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A Plasma Modeling Hierarchy and Verification Approach

Richard M. J. Kramer
Eric C. Cyr
Sean T. Miller
Edward G. Phillips
Gregg A. Radtke
Allen C. Robinson
John N. Shadid

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185
Livermore, California 94550

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ABSTRACT

This report reviews a hierarchy of formal mathematical models for describing plasma phenomena. Starting with the Boltzmann equation, a sequence of approximations and modeling assumptions can be made that progressively reduce to the equations for magnetohydrodynamics. Understanding the assumptions behind each of these models and their mathematical form is essential to appropriate use of each level of the hierarchy. A sequence of moment models of the Boltzmann equation are presented, then focused into a generalized three-fluid model for neutral species, electrons, and ions. This model is then further reduced to a two-fluid model, for which Braginskii described a useful closure. Further reduction of the two-fluid model yields a Generalized Ohm's Law model, which provides a connection to magnetohydrodynamic approaches. A verification approach based on linear plasma waves is presented alongside the model hierarchy, which is intended as an initial and necessary but not sufficient step for verification of plasma models within this hierarchy.

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1. INTRODUCTION

Modeling of plasmas at Sandia has historically concentrated on two distinct regimes in number density. On the one hand, the low-density plasma regime has been modeled using the Particle-In-Cell (PIC) approach, where the plasma is represented as a collection of charged particles. On the other hand, interactions of continuum-scale materials with electromagnetics has been modeled using magnetohydrodynamic (MHD) approximations, which couples material motion and electromagnetic fields through Lorentz forces. This report provides a bridge between these extremes by presenting a sequence of continuum-based models for describing multi-species electromagnetically coupled plasmas.

Figure 1-1 presents a useful pictorial description of the spectrum of plasma modeling addressed in this report, developed by Shumlak et. al. [19]. The horizontal axis shows the Knudsen number, Kn, which is the ratio of the mean free path to a length scale of interest; this primarily defines where the continuum approximation is valid. The vertical axis shows the charge separation distance, Λ_d, which represents the degree to which electrical charge is relevant. The limit $\Lambda_d \to \infty$ is the neutral limit (e.g., standard fluid mechanics) and the limit $\Lambda_d \to 0$ represents the bulk effect limit of magnetohydrodynamics. Our historical modeling experience has been in one of these limits, but the current challenge is to extend expertise and capabilities into intermediate charge separation distances. Similarly, it is desirable to be able to efficiently model problems of interest that range from the highly dense and collisional limit where fluid modeling is appropriate to non-continuum (large Knudsen number) problems where PIC modeling utilizing the Vlasov or Boltzmann or equations is appropriate.

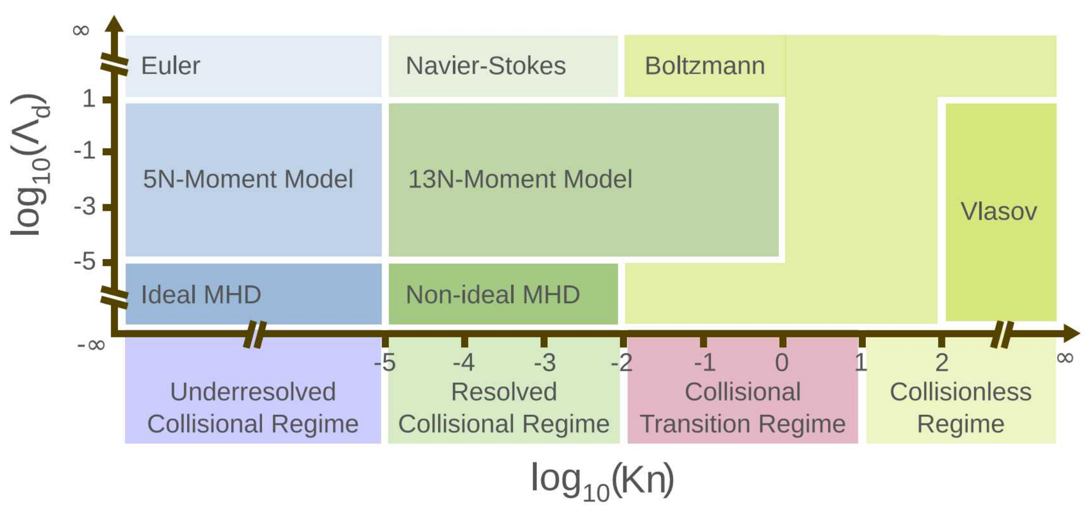

Figure 1-1. The plasma modeling spectrum, shown on axes of Knudsen Number (Kn) against charge separation distance (Λ_d). Courtesy of Uri Shumlak.

This report provides a summary of a hierarchy of models available for investigating this problem space. The report begins with the Boltzmann equation in Section 2, which together with Maxwell's equations, describes the motion of charged particles. The Boltzmann equation has seven independent variables: time, three components in space, and three components in velocity. For low-density plasmas, this system is feasibly solved directly using particle methods, but in the relatively high-density regime of interest, it becomes computationally prohibitive to solve due to the number of particles required, the complex interactions that occur between particles, and the seven-dimensional space they occupy.

Continuum plasma models are derived by taking moments of the Boltzmann equation over velocity space. This procedure reduces the problem to the familiar four independent variables of time and three-dimensional space. The partial differential equations that result from the first three moments of the Boltzmann equation are the familiar continuity, momentum, and energy equations of fluid mechanics, with generalized closure terms for stress and heat conduction. With appropriate simplifications, these reduce to the Euler or generalized Navier-Stokes equations that contain additional Lorentz force terms. This is termed the 5N-Moment model, referring to the number of equations in the system: one for density, three for momentum, and one for energy.

Further moments of the Boltzmann equation can be taken, which yield more complex models that have a greater range of validity (see Figure 1-1). The 13N-Moment model, for example, retains a tensor energy equation which is closed by reducing the third moment of the Boltzmann equation to a vector equation, for a total of 13 PDEs: one equation for density, three for momentum, six for energy, and three for the vector heat flux. While the numerical behavior of Euler or Navier-Stokes systems is well developed and relatively well understood, this is not the case for the higher-moment system. Shumlak and his coworkers [20] have demonstrated some success in this field, however.

In this report, we focus on variations of the 5N-moment model in Sections 3 and 4. As derived from the Boltzmann equation, the model is incomplete without closures for not only stress and heat flux, but also for collision terms. These collision terms capture the effects of creation and destruction of species (e.g., by ionization and recombination in a neutral-ion-electron plasma), and momentum and energy transfers between species due to collisions. Modeling collision terms is particularly challenging, because the effects they represent are difficult to integrate from the microscopic (particle) scale to the macroscopic (continuum) scale. For two-fluid ion-electron plasmas, a canonical closure is the Braginskii model [5], which uses various simplifying assumptions to derive an analytic closure. Even for more complex collisional plasma modeling, terms from the Braginskii model will often be used.

These multi-fluid plasma models directly specify the motion of charged fluids. Electrical current is therefore defined directly and exactly as the net charge flux of the fluids. However, because the mass of an electron is approximately 1/1800 of that of a proton (the lightest possible ion), the coupled system of fluid plasma equations can be challenging to solve numerically. In fact, in many situations, the mass and inertia of the electrons can be neglected as small compared to that of the ion species. Deriving a consistent set of plasma equations in this limit begins by writing the two-fluid equations in terms of a center-of-mass velocity and the current density. Using these variables instead of the separate fluid velocities allows the approximation $m_i \gg m_e$ to be consistently applied, and yields a partial differential equation for the current density $\mathbf{J}$. This is the most general form of generalized Ohm's Law plasma modeling, as only the approximation of negligible electron mass has been applied. These equations are derived in Section 5. Note that the term "generalized Ohm's Law" may be somewhat misleading, because while Ohm's Law is a constitutive relation that defines **J** in terms of the electric field, the PDE for **J** that results in the "generalized" law is directly defined from the fundamental fluid equations for charged particles. Ultimately, pursuing the limit of negligible charge separation will yield the equations of MHD.

Each of the continuum plasma models described describes a set of transport equations coupled to Maxwell's equations. These systems are primarily hyperbolic in nature, with complex dispersion relations for multiple propagating waves. Further, simplifications of the multi-fluid equations will not capture all of these waves, particularly in limits that may be of interest. Section 6 uses the wave properties of the two-fluid equations to define a series of verification problems to ensure that numerical models can capture the appropriate wave behavior of the system.

2. THE BOLTZMANN EQUATION AND ITS MOMENTS

2.1. Particle Motion

At a fundamental level, a plasma can be modeled as a set of charged particles moving in space under the influence of an electromagnetic field. The following presentation follows [17]. Kinetic theory defines an equation of motion for each particle in the plasma, from the density $N_p(\mathbf{x},\mathbf{v},t)$ of a single particle in phase space $(\mathbf{x},\mathbf{v},t)$,

$$N_p(\mathbf{x},\mathbf{v},t) = \delta(\mathbf{x}-\mathbf{X}_p(t))\delta(\mathbf{v}-\mathbf{V}_p(t)), \tag{2.1}$$

where $\mathbf{X}_p(t)$ and $\mathbf{V}_p(t)$ are the Lagrangian coordinates of the particle. The total density of a particular particle species s (e.g., electron, ion, or neutral) is given by the summation over all particles of that species:

$$N_s(\mathbf{x},\mathbf{v},t) = \sum_p \delta(\mathbf{x}-\mathbf{X}_p(t))\delta(\mathbf{v}-\mathbf{V}_p(t)). \tag{2.2}$$

Similarly, the microscopic charge density and microscopic current density are given by

$$q^m(\mathbf{x},t) = \sum_s q_s \int N_s(\mathbf{x},\mathbf{v},t))d^3\mathbf{v} \tag{2.3}$$

$$\mathbf{J}^m(\mathbf{x},t) = \sum_s q_s \int \mathbf{v}N_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} \tag{2.4}$$

where the superscript m indicate microscopic quantities and these values are used to evolve the microscopic Maxwell equations. Taking the time derivative of the species number density, we obtain

$$\frac{\partial N_s}{\partial t}(\mathbf{x},\mathbf{v},t) = \sum_p(-\frac{\mathrm{d}\mathbf{X}_p}{\mathrm{d}t})\cdot\frac{\partial\delta(\mathbf{x}-\mathbf{X}_p(t))}{\partial\mathbf{x}}\delta(\mathbf{v}-\mathbf{V}_p(t)) + \sum_p \delta(\mathbf{x}-\mathbf{X}_p(t))(-\frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t})\cdot\frac{\partial\delta(\mathbf{v}-\mathbf{V}_p(t))}{\partial\mathbf{v}}. \tag{2.5}$$

Observing the identity

$$\frac{\mathrm{d}\mathbf{X}_p}{\mathrm{d}t} = \mathbf{V}_p, \tag{2.6}$$

and the Lorentz-force Law (neglecting relativistic effects)

$$\frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} = \frac{q_s}{m_s}(\mathbf{E}^m + \mathbf{V}_p\times\mathbf{B}^m), \tag{2.7}$$

we can utilize the delta function identity $\alpha\delta(\alpha-\beta)=\beta\delta(\alpha-\beta)$ and the fact the ith delta function velocity derivative does not interact with the ith velocity in the cross product term to obtain the Klimontovich equation,

$$\frac{\partial N_s}{\partial t}+\mathbf{v}\cdot\frac{\partial N_s}{\partial \mathbf{x}}+\frac{q_s}{m_s}\left(\mathbf{E}^m+\mathbf{v}\times\mathbf{B}^m\right)\cdot\frac{\partial N_s}{\partial \mathbf{v}}=0, \tag{2.8}$$

which when coupled to Maxwell's equations gives the exact microscopic dynamical description of the plasma [23, Chapter 2].

