multilevel programming problems (Faisca et al., 2007, 2006).

Expanding on their earlier works, the approach presented here is based upon a recently proposed Multi-parametric Mixed-integer Linear Programming (mp-MILP) algorithm of Oberdieck et al., 2014 Oberdieck et al. (2014) summarized in Appendix I, and new theory for binary parameters in multi-parametric programming problems (Oberdieck et al., 2017). The proposed methodology will be firstly introduced through the general form of the B-MILP problem (2), and then illustrated through 3 numerical examples.

$$\begin{aligned} \min_{x_1,y_1} \quad & F_1(x,y) = {c_1}^T x + {d_1}^T y \\ \text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\ & x_2, y_2 \in \arg\min_{x_2,y_2} \{F_2(x,y) = {c_2}^T x + {d_2}^T y : A_2 x + B_2 y \leq b_2\} \\ & x = [{x_1}^T \ {x_2}^T]^T, \quad y = [{y_1}^T \ {y_2}^T]^T \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in Y \subseteq \mathbb{Z}^m \end{aligned} \tag{2}$$

where c_1, d_1, A_1, B_1, b_1 are constant coefficient matrices in the upper level (leader) problem, and c_2, d_2, A_2, B_2, b_2 are constant coefficient matrices in the lower level (follower) problem, and X and Y are compact polyhedral convex sets of dimensions n and m, respectively.

As a first step, we establish bounds for all integer and continuous variables, by solving problems (3) to (6) for upper level variables $x_{1,\alpha}$ and $y_{1,\beta}$, for all $\alpha \in \{1, ..n_1\}$ and $\beta \in \{1, ..., n_2\}$, where n_1 and n_2 are the dimensions of vectors x_1 and y_1 respectively. Similar problems are solved for the lower level variables $x_{1,\gamma}$ and $y_{2,\delta}$, for all $\gamma \in \{1, ..n_3\}$ and $\delta \in \{1, ..n_4\}$ (where n_3 and n_4 are the dimensions of vectors x_2 and y_2 respectively), in order to obtain bounds on both x, $x^L \leq x \leq x^U$, and y, $y^L \leq y \leq y^U$.

$$\begin{aligned} {x_{1,\alpha}}^L = \min \quad & x_{1,\alpha} \\ \text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\ & A_2 x + B_2 y \leq b_2 \end{aligned} \tag{3}$$

$$
\begin{aligned}
{x_{1,\alpha}}^U = \min \quad & -x_{1,\alpha} \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& A_2 x + B_2 y \leq b_2
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
{y_{1,\beta}}^L = \min \quad & y_{1,\beta} \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& A_2 x + B_2 y \leq b_2
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
{y_{1,\beta}}^U = \min \quad & -y_{1,\beta} \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& A_2 x + B_2 y \leq b_2
\end{aligned} \tag{6}
$$

Then, the B-MILP is transformed into a binary B-MILP by expressing integer variables, $y_{1,1}....y_{1,n_2}$ and $y_{2,1}....y_{2,n_4}$, in terms of binary 0-1 variables, $\widehat{y_{1,\beta,1}}, ..., \widehat{y_{1,\beta,n_5}} \in \{0,1\}$ for all β and $\widehat{y_{2,\delta,1}}, ..., \widehat{y_{2,\delta,n_6}} \in \{0,1\}$ for all δ, by following the formulas presented in Floudas (1995) (Section 6.2.1 - Remark 1). The hat accent will be omitted in the following steps for simplicity.

As a next step, the lower level problem of the binary B-MILP, is transformed into a mp-MILP problem (7), in which the optimization variables of the upper level problem, x_1 and y_2 that appear in the lower level problem, are considered as parameters for the lower level.

$$
\begin{aligned}
\min_{x_2, y_2} \quad & {d_2}^T y + {c_2}^T x \\
\text{s.t.} \quad & B_2 y \leq b_2 - A_2 x \\
& x^L \leq x \leq x^U
\end{aligned} \tag{7}
$$

The solution of (7) using multi-parametric mixed-integer programming (mp-MILP) solution algorithms, such as Bank and Hansel (1984); Bank and Mandel (1988); Baotic et al. (2006); Bemporad et al. (2000); Crema (2002); Dua and Pistikopoulos (2000); Dua et al. (2002); Jia and Ierapetritou (2006); Li and Ierapetritou (2007); Oberdieck and Pistikopoulos (2015); Oberdieck et al. (2014); Wittmann-Hohlbein and Pistikopoulos (2012, 2013), provide the complete profile of optimal solutions of the lower level problem as explicit functions of the variables of the higher level problem with corresponding expressions (8).

POP® , the Parametric Optimization toolbox (Oberdieck et al., 2016b), can be used at this step to get the optimal solutions (8). POP® toolbox features a state-of-the-art multi-parametric programming solver for continuous and mixed-integer linear and quadratic problems. The toolbox is freely available for download in *parametric.tamu.edu* website. Appendix I illustrates the main steps for the solution of mp-MILP problems through POP® toolbox.

$$x_2, y_2 = \begin{cases} \xi_1 = p_1 + q_1 x_1 + r_1 y_1, \quad \psi_1 & \text{if } H_1[x_1^T y_1^T]^T \leq h_1 \\ \xi_2 = p_2 + q_2 x_1 + r_2 y_1, \quad \psi_2 & \text{if } H_2[x_1^T y_1^T]^T \leq h_2 \\ \vdots & \vdots \\ \xi_k = p_k + q_k x_1 + r_k y_1, \quad \psi_k & \text{if } H_k[x_1^T y_1^T]^T \leq h_k \end{cases} \tag{8}$$

where ξ_i are vectors of the lower level (follower) continuous variables and ψ_i are vectors of the lower level integer variables, p_i, q_i and r_i are all constant vectors, $H_k[x_1^T y_1^T]^T \leq h_k$ is referred to as critical region(CR^k), and k denotes the number of computed critical regions.

Remark 1: Solutions (8) comprise of all *optimal* solutions in the feasible space of the upper level variables and not all feasible solutions.

The computed solutions (8) are then substituted into the upper level problem, which can be solved as a set of single-level deterministic mixed-integer programming problems, (9). More specifically, the functions ξ expressing the lower level variables (x_2) in terms of the upper level variables (x_1 and y_1), are substituted in the place of lower level variables (x_2 and y_2) in the upper level problem, eliminating in this way the lower level variables form the upper level problem. Moreover, the critical region definitions are added to the corresponding single level problems as an additional set of constraints.

$$
\begin{aligned}
z_1 = \min_{x_1,y_1} \quad & {c_1}^T[{x_1}^T \xi_1(x_1,y_1)^T]^T + {d_1}^T[{y_1}^T {\psi_1}^T]^T \\
\text{s.t.} \quad & A_1[{x_1}^T \xi_1(x_1,y_1)^T]^T + B_1[{y_1}^T {\psi_1}^T]^T \leq b_1 \\
& H_1[x_1^T y_1^T]^T \leq h_1 \\
\\
z_2 = \min_{x_1,y_1} \quad & {c_1}^T[{x_1}^T \xi_2(x_1,y_1)^T]^T + {d_1}^T[{y_1}^T {\psi_2}^T]^T \\
\text{s.t.} \quad & A_1[{x_1}^T \xi_2(x_1,y_1)^T]^T + B_1[{y_1}^T {\psi_2}^T]^T \leq b_1 \\
& H_2[x_1^T y_1^T]^T \leq h_2 \\
& \vdots \\
z_k = \min_{x_1,y_1} \quad & {c_1}^T[{x_1}^T \xi_k(x_1,y_1)^T]^T + {d_1}^T[{y_1}^T {\psi_k}^T]^T \\
\text{s.t.} \quad & A_1[{x_1}^T \xi_k(x_1,y_1)^T]^T + B_1[{y_1}^T {\psi_k}^T]^T \leq b_1 \\
& H_k[x_1^T y_1^T]^T \leq h_k
\end{aligned}
\tag{9}
$$

The single-level, deterministic programming problems (9) are independent of each other, making it possible to use parallel programming to solve them simultaneously.

The solutions of the above single level MILP problems correspond to different sub-optimal solutions of the original B-MILP. The final step of the algorithm is to compare all the sub-optimal solutions to obtain the minimum z that would correspond to the exact and global optimum, z^*, of the original bi-level problem.

The proposed algorithm is summarized in Algorithm 1.

Remark 2: *Pessimistic and Optimistic Solutions:*

When the optimal solution of the lower level problem is not unique for the set of optimal upper level variables the decision maker can take a pessimistic or an optimistic decision.This degeneracy can result either because of the lower level integer variables or because of the lower level continuous variables.

For the cases where a degeneracy results because of the lower level integer variables the solution method described above is able to capture all degenerate solutions and therefore supply the decision maker with both the

Algorithm 1 Multi-parametric algorithm for the solution of Bilevel Mixed-Integer Linear Programming problems

Establish integer and continuous variable bounds
Express integer variables into binary and substitute in (2)
Formulate the mp-MILP problem (7)
Solve (7) and obtain solution $[x_2 \quad y_2]^T = \mathcal{F}_i(x_1, y_1)$ defined over CR_i.
for $i \leftarrow 1, ..., \#CR_i$ **do**
    Formulate MILP (9-i)
    Solve (9-i) to get candidate solution z_i
end for
return z_i with minimum value

pessimistic and optimistic solutions.

For the cases where a degeneracy results because of the lower level continuous variables the multi-parametric solution via POP® toolbox is not able to supply the decision maker with the full range of degenerate solutions. Even though there are techniques to handle degeneracy in multi-parametric problems (Gal and Nedoma, 1972; Jones et al., 2007; Olaru and Dumur, 2006; Spjotvold et al., 2005), those are not yet implemented in the approach described above.

Therefore, it is assumed that there is a unique optimal solution for the continuous lower level variables corresponding to the upper level optimal solution.

2.1. Numerical Examples

Three B-MILP numerical examples will be solved to illustrate the use of the proposed algorithm.

2.1.1. Example 1: LP-ILP

Consider the following Type 3 example taken from Dempe (2001):

$$
\begin{aligned}
\min_{y} \quad & -x_1 - 2x_2 + 3y_1 + 3.2y_2 \\
\text{s.t.} \quad & -y_1 - y_2 \leq 2 \\
& y_1 + y_2 \leq 2 \\
& -2 \leq y_{1,2} \leq 2 \\
& \min_{x} - x_1 y_1 - x_2 y_2 \\
& \text{s.t.} - x_1 + 3x_2 \leq 3 \\
& \qquad x_1 - x_2 \leq 1 \\
& \qquad -x_1 - x_2 \leq -2 \\
& y \in \mathbb{R}^n, \quad x \in \mathbb{Z}^{+m}
\end{aligned}
\tag{10}
$$

Step 1: Bounds are established for the unbounded integer variables x_1 and x_2 resulting in $1 \leq x_1 \leq 3$ and $1 \leq x_2 \leq 2$.

Step 2: The problem is transformed into a 0-1 binary B-MILP. Following Floudas (1995), the integer variables x_1 and x_2 can be expressed through binary variables as $x_1 = 1 + x_{1a} + 2x_{1b}$ and $x_2 = 1 + x_{2a}$. Therefore, formulation (10) can be reformulated as (11).

$$
\begin{aligned}
\min_{y} \quad & -x_{1a} - 2x_{1b} - 2x_{2a} + 3y_1 + 3.2y_2 - 3 \\
\text{s.t.} \quad & -y_1 - y_2 \leq 2 \\
& y_1 + y_2 \leq 2 \\
& -2 \leq y_{1,2} \leq 2 \\
& (x_{1a}, x_{1b}, x_{2a}) \in \arg\{ \min_{x_{1a}, x_{1b}, x_{2a}} (x_{1a} + 2x_{1b} + 1)y_1 + (x_{2a} + 1)y_2 \\
& \qquad \text{s.t.} \quad - x_{1a} - 2x_{1b} + 3x_{2a} \leq 1 \\
& \qquad \qquad x_{1a} + 2x_{1b} - x_{2a} \leq 1 \\
& \qquad \qquad -x_{1a} - 2x_{1b} - x_{2a} \leq 0 \quad \} \\
& y \in \mathbb{R}^2, \quad x_{1a}, x_{1b}, x_{2a} \in \{0, 1\}^3
\end{aligned}
\tag{11}
$$

Step 3: The lower level problem is reformulated as a mp-ILP, in which the optimization variables of the upper level problem y_1 and y_2 are considered as

Table 8: Lower level problem solution of Example 1

Critical Region	Definition	Objective Function	Variable value
CR1	$-2 \leq y_{1,2} \leq 2$	$y_1 + y_2$	$x_{1a} = 0, \quad x_{1ab} = 0, \quad x_{2a} = 0$

parameters.

$$
\begin{aligned}
\min_{x_{1a}, x_{1b}, x_{2a}} \quad & \left[\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} y \right]^T \begin{bmatrix} x_{1a} \\ x_{1b} \\ x_{2a} \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} y \\
\text{s.t.} \quad & \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & -1 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_{1a} \\ x_{1b} \\ x_{2a} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
& -2 \leq y_{1,2} \leq 2 \\
& x_{1a}, x_{1b}, x_{2a} \in \{0, 1\}^3
\end{aligned} \tag{12}
$$

Step 4: The above problem is then solved using a mp-ILP algorithm and yields the optimal parametric solution given in Table 8. In this example the parametric solution consists of only one critical region.

Step 5: The solution obtained is then substituted into the upper level problem to formulate one new single-level deterministic linear programming (LP) problem.

$$
\begin{aligned}
\min_{y} \quad & -3 + 3y_1 + 3.2y_2 \\
\text{s.t.} \quad & -y_1 - y_2 \leq 2 \\
& y_1 + y_2 \leq 2 \\
& -2 \leq y_{1,2} \leq 2 \\
& y \in \mathbb{R}^2
\end{aligned} \tag{13}
$$

Step 6: Problem (13) is solved using CPLEX linear programming solver, and results to the solution presented in Table 9. Since only one solution is derived no comparison procedure in this step is needed and the solution listed

Table 9: Solution of the single level problem formulated in Example 1

Objective Function	**Continues Variables**	**Discrete Variables**
-9.4	$y_1 = 0, \quad y_2 = -2$	$x_1 = 1, \quad x_2 = 1$

in Table 9 is the exact and global optimal solution of Example 1.

2.1.2. Example 2: ILP-ILP

Consider the following Type 2 class example taken from Moore and Bard (1990):

$$
\begin{aligned}
\min_{x} \quad & -x - 10y \\
\text{s.t.} \quad & y \in \arg\{\min_{y} \quad y \\
& \qquad\qquad \text{s.t.} \quad -25x + 20y \leq 30 \\
& \qquad\qquad\qquad x + 2y \leq 10 \\
& \qquad\qquad\qquad 2x - y \leq 15 \\
& \qquad\qquad\qquad -2x - 10y \leq -15\} \\
& x, y \in \mathbb{Z}^{+2}
\end{aligned}
\tag{14}
$$

Step 1 & **2:** Bounds are established for all the variables, resulting in $1 \leq x \leq 8$ and $1 \leq y \leq 4$. The problem is then transformed into a 0-1 binary B-ILP problem (4.1), by expressing the integer variables x and y through the binary variables x_1, x_2, x_3, y_1 and y_2 as $x = 1 + x_1 + 2x_2 + 4x_3$ and $y = 1 + y_1 + 2y_2$

Table 10: Lower level problem solution of Example 2

Critical Region	**Definition**	**Objective**	**Variables**
CR1	$x_2 = 0, \quad x_3 = 0$	2	$y_1 = 1, \quad y_2 = 0$
CR2	$-x_2 - x_3 \leq -1$	1	$y_1 = 0, \quad y_2 = 0$

Floudas (1995).

$$
\begin{aligned}
\min_{x_1,x_2,x_3} \quad & -x_1 - 2x_2 - 4x_3 - 10y_1 - 20y_2 - 11 \\
\text{s.t.} \quad & (y_1, y_2) \in \arg\{\min_{y_1,y_2} \quad y_1 + 2y_2 + 1 \\
& \qquad \text{s.t.} \quad -25x_1 - 50x_2 - 100x_3 + 20y_1 + 40y_2 \leq 35 \\
& \qquad\qquad x_1 + 2x_2 + 4x_3 + 2y_1 + 4y_2 \leq 7 \\
& \qquad\qquad 2x_1 + 4x_2 + 8x_3 - y_1 - 2y_2 \leq 14 \\
& \qquad\qquad -2x_1 - 4x_2 - 8x_3 - 10y_1 + 20y_2 \leq -3\} \\
& x_1, x_2, x_3 \in \{1,0\}^3, y_1, y_2 \in \{0,1\}^2
\end{aligned}
\tag{15}
$$

Step 3: The lower level problem is then reformulated as a mp-MILP (16), in which the optimization variables of the upper level problem, x_1, x_2 and x_3, are considered as parameters.

$$
\begin{aligned}
\min_{y_1,y_2} \quad & y_1 + 2y_2 + 1 \\
\text{s.t.} \quad & 25x_1 - 50x_2 - 100x_3 + 20y_1 + 40y_2 \leq 35 \\
& x_1 + 2x_2 + 4x_3 + 2y_1 + 4y_2 \leq 7 \\
& 2x_1 + 4x_2 + 8x_3 - y_1 - 2y_2 \leq 14 \\
& -2x_1 - 4x_2 - 8x_3 - 10y_1 + 20y_2 \leq -3 \\
& x_1, x_2, x_3 \in \{1,0\}^3, y_1, y_2 \in \{0,1\}^2
\end{aligned}
\tag{16}
$$

Step 4: The above problem is then solved using the theory presented in Oberdieck et al. (2017) for binary parameters in multi-parametric problems, and yields to the optimal parametric solution presented in Table 10.