It is rarely useful, however, to determine all particle trajectories in the plasma. Instead, the Boltzmann equation is used to track species distribution functions, $f_s(\mathbf{x},\mathbf{v},t)$, for each particle species with an average macroscopic electromagnetic field. The distribution function describes the probability that a number of particles occupy a differential volume in phase space and statistically describes the plasma motion by the Boltzmann equation:

$$\frac{\partial f_s}{\partial t}+\mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{x}}+\frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\frac{\partial f_s}{\partial \mathbf{v}}=\left.\frac{\partial f_s}{\partial t}\right|_c. \tag{2.9}$$

The right-hand side term represents the effective source and sink term due to Coulomb collisions, which arises from the averaging process. We will not dwell on collisions here, except to note that the collision operator can be extended for more general collision chemistry. Specific forms of the operator yield the Fokker-Planck equation [21], or if neglected, the Vlasov equation (for collisionless plasmas). The plasma dynamics are completed by coupling $\mathbf{E}$ and $\mathbf{B}$ with Maxwell's equations (in this form, from [11]):

$$\nabla\cdot\mathbf{D}=q, \tag{2.10}$$

$$\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J}, \tag{2.11}$$

$$\nabla\cdot\mathbf{B}=0, \tag{2.12}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{2.13}$$

where

$$\mathbf{D}=\varepsilon_0\mathbf{E} \tag{2.14}$$

$$\mathbf{H}=\frac{1}{\mu_0}\mathbf{B} \tag{2.15}$$

$$q\equiv\sum_s q_s n_s, \tag{2.16}$$

$$\mathbf{J}\equiv\sum_s q_s n_s \mathbf{v}_s, \tag{2.17}$$

and where the particle number density n_s is defined as the zeroth moment of the distribution function,

$$n_s\equiv\int f_s \mathrm{d}\mathbf{v}. \tag{2.18}$$

For the remainder of this section, we switch to indicial notation, which is more convenient for this analysis. The Boltzmann equation written as such is:

$$\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x_k} + \frac{q}{m}\left(E_i + \varepsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i} = \left.\frac{\partial f}{\partial t}\right|_c, \tag{2.19}$$

where the species subscript s has been dropped as implicit to avoid confusion with the tensorial indices.

The Klimontovich equation may be implemented directly to simulate very low density plasma dynamics, where volumetric averages are not accurate. This is the formal starting point for the derivation of Particle-in-Cell (PIC) methods, which are formulated to include the continuous Boltzmann solution, and effectively by discretization perform a similar smoothing. Therefore low-density plasmas may be modeled by this approach with a much smaller number of discrete particles than exist in the real plasma. However, as the number of particles becomes large, it becomes computationally prohibitive to track all the particles and their interactions across seven-dimensional space, so a continuum approximation becomes useful. This is the case that the remainder of this section will address.

2.2. Moments of the Boltzmann Equation

Because the seven-dimensional space of the Boltzmann equation is unwieldy, dense plasma models are developed by taking moments in velocity space. Depending on the number of moments taken, increasingly complex models may be developed. The zeroth and first moments are the familiar continuity and momentum equations (for each species), but depending on the complexity of the model, the energy equation may be a scalar or a tensorial equation.

First, we identify moments of the distribution function, f, to define the fluid variables for each species. Integrating over the volume element $\mathrm{d}\mathbf{v} = \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}v_3$ in velocity space, the zeroth through third moments are defined as:

$$\rho = m \int f(\mathbf{v})\mathrm{d}\mathbf{v}, \tag{2.20}$$

$$\rho u_i = m \int v_i f(\mathbf{v})\mathrm{d}\mathbf{v}, \tag{2.21}$$

$$\mathscr{P}_{ij} = m \int v_i v_j f(\mathbf{v})\mathrm{d}\mathbf{v}, \tag{2.22}$$

$$\mathscr{Q}_{ijk} = m \int v_i v_j v_k f(\mathbf{v})\mathrm{d}\mathbf{v}. \tag{2.23}$$

Obviously, (2.20) is the species density, and (2.21) is the ith component of the species momentum. Occasionally, the number density, $n = \rho/m$ is also useful. The tensorial quantities $\mathscr{P}_{ij}$ and $\mathscr{Q}_{ijk}$ as defined in (2.22) and (2.23) do not have convenient physical interpretations, so instead, the moments are taken with respect to the relative velocity $\mathbf{w} \equiv \mathbf{v} - \mathbf{u}$:

$$P_{ij} = m \int w_i w_j f(\mathbf{v})\mathrm{d}\mathbf{v}, \tag{2.24}$$

$$Q_{ijk} = m \int w_i w_j w_k f(\mathbf{v})\mathrm{d}\mathbf{v}. \tag{2.25}$$

Then, P_{ij} is a fluid stress tensor, and Q_{ijk} is a heat flux tensor. With a little manipulation, it is straightforward to show that

$$\mathscr{P}_{ij} = P_{ij} + \rho u_i u_j, \tag{2.26}$$
$$\mathscr{Q}_{ijk} = Q_{ijk} + u_i P_{jk} + u_j P_{ik} + u_k P_{ij} + \rho u_i u_j u_k. \tag{2.27}$$

Note that because the indices of the velocity components in the moment definitions are arbitrary, P_{ij} and Q_{ijk} are symmetric tensors, with six and and ten independent components, respectively.

The fluid equations are derived by taking moments of the Boltzmann equation, multiplying (2.19) by velocity and integrating over (all) velocity space. In general, and somewhat abusing notation in the velocity vector exponents for the purpose of illustration,

$$m\int \mathbf{v}^n \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} + m\int \mathbf{v}^{n+1}\frac{\partial f}{\partial \mathbf{x}}\mathrm{d}\mathbf{v} = -q\int \mathbf{v}^n \left(E_i + \varepsilon_{ijk}v_j B_k\right)\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} + \dots,$$

we see that convective term for the evolution of a given moment variable will introduce the next order moment variable. This presents a closure problem, and each plasma model chooses a different level at which to approximate the higher-order moment term. Before introducing these approximations, however, we will show the complete ('exact') set of equations in order to properly illustrate how the system is simplified.

First, we derive the continuity equation. Integrating (2.19) over velocity space and multiplying by the species mass, we have,

$$m\int \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} + m\int v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} + q\int \left(E_i + \varepsilon_{ijk}v_j B_k\right)\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} = m\int \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v}\bigg|_c.$$

In this derivation and all that follow, we will assume that f is sufficiently smooth for the manipulations required. Proceeding term by term, the convective terms are straightforward:

$$m\int \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} = \frac{\partial}{\partial t}\left(m\int f\mathrm{d}\mathbf{v}\right) = \frac{\partial \rho}{\partial t},$$
$$m\int v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} = \frac{\partial}{\partial x_k}\left(m\int v_k f\mathrm{d}\mathbf{v}\right) = \frac{\partial(\rho u_k)}{\partial x_k}.$$

To evaluate the Lorentz force term, we integrate by parts:

$$\begin{aligned} q\int \left(E_i + \varepsilon_{ijk}v_j B_k\right)\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} &= qE_i\int \frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} + q\varepsilon_{ijk}B_k\int v_j\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v}, \\ &= qE_i\int f n_i \mathrm{d}\Gamma + q\varepsilon_{ijk}B_k\left(\int f v_j n_i \mathrm{d}\Gamma - \int f\delta_{ij}\mathrm{d}\mathbf{v}\right). \end{aligned}$$

Here, Γ is the boundary of the velocity space, where we can assume that $f \to 0$ because the distribution vanishes at extreme velocities. Therefore, these boundary terms disappear, leaving only the last term, where we note that $\varepsilon_{ijk}\delta_{ij} = 0$, and so it too disappears. This yields the familiar continuity equation, with a source term on the right-hand side:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = \frac{\partial \rho}{\partial t}\bigg|_c. \tag{2.28}$$

To derive the momentum equation, we multiply (2.19) by v_m and again integrate over velocity space:

$$m\int v_m \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} + m\int v_m v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} + q\int v_m \left(E_i + \varepsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} = m\int v_m \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v}\bigg|_c .$$

As before, the convective terms are straightforward:

$$m\int v_m \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} = \frac{\partial}{\partial t}\left(m\int v_m f \mathrm{d}\mathbf{v}\right) = \frac{\partial}{\partial t}\left(\rho u_m\right),$$
$$m\int v_m v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} = \frac{\partial}{\partial x_k}\left(m\int v_m v_k f \mathrm{d}\mathbf{v}\right) = \frac{\partial \mathscr{P}_{mk}}{\partial x_k}.$$

Integrating the Lorentz term by parts, and dropping the boundary terms as before, we have

$$\begin{aligned} q\int v_m \left(E_i + \varepsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} &= qE_i\int v_m \frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} + q\varepsilon_{ijk}B_k\int v_m v_j \frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v}, \\ &= -qE_i\int f\delta_{im}\mathrm{d}\mathbf{v} - q\varepsilon_{ijk}B_k\int f\left(v_j\delta_{im} + v_m\delta_{ij}\right)\mathrm{d}\mathbf{v} \\ &= -\frac{\rho q}{m}\left(E_m + \varepsilon_{mjk}u_j B_k\right). \end{aligned}$$

Putting the terms together, using (2.26), and switching the independent index to i, we obtain the momentum equation with a generalized stress tensor P_{ij}:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + P_{ij}\right) = \frac{\rho q}{m}\left(E_i + \varepsilon_{ijk}u_j B_k\right) + \frac{\partial(\rho u_i)}{\partial t}\bigg|_c . \tag{2.29}$$

The next moment equation, for energy, is derived by multiplying (2.19) by $v_m v_n$ and again integrating over velocity space:

$$m\int v_m v_n \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} + m\int v_m v_n v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} = -q\int v_m v_n \left(E_i + \varepsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} + m\int v_m v_n \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v}\bigg|_c .$$

Proceeding as before, for the left-hand side we have

$$m\int v_m v_n \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} = \frac{\partial}{\partial t}\left(m\int v_m v_n f \mathrm{d}\mathbf{v}\right) = \frac{\partial \mathscr{P}_{mn}}{\partial t},$$
$$m\int v_m v_n v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} = \frac{\partial}{\partial x_k}\left(m\int v_k v_m v_n f \mathrm{d}\mathbf{v}\right) = \frac{\partial \mathscr{Q}_{kmn}}{\partial x_k},$$

and for the Lorentz term,

$$\begin{aligned} -q\int v_m v_n E_i \frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} &= qE_i\int f\left(v_m\delta_{in} + v_n\delta_{im}\right)\mathrm{d}\mathbf{v} \\ &= \frac{q}{m}E_i\left(\delta_{in}\rho u_m + \delta_{im}\rho u_n\right) \\ &= \frac{\rho q}{m}\left(E_n u_m + E_m u_n\right). \end{aligned}$$

$$-q\int \varepsilon_{ijk}B_k v_j v_m v_n \frac{\partial f}{\partial v_i}\mathrm{d}\mathbf{v} = q\varepsilon_{ijk}B_k \int f\left(v_m v_n \delta_{ij} + v_j v_n \delta_{im} + v_j v_m \delta_{in}\right)\mathrm{d}\mathbf{v}$$
$$= \frac{q}{m}\varepsilon_{ijk}B_k\left(\mathscr{P}_{mn}\delta_{ij} + \mathscr{P}_{jn}\delta_{im} + \mathscr{P}_{jm}\delta_{in}\right)$$
$$= \frac{q}{m}B_k\left(\varepsilon_{mjk}\mathscr{P}_{jn} + \varepsilon_{njk}\mathscr{P}_{jm}\right).$$

Assembling the terms, and switching the independent indices to i and j, we obtain an equation for $\mathscr{P}_{ij}$,

$$\frac{\partial \mathscr{P}_{ij}}{\partial t} + \frac{\partial \mathscr{Q}_{ijk}}{\partial x_k} = \frac{\rho q}{m}\left(E_i u_j + E_j u_i\right) + \frac{q}{m}\left(\varepsilon_{ikl}\mathscr{P}_{kj}B_l + \varepsilon_{jkl}\mathscr{P}_{ik}B_l\right) + \left.\frac{\partial \mathscr{P}_{ij}}{\partial t}\right|_c. \tag{2.30}$$

The third moment equation is derived in the same way, but since it is usually approximated, the derivation of the complete tensor equation is left out for brevity. It introduces the next moment of the distribution function,

$$\mathscr{K}_{ijkl} = m\int v_i v_j v_k v_l f(\mathbf{v})\mathrm{d}\mathbf{v}, \tag{2.31}$$

and is quoted here from [9]:

$$\frac{\partial \mathscr{Q}_{ijk}}{\partial t} + \frac{\partial \mathscr{K}_{ijkl}}{\partial x_l} = \frac{q}{m}\Big(E_i\mathscr{P}_{jk} + E_j\mathscr{P}_{ik} + E_k\mathscr{P}_{ij} + \left[\varepsilon_{ilm}\mathscr{Q}_{ljk} + \varepsilon_{jlm}\mathscr{Q}_{ilk} + \varepsilon_{klm}\mathscr{Q}_{ijl}\right]B_m\Big) + \left.\frac{\partial \mathscr{Q}_{ijk}}{\partial t}\right|_c. \tag{2.32}$$

Thus we have derived the set of 'exact' moment equations, (2.28), (2.29), (2.30) and (2.32), corresponding to Eqs. (19)–(22) of [9]. These correspond to $(1+3+6+10) = 20$ equations in $(1+3+6+10+15) = 35$ independent unknowns (due to the symmetry of the tensor quantities). Closure of the system is achieved by approximating various terms in these equations. Three standard approaches will be presented in the following sections.

2.3. The 5N-Moment Model

A standard reduction of the moment equations is to use a tensor contraction of the second moment [21, 8], to define a scalar fluid pressure for each species:

$$p_s \equiv \frac{1}{3}m_s\int w^2 f(\mathbf{v})\mathrm{d}\mathbf{v}, \tag{2.33}$$

where $w^2 = |\mathbf{v}-\mathbf{u}|^2$. Dropping the species subscript for clarity as before, note that

$$p = \frac{1}{3}\mathrm{tr}(\mathbf{P}) = \frac{1}{3}P_{ii},$$

with summation over the repeated index. Cross terms in the stress tensor will still appear in the equation system, so we define

$$P_{ij} = p\delta_{ij} + \Pi_{ij}, \tag{2.34}$$

and note that $\Pi_{ii} = 0$.

The continuity equation is unchanged, and the momentum equation is trivially rewritten using the definition (2.34). To proceed with the second moment, we need an additional contraction to define a vector heat flux,

$$h_i = \frac{1}{2} m \int w^2 w_i f(\mathbf{v}) \mathrm{d}\mathbf{v}. \tag{2.35}$$

The second moment equation must be rederived now, so multiplying (2.19) by $v_m v_m \equiv v_m^2$ (with implicit summation over the index in the squared term), and integrating gives:

$$m \int v_m^2 \frac{\partial f}{\partial t} \mathrm{d}\mathbf{v} + m \int v_m^2 v_k \frac{\partial f}{\partial x_k} \mathrm{d}\mathbf{v} = -q \int v_m^2 \left(E_i + \varepsilon_{ijk} v_j B_k \right) \frac{\partial f}{\partial v_i} \mathrm{d}\mathbf{v} + m \int v_m^2 \frac{\partial f}{\partial t} \mathrm{d}\mathbf{v} \bigg|_c .$$

Proceeding as before,

$$\begin{aligned} m \int v_m^2 \frac{\partial f}{\partial t} \mathrm{d}\mathbf{v} &= \frac{\partial}{\partial t} \left(m \int f (w_m^2 + 2 v_m u_m - u_m^2) \mathrm{d}\mathbf{v} \right) \\ &= \frac{\partial}{\partial t} \left(m \int f w^2 \mathrm{d}\mathbf{v} + 2 u_m m \int f v_m \mathrm{d}\mathbf{v} - u_m^2 m \int f \mathrm{d}\mathbf{v} \right), \\ &= \frac{\partial}{\partial t} \left(3p + \rho u_m^2 \right), \end{aligned}$$

$$\begin{aligned} m \int v_m^2 v_k \frac{\partial f}{\partial x_k} \mathrm{d}\mathbf{v} &= \frac{\partial}{\partial x_k} \left(m \int f \left(w_k w_m^2 + u_k w_m^2 + 2 u_m v_k v_m - u_m^2 v_k \right) \mathrm{d}\mathbf{v} \right) \\ &= \frac{\partial}{\partial x_k} \left(m \int f w_k w_m^2 \mathrm{d}\mathbf{v} + u_k m \int f w_m^2 \mathrm{d}\mathbf{v} + 2 u_m m \int f v_k v_m \mathrm{d}\mathbf{v} - u_m^2 m \int f v_k \mathrm{d}\mathbf{v} \right) \\ &= \frac{\partial}{\partial x_k} \left(2 h_k + 3 u_k p + 2 u_m \mathscr{P}_{km} - \rho u_m^2 u_k \right) \\ &= \frac{\partial}{\partial x_k} \left(2 h_k + 5 u_k p + 2 u_m \Pi_{km} + \rho u_m^2 u_k \right). \end{aligned}$$

For the Lorentz term,

$$\begin{aligned} -q \int v_m^2 E_i \frac{\partial f}{\partial v_i} \mathrm{d}\mathbf{v} &= 2 q E_i \int f v_i \mathrm{d}\mathbf{v} \\ &= 2 \frac{\rho q}{m} E_i u_i. \end{aligned}$$

$$\begin{aligned} -q \int \varepsilon_{ijk} B_k v_j v_m^2 \frac{\partial f}{\partial v_i} \mathrm{d}\mathbf{v} &= q \varepsilon_{ijk} B_k \int f \left(v_m^2 \delta_{ij} + 2 v_i v_j \right) \mathrm{d}\mathbf{v} \\ &= 2 \frac{q}{m} \varepsilon_{ijk} B_k \mathscr{P}_{ij} \\ &= 2 \frac{q}{m} \varepsilon_{ijk} B_k \left(p \delta_{ij} + \Pi_{ij} + \rho u_i u_j \right) \\ &= 0, \end{aligned}$$

by symmetry. Dividing the reassembled system by two, this yields an energy-type equation of the familiar form:

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p+\frac{1}{2}\rho u_i^2\right)+\frac{\partial}{\partial x_j}\left(\left[\frac{5}{2}p+\frac{1}{2}\rho u_i^2\right]u_j+u_i\Pi_{ij}+h_j\right)=\frac{\rho q}{m}E_iu_i+\text{src.} \tag{2.36}$$

A scalar energy can also be defined,

$$\mathscr{E}_s=\frac{3}{2}p_s+\frac{1}{2}\rho_s u_i^2, \tag{2.37}$$

which is often generalized with the pressure coefficient as $1/(\gamma-1)$, implying here that $\gamma=5/3$ for this presentation. The proper generalization for nonatomic particles with other values for γ occurs in the definition of pressure as a tensor contraction of the second moment of the particle distribution, i.e., in equation (2.33).

The complete set of equations for the 5N model, so named because there are $(1+3+1)=5$ equations in the system, are therefore (for each species):

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho u_k)=\left.\frac{\partial\rho}{\partial t}\right|_c, \tag{2.38}$$

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho u_iu_j+p\delta_{ij}+\Pi_{ij}\right)=\frac{\rho q}{m}\left(E_i+\varepsilon_{ijk}u_jB_k\right)+\left.\frac{\partial(\rho u_i)}{\partial t}\right|_c, \tag{2.39}$$

$$\frac{\partial\mathscr{E}}{\partial t}+\frac{\partial}{\partial x_j}\left((\mathscr{E}+p)u_j+u_i\Pi_{ij}+h_j\right)=\frac{\rho q}{m}E_iu_i+\left.\frac{\partial\mathscr{E}}{\partial t}\right|_c. \tag{2.40}$$

The system is closed by introducing models for the stress Π_{ij} and the heat flux h_i. This model is essentially a standard formulation of the continuity, momentum and energy equations for each component of the fluid, with the addition of Lorentz forces, and standard assumptions regarding the fluid stress tensor and heat flux can apply. Note, for example, that h_i is consistent with a Fourier conduction approximation ($\mathbf{h}=\kappa\nabla T$).

Due to the familiarity of this system, it is often convenient to express the 5N equations in vector notation, as follows:

$$\frac{\partial\rho_s}{\partial t}+\nabla\cdot(\rho_s\mathbf{u}_s)=\left.\frac{\partial\rho_s}{\partial t}\right|_c \tag{2.41}$$

$$\frac{\partial(\rho_s\mathbf{u}_s)}{\partial t}+\nabla\cdot(\rho_s\mathbf{u}_s\otimes\mathbf{u}_s+p_sI+\Pi_s)=\frac{\rho_sq_s}{m_s}(\mathbf{E}+\mathbf{u}_s\times\mathbf{B})+\left.\frac{\partial(\rho_s\mathbf{u}_s)}{\partial t}\right|_c \tag{2.42}$$

$$\frac{\partial\mathscr{E}_s}{\partial t}+\nabla\cdot((\mathscr{E}_s+p_s)\mathbf{u}_s+\mathbf{u}_s\cdot\Pi_s+\mathbf{h}_s)=\frac{\rho_sq_s}{m_s}\mathbf{E}\cdot\mathbf{u}_s+\left.\frac{\partial\mathscr{E}_s}{\partial t}\right|_c \tag{2.43}$$

2.4. The 10N-Moment Model

A more complex plasma fluid model is presented by [9]. This retains the full symmetric tensor form of the second moment equation, but closes the system by assuming that the divergence of

Q_{ijk} disappears, which removes it as a variable from the system. Justification for this simplification is provided by assuming a Gaussian distribution function, and it yields $(1+3+6) = 10$ equations in the following system:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = \left.\frac{\partial \rho}{\partial t}\right|_c, \tag{2.44}$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + P_{ij}\right) = \frac{\rho q}{m}\left(E_i + \varepsilon_{ijk}u_j B_k\right) + \left.\frac{\partial(\rho u_i)}{\partial t}\right|_c, \tag{2.45}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(P_{ij} + \rho u_i u_j\right) &+ \frac{\partial}{\partial x_k}\left(u_k P_{ij} + u_j P_{ik} + u_i P_{jk} + \rho u_i u_j u_k\right) \\ &= \frac{\rho q}{m}\left(E_i u_j + E_j u_i\right) + \frac{q}{m}\left(\varepsilon_{ikl}\mathscr{P}_{kj}B_l + \varepsilon_{jkl}\mathscr{P}_{ik}B_l\right) + \left.\frac{\partial \mathscr{P}_{ij}}{\partial t}\right|_c.\end{aligned} \tag{2.46}$$

This model spans the space between the 5N and 13N models, and is motivated by a need to capture finite Larmor radius effects. The system remains primarily hyperbolic in character, with a strong dispersive effect caused by the collisional source terms that results in a complex dispersion relation.

2.5. The 13N-Moment Model

Following [20], the third moment equation can be added to the system with a closure approach very similar to that employed in the 5N model. As in the 10N model, the full second moment equation will be used, but Q_{ijk} will be retained and a reduced fourth moment will close the third moment equation:

$$N_{ij} = m_s \int w^2 w_i w_j f(\mathbf{v})\mathrm{d}\mathbf{v}. \tag{2.47}$$

For this derivation, we will also refer to the reductions defined for the 5N model, (2.33), (2.34), and (2.35).