Step 5 & **6:** The solution obtained is then substituted into the upper level problem to formulate two new single-level ILP problems corresponding to each

Table 11: Solution of the single level problem formulated in Example 2

Critical Region	**Objective Function**	**Transformed Variables**	**Original Variables**
CR1	-22	$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$	$x = 2, \quad y = 2$
CR2	-18	$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1$	$x = 8, \quad y = 1$

critical region. Solving this single level problems using CPLEX results to the solution presented in Table 11.

After the comparison procedure the global optimum is found to be -22 with $x = 2$ and $y = 2$.

2.1.3. Example 3: MILP-MILP

Consider the following Type 4 class example:

$$
\begin{aligned}
\min_{x_{1,2},y_3} \quad & 4x_1 - x_2 + x_3 + 5y_1 - 6y_3 \\
\text{s.t.} \quad & (x_3, y_{1,2}) \in \arg\{\min_{x_3,y_{1,2}} -x_1 + x_2 - 2x_3 - y_1 + 5y_2 + y_3 \\
& \qquad\qquad \text{s.t.} \quad 6.4x_1 + 7.2x_2 + 2.5x_3 \leq 11.5 \\
& \qquad\qquad\qquad -8x_1 - 4.9x_2 - 3.2x_3 \leq 5 \\
& \qquad\qquad\qquad 3.3x_1 + 4.1x_2 + 0.02x_3 + 4y_1 + 4.5y_2 + 0.5y_3 \leq 1 \\
& \qquad\qquad\qquad -10 \leq x_{1,2} \leq 10 \quad \} \\
& x_1, x_2, x_3 \in \mathbb{R}^3, \quad y_1, y_2, y_3 \in \{0,1\}^3
\end{aligned}
\tag{17}
$$

Step 1 & **2:** This example is already bounded and in terms of binary $0-1$ variables, therefore we can go directly to Step 3.

Step 3: Considering only the lower level problem, and treating x_1, x_2 and y_3 (upper level variables) as parameters, the lower level problem is transformed to a mp-MILP (18).

$$
\begin{aligned}
\min_{x_3, y_{1,2}} \quad & -x_1 - y_1 + 5y_2 + x_2 - 2x_3 + y_3 \\
\text{s.t.} \quad & 6.4x_1 \leq 11.5 - 7.2x_2 - 2.5x_3 \\
& -8x_1 \leq 5 + 4.9x_2 + 3.2x_3 \\
& 3.3x_1 + 4y_1 + 4.5y_2 \leq 1 - 4.1x_2 - 0.02x_3 - 0.5y_3 \\
& -10 \leq x_{1,2} \leq 10
\end{aligned}
\tag{18}
$$

Step 4: Problem (18) is then solved using POP toolbox and the theory presented in Oberdieck et al. (2017), and yields the optimal parametric solution shown in Table 12.

Step 5: Each solution was then substituted into the upper level problem, resulting into 8 single level linear programming problems, (19), corresponding to each critical region.

$$
\begin{aligned}
z_1 = \min_{x_{1,2}} \quad & -161x_1 - 206x_2 + 50 \\
\text{s.t.} \quad & -0.624x_1 - 0.780x_2 \leq -0.175 \\
& 0.624x_1 + 0.781x_2 \leq 0.198 \\
& x_1 \leq 10 \\
& \vdots \\
z_8 = \min_{x_{1,2}} \quad & 1.44x_1 - 3.88x_2 - 1.4 \\
\text{s.t.} \quad & 0.626x_1 + 0.779x_2 \leq 0.0787 \\
& 0.044x_1 + 0.999x_2 \leq 4.565 \\
& -0.6241x_1 - 0.7814x_2 \leq 0.666 \\
& -10 \leq x_1 \leq 10 \\
& -x_2 \leq 10
\end{aligned}
\tag{19}
$$

Remark 3: Mixed integer linear or quadratic bilevel problems with **all** of the binary variables appearing in the lower level problem will result in pure continuous single-level programming problems at **Step 5** of the algorithm.

Step 6: All 8 linear programming problems (19) were solved using CPLEX solver and their solution is reported in Table 13.

Table 12: Example 3: Parametric solution of the lower level problem

CR	Definition	Variables
1	$-0.624x_1 - 0.780x_2 \leq -0.175$ $0.624x_1 + 0.781x_2 \leq 0.198$ $x_1 \leq 10, \quad y_3 = 0$	$x_3 = -165x_1 - 205x_2 + 50$ $y_1 = 0$ $y_2 = 0$
2	$0.624x_1 + 0.781x_2 \leq -0.570$ $-0.624x_1 - 0.780x_2 \leq 0.594$ $x_1 \leq 10, \quad y_3 = 0$	$x_3 = -2.56x_1 - 2.88x_2 + 4.6$ $y_1 = 0$ $y_2 = 0$
3	$-0.626x_1 - 0.780x_2 \leq 0.596$ $0.624x_1 + 0.781x_2 \leq -0.570$ $-0.626x_1 - 0.780x_2 \leq 0.594$ $x_1 \leq 10, \quad y_3 = 0$	$x_3 = -165x_1 - 205x_2 + 50$ $y_1 = 1$ $y_2 = 0$
n	...	...
7	$0.626x_1 + 0.779x_2 \leq -0.693$ $0.044x_1 + 0.999x_2 \leq 4.565$ $-10 \leq x_1 \leq 10$ $-x_2 \leq 10, \quad y_3 = 1$	$x_3 = -2.56x_1 - 2.88x_2 + 4.6$ $y_1 = 1$ $y_2 = 0$
8	$0.626x_1 + 0.779x_2 \leq 0.0787$ $0.044x_1 + 0.999x_2 \leq 4.565$ $-0.6241x_1 - 0.7814x_2 \leq 0.666$ $-10 \leq x_1 \leq 10$ $-x_2 \leq 10, \quad y_3 = 1$	$x_3 = -2.56x_1 - 2.88x_2 + 4.6$ $y_1 = 0$ $y_2 = 0$

Table 13: Solution of the single level problems generated in Example 3

Critical Region	**Objective**	x_1	x_2	x_3	y_1	y_2	y_3
CR1	-38.115	-10	8.243	10.128	0	0	0
CR2	-37.969	-10	7.26	9.291	0	0	0
CR3	173.636	-8.835	6.329	210.306	1	0	0
CR4	-24.457	-7.032	4.879	8.549	1	0	0
CR5	-24.438	-7.020	4.879	8.520	0	0	0
CR6	61.086	-2.736	2.055	80.083	0	0	1
CR7	-25.708	-7.187	4.886	8.928	1	0	1
CR8	-30.704	-7.185	4.886	8.921	0	0	1

After the comparison procedure the solution with the minimum objective value was chosen as the global solution of the bilevel programming problem (14), lying in critical region 1, with $x_1 = -10, x_2 = 8.243, x_3 = 10.128$ and $y_{1,2,3} = 0$.

3. Bilevel Mixed-integer Quadratic programming problems

The algorithm presented in Section 2 is extended for mixed-integer quadratic programming problems of the following general form (20), belonging to problem class Type 4.

$$\begin{aligned}
\min_{x_1,y_1} \quad & ({Q_1}^T\omega + c_1)^T\omega + c_{c1} \\
\text{s.t.} \quad & A_1x + B_1y \le b_1 \\
& (x_2, y_2) \in \arg\{\min_{x_2,y_2} \quad ({Q_2}^T\omega + c_2)^T\omega + c_{c2} \\
& \qquad\qquad\qquad \text{s.t.} \quad A_2x + B_2y \le b_2 \quad \} \\
& x \in X \subseteq \mathbb{R}^n, \quad y \in Y \subseteq \mathbb{Z}^p \\
& x = [x_1^T x_2^T]^T, \quad y = [y_1^T y_2^T]^T, \quad \omega = [x^T y^T]^T
\end{aligned} \tag{20}$$

where Q_1, c_1,c_{c1} A_1, B_1, b_1 are constant coefficient matrices in the upper level (leader) problem, and Q_2, c_2, c_{c2}, A_2, B_2, b_2 are constant coefficient matrices in the lower level (follower) problem. Q_2 is positive definite, and X and Y are compact polyhedral convex sets of dimensions n and p respectively.

The main idea and methodology for solving this type of problems follows the methodology proposed in Section 2, and is based on a recently developed mp-MIQP algorithm by Oberdieck and Pistikopoulos (2015), summarized in Appendix II. The proposed methodology will be firstly introduced through the general form of the B-MIQP problem (20), and then illustrated through two numerical examples.

The first three steps of the B-MIQP algorithm are similar to the first three steps of the B-MILP algorithm. In Step 1 integer and continuous variable bounds are established and in Step 2 integer variables are transformed into binary variables similarly to Steps 1 and 2 of the B-MILP algorithm. In Step 3 the lower level problem of the reformulated B-MIQP is transformed into a mp-MIQP problem (21), in which the optimization variables of the upper level problem that appear in the lower level problem, x_1 and y_1, are considered as parameters for the lower level problem.

$$\begin{aligned} \min_{x_2,y_2} \quad & \left({Q_2}^T\omega + c_2\right)^T\omega + c_{c2} \\ \text{s.t.} \quad & A_2x + B_2y \leq b_2 \\ & {x_1}^L \leq x_1 \leq {x_1}^U \end{aligned} \tag{21}$$

The solution of the mp-MIQP problem (21), using mp-MIQP algorithms through POP toolbox, will result to the complete profile of optimal solutions of the lower level problem as explicit functions of the variables of the upper level problem with corresponding critical regions. Critical regions for mp-MIQP problems can be non-convex. The final step of the mp-MIQP algorithm is a comparison procedure (minmax, affine, exact) for overlapping critical regions that are created by the integer terms (Oberdieck et al., 2016b). This comparison step is essential as only one critical region can be optimal at any given point in the space. The quadratic objective function of the lower level problem can therefore make the final critical regions non-convex, by the creation of non-linear inequalities for the definition of the final critical regions (22) (Oberdieck and

Pistikopoulos, 2015).

$$[x_2, y_2] = \begin{cases} \xi_1 = p_1 + q_1 x_1 + {x_1}^T r_1 x_1, \psi_1 & \text{if } H_1 x_1 \leq h_1, g_1(x_1) \leq g_1 \\ \xi_2 = p_2 + q_2 x_1 + {x_1}^T r_2 x_1, \psi_2 & \text{if } H_2 x_1 \leq H_2, g_2(x_1) \leq g_2 \\ \vdots & \vdots \\ \xi_k = p_k + q_k x_1 + {x_1}^T r_k x_1, \psi_k & \text{if } H_k x_1 \leq h_k, g_k(x_1) \leq g_k \end{cases} \tag{22}$$

Therefore, in Step 5 we substitute the multi-parametric solution into the upper level MIQP problem to formulate single level MIQP or MINLP problems. In Step 6 the single level problems are solved using appropriate mixed-integer linear, quadratic or non-linear global optimization solvers, and their solutions are compared to select the global optimum solution.

The proposed algorithm in summarized in Algorithm 2.

Algorithm 2 Multi-parametric algorithm for the solution of Bilevel Mixed-Integer Quadratic Programming problems

---

1: Establish integer and continuous variable bounds
2: Express integer variables into binary and substitute in (20)
3: Formulate the mp-MIQP problem (21)
4: Solve (21) and obtain solution $[x_2 \quad y_2]^T = \mathcal{F}_i(x_1, y_1)$ defined over CR_i.
5: **for** $i \leftarrow 1, ..., \#CR_i$ **do**
6:     Formulate MIQP or MINLP using (20) and (22)
7:     Solve MIQP or MINLP to get candidate solution z_i
8: **end for**
9: **return** z_i with minimum value

Remark 4: This algorithm achieves exact and global optimal solutions when $Q_1 \succ 0$. For problem cases where this property does not hold, this algorithm is able to achieve approximate global optimum solutions.

3.1. Numerical Examples

Two B-MIQP numerical examples will be solved to illustrate the use of the proposed algorithm.

3.1.1. Example 4: QP-IQP

Consider the following Type 4 class example taken from Edmunds and Bard (1992):

$$\begin{array}{lll} \min\limits_{x} & (x-2)^2 + (y-2)^2 & \\ \text{s.t.} & \min\limits_{y} \quad y^2 & \\ & \text{s.t.} \quad -2x - 2y \leq -5 & \\ & \qquad x - y \leq 1 & \\ & \qquad 3x + 2y \leq 8 & \\ & x \in \mathbb{R}, \quad y \in \{0,1,2\} & \end{array} \tag{23}$$

Step 1: Bounds are established for the unbounded continuous variable x (y is already bounded), resulting in $\frac{1}{2} \leq x \leq \frac{8}{3}$.

Step 2: The problem is reformulated into a 0-1 binary B-MIQP (24) by expressing the integer variable y as a linear function of new binary variables y_1 and y_2, $y = y_1 + 2y_2$.

$$\begin{array}{lll} \mathop{min}\limits_{x} & (x-2)^2 + (y_1 + 2y_2 - 2)^2 & \\ \text{s.t.} & \mathop{min}\limits_{y_1,y_2} \quad (y_1 + 2y_2)^2 & \\ & \text{s.t.} \quad -2x - 2y_1 - 4y_2 \leq -5 & \\ & \qquad x - y_1 - 2y_2 \leq 1 & \\ & \qquad 3x + 2y_1 + 4y_2 \leq 8 & \\ & x \in \mathbb{R}, \quad y_1, y_2 \in \{0,1\}^2 & \end{array} \tag{24}$$

Step 3: The lower level problem is then reformulated as a mp-MIQP problem (25), by considering the upper level optimization variable, x, as a parameter.

Table 14: Lower level problem solution of Example 4

Critical Region	**Definition**	**Objective Function**	**Variables**
CR1	$1.5 \leq x \leq 2$	1	$y_1 = 1, \quad y_2 = 0$
CR2	$0.5 \leq x \leq 4/3$	4	$y_1 = 0, \quad y_2 = 1$

$$\begin{aligned} \min_{y_1, y_2} \quad & (y_1 + 2y_2)^2 \\ \text{s.t.} \quad & -2y_1 - 4y_2 \leq 2x - 5 \\ & -y_1 - 2y_2 \leq -x + 1 \\ & 2y_1 + 4y_2 \leq -3x + 8 \\ & \tfrac{1}{2} \leq x \leq \tfrac{8}{3} \end{aligned} \tag{25}$$

Step 4: The resulting mp-MIQP problem (25) is solved using POP toolbox, resulting in the optimal solution presented in Table 14.

Step 5: The two solutions were then substituted into the upper level problem, resulting into two single level quadratic programming problems (see *Remark 1*) corresponding to each critical region.

$$\begin{aligned} z_1 = \min_{x} \quad & (x-2)^2 + 1 \\ \text{s.t.} \quad & 1.5 \leq x \leq 2 \end{aligned} \tag{26}$$

$$\begin{aligned} z_2 = \min_{x} \quad & (x-2)^2 \\ \text{s.t.} \quad & 0.5 \leq x \leq 4/3 \end{aligned} \tag{27}$$

Step 6: The resulting problems are convex quadratic programming problems therefore CPLEX solver was used for their solution (Table 15). After comparison the global solution of the problem was found to be at $x = 4/3$ and $y = 2$ with the objective value of $4/9$.

Table 15: Solution of the single level problem formulated in Example 4

Critical Region	**Objective Function**	**Variables**
CR1	5	$x = 2, \quad y = 1$
CR2	4/9	$x = 4/3, \quad y = 2$

3.1.2. Example 5: MIQP-MIQP

Consider the following Type 4 class example problem:

$$\begin{aligned}
\min_{x_1,x_2,y_3} \quad & 4x_1^2 - x_2^2 + 2x_2 + x_3 + 5y_1 + 6y_3 \\
\text{s.t.} \quad & -y_1 - y_2 - y_3 \le -1 \\
& \min_{x_3,y_1,y_2} 4x_3^2 + y_1^2 + 5y_2 + x_2y_1 - x_2y_2 - 5x_3 - 15y_1 - 16y_2 \\
& \text{s.t.} \quad 6.4x_1 + 7.2x_2 + 2.5x_3 \le 11.5 \\
& \qquad -8x_1 - 4.9x_2 - 3.2x_3 \le 5 \\
& \qquad 3.3x_1 + 4.1x_2 + 0.02x_3 + 4y_1 + 4.5y_2 + 0.5y_3 \le 1 \\
& \qquad -10 \le x_1 \le 10 \\
& \qquad -10 \le x_2 \le 10 \\
& x_1, x_2, x_3 \in \mathbb{R}^3, \quad y_1, y_2, y_3 \in \{0,1\}^3
\end{aligned} \tag{28}$$

Step 1 & 2: The problem is already bounded and in the form of a binary $0-1$ B-MIQP problem, therefore we can directly proceed to Step 3.