We now proceed to derive the reduced third moment equation, multiplying (2.19) by $v_i v_j^2$ and integrating for:

$$m\int v_i v_j^2 \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} + m\int v_i v_j^2 v_k \frac{\partial f}{\partial x_k}\mathrm{d}\mathbf{v} = -q\int v_i v_j^2\left(E_k + \varepsilon_{kmn}v_m B_n\right)\frac{\partial f}{\partial v_k}\mathrm{d}\mathbf{v} + m\int v_i v_j^2 \left.\frac{\partial f}{\partial t}\mathrm{d}\mathbf{v}\right|_c.$$

Proceeding as before, we have

$$\begin{aligned} m\int v_i v_j^2 \frac{\partial f}{\partial t}\mathrm{d}\mathbf{v} &= \frac{\partial}{\partial t}\left(m\int f(w_i w_j^2 + 2w_i u_j w_j + w_i u_j^2 + u_i v_j^2)\mathrm{d}\mathbf{v}\right) \\ &= \frac{\partial}{\partial t}\left(m\int f w_i w_j^2 \mathrm{d}\mathbf{v} + 2u_j m\int f w_i w_j \mathrm{d}\mathbf{v} + u_j^2 m\int f w_i \mathrm{d}\mathbf{v} + u_i m\int f v_j^2 \mathrm{d}\mathbf{v}\right), \\ &= \frac{\partial}{\partial t}\left(2h_i + 2u_j P_{ij} + u_i \mathscr{P}_{jj}\right), \\ &= \frac{\partial}{\partial t}\left((\rho u_j^2 + 3p)u_i + 2P_{ij}u_j + 2h_i\right), \end{aligned}$$

where $\mathscr{P}_{jj} = 3p + \rho u_j^2$ by retaining the definition of p from (2.33); and,

$$
\begin{aligned}
m \int v_i v_j^2 v_k \frac{\partial f}{\partial x_k} \mathrm{d}\mathbf{v} = \frac{\partial}{\partial x_k} \Big(m \int f \big(& w_i w_j^2 w_k - u_i u_j^2 u_k + u_j^2 u_k v_i + u_j^2 u_i v_k + 2 u_i u_j u_k v_j \\
& - 2 u_j u_k v_i v_j - u_i u_k v_j^2 - u_j^2 v_i v_k - 2 u_i u_j v_j v_k \\
& + u_k v_i v_j^2 + 2 u_j v_i v_j v_k + u_i v_j^2 v_k \big) \mathrm{d}\mathbf{v} \Big) \\
= \frac{\partial}{\partial x_k} \Big(& N_{ik} + 3\rho u_i u_j^2 u_k - 2 u_j u_k \mathscr{P}_{ij} - u_i u_k \mathscr{P}_{jj} - u_j^2 \mathscr{P}_{ik} - 2 u_i u_j \mathscr{P}_{jk} + 2 u_j Q_{ijk} \\
& + u_k \left[(\rho u_j^2 + 3p) u_i + 2 P_{ij} u_j + 2 h_i \right] + u_i \left[(\rho u_j^2 + 3p) u_k + 2 P_{kj} u_j + 2 h_k \right] \Big) \\
= \frac{\partial}{\partial x_k} \Big(& N_{ik} + 3\rho u_i u_j^2 u_k - 2 u_j u_k \mathscr{P}_{ij} - u_i u_k \mathscr{P}_{jj} - u_j^2 \mathscr{P}_{ik} - 2 u_i u_j \mathscr{P}_{jk} + 2 u_j Q_{ijk} \\
& + u_k \left[(\rho u_j^2 + 3p) u_i + 2 P_{ij} u_j + 2 h_i \right] + u_i \left[(\rho u_j^2 + 3p) u_k + 2 P_{kj} u_j + 2 h_k \right] \Big) \\
= \frac{\partial}{\partial x_k} \big(& N_{ik} + (\rho u_j^2 + 3p) u_i u_k + 4 u_k P_{ij} u_j + u_j^2 \mathscr{P}_{ik} + u_k h_i + u_i h_k + 2 u_j Q_{ijk} \big).
\end{aligned}
$$

For the Lorentz term, and following the previous arguments, we have

$$
\begin{aligned}
-m \int v_i v_j^2 \frac{\partial f}{\partial v_k} \mathrm{d}\mathbf{v} &= \delta_{ik} m \int f v_j^2 \mathrm{d}\mathbf{v} + 2m \int f v_i v_k \mathrm{d}\mathbf{v} \\
&= \delta_{ik} (\rho u_j^2 + 3p) + 2 \left(P_{ik} + \rho u_i u_k \right),
\end{aligned}
$$

and for the $\mathbf{v} \times \mathbf{B}$ part,

$$
\begin{aligned}
-m \varepsilon_{kmn} \int v_i v_j^2 v_m \frac{\partial f}{\partial v_k} \mathrm{d}\mathbf{v} &= \varepsilon_{kmn} \int f \left(\delta_{km} v_i v_j^2 + \delta_{ik} v_m v_j^2 + 2 v_i v_k v_m \right) \mathrm{d}\mathbf{v} \\
&= 0 + \varepsilon_{imn} \int f v_m v_j^2 \mathrm{d}\mathbf{v} + 2 \varepsilon_{kmn} \int f v_i v_k v_m \mathrm{d}\mathbf{v} \\
&= \varepsilon_{imn} u_m (\rho u_j^2 + 3p) + 2 \varepsilon_{imn} (P_{mj} u_j + h_m) \\
&\quad + 2 \varepsilon_{kmn} (u_i P_{km} + u_k P_{im} + u_m P_{ik} + Q_{ikm}),
\end{aligned}
$$

This simplifies by the symmetry of P when multiplied by B_n, because

$$
\varepsilon_{kmn} u_m P_{ik} B_n = \varepsilon_{mkn} u_k P_{im} B_n = -\varepsilon_{kmn} u_k P_{im} B_n,
$$

such that:

$$
\begin{aligned}
-m \varepsilon_{kmn} B_n \int v_i v_j^2 v_m \frac{\partial f}{\partial v_k} \mathrm{d}\mathbf{v} = \varepsilon_{imn} & (\rho u_j^2 + 3p) u_m B_n \\
& + 2 \left(\varepsilon_{kmn} Q_{ikm} + \varepsilon_{imn} h_m + \varepsilon_{imn} P_{mj} u_j \right) B_n.
\end{aligned}
$$

Collecting all the terms, and dividing by two, we have:

$$\frac{\partial}{\partial t}\left(\mathscr{E}u_i + P_{ij}u_j + h_i\right) + \frac{\partial}{\partial x_k}\left(\mathscr{E}u_iu_k + \frac{1}{2}u_j^2P_{ik} + 2u_kP_{ij}u_j + 2h_iu_k + Q_{ijk}u_j + \frac{1}{2}N_{ik}\right)$$
$$= \frac{q}{m}\left(\mathscr{E}E_i + (P_{ik} + \rho u_iu_k)E_k\right)$$
$$+ \frac{q}{m}\left(\varepsilon_{imn}(\mathscr{E}u_m + P_{mj}u_j + h_m)B_n + \varepsilon_{kmn}Q_{ikm}B_n\right) + \text{sources}, \quad (2.48)$$

using the scalar energy definition (2.37).

The complete set of equations for the 13N-moment model, so named because there are $(1+3+6+3) = 13$ equations in the system, are therefore (for each species):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = \left.\frac{\partial\rho}{\partial t}\right|_c, \quad (2.49)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_iu_j + P_{ij}\right) = \frac{\rho q}{m}\left(E_i + \varepsilon_{ijk}u_jB_k\right) + \left.\frac{\partial(\rho u_i)}{\partial t}\right|_c, \quad (2.50)$$

$$\frac{\partial}{\partial t}\left(P_{ij} + \rho u_iu_j\right) + \frac{\partial}{\partial x_k}\left(Q_{ijk} + u_iP_{jk} + u_jP_{ik} + u_kP_{ij} + \rho u_iu_ju_k\right)$$
$$= \frac{\rho q}{m}(E_iu_j + E_ju_i) + \frac{q}{m}\left(\varepsilon_{ikm}\mathscr{P}_{kj} + \varepsilon_{jkm}\mathscr{P}_{ki}\right)B_m \quad (2.51)$$
$$+ \text{sources},$$

$$\frac{\partial}{\partial t}\left(\mathscr{E}u_i + P_{ij}u_j + h_i\right) + \frac{\partial}{\partial x_k}\left(\mathscr{E}u_iu_k + \frac{1}{2}u_j^2P_{ik} + 2u_kP_{ij}u_j + 2h_iu_k + Q_{ijk}u_j + \frac{1}{2}N_{ik}\right)$$
$$= \frac{q}{m}\left(\mathscr{E}E_i + \mathscr{P}_{ik}E_k + \left[\varepsilon_{imn}(\mathscr{E}u_m + P_{mj}u_j + h_m) + \varepsilon_{kmn}Q_{ikm}\right]B_n\right) \quad (2.52)$$
$$+ \text{sources}.$$

As in the 5N model, this system is closed by introducing models for Q_{ijk} and N_{ik}, leaving the variables to be solved for as ρ, u_i, P_{ij} and h_i.

In the same way that the 5N model relates to the Navier-Stokes equations, the 13N model relates to Grad's equations [24]. Formal analysis will show that this system is second order in Knudsen number, and hence extends the region of validity of the continuum approximation beyond that accessible with the standard Navier-Stokes truncation. The 13N equations are generally considered highly challenging to solve numerically in terms of closure models, boundary conditions, and stability.

3. THE GENERAL THREE-FLUID MODEL

3.1. Equations of the Model

For the vast majority of plasma modeling, the assumptions introduced by the 5N-moment equations—essentially, isotropic heat flux—are not restrictive. Also, because the 5N system is equivalent to the multi-species Euler or Navier-Stokes equations with a Lorentz force term, it is much better understood numerically than the higher-order moment systems. In fact, significant progress can be made in plasma modeling with even simpler systems of equations. The principal modeling question then becomes the choice of representation for the components of the fluid system.

First, however, we will consider the three-component fluid model in detail, starting with the 5N system to account for interaction of ions, electrons and neutrals, largely following [15], but neglecting resonant charge exchange terms. This system is generalized for a particular plasma system by adding ion and neutral equations for each species of interest, because no assumption has been made regarding the mass of an electron relative to other species, and each species is considered to have distinct dynamics. This implies tracking of only integer ionization states, i.e., $q_i = eZ$ where Z is a positive or negative integer, with a separate set of equations for each state (e.g., N^+, N^{2+}). We note that for plasmas with many species (e.g., an air plasma) this can quickly become very expensive to solve numerically, analogous to a chemically reacting flow/combustion model that attempts to follow a large number of species undergoing a significant number of chemical reactions, but may be necessary for some applications.

In the three-fluid system, processes that ionize neutrals and recombine electrons and ions will transfer mass between each species system. From (2.38), the continuity equations become:

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = m_i \Gamma^{ion} - m_i \Gamma^{rec}, \tag{3.1}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e) = m_e \Gamma^{ion} - m_e \Gamma^{rec}, \tag{3.2}$$

$$\frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \mathbf{u}_n) = m_n \Gamma^{rec} - m_n \Gamma^{ion}, \tag{3.3}$$

where $\rho_i = m_i n_i, \rho_e = m_e n_e, \rho_n = (m_i + m_e) n_n = m_n n_n$ define, respectively, the ion, electron and neutral mass density in terms of the particle masses and the number densities. Note that this definition implies singly ionized particles ($Z = 1$), but is easily generalized. The source terms, Γ^{ion} and Γ^{rec}, represent the ionization rate per unit volume and recombination rate per unit volume, respectively.

The momentum equations follow similarly, with the addition of momentum transfer (drag) terms that arise due to particle collisions. Using the general stress tensor P_a, this gives:

$$\frac{\partial \rho_i \mathbf{u}_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i \otimes \mathbf{u}_i + P_i) = q_i n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \mathbf{R}^{ie} + \mathbf{R}^{in} + \Gamma^{ion} m_i \mathbf{u}_n - \Gamma^{rec} m_i \mathbf{u}_i, \tag{3.4}$$

$$\frac{\partial \rho_e \mathbf{u}_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_i \otimes \mathbf{u}_e + P_e) = q_e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \mathbf{R}^{ie} + \mathbf{R}^{en} + \Gamma^{ion} m_e \mathbf{u}_n - \Gamma^{rec} m_e \mathbf{u}_e, \tag{3.5}$$

$$\frac{\partial \rho_n \mathbf{u}_n}{\partial t} + \nabla \cdot (\rho_n \mathbf{u}_n \otimes \mathbf{u}_n + P_n) = -\mathbf{R}^{in} - \mathbf{R}^{en} - \Gamma^{ion} m_n \mathbf{u}_n + \Gamma^{rec}(m_i \mathbf{u}_i + m_e \mathbf{u}_e), \tag{3.6}$$

where $\mathbf{R}^{ab}$ is the source force density on species a due to scattering collisions with species b. In order to conserve momentum, $\mathbf{R}^{ab}$ must be anti-symmetric in the a and b indices.