Step 3: The lower level problem is reformulated as a mp-MIQP problem by considering the upper level optimization variables that appear in the lower level (x_1, x_2, y_3) as parameters.

Step 4: The existence of bilinear terms introduces another step for the solution of this problem, as a z-transformation to eliminate those terms is required. This transformation can be done through POP toolbox, and the resulting mp-MIQP problem is solved again using POP toolbox and the theory presented in Oberdieck et al. (2017), resulting in the optimal parametric solution presented in Table 16 and Figure 2.

Step 5: All 12 critical regions that form the parametric solution were then substituted into the upper level problem, formulating twelve single level MIQP

Table 16: Example 5: Solution of the lower level problem

CR	Definition	Variables
1	$0.624x_1 + 0.781x_2 \leq 0.198$ $-0.627x_1 - 0.779x_2 \leq -0.185$ $x_1 \leq 10$	$x_3 = -165x_1 - 205x_2 + 50$ $y_1 = 0$ $y_2 = 0$
2	$0.044x_1 + 0.999x_2 \leq 4.565$ $0.624x_1 + 0.781x_2 \leq 0.198$ $0.853x_1 + 522x_2 \leq -0.959$ $-0.6241x_1 - 0.7814x_2 \leq 0.57$ $-x_1 \leq 10$ $-x_2 \leq 10$	$x_3 = -2.5x_1 - 1.531x_2 - 1.563$ $y_1 = 0$ $y_2 = 0$
3	$0.624x_1 + 0.781x_2 \leq -0.57$ $-0.627x_1 - 0.779x_2 \leq -0.575$ $x_1 \leq 10$	$x_3 = 1.25$ $y_1 = 0$ $y_2 = 0$
4	$-0.853x_1 - 0.5223x_2 \leq 0.959$ $0.627x_1 + 0.779x_2 \leq 0.185$ $-0.624x_1 - 0.781x_2 \leq 0.57$ $x_1 \leq 10$ $-x_2 \leq 10$	$x_3 = 1.25$ $y_1 = 0$ $y_2 = 0$
n	...	...
11	$0.624x_1 + 0.7814x_2 \leq -1.434$ $-0.627x_1 - 0.779x_2 \leq 1.430$ $x_1 \leq 10$	$x_3 = 1.25$ $y_1 = 0$ $y_2 = 1$
12	$0.627x_1 + 0.779x_2 \leq -0.670$ $-0.853x_1 - 0.522x_2 \leq 0.959$ $-0.624x_1 - 0.781x_2 \leq 1.434$ $x_1 \leq 10$ $-x_2 \leq 10$	$x_3 = 1.25$ $y_1 = 0$ $y_2 = 1$

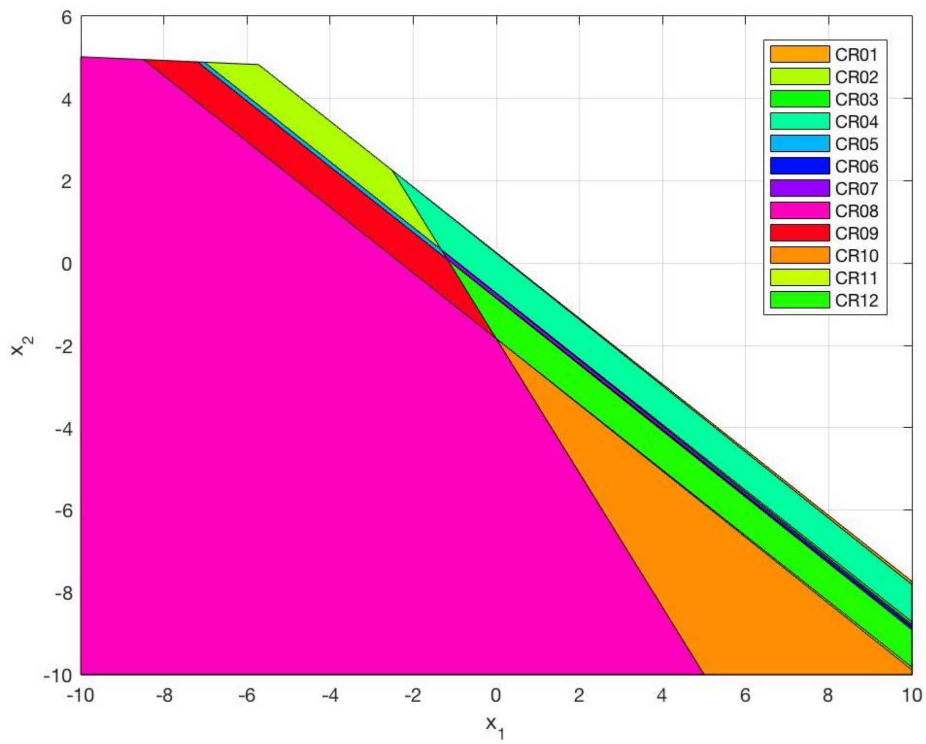

Figure 2: Example 5: Graphical representation of the parametric solution of the mp-MIQP problem

problems corresponding to each critical region.

Step 6: The resulting problems where then solved using CPLEX solver and their solution is presented in Table 17. After the comparison procedure the global optimum was found to be in critical region 11 with an upper level objective function of -1.742.

4. Mixed integer bilevel programming with right hand side uncertainty

For applications that involve constantly changing and unpredictable conditions it is of high importance to consider the effect of uncertainties in programming problems. When considering bilevel programming formulations, uncertainties can be both integer or continuous, and can arise in both optimization levels. Such examples include *i) supply chain planning:* unstable business environment, with constantly changing market conditions and customer needs and expectations (Gupta and Maranas, 2000; Jung et al., 2004; Ryu et al., 2007), *ii) hierarchical model predictive control:* constantly changing system states and unpredicted system disturbances (Sakizlis et al., 2004) and *iii) economic planning:* world trade, politics, weather etc. (Radner and Portes, 1975).

The presence of uncertainty in bilevel problems has been addressed before for the continuous linear case Ryu et al. (2004) and the continuous quadratic case (Faisca et al., 2007). In this work we present an extension of our earlier work that covers both mixed integer linear and quadratic cases. We are considering the uncertainty to be unstructured but bounded, and can appear in one or both optimization levels.

We are addressing the following bilevel programming problem with right

Table 17: Example 5: Single level solutions

CR	**Variables**	**Obj. Level 1**
1	$x_1 = 1.283$, $x_2 = -0.771$, $y_3 = 1$, $x_3 = -3.589$, $y_1 = 0$, $y_2 = 0$	3.913
2	$x_1 = -1.328$, $x_2 = 0.331$, $y_3 = 1$, $x_3 = 1.25$, $y_1 = 0$, $y_2 = 0$	10.951
3	$x_1 = 0.565$, $x_2 = -1.193$, $y_3 = 1$, $x_3 = 1.25$, $y_1 = 0$, $y_2 = 0$	6.790
4	$x_1 = 0.563$, $x_2 = -1.179$, $y_3 = 1$, $x_3 = 1.25$, $y_1 = 0$, $y_2 = 0$	6.810
5	$x_1 = -1.180$, $x_2 = 0.090$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 1$, $y_2 = 0$	7.825
6	$x_1 = 0.544$, $x_2 = -1.298$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 1$, $y_2 = 0$	4.372
7	$x_1 = 0.542$, $x_2 = -1.285$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 1$, $y_2 = 0$	4.386
8	$x_1 = 0.530$, $x_2 = -2.702$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 1$, $y_2 = 1$	2.952
9	$x_1 = -0.002$, $x_2 = -1.834$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 0$, $y_2 = 1$	-1.577
10	$x_1 = 0.530$, $x_2 = -2.702$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 1$, $y_2 = 1$	2.952
11	$x_1 = 0.375$, $x_2 = -2.137$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 0$, $y_2 = 1$	**-1.742**
12	$x_1 = 0.373$, $x_2 = -2.133$, $y_3 = 0$, $x_3 = 1.25$, $y_1 = 0$, $y_2 = 1$	-1.740

hand side uncertainty θ.

$$
\begin{aligned}
\min_{x_1,y_1} \quad & \left({Q_1}^T\omega + H_{t1}\theta + c_1\right)^T \omega + (Q_{t1}\theta + c_{t1})^T\theta + c_{c1} \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 + F_1\theta \\
& \min_{x_2,y_2} \left({Q_2}^T\omega + H_{t2}\theta + c_2\right)^T \omega + (Q_{t2}\theta + c_{t2})^T\theta + c_{c2} \\
& \text{s.t.} \quad A_2 x + B_2 y \leq b_2 + F_2\theta \\
& x = [x_1^T x_2^T]^T, \quad x \in \mathbb{R}^n \\
& y = [y_1^T y_2^T]^T, \quad y \in \mathbb{Z}^p \\
& \omega = [x^T y^T]^T \\
& \theta \in \Theta := \{\theta \in \mathbb{R}^q | M\theta \leq d\}
\end{aligned}
\tag{29}
$$

where Q_1, H_{t1}, c_1, Q_{t1}, c_{t1}, c_{c1} A_1, B_1, b_1, F_1 are constant coefficient matrices in the upper level (leader) problem, and Q_2, H_{t2}, c_2, Q_{t2}, c_{t2}, c_{c2} A_2, B_2, b_2, F_2 are constant coefficient matrices in the lower level (follower) problem. Q_1 and Q_2 are positive definite, and it is assumed that upper level optimization variables that appear in the lower level problem, and lower level integer variables, are bounded, or their bounds can be derived through the problem constraints.

For the solution of this problem we follow the following steps:

Step 1: Similarly to the BMILP and BMIQP algorithm, integer and continuous variable bounds are established for the variables that appear in the lower level problem.

Step 2: Integer variables are transformed into binary 0-1 variables.

Step 3: The lower level problem is transformed into a mp-MIQP or mp-MILP, considering as parameters both the upper level variables that appear in the lower lever (x_1, y_1), and the uncertainty θ.

Step 4: The resulting multi-parametric problems are solved using POP toolbox.

Step 5: Each critical region is substituted into the upper level problem to result into k single level multi-parametric problems, considering the uncertainty θ as parameters.

Step 6: The resulting k multi-parametric problems are solved using POP toolbox.

Step 7: All k parametric solutions are combined. For overlapping regions the exact comparison procedure implemented in POP toolbox and presented in Oberdieck and Pistikopoulos (2015) is used to result to the final exact and global parametric solution of the original bilevel problem.

4.1. Numerical Example

4.1.1. Example 6: mp-MIQP-MILP

Consider the following Type 4 class example with right hand side uncertainty θ.

$$
\begin{aligned}
\min_{x_1, y_3} \quad & 4{x_1}^2 + x_3 y_3 + 5y_1 - 6y_3 - \theta^2 + 2\theta \\
\text{s.t.} \quad & y_1 + y_2 + y_3 \leq 1 \\
& \min_{x_2, y_{1,2}} - x_1 - 2x_2 - y_1 + 5y_2 + \theta \\
& \text{s.t.} \quad 6.4x_1 + 2.5x_2 \leq 11.5 - 7.2\theta \\
& \qquad -8x_1 - 3.2x_2 \leq 5 + 4.9\theta \\
& \qquad 3.3x_1 + 0.02x_2 + 4y_1 + 4.5y_2 \leq 1 - 4.1\theta \\
& \qquad -10 \leq x_1 \leq 10 \\
& \qquad -10 \leq \theta \leq 10 \\
& x_1, x_2 \in \mathbb{R}^2, \quad y_1, y_2, y_3 \in \{0, 1\}^2
\end{aligned}
\tag{30}
$$

Steps 1 & 2: The problem is already bounded and in a binary form.

Step 3: The lower level problem is transformed into a mp-MILP problem. Both the upper level variables that appear in the lower level (x_1) and uncertainty (θ) are being treated as parameters for the lower level problem.

Step 4: The problem is then solved using POP toolbox, and yields to the optimal parametric solution presented in Table 18.

Step 5: The solutions obtained for every critical region are then substituted into the upper level problem to formulate five new single level mp-MIQP problems. More specifically, the functions of the optimization variables of the lower level, x_2, y_1 and y_2, in terms of the upper level optimization variables, x_1 and θ, are substituted in the upper level problem. The definition of each critical region is added to each new single level problem as a new set of constraints.

Table 18: Example 6: Solution of the lower level problem

CR	Definition	Variables
1	$-0.624x_1 - 0.780\theta \leq -0.175$ $0.624x_1 + 0.781\theta \leq 0.198$ $x_1 \leq 10$	$x_2 = -165x_1 - 205\theta + 50$ $y_1 = 0$ $y_2 = 0$
2	$0.624x_1 + 0.781\theta \leq -0.570$ $-0.624x_1 - 0.780\theta \leq 0.594$ $x_1 \leq 10$	$x_2 = -2.56x_1 - 2.88\theta + 4.6$ $y_1 = 0$ $y_2 = 0$
3	$-0.626x_1 - 0.780\theta \leq 0.596$ $0.624x_1 + 0.781\theta \leq -0.570$ $-0.626x_1 - 0.780\theta \leq 0.594$ $x_1 \leq 10$	$x_2 = -165x_1 - 205\theta + 50$ $y_1 = 1$ $y_2 = 0$
4	$0.626x_1 + 0.780\theta \leq -0.596$ $0.044x_1 + 0.999\theta \leq 4.565$ $-10 \leq x_1 \leq 10$ $-\theta \leq 10$	$x_2 = -2.56x_1 - 2.88\theta + 4.6$ $y_1 = 1$ $y_2 = 0$
5	$0.044x_1 + 0.999\theta \leq 4.565$ $0.626x_1 + 0.780\theta \leq 0.175$ $-0.624x_1 - 0.781\theta \leq 0.570$ $-10 \leq x_1 \leq 10$ $-\theta \leq 10$	$x_2 = -2.56x_1 - 2.88\theta + 4.6$ $y_1 = 0$ $y_2 = 0$

Table 19: Example 6: Solution of the single level mp-MIQPs

CR	**Definition**	**Objective**
1.1	$-4.824 \leq \theta \leq 7.733$	$2.136\theta^2 - 408.010\theta - 8.154$
1.2	$7.733 \leq \theta \leq 7.812$	$-\theta^2 - 203\theta - 1406$
...	...	...
5.1	$0.290 \leq \theta \leq 1.241$	$-\theta^2 - 0.880\theta - 2.219$
5.2	$-4.824 \leq \theta \leq 0.290$	$2.096\theta^2 - 2.674\theta - 1.959$
5.3	$-4.882 \leq \theta \leq -4.824$	$0.001\theta^2 + 0.0092\theta + 0.0002$
5.4	$1.2407 \leq \theta \leq 8.7160$	$2.136\theta^2 - 8.661\theta - 2.607$

Step 6: The five resulting single level problems are in the form of mp-MIQP problems, with the uncertainty θ being a parameter of the single level problems. Therefore, the POP toolbox was used for their solution. Each critical region formed in Step 4 is now divided into smaller regions as another parametric programming problem is solved within the original regions.

A summary of the resulting parametric solutions of all five problems is presented in Table 19. Figure 3 graphically illustrates the objective function versus the uncertain parameter θ for all the critical regions derived in Step 6.

Step 7: As a last step, the solutions generated from each critical region are compared and the parametric solutions resulting to the minimum objective through the parametric space are chosen as the final solution of the mixed integer bi-level programming problem with uncertainty. Table 20 summarizes the final solution of this problem. The solution can also be seen in Figure 3 as it consists of the lines forming the lower value of the objective function in the space of the parameter θ.

5. Computational Implementation

The presented B-MILP and B-MIQP algorithms have been implemented in our new B-POP toolbox (Avraamidou and Pistikopoulos, 2018a), a MATLAB toolbox for bilevel programming, an extension to our already developed POP

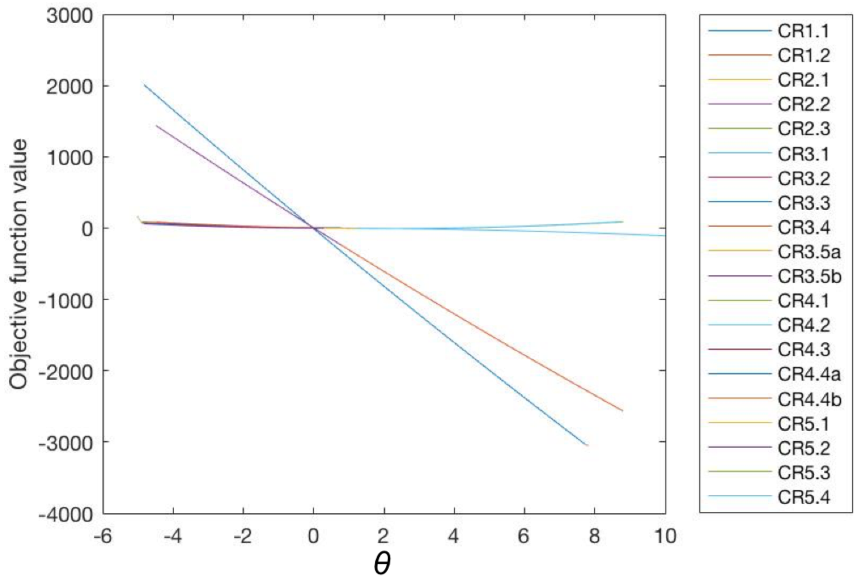

Figure 3: Example 6: Graphical representation of the parametric solution of the single level mp-MIQP problems

toolbox. The toolbox features i) bilevel programming solvers for linear and quadratic programming problems and their mixed-integer counter-parts, ii) a versatile problem generator capable of creating random bilevel problems of arbitrary size, and iii) a library of bilevel programming test problems.