The equations for the total energy, $\mathscr{E}_s$, account for energy exchange directly due to the previously mentioned collisional effects, and allow for additional energy changes caused by elastic or inelastic collisions:

$$\begin{aligned} \frac{\partial \mathscr{E}_i}{\partial t} + \nabla \cdot (\mathscr{E}_i \mathbf{u}_i + \mathbf{u}_i \cdot P_i + \mathbf{h}_i) &= q_i n_i \mathbf{E} \cdot \mathbf{u}_i + \mathbf{u}_i \cdot \mathbf{R}^{ie} + \mathbf{u}_i \cdot \mathbf{R}^{in} \\ &+ \Gamma^{ion} \frac{1}{2} m_i u_n^2 + m_i Q_n^{ion}/m_n - \Gamma^{rec} \frac{1}{2} m_i u_i^2 - Q_i^{rec} + Q^{ie} + Q^{in}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial \mathscr{E}_e}{\partial t} + \nabla \cdot (\mathscr{E}_e \mathbf{u}_e + \mathbf{u}_e \cdot P_e + \mathbf{h}_e) &= q_e n_e \mathbf{E} \cdot \mathbf{u}_e - \mathbf{u}_e \cdot \mathbf{R}^{ie} + \mathbf{u}_e \cdot \mathbf{R}^{en} \\ &+ \Gamma^{ion} (\frac{1}{2} m_e u_n^2 - \phi^{ion}) + m_e Q_n^{ion}/m_n - \Gamma^{rec} \frac{1}{2} m_e u_e^2 - Q_e^{rec} - Q^{ie} + Q^{en}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \frac{\partial \mathscr{E}_n}{\partial t} + \nabla \cdot (\mathscr{E}_n \mathbf{u}_n + \mathbf{u}_n \cdot P_n + \mathbf{h}_n) &= -\mathbf{u}_n \cdot \mathbf{R}^{in} - \mathbf{u}_n \cdot \mathbf{R}^{en} \\ &- \Gamma^{ion} \frac{1}{2} m_n u_n^2 - Q_n^{ion} + \Gamma^{rec} (\frac{1}{2} m_i u_i^2 + \frac{1}{2} m_e u_e^2) + Q_i^{rec} + Q_e^{rec} - Q^{in} - Q^{en}, \end{aligned} \tag{3.9}$$

where Q_n^{ion} is the thermal energy associated with neutrals at ionization and Q_a^{rec} is the thermal energy associated with species a upon recombination. The Q^{ab} terms are the rates of transfer of thermal energy from species b to a due to collisional effects, and are anti-symmetric in a and b. The ionization energy term, ϕ^{ion} represents energy required to create an additional free electron via an electron-neutral collision. This energy is correspondingly removed from the electron energy equation. Upon recombination, a photon of energy $h\nu = \phi^{ion}$ is produced, and is assumed lost from the system (it may be tallied separately). Note that source terms presented here implicitly assume single ionization of the neutral species, i.e., $q_i = e$ and $q_e = -e$.

The equations are also coupled via Maxwell's equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{3.10}$$

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = -\mathbf{J}, \tag{3.11}$$

where **D** and **H** are the electric displacement and magnetic field as defined previously. The fluid plasma equations couple into Maxwell's equations through the total charge density,

$$q = q_i n_i + q_e n_e, \tag{3.12}$$

and the total current density,

$$\mathbf{J} = \mathbf{J}_i + \mathbf{J}_e = q_i n_i \mathbf{u}_i + q_e n_e \mathbf{u}_e. \tag{3.13}$$

Familiar observations regarding conservation can be made by summing the equations for each species. From the continuity equations, we recover the statement of mass conservation for the closed system:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

where the total density is defined as $\rho = \rho_i + \rho_e + \rho_n$ and the momentum as $\rho \mathbf{u} = \rho_i \mathbf{u}_i + \rho_e \mathbf{u}_e + \rho_n \mathbf{u}_n$. Similar arguments for total momentum and energy impose the symmetry constraints on the source terms. Also, from the right-hand sides of the momentum equations, the familiar definition for the electrodynamic force density is recovered: $q\mathbf{E} + \mathbf{J} \times \mathbf{B}$.

Generalization of this system for multiple species and ionization levels modifies the relationship between m_n, m_i, and m_e. Note that the momentum and energy equations as written assume single ionization of the neutral species, i.e., $m_n = m_i + m_e$. This assumption appears in the terms that account for momentum and kinetic energy transfers due to Γ^{ion} and Γ^{rec}.

3.2. Closure Models

The three-fluid system described by equations (3.1)–(3.11) remains to be closed with appropriate closure relations for the plasma being modeled. For most plasmas, and all those that will be considered in this report, an ideal gas approximation is usually suitable. For a monatomic gas, this takes the form

$$\mathscr{E}_a = \frac{3}{2} n_a k_B T_a + \frac{1}{2} \rho_a u_a^2, \tag{3.14}$$

$$e_a = \frac{3}{2} n_a k_B T_a, \tag{3.15}$$

$$p_a = n_a k_B T_a, \tag{3.16}$$

where e_a, T_a, p_a are the internal energy, temperature, and pressure, respectively, of species a, and k_B is the Boltzmann constant (cf. these definitions with the derivation of the 5M equations). Note that the Boltzmann constant is occasionally omitted (giving $p = nT$) when temperature is specified in units of energy, but we prefer to retain k_B in this presentation for consistency and to remain in SI units. The ideal gas equations may be generalized for non-monatomic gases by replacing the factor of $3/2$ with $1/(\gamma - 1)$, where γ is the usual heat capacity ratio that is related to the number of degrees of freedom of the molecule. In this case, a monatomic gas is specified using $\gamma = 5/3$.

Modeling of the deviatoric fluid stress tensor, Π, and the heat flux vector, $\mathbf{h}$, typically follows standard practice in fluid dynamics. An ideal fluid approximation ignores any dissipative effects and drops both terms, which is the basis of the ideal two-fluid model described in the next section, and yields an Euler-like system coupled to Maxwell's equations. If dissipative effects are expected to be significant, a Newtonian fluid model can be incorporated by relating Π to the velocity gradient, and Fourier heat conduction can be incorporated by relating $\mathbf{h}$ to the temperature gradient. This will yield a Navier-Stokes–like system coupled to Maxwell's equations.

In this case, the stress tensor may be simply approximated by

$$\Pi_a = -\zeta_a \left(\nabla \mathbf{u}_a + (\nabla \mathbf{u}_a)^T\right), \tag{3.17}$$

where ζ_a is an isotropic dynamic viscosity for species a. For the neutral fluid, ζ_n can be approximated using a rigid elastic sphere model following standard gas theory [15]. For the charged particles, [5] presents an isotropic dynamic viscosity model for an incompressible unmagnetized plasma, and a more complex anisotropic model for a plasma in the presence of a strong magnetic field.

Assuming that scattering collisions dominate the particle motion, standard models may also be used for the heat flux. For the neutral fluid, a standard Fourier-type model can be used, of the form

$$\mathbf{h}_n = -\kappa_n \nabla T_n, \tag{3.18}$$

where the conductivity κ_n can be derived using a rigid elastic sphere model consistent with the dynamic viscosity. For charged particles, the heat fluxes are strongly dependent on the magnetic field, taking the form

$$\begin{aligned} \mathbf{h}_a &= -\left(\kappa_\| \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} + \kappa_\perp (I - \hat{\mathbf{b}} \otimes \hat{\mathbf{b}})\right) \cdot \nabla T_a - \frac{\gamma p_e \mathbf{J}}{n_a q_e (\gamma - 1)}, \\ &= -\kappa_\| \nabla_\| T_a - \kappa_\perp \nabla_\perp T_a - \frac{\gamma}{\gamma - 1} \frac{k_B T_e}{n_a n_e q_e} \mathbf{J}, \end{aligned} \tag{3.19}$$

where $\hat{\mathbf{b}} \equiv \mathbf{B}/|\mathbf{B}|$ is the unit normal of the magnetic field direction, and the conductivities $\kappa_\|$ and $\kappa_\perp$ are parallel and perpendicular to the magnetic field direction, respectively. Braginskii [5] again provides the relevant values; see Section 4.2 for full details of this model.

3.3. An Analytic Elastic Collision Model

Closure of the collisional source terms presents the greatest challenge for solving the multi-fluid plasma system. A common form of the momentum transfer term is

$$\mathbf{R}^{ie} = -m_e n_e (\mathbf{u}_i - \mathbf{u}_e) \nu_{ie} = -\mathbf{R}^{ei},$$

where ν_{ie} is a collision frequency, a complicated function of velocity and temperature derived by making appropriate assumptions regarding the statistical particle distribution and collisions. One analytic example of this function is adapted from [16].

Starting from the 5M equations, and neglecting ionization and recombination effects (which are inherently inelastic collision effects), we can use the momentum equation to eliminate the kinetic part of the energy equation to write the following system for each species a:

$$\begin{aligned}
&\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}_a) = 0,\\
&\frac{\partial (\rho_a \mathbf{u}_a)}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}_a \otimes \mathbf{u}_a + p_a I + \Pi_a) - \frac{\rho_a q_a}{m_a} (\mathbf{E} + \mathbf{u}_a \times \mathbf{B}) = \left.\frac{\partial (\rho_a \mathbf{u}_a)}{\partial t}\right|_c,\\
\frac{3}{2}&\frac{\partial (n_a k_B T_a)}{\partial t} + \nabla \cdot \left(\frac{3}{2} n_a k_B T_a \mathbf{u}_a + \mathbf{h}_a\right) + p_a \nabla \cdot \mathbf{u}_a + \Pi_a : \nabla \mathbf{u}_a = \frac{3}{2} n_a k_B \left.\frac{\partial T_a}{\partial t}\right|_c.
\end{aligned}$$

Note that the right hand side of the temperature equation has been simplified because there is no source for n_a under these assumptions. To write closed analytic expressions for the source terms, a very simple model for the collision of a particle of species a with a particle of species b is required. We assume purely elastic collisions that result only in a change in direction of the relative particle velocity, $\mathbf{g} = \mathbf{v}_a - \mathbf{v}_b$, such that $|\mathbf{g}'| = |\mathbf{g}|$. The change in angle is characterized by a scattering cross section, and for these elastic collisions it is the total momentum transfer cross section, $\sigma(g)$, which is of interest. This is defined by integrating a differential scattering cross section I over all scattering angles $\mathrm{d}\omega$ (e.g., from [6]),

$$\sigma(g)\mathbf{g} = -\int (\mathbf{g}' - \mathbf{g}) I(g, \omega) \mathrm{d}\omega. \tag{3.20}$$

Note that contained in this definition of σ is the assumption that the collisions have azimuthally symmetric scattering.

The general form of the classical Boltzmann term for elastic collisions is

$$\left.\frac{\partial f_a}{\partial t}\right|_c = \int \mathrm{d}\mathbf{v}_b \int \mathrm{d}\omega \left(f_a(\mathbf{v}_a') f_b(\mathbf{v}_b') - f_a(\mathbf{v}_a) f_b(\mathbf{v}_b)\right) g I(g, \omega). \tag{3.21}$$

Moments are taken of this equation in the usual way, but a generalized expression may be written for the collision term for a macroscopic variable Ψ in terms of the microscopic variable ψ:

$$\left.\frac{\partial \Psi}{\partial t}\right|_c = \int \mathrm{d}\mathbf{v}_a \int \mathrm{d}\mathbf{v}_b f_a f_b g \int \mathrm{d}\omega I(g, \omega) \left(\psi' - \psi\right). \tag{3.22}$$

For density, $\psi = 1 = \psi'$ by conservation of particles, so as has already been assumed, there is no continuity source term. For momentum, $\psi = m_a \mathbf{v}_a$, and for temperature, $\psi = m_a (\mathbf{v}_a - \mathbf{u}_a)^2/2$. To make further progress, a specific form for f_a is required. Here, we will assume a local drifting Maxwellian distribution for each species,

$$f_a(\mathbf{x}, \mathbf{v}, t) = n_a(\mathbf{x}, t) \left(\frac{m_a}{2\pi k_B T_a(\mathbf{x}, t)}\right)^{3/2} \exp\left[-\frac{m_a (\mathbf{v} - \mathbf{u}_a(\mathbf{x}, t))^2}{2 k_B T_a(\mathbf{x}, t)}\right] \tag{3.23}$$

Note that integrating this distribution according to the macroscopic quantity definitions, (2.20) etc., will recover the expected quantities.

After extensive work, including a variable change that introduces the macroscopic velocity difference $\mathbf{U} = \mathbf{u}_a - \mathbf{u}_b$, the general form of the momentum term is found to be

$$\left.\frac{\partial (\rho_a \mathbf{u}_a)}{\partial t}\right|_c = -m_{ab} n_a \mathbf{U} \nu_{ab}^M, \tag{3.24}$$

where $\nu_{ab}^M = \nu_{ab}^M(n_b, v_{ab}, U)$ is a complicated function that defines the momentum transfer collision frequency, and

$$m_{ab} = \frac{m_a m_b}{m_a + m_b}, \quad \text{and} \quad v_{ab} = \sqrt{k_B \left(\frac{T_a}{m_a} + \frac{T_b}{m_b} \right)}.$$

Similarly for the temperature source term,

$$\frac{3}{2} \frac{\partial T_a}{\partial t} \bigg|_c = -(T_a - T_b) \frac{m_{ab}}{m_a + m_b} \nu_{ab}^E + \frac{m_b}{k_B T_b} \left(\frac{m_a}{T_a} + \frac{m_b}{T_b} \right)^{-1} m_{ab} U^2 \nu_{ab}^M, \tag{3.25}$$

where $\nu_{ab}^E = \nu_{ab}^E(n_b, v_{ab}, U)$ is the temperature equilibration collision frequency.

A closed form expression for these two frequencies can be written down for Coulomb collisions, which are typically the dominant process in plasmas. The momentum transfer cross section is given by $\sigma(g) = \sigma_0 / g^4$, with

$$\sigma_0 = \frac{1}{4\pi} \left(\frac{q_a q_b}{\varepsilon_0 m_{ab}} \right)^2 \ln \Lambda, \tag{3.26}$$

where $\ln \Lambda$ is the Coulomb logarithm, which is an order unity quantity related to the Debye length. Physically, $\sigma_0 \sim O(10^7)$ in SI units. This expression allows the frequency integrals to be computed analytically, giving

$$\nu_{ab}^M(n_b, v_{ab}, U) = n_b \frac{\sigma_0}{U^3} \left[\operatorname{erf}\left(\frac{U}{v_{ab}\sqrt{2}} \right) - \sqrt{\frac{2}{\pi}} \frac{U}{v_{ab}} \exp\left(-\frac{1}{2} \frac{U^2}{v_{ab}^2} \right) \right], \tag{3.27}$$

$$\nu_{ab}^E(n_b, v_{ab}, U) = n_b \frac{\sigma_0}{U v_{ab}^2} \operatorname{erf}\left(\frac{U}{v_{ab}\sqrt{2}} \right). \tag{3.28}$$

The key observation here is that these terms are functions of the field variables, not their derivatives. This appears to be true even for more complicated models (e.g., see [9]). In terms of asymptotic behavior, as $U \to 0$, ν^M and ν^E go to a constant multiplied by the number density, and as $U \to \infty$, both go to zero (both functions are in fact monotonically decreasing functions of U).

Finally, for completeness, the source terms for the momentum and energy equations in their original forms are:

$$\frac{\partial(\rho_a \mathbf{u}_a)}{\partial t} \bigg|_c = -m_{ab} n_a (\mathbf{u}_a - \mathbf{u}_b) \nu_{ab}^M, \tag{3.29}$$

$$\begin{aligned} \frac{\partial \mathscr{E}_a}{\partial t} \bigg|_c &= -\frac{m_{ab}}{m_a + m_b} (T_a - T_b) n_a k_B \nu_{ab}^E \\ &\quad + \left[\frac{m_b}{T_b} \left(\frac{m_a}{T_a} + \frac{m_b}{T_b} \right)^{-1} U^2 - \mathbf{u}_a \cdot (\mathbf{u}_a - \mathbf{u}_b) \right] m_{ab} n_a \nu_{ab}^M, \end{aligned} \tag{3.30}$$

where the collision frequencies ν_{ab}^M and ν_{ab}^E are as given in (3.27) and (3.28).

3.4. A Generalized Form of the Collision Terms

For computation of a variety of plasma and collision models, a generalized form of the collision terms in the multi-fluid equations will be useful. The following form is proposed, with a view to supporting a simple physical explanation of the nature of each term: for each species a, the cross-section for an interaction with a particle of species b, σ^*, is integrated to define a collision frequency per unit volume, $\bar{\nu}^*_{ab} = \langle \sigma^* \rangle$. In general, $\bar{\nu}^*_{ab} \neq \bar{\nu}^*_{ba}$, because the partitioning between particles of different mass is rarely equal, but usually a simple relationship will exist between them. The collision frequency as written is typically a function of the species masses, velocities, and temperatures, (e.g., (3.27)–(3.28), although for this section the notation is generalized to explicitly show the dependence on both species number densities).

For the likely collision terms of interest, we define four collision frequencies, per unit volume:

1. $\bar{\nu}^+_{ab}$, the frequency at which species a is created from species b; e.g., ionization for an ion/neutral interaction,

2. $\bar{\nu}^-_{ab}$, the frequency at which species a is destroyed to become species b; e.g., recombination for an ion/neutral interaction,

3. $\bar{\nu}^M_{ab}$, the momentum transfer collision frequency tensor, a generalization of (3.27),

4. $\bar{\nu}^E_{ab}$, the energy equilibration collision frequency, a generalization of (3.28).

Each of these frequencies may be a function of some combination of number density, velocity, magnetic field strength, and temperature. Also, most generally, the momentum term $\bar{\nu}^M_{ab}$ is an anisotropic tensor. The source terms are then as follows:

$$\left.\frac{\partial \rho_a}{\partial t}\right|_c = \sum_{b\neq a} n_a m_b n_b \bar{\nu}^+_{ab} - \sum_{b\neq a} n_b m_a n_a \bar{\nu}^-_{ab}, \tag{3.31}$$

$$\left.\frac{\partial (\rho_a \mathbf{u}_a)}{\partial t}\right|_c = -\sum_{b\neq a} m_a n_a (\mathbf{u}_a - \mathbf{u}_b) n_b \bar{\nu}^M_{ab} + \sum_{b\neq a} m_b n_b \mathbf{u}_b n_a \bar{\nu}^+_{ab} - \sum_{b\neq a} m_a n_a \mathbf{u}_a n_b \bar{\nu}^-_{ab}, \tag{3.32}$$

$$\begin{aligned}\left.\frac{\partial \mathscr{E}_a}{\partial t}\right|_c = &-\sum_{b\neq a} (T_a - T_b) k_B n_a n_b \bar{\nu}^E_{ab} - \sum_{b\neq a} m_a n_a \mathbf{u}_a \cdot (\mathbf{u}_a - \mathbf{u}_b) n_b \bar{\nu}^M_{ab} \\ &+ \sum_{b\neq a} n_a \bar{\nu}^+_{ab} \mathscr{E}_b - \sum_{b\neq a} n_b \bar{\nu}^-_{ab} \mathscr{E}_a + Q^{src}_a. \end{aligned} \tag{3.33}$$

Note that in the momentum and energy sources, the momentum and kinetic energy gained by species a from the creation of that species is the momentum and kinetic energy of species b. The choice of sign should ensure that the collision frequencies are all positive quantities. An additional term, Q^{src}, is added to the energy equation to capture additional energy sources (e.g., energy released by ionization) that do not fit into the aforementioned categories, and may be used for other external energy sources acting on the system. The general multi-fluid plasma equations with these source terms is summarized in Table 3-1 at the end of this chapter.

Comparing these source terms to the analytic collisional model given by (3.29)–(3.30), we see that:

$$\bar{\nu}_{ab}^{+} = \bar{\nu}_{ab}^{-} = 0, \tag{3.34}$$

$$\bar{\nu}_{ab}^{M} = \frac{1}{n_b}\frac{m_b}{m_a+m_b}\nu_{ab}^{M}, \tag{3.35}$$

$$\bar{\nu}_{ab}^{E} = \frac{1}{n_b}\frac{m_a m_b}{(m_a+m_b)^2}\nu_{ab}^{E}, \tag{3.36}$$

$$Q_a^{src} = \frac{m_a m_b}{m_a+m_b}\frac{m_b}{T_b}\left(\frac{m_a}{T_a}+\frac{m_b}{T_b}\right)^{-1} U^2 n_a \nu_{ab}^{M}, \tag{3.37}$$

where ν_{ab}^{M} and ν_{ab}^{E} are as defined in (3.27) and (3.28). The more general three-fluid model defined at the beginning of this section can also be expressed in this form. Comparing notation, we have for continuity and momentum:

$$\Gamma^{ion} = n_i n_n \frac{m_n}{m_i}\bar{\nu}_{in}^{+}, \tag{3.38}$$

$$\Gamma^{rec} = n_i n_n \bar{\nu}_{in}^{-}, \tag{3.39}$$

$$\mathbf{R}_i^{ie} = -m_i n_i n_e (\mathbf{u}_i - \mathbf{u}_e)\bar{\nu}_{ie}^{M}, \tag{3.40}$$

$$\mathbf{R}_i^{in} = -m_i n_i n_n (\mathbf{u}_i - \mathbf{u}_n)\bar{\nu}_{in}^{M}, \tag{3.41}$$

$$\mathbf{R}_e^{en} = -m_e n_e n_n (\mathbf{u}_e - \mathbf{u}_n)\bar{\nu}_{en}^{M}. \tag{3.42}$$

In determining the above relationships, note that the following equivalences hold:

$$\bar{\nu}_{en}^{+} = \frac{m_e}{m_i}\frac{n_i}{n_e}\bar{\nu}_{in}^{+}, \qquad \bar{\nu}_{ni}^{+} = \bar{\nu}_{in}^{-}, \qquad \bar{\nu}_{ne}^{+} = \frac{n_i}{n_e}\bar{\nu}_{in}^{-},$$

$$\bar{\nu}_{en}^{-} = \frac{n_i}{n_e}\bar{\nu}_{in}^{-}, \qquad \bar{\nu}_{ni}^{-} = \bar{\nu}_{in}^{+}, \qquad \bar{\nu}_{ne}^{-} = \frac{m_e}{m_i}\frac{n_i}{n_e}\bar{\nu}_{in}^{+},$$

$$\bar{\nu}_{ei}^{M} = \frac{m_i}{m_e}\bar{\nu}_{ie}^{M}, \qquad \bar{\nu}_{ni}^{M} = \frac{m_i}{m_n}\bar{\nu}_{in}^{M}, \qquad \bar{\nu}_{ne}^{M} = \frac{m_e}{m_n}\bar{\nu}_{en}^{M},$$

where we presume that $\bar{\nu}_{in}^{+}$, $\bar{\nu}_{in}^{-}$, $\bar{\nu}_{ie}^{M}$, $\bar{\nu}_{in}^{M}$, and $\bar{\nu}_{en}^{M}$ are given. For the source terms in the energy equation, a close reading of [15] shows that

$$Q_n^{ion} = \Gamma^{ion}\frac{3}{2}k_B T_n,$$

$$Q_i^{rec} = \Gamma^{rec}\frac{3}{2}k_B T_i,$$

which arise because of the high electron thermal speed assumption. A little manipulation using the above equations and the definition of $\mathscr{E}$ yields consistent ionization and recombination terms, except for Q_e^{ion}. This term does not simply reduce like Q_i^{ion}, because the moment integral cannot be easily separated and related to usual thermodynamic quantities. Using our generalized notation, this discrepancy, and the additional energy related to the ionization potential, can be handled by the source term Q_e^{src}, as follows:

$$Q_e^{src} = -\Gamma^{ion}\phi_{ion} - \left(Q_e^{rec} - \frac{3}{2}\Gamma^{ion}k_B T_e\right). \tag{3.43}$$

The remaining interaction terms in the energy equation (Q_i^{ie}, Q_i^{in}, etc.) can be arbitrarily related to the $\bar{\nu}^E$ energy transfer terms (note that [15] and [14] do not provide further information on these terms).

3.5. Collision Coefficients

Cross-section coefficients are available for some elements based on direct Boltzmann calculation calibrations for simplified configurations. Coefficients and transitions energies can be found in the NIST databases [3]. The LXcat project provides electron scattering cross-sections and "swarm" parameters as functions of the reduced electric field strength (E/n = ratio of electric field to number density) [2, 18]. The CANTERA project provides open source information and code to support chemical reaction networks [1].

Models for ionization and recombination are recommended by Meier [14, 15]. For ionization, the model by Voronov [25] provides a formula for the first 28 elements, which in our notation is

$$\frac{m_i}{m_n}\bar{\nu}_{in}^{+} \approx \bar{\nu}_{in}^{+} = A\frac{1+P\sqrt{\phi_{ion}/T_e}}{X+\phi_{ion}/T_e}\left(\frac{\phi_{ion}}{T_e}\right)^K \exp(-\phi_{ion}/T_e)\ \mathrm{m}^3/\mathrm{s}, \tag{3.44}$$

where the effective ionization potential ϕ_{ion} and the electron temperature T_e have the same energy units (e.g., eV), and the coefficients A, P, X, and K are tabulated for each ionization. For example, for hydrogen, $\phi_{ion} = 13.6$ eV, and

$$A = 2.91\times 10^{-14}\ \mathrm{m}^3/\mathrm{s}, \qquad P = 0, \qquad X = 0.232, \qquad K = 0.39,$$

which generates a formula that is accurate to within 5% for electron temperatures from 1 eV to 20 keV. For recombination of an ion of charge Z to $Z-1$, the model by McWhirter [13] specifies (in our notation)

$$\frac{n_n}{n_e}\bar{\nu}_{in}^{-} = 2.6\times 10^{-19}\frac{Z^2}{\sqrt{T_e}}\ \mathrm{m}^3/\mathrm{s}, \tag{3.45}$$

where T_e is the electron temperature in eV. This model is claimed to be accurate to within 100% over a similar temperature range to the ionization model.

For the frictional collision terms ($\mathbf{R}$ or $\bar{\nu}^M$), it is claimed [15, 7] that for plasmas with any more than a few percent ionization, the ion-neutral and electron-neutral collision effects are negligible compared to Coulomb collisions between charged particles. Models exist for weakly ionized collision terms, if required, but in many cases it can be assumed that $\bar{\nu}_{in}^M = \bar{\nu}_{en}^M = 0$. The remaining frictional term is the Coulomb collision term, $\mathbf{R}^{ie}$. An analytic model for this term is presented in Section 3.3, but other approximate models exist that replace $\mathbf{R}^{ie}$ with an anisotropic resistivity of the form $\hat{\eta}\cdot\mathbf{J}$, with tensor $\hat{\eta}$, or even more simply with an isotropic resistivity, $\eta\mathbf{J}$.

If neutral-charged particle collisions are ignored, the only remaining terms to consider in the energy equations are the Coulomb collision transfers, $\bar{\nu}_{ie}^E$, and direct source terms, Q_a^{src}. We have an analytic model for $\bar{\nu}_{ie}^E$ in (3.28). The only direct source terms likely to be encountered are those related to ionization, appearing in Q_e^{src}. Models for the ionization potential exist, and [15] suggest that parameterization of Q_e^{rec} in terms of T_e should feasible if this term is required.

3.6. Summary of the Multi-fluid Model

Density	$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}_a) = \sum_{b \neq a} (n_a \rho_b \bar{\mathbf{v}}_{ab}^{+} - n_b \rho_a \bar{\mathbf{v}}_{ab}^{-})$
Momentum	$\frac{\partial (\rho_a \mathbf{u}_a)}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}_a \otimes \mathbf{u}_a + p_a I + \Pi_a) = q_a n_a (\mathbf{E} + \mathbf{u}_a \times \mathbf{B})$ $- \sum_{b \neq a} \left[\rho_a (\mathbf{u}_a - \mathbf{u}_b) n_b \bar{\mathbf{v}}_{ab}^{M} + \rho_b \mathbf{u}_b n_a \bar{\mathbf{v}}_{ab}^{+} - \rho_a \mathbf{u}_a n_b \bar{\mathbf{v}}_{ab}^{-} \right]$
Energy	$\frac{\partial \mathscr{E}_a}{\partial t} + \nabla \cdot ((\mathscr{E}_a + p_a) \mathbf{u}_a + \Pi_a \cdot \mathbf{u}_a + \mathbf{h}_a) = q_a n_a \mathbf{u}_a \cdot \mathbf{E} + Q_a^{src}$ $- \sum_{b \neq a} \left[(T_a - T_b) k_B \bar{\mathbf{v}}_{ab}^{E} - \rho_a \mathbf{u}_a \cdot (\mathbf{u}_a - \mathbf{u}_b) n_b \bar{\mathbf{v}}_{ab}^{M} - n_a \bar{\mathbf{v}}_{ab}^{+} \mathscr{E}_b + n_b \bar{\mathbf{v}}_{ab}^{-} \mathscr{E}_a \right]$
Charge and Current Density	$q = \sum_k q_k n_k$ $\qquad$ $\mathbf{J} = \sum_k q_k n_k \mathbf{u}_k$
Maxwell's Equations	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$ $\qquad$ $\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}$ $\nabla \cdot \mathbf{B} = 0$ $\qquad$ $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
EOS	$\mathscr{E}_a = \frac{p_a}{\gamma - 1} + \frac{1}{2} \rho_a u_a^2$

Table 3-1. The multi-fluid 5-moment equations of plasma dynamics, including collisional source terms.

4. TWO-FLUID PLASMA MODELS

4.1. The General Two-fluid Model

Analytic progress using the three-fluid model is difficult in general. Every opportunity is therefore taken to simplify the model, and one of the most obvious simplifications is to drop the neutral fluid. It is argued by [15] that Coulomb collisions are the dominant collisional effect in moderately ionized plasmas, which leaves only ionization and recombination reactions as coupling terms between the charged particles and the neutral fluid. If even these reactions are ignored (e.g., because the source of ionization is external), or the fluid can be treated as fully ionized, then the problem can be reduced to a two-fluid model. Further, if the plasma can be treated as a fully ionized gas having only electrons and one ion species, then the particle dynamics can be treated using a perturbed Maxwellian distribution, and analytic closures can be computed for each fluid. This forms the basis of the Braginskii closure model, described in the following section.

Dropping the neutral fluid and ionization/recombination reactions from the three-fluid model, the two-fluid model for a simple plasma may be written as:

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = 0, \tag{4.1}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e) = 0, \tag{4.2}$$

$$\frac{\partial (\rho_i \mathbf{u}_i)}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i \otimes \mathbf{u}_i + p_i I + \Pi_i) = \frac{\rho_i q_i}{m_i} (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \mathbf{R}, \tag{4.3}$$

$$\frac{\partial (\rho_e \mathbf{u}_e)}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e \otimes \mathbf{u}_e + p_e I + \Pi_e) = \frac{\rho_e q_e}{m_e} (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \mathbf{R}, \tag{4.4}$$

$$\frac{3}{2}\frac{\partial (n_i k_B T_i)}{\partial t} + \nabla \cdot \left(\frac{3}{2} n_i k_B T_i \mathbf{u}_i + \mathbf{h}_i \right) + p_i \nabla \cdot \mathbf{u}_i + \nabla \mathbf{u}_i : \Pi_i = Q_i, \tag{4.5}$$

$$\frac{3}{2}\frac{\partial (n_e k_B T_e)}{\partial t} + \nabla \cdot \left(\frac{3}{2} n_i k_B T_e \mathbf{u}_e + \mathbf{h}_e \right) + p_e \nabla \cdot \mathbf{u}_e + \nabla \mathbf{u}_e : \Pi_e = Q_e, \tag{4.6}$$

where the energy equations have been expressed as temperature equations. Note that these equations describe an electron-proton plasma, with $Z = 1$. Modifications for general Z are provided in [5]. Note also that the analytic closure presented in Section 3.3 is compatible with this model, as is the generalized form of the collision terms in Section 3.4.

4.2. The Braginskii Closure

The canonical paper on plasma modeling by Braginskii, [5], contains a complete closure for the simple two-fluid plasma. Here are reproduced the closure terms from that paper, converted to SI units. The only exception is the conductivity model, which is instead taken from the NRL Plasma Formulary [10]. Note that the neutral plasma assumption is built-in, i.e., $n_i = n_e = n$, and the small electron-ion mass ratio ($m_e/m_i \ll 1$) has been used in these derivations.

The cyclotron frequencies for the electrons and ions are

$$\omega_{ce} = \frac{q_e B}{m_e}, \qquad \text{and} \qquad \omega_{ci} = \frac{q_i B}{m_i}, \tag{4.7}$$

where $B = |\mathbf{B}|$ is the magnitude of the magnetic field. Characteristic time scales are the electron and ion collision times,

$$\tau_e = \frac{6\pi\sqrt{20\pi m_e}(k_B T_e)^{3/2}\varepsilon_0^2}{10^{11}\lambda q_e^4 n} \approx 8.71\times 10^{-6}\frac{T_e^{3/2}}{\lambda n}, \tag{4.8}$$

$$\tau_i = \frac{6\pi\sqrt{10\pi m_i}(k_B T_i)^{3/2}\varepsilon_0^2}{10^{11}\lambda q_e^4 n} \approx 2.64\times 10^{-4}\sqrt{\frac{m_i}{m_p}}\frac{T_i^{3/2}}{\lambda n}, \tag{4.9}$$

where $\lambda = \ln\Lambda$ is the Coulomb Logarithm, for which dynamic models exist but $\lambda \sim 10$ is usually adequate. Note that the numerical coefficient arises from the conversion from Gaussian/cgs units to SI, where n has units of m^{-3} and T has units of Kelvin.

The Braginskii model is applicable to a plasma that satisfies all of the following conditions [5, 10]:

1. the rate of change of quantities is slow compared to the collision rate, i.e., $\mathrm{d}/\mathrm{d}t \ll 1/\tau$,
2. the plasma macroscopic length scale is large compared to the mean free path in either fluid,
3. the Coulomb logarithm $\lambda \gg 1$,
4. the electron gyroradius is much larger than the Debye length, i.e., $2n_e m_e \gg \varepsilon_0 B^2$,
5. the relative drift velocity is small compared to the thermal velocities, i.e., $|\mathbf{u}_e - \mathbf{u}_i|^2 \ll k_B T_e/m_e, k_B T_i/m_i$,
6. anomalous transport effects are negligible.

Further, the product $\omega_c\tau$ should be either zero (no magnetic field) or large. The coefficients that depend on B or the magnetic field direction $\hat{\mathbf{b}}$ apply for $\omega_c\tau \gg 1$; if $B = 0$, the transport coefficients are given by the values parallel to B alone. Also, if $\omega_{ce}\tau_e \gg 1 \gg \omega_{ci}\tau_i$, the electron closure for large B can be used alongside the ion closure for $B = 0$.

The interaction force $\mathbf{R}$ is partitioned into frictional and thermal parts, $\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T$ where the frictional part is given by

$$\mathbf{R}_u = nq_e\left(\frac{\mathbf{J}_\parallel}{\sigma_\parallel} + \frac{\mathbf{J}_\perp}{\sigma_\perp}\right), \tag{4.10}$$

and the thermal part by

$$\mathbf{R}_T = -0.71 n \nabla_{\parallel}(k_B T_e) - \frac{3n}{2\omega_{ce}\tau_e}\hat{\mathbf{b}} \times \nabla_{\perp}(k_B T_e), \tag{4.11}$$

where the parallel and perpendicular direction derivatives are relative to the magnetic field direction, $\hat{\mathbf{b}}$. The conductivities are [10]:

$$\sigma_{\parallel} = 1.96\sigma_0, \qquad \text{and} \qquad \sigma_{\perp} = \sigma_0(\omega_{ce}\tau_e)^{-2}, \tag{4.12}$$

where

$$\sigma_0 = \frac{n q_e^2 \tau_e}{m_e}.$$

For the ions, the heat generated by collisions with electrons is

$$Q_i = \frac{3m_e}{m_i}\frac{nk_B}{\tau_e}(T_e - T_i). \tag{4.13}$$

For the electrons, their small mass relative to the ions adds a frictional term:

$$Q_e = -Q_i - \mathbf{R}\cdot(\mathbf{u}_e - \mathbf{u}_i). \tag{4.14}$$

Closure for the fluid model provides values for the stress tensor and heat flux vector. The stress model is a simple viscous model, such that the stress is linearly proportional to the rate of strain tensor,

$$W_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}. \tag{4.15}$$

In the absence of a magnetic field, this is particularly simple,

$$\Pi_{ij} = -\eta_0 W_{ij}, \tag{4.16}$$

but in a strong magnetic field ($\omega_c\tau \gg 1$), the stress tensor becomes aligned with $\hat{\mathbf{b}}$. In the coordinate system with the third axis parallel to the magnetic field, the components of the stress tensor are:

$$\begin{aligned}
\Pi_{11} &= -\frac{1}{2}\eta_0(W_{11}+W_{22}) - \frac{1}{2}\eta_1(W_{11}-W_{22}) - \eta_3 W_{12},\\
\Pi_{12} &= -\eta_1 W_{12} + \frac{1}{2}\eta_3(W_{11}-W_{22}) = \Pi_{21},\\
\Pi_{13} &= -\eta_2 W_{13} - \eta_4 W_{23} = \Pi_{31},\\
\Pi_{22} &= -\frac{1}{2}\eta_0(W_{11}+W_{22}) - \frac{1}{2}\eta_1(W_{22}-W_{11}) + \eta_3 W_{12},\\
\Pi_{23} &= -\eta_2 W_{23} + \eta_4 W_{13} = \Pi_{32},\\
\Pi_{33} &= -\eta_0 W_{33}.
\end{aligned} \tag{4.17}$$