In B-POP we consider the following bilevel programming problem:

$$
\begin{aligned}
\min_{x_1,y_1} \quad & {(Q_1{}^T\omega + c_1)}^T\omega + c_{c1} \\
\text{s.t.} \quad & A_1x + B_1y \leq b_1 \\
& (x_2, y_2) \in \arg\{\min_{x_2,y_2} \quad {(Q_2{}^T\omega + c_2)}^T\omega + c_{c2} \\
& \qquad\qquad\qquad\quad \text{s.t.} \quad A_2x + B_2y \leq b_2 \quad \} \\
& x \in \mathbb{R}^n, \quad y \in \mathbb{Z}^p \\
& x = [x_1^T x_2^T]^T, \quad y = [y_1^T y_2^T]^T, \quad \omega = {[x^T y^T]}^T
\end{aligned}
\tag{31}
$$

To our knowledge, currently available bi-level solution toolboxes exist for the solution of continuous bi-level programming problems, and this include YALMIP® (Lofberg, 2004) and JAMS® (JAMS) for convex continuous prob-

Table 20: Example 6: Final solution

CR	**Definition**	**Objective**
4.1	$-5.014 \leq \theta \leq -4.882$	$0.011\theta^2 + 0.009\theta + 0.0002$
3.1	$-4.882 \leq \theta \leq -4.840$	$2.136\theta^2 - 2.575\theta + 6.669$
5.3	$-4.840 \leq \theta \leq -4.824$	$0.011\theta^2 + 0.009\theta + 0.0002$
5.2	$-4.824 \leq \theta \leq -0.015$	$2.096\theta^2 - 2.674\theta - 1.959$
1.1	$-0.015 \leq \theta \leq 7.733$	$2.136\theta^2 - 408.010\theta - 8.154$
1.2	$7.733 \leq \theta \leq 7.812$	$-\theta^2 - 203\theta - 1406$
3.4	$7.812 \leq \theta \leq 8.799$	$2.096\theta^2 - 310.374\theta + 4.065$
3.3	$8.799 \leq \theta \leq 8.802$	$-\theta^2 - 203\theta - 701$
4.2	$8.802 \leq \theta \leq 10$	$-\theta^2 - 0.880\theta - 1.095$

lems, and BLEAQ/N-BLEA® (Sinha, 2003) for non-convex continuous problems. A new toolbox for the solution of bilevel mixed-integer linear problems is available by Matteo Fischetti et al., and to our knowledge B-POP is currently the only freely accessible toolbox for bilevel mixed-integer quadratic programming problems.

5.1. Problem solution

Multi-parametric programming: For the lower level multi-parametric programming problems, B-POP utilizes POP toolbox to solve the problem. The user can specify the solution method that can be either geometrical, combinatorial or connected graph algorithms, or utilize POPs interface with the solver in MPT Toolbox (more information in Oberdieck et al. (2016b)).

Sinlge level deterministic problems: For the resulting transformed single level problems, being either LP, QP, MI(N)LP programming problems, B-POP features links to CPLEX, NAG or MATLAB solvers as LP or QP solvers, and CPLEX or MATLAB for MI(N)LP problems.

5.2. Computational Performance

A small set of problems was solved to show the capabilities of the presented algorithms and B-POP toolbox. More computational results and in-depth discussion on the cababilities of B-POP toolbox for different classes of multi-level problems are discussed in Avraamidou and Pistikopoulos (2018a,b) .

The computations were carried out on a 2-core machine with an Intel Core i7 at 3.1 GHz and 16 GB of RAM, MATLAB R2016a, and IBM ILOG CPLEX Optimization Studio 12.6.3.

In order to highlight the applicability of B-POP as well as its scaling capabilities, we are considering both mixed-integer linear and quadratic bilevel problems of different sizes and structures. The results of the computations are presented in Table 21 for the B-MILP problems and in Table 22 for the B-MIQP problems, where x_1 is the number of upper level continuous variables that also appear in the lower level, y_1 is the number of upper level binary $0-1$ variables, x_2 is the number of lower level continuous variables, y_2 is the number of lower level binary $0-1$ variables, m is the number of constraints (excluding boundary constraints), "mp-Level 2 " is the time required to solve the lower level problem multi-parametrically (i.e *Step 4*), "Single Level" is the time required by CPLEX to solve the single level transformed problems and select the minimum one, and finally "Total time" is the total time required for the solution of the bilevel problem. The test problems in Tables 21 and 22 are available at parametric.tamu.edu\POP\ under the names 'BPOP_BMILP' and 'BPOP_BMIQP'.

5.3. Discussion on the computational results

The proposed algorithm can be used to solve the randomly generated B-MILP problems with up to 480 variables (a total of all integer and continuous) and B-MIQP problems with up to 65 variables (a total of all integer and continuous) in less than 4 minutes for 92% of the problems. The size and complexity of the bi-level problems that can be solved through the proposed approach is constrained by the capabilities of the multi-parametric mixed integer solution

Table 21: Computational results: B-MILP

Problem	x_1	y_1	x_2	y_2	m	**mp-Level 2 (s)**	**Single Level (s)**	**Total (s)**
test8	40	40	2	2	22	291.130	0.1731	291.3035
test9	45	45	2	2	25	112.6904	0.2437	112.93.42
test10	50	50	2	2	27	455.0280	0.9539	455.09819
test25	2	2	75	75	40	4.1390	0.0026	4.1416
test26	2	2	80	80	42	3.8449	0.0023	3.8472
test27	2	2	85	85	45	5.6016	0.0031	5.6047
test28	2	2	90	90	47	5.4821	0.0032	5.4853
test29	2	2	95	95	50	5.9115	0.0029	5.9144
test30	2	2	100	100	52	8.1234	0.0029	8.1263
test44	40	40	40	40	40	5.4329	0.0025	5.4354
test45	45	45	45	45	45	4.3939	0.0021	4.3960
test46	50	50	50	50	50	8.3232	0.0099	8.3331
test47	55	55	55	55	55	8.9885	0.0034	8.9918
test48	60	60	60	60	60	17.5849	0.0109	17.5957
test49	65	65	65	65	65	10.2242	0.0046	10.2288
test50	70	70	70	70	70	24.5837	0.0123	24.5960
test51	75	75	75	75	75	18.4030	0.0057	18.4087
test52	80	80	80	80	80	13.8105	0.0069	13.8175
test53	85	85	85	85	85	19.7149	0.0142	19.7291
test54	90	90	90	90	90	37.1772	0.0080	37.1852
test55	95	95	95	95	95	59.2469	0.0159	59.2628
test56	100	100	100	100	100	55.3647	0.0080	55.3727
test57	105	105	105	105	105	45.3738	0.0095	45.3833
test58	110	110	110	110	110	68.0360	0.0074	68.0434
test59	115	115	115	115	115	100.8653	0.0119	100.8772
test60	120	120	120	120	120	191.6486	0.0477	191.6963

Table 22: Computational results: B-MIQP

Problem	x_1	y_1	x_2	y_2	m	**mp-Level 2 (s)**	**Single Level (s)**	**Total (s)**
test1	5	5	2	2	5	4.1001	0.1238	4.2239
test2	10	10	2	2	7	2.6959	0.0377	2.7336
test3	15	15	2	2	10	152.1813	0.4648	152.6460
test4	20	20	2	2	12	201.1591	0.5662	201.7052
test5	25	25	2	2	15	175.0922	0.9555	176.0477
test6	2	2	5	5	5	79.3080	0.0730	79.3742
test7	2	2	10	10	7	257.4909	0.0060	257.4969
test8	5	2	5	2	3	8.5233	0.1221	8.6353
test9	10	2	10	2	5	33.0615	0.0601	33.6625
test10	15	2	15	2	6	32.3564	0.0532	32.4096
test11	20	2	20	2	7	47.3702	0.5851	47.9553
test12	25	2	25	2	8	4.5226	0.0345	4.5571
test13	20	5	20	2	5	165.4143	0.7007	166.1150
test14	10	10	30	5	1	210.9560	0.0386	210.9946
test15	5	2	25	5	1	183.3612	0.0856	183.4468

algorithm as the time required for the solution of the single level MIPs is always much smaller than the solution time required for the solution of the lower level multi-parametric problems. It is evident that this algorithm is not intended for the solution of larger scale problems and will not be able to handle problems much bigger than those presented in the computational studies, especially for the case of B-MIQP problems.