Both fluids have this form, with the following viscous coefficients: for the ion species,

$$\begin{aligned}
&\eta_0^i = 0.96 n_i k_B T_i \tau_i,\\
&\eta_1^i = \frac{3}{10}\frac{n_i k_B T_i}{\omega_{ci}^2 \tau_i}, \quad \eta_2^i = 4\eta_1^i, \quad \eta_3^i = \frac{1}{2}\frac{n_i k_B T_i}{\omega_{ci}}, \quad \eta_4^i = 2\eta_3^i,
\end{aligned} \tag{4.18}$$

and for the electron species,

$$\eta_0^e = 0.73 n_e k_B T_e \tau_e,$$
$$\eta_1^e = 0.51 \frac{n_e k_B T_e}{\omega_{ce}^2 \tau_e}, \quad \eta_2^e = 4\eta_1^e, \quad \eta_3^e = -\frac{1}{2}\frac{n_e k_B T_e}{\omega_{ce}}, \quad \eta_4^e = 2\eta_3^e. \tag{4.19}$$

The heat flux vector is also dependent on the magnetic field direction. The ion heat flux has a Fourier-type form (for $\omega_{ci}\tau_i \gg 1$):

$$\mathbf{h}_i = -\kappa_\parallel^i \nabla_\parallel T_i - \kappa_\perp^i \nabla_\perp T_i + \kappa_\wedge^i \hat{\mathbf{b}} \times \nabla_\perp T_i, \tag{4.20}$$

with the conductivities

$$\kappa_\parallel^i = 3.9 \frac{n_i k_B^2 T_i \tau_i}{m_i}, \quad \kappa_\perp^i = 2 \frac{n_i k_B^2 T_i}{m_i \omega_{ci}^2 \tau_i}, \quad \kappa_\wedge^i = \frac{5}{2}\frac{n_i k_B^2 T_i}{m_i \omega_{ci}}. \tag{4.21}$$

The electron heat flux is split into drift and thermal components, $\mathbf{h}_e = \mathbf{h}_e^u + \mathbf{h}_e^T$, which take the forms

$$\mathbf{h}_e^u = 0.71 n_e k_B T_e \mathbf{u}_\parallel + \frac{3}{2}\frac{n_e k_B T_e}{\omega_{ce}\tau_e} \hat{\mathbf{b}} \times \mathbf{u}_\perp, \tag{4.22}$$
$$\mathbf{h}_e^T = -\kappa_\parallel^e \nabla_\parallel T_e - \kappa_\perp^e \nabla_\perp T_e + \kappa_\wedge^e \hat{\mathbf{b}} \times \nabla_\perp T_e, \tag{4.23}$$

where $\mathbf{u} = \mathbf{u}_e - \mathbf{u}_i$. The electron thermal conductivities are

$$\kappa_\parallel^e = 3.16 \frac{n_e k_B^2 T_e \tau_e}{m_e}, \quad \kappa_\perp^e = 4.66 \frac{n_e k_B^2 T_e}{m_e \omega_{ce}^2 \tau_e}, \quad \kappa_\wedge^e = \frac{5}{2}\frac{n_e k_B^2 T_e}{m_e \omega_{ce}}. \tag{4.24}$$

In order to implement the Braginskii model numerically, care is required to properly handle the extreme values in the coefficients without a severe loss of precision. Also, the terms that depend on the magnetic field magnitude, B, must transition smoothly for any field value. The only estimate for field magnitude provided by the model is the condition $\omega\tau \gg 1$ for magnetic effects to be present. Notably, coefficients in the model that depend on B are (or can be related to) functions of $\omega\tau$. To smoothly transition these numerically, we arbitrarily choose $\omega\tau = 1$ as condition at which magnetic field effects become relevant (although the model may not be accurate at such intermediate magnitudes). The complete set of closure terms expressed in this form is summarized in Table 4-1.

4.3. Heavy/Light Fluid System

Another approach for simplifying the complexity of source terms is to consider a fluid description with some average ionization and recombination rates and ionization state. A gas ionized to a maximum state Z can be described using $(Z+2)$ fluids (assuming no negatively charged ions), where each ionization state of the heavy molecule is numbered $0 \leq i \leq Z$ (zero being the neutral state), and the electrons are labeled e. The charge of each heavy fluid is given by ie where i is the

$\omega_{ci}\tau_i$	$4.229 \times 10^{-23} \dfrac{BT_i^{3/2}\sqrt{\tilde{m}_i}}{\lambda \rho_i}$	$\omega_{ce}\tau_e$	$1.395 \times 10^{-24} \dfrac{BT_e^{3/2}}{\lambda \rho_e}$
$\dfrac{nq_e}{\sigma_0}$	$7.166 \times 10^{23} \dfrac{\lambda \rho_e}{T_e^{3/2}}$		
$\mathbf{R}_{u_\parallel}$	$0.5102 \dfrac{nq_e}{\sigma_0}$	$\mathbf{R}_{u_\perp}$	$(\omega_{ce}\tau_e)^2 \dfrac{nq_e}{\sigma_0}$
$\mathbf{R}_{T_\parallel}$	$1.0764 \times 10^7 \rho_e$	$\mathbf{R}_{T_\perp}$	$2.274 \times 10^7 \dfrac{\rho_e}{\omega_{ce}\tau_e}$
Q_i	$3.121 \times 10^{39} \dfrac{\lambda \rho_e^2}{\tilde{m}_i T_e^{3/2}}$		
η_0^i	$3.500 \times 10^{-27} \dfrac{T_i^{5/2}\sqrt{\tilde{m}_i}}{\lambda}$	η_0^e	$8.781 \times 10^{-29} \dfrac{T_e^{5/2}}{\lambda}$
η_1^i	$0.3125 \dfrac{\eta_0^i}{(\omega_{ci}\tau_i)^2}$	η_1^e	$0.6986 \dfrac{\eta_0^e}{(\omega_{ce}\tau_e)^2}$
η_3^i	$0.5208 \dfrac{\eta_0^i}{\omega_{ci}\tau_i}$	η_3^e	$-0.6849 \dfrac{\eta_0^e}{\omega_{ce}\tau_e}$
$\kappa_\parallel^i$	$1.1735 \times 10^{-22} \dfrac{T_i^{5/2}}{\lambda\sqrt{\tilde{m}_i}}$	$\kappa_\parallel^e$	$5.763 \times 10^{-21} \dfrac{T_e^{5/2}}{\lambda}$
$\kappa_\perp^i$	$0.5128 \dfrac{\kappa_\parallel^i}{(\omega_{ci}\tau_i)^2}$	$\kappa_\perp^e$	$1.4747 \dfrac{\kappa_\parallel^e}{(\omega_{ce}\tau_e)^2}$
$\kappa_\wedge^i$	$0.6410 \dfrac{\kappa_\parallel^i}{\omega_{ci}\tau_i}$	$\kappa_\wedge^e$	$0.7911 \dfrac{\kappa_\parallel^e}{\omega_{ce}\tau_e}$
		$\kappa_{u_\parallel}^e$	$1.0764 \times 10^7 T_e \rho_e$
		$\kappa_{u_\perp}^e$	$2.113 \dfrac{\kappa_{u_\parallel}^e}{\omega_{ce}\tau_e}$

Table 4-1. List of the Braginskii model terms with physical constants evaluated. The electron density is used in preference to the ion density to define the number density as applicable. The ion mass is expressed in proton mass units, i.e., $\tilde{m}_i = m_i/m_p$.

ionization state and e is the elementary charge. A center of mass coordinate for species $0 \le i \le Z$ is defined by

$$m_\alpha n_\alpha = \sum_{i=0}^{Z} m_i n_i, \qquad m_\alpha n_\alpha \mathbf{u}_\alpha = \sum_{i=0}^{Z} m_i n_i \mathbf{u}_i, \qquad \mathscr{E}_\alpha = \sum_{i=0}^{Z} \mathscr{E}_i. \tag{4.25}$$

Additional mass-averaged quantities will be introduced as necessary. Summing over the atomic conservation of mass equations we have

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{i=0}^{Z} m_i n_i + \nabla\cdot\sum_{i=0}^{Z} m_i n_i \mathbf{u}_i = &\sum_{i=0}^{Z-1}\sum_{j\neq i} n_i n_j (m_j \bar{\nu}_{ij}^{+} - m_i \bar{\nu}_{ij}^{-}) \\ &+ \sum_{i=0}^{Z} n_i n_e (m_e \bar{\nu}_{ie}^{+} - m_i \bar{\nu}_{ie}^{-}),\end{aligned} \tag{4.26}$$

where conservation of mass implies that

$$\bar{\nu}_{ij}^{+} = \bar{\nu}_{ji}^{-}. \tag{4.27}$$

Noting that the ionization and recombination terms now vanish and we are left with

$$\frac{\partial}{\partial t} m_\alpha n_\alpha + \nabla\cdot m_\alpha n_\alpha \mathbf{u}_\alpha = \sum_{i=0}^{Z} n_i n_e (m_e \bar{\nu}_{ie}^{+} - m_i \bar{\nu}_{ei}^{-}), \tag{4.28}$$

we then introduce the average ionization and recombination rates as

$$m_\alpha n_\alpha \bar{\nu}_{\alpha e}^{\pm} = \sum_{i=0}^{Z} m_i n_i \bar{\nu}_{ie}^{\pm}. \tag{4.29}$$

This yields the atomic and electron conservation of mass equations

$$\frac{\partial}{\partial t} m_\alpha n_\alpha + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha) = n_\alpha n_e (m_e \bar{\nu}_{\alpha e}^{+} - m_\alpha \bar{\nu}_{\alpha e}^{-}), \tag{4.30}$$

$$\frac{\partial}{\partial t} m_e n_e + \nabla\cdot(m_e n_e \mathbf{u}_e) = n_\alpha n_e (m_\alpha \bar{\nu}_{\alpha e}^{-} - m_e \bar{\nu}_{\alpha e}^{+}). \tag{4.31}$$

For the averaged momentum equation we will assume that $\mathbf{u}_i = \mathbf{u}_\alpha$ for $0 \le i \le Z$, i.e., that the various ions do not drift apart under ionization, which is reasonable for a well-mixed relatively heavy material. Summing conservation of momentum for each species we then have

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{i=0}^{Z} n_i m_i \mathbf{u}_\alpha + \nabla\cdot\sum_{i=0}^{Z}(n_i m_i \mathbf{u}_\alpha\otimes\mathbf{u}_\alpha + p_i \mathbf{I} + \Pi_i) \\ &\quad= \sum_{i=0}^{Z} e i n_i (\mathbf{E} + \mathbf{u}_\alpha\times\mathbf{B}) + \sum_{i=0}^{Z-1}\sum_{j\neq i} n_i n_j (m_j \bar{\nu}_{ij}^{+} - m_i \bar{\nu}_{ij}^{-})\mathbf{u}_\alpha \\ &\quad+ \sum_{i=0}^{Z} n_i n_e (m_e \bar{\nu}_{ie}^{+}\mathbf{u}_e - m_i \bar{\nu}_{ie}^{-}\mathbf{u}_\alpha) + \sum_{i=0}^{Z} m_i n_i \bar{\nu}_{ie}^{M}(\mathbf{u}_\alpha - \mathbf{u}_e).\end{aligned} \tag{4.32}$$

We now define the average ionization state, which we call $\bar{Z}$ by the identity

$$\bar{Z}n_\alpha = \sum_{i=0}^{Z} in_i. \tag{4.33}$$

To complete the system, we define the total pressure and deviatoric stress by

$$p_\alpha = \sum_{i=0}^{Z} p_i, \qquad \Pi_\alpha = \sum_{i=0}^{Z} \Pi_i, \tag{4.34}$$

and introduce the average friction given by

$$m_\alpha n_\alpha \bar{\nu}_{\alpha e}^M(\mathbf{u}_\alpha - \mathbf{u}_e) = \sum_{i=0}^{Z} m_i n_i \bar{\nu}_{ie}^M(\mathbf{u}_\alpha - \mathbf{u}_e). \tag{4.35}$$

Simplifying, we are left with atomic and electron conservation of momentum:

$$\frac{\partial}{\partial t}m_\alpha n_\alpha \mathbf{u}_\alpha + \nabla\cdot(m_\alpha n_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + p_\alpha \mathbf{I} + \Pi_\alpha) = e\bar{Z