6. Conclusion

This paper introduces novel algorithms for the exact global solution of a range of classes of mixed integer bi-level programming problems that contain integer and continuous variables in both optimization levels. The algorithms utilize multi-parametric programming to solve the lower level problem as a function of the upper level variables, and are able to supply the decision maker with the exact and global solution of the bi-level problem. Furthermore an extension of the algorithms is introduced for the parametric solution of bi-level mixed-integer problems under uncertainty.

The proposed approaches has been successfully implemented into a MATLAB toolbox, B-POP, and their performance and efficiency was assessed through a set of test problems, suggesting that the limiting step is the solution of the multi-parametric program in the lower optimization level.

Ongoing work involves the improvement of the computational performance of the presented algorithm by developing methodologies that eliminating the need for exploration of the full parametric space.

Further work also involves the extension of the presented algorithm for the solution of multi-level problems, as well as more general non-linear bilevel problems. Also bilevel problems with right hand side uncertainty will be explored further, and the presented algorithm for their solution will be implemented into B-POP toolbox.

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Appendix I - Algorithm for the solution of mp-MILP problems

A schematic representation showing the steps of the algorithm for the solution of mp-MILP problems (12) by Oberdieck et al. (2014) is shown below.

$$\begin{aligned} \min_{x,y} \quad & Q\omega + H\theta \\ \text{s.t.} \quad & Ax + Ey \leq b + F\theta \\ & x \in \mathbb{R}^n, \quad y \in \{0,1\}^p, \omega = [x^T y^T] \\ & \theta \in \Theta := \{\theta \in \mathbb{R}^q | CR_A \theta \leq CR_b\}, \end{aligned} \tag{32}$$

where $Q \in \mathbb{R}^{(n+p)\times(n+p)} \succ 0$, $H \in \mathbb{R}^{(n+p)\times q}$, $A \in \Re^{m\times n}$, $E \in \mathbb{R}^{m\times p}$, $b \in \mathbb{R}^m$, $F \in \mathbb{R}^{m\times q}$ and Θ is compact.

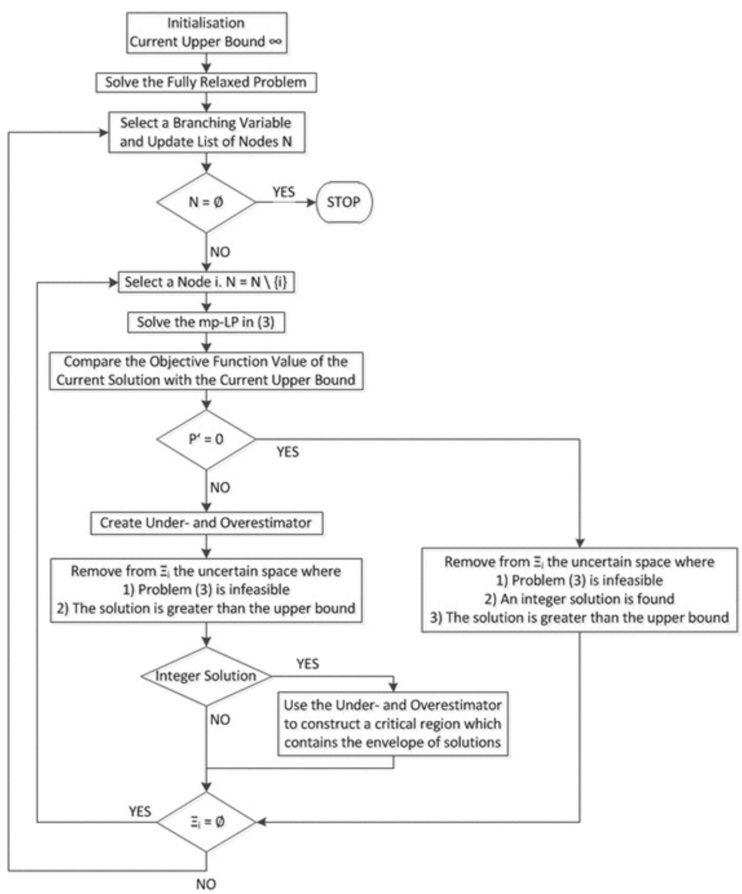

Figure 4: The mp-MILP algorithm proposed by Oberdieck et al. (2014)

Appendix II - Exact algorithm for the solution of mp-MIQP problems

The algorithm of Oberdieck and Pistikopoulos (2015) for the solution of problems with the general formulation of (13) is summarized below. It is based on the decomposition algorithm shown graphically in Figure 5.

$$\begin{aligned} \min_{x,y} \quad & (Q\omega + H\theta + c)^T \omega \\ \text{s.t.} \quad & Ax + Ey \leq b + F\theta \\ & x \in \mathbb{R}^n, \quad y \in \{0,1\}^p, \omega = [x^T y^T] \\ & \theta \in \Theta := \{\theta \in \mathbb{R}^q | CR_A \theta \leq CR_b\}, \end{aligned} \tag{33}$$

where $Q \in \mathbb{R}^{(n+p)\times(n+p)} \succ 0$, $H \in \mathbb{R}^{(n+p)\times q}$, $c \in \mathbb{R}^{n+p}$, $A \in \mathbb{R}^{m\times n}$, $E \in \mathbb{R}^{m\times p}$, $b \in \mathbb{R}^m$, $F \in \mathbb{R}^{m\times q}$ and Θ is compact.

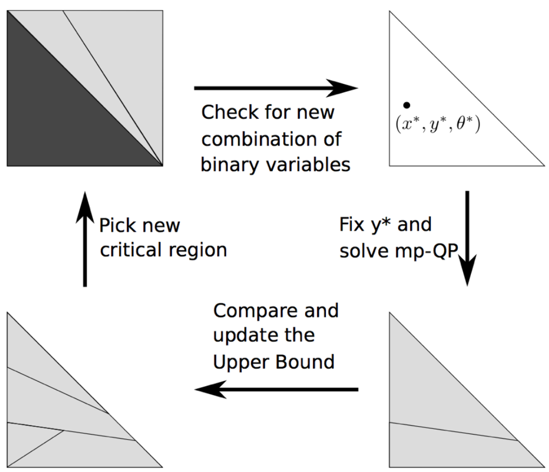

Figure 5: A graphical representation of the decomposition algorithm (Dua et al., 2002).

Initialization: A candidate solution for the binary variables is found by solving the MIQP problem formed when considering parameters as optimization variables. A binary solution is obtained and subsequently fixed in the original problem, thus resulting in a mp-QP problem. This problem can be solved using the algorithm presented in Dua et al. (2002), which results in an initial partitioning of the parameter space and provides a parametric upper bound to the solution. The upper bound for the remaining part of the parameter space which has not yet been explored is set to infinity.

Step 1: A candidate solution for the binary variables is found by considering parameters as optimization variables and solving the resulting MIQP problem.

Step 2: Create an affine outer approximation by employing McCormick relaxations (McCormick (1976)) for each bilinear or quadratic term in the constraints. Since the nonlinearities in the constraints only arise from comparison procedures, these relaxations are calculated during the comparison procedure.

Step 3: The candidate solution of the binary variables is substituted into the initial problem, thus resulting in a mp-QP. This mp-QP problem can be solved using mp-QP algorithms by Dua et al. (2002).

Step 4: This and all subsequent steps have to be performed for each critical region. Compare solution with the current upper bound. Here the explicit solution of the problem is considered and thus two new critical regions are created.

Step 5: Calculate appropriate relaxations in order to create the outer approximation for the next iteration.

Step 6: The original inequalities from the current critical region are reintroduced to each newly formed critical region, while the relaxations used before are removed. The newly formed critical regions are returned to Step 1 thus resuming the iteration.

Termination: The algorithm terminates as soon as problem in Step 1 is infeasible for all critical regions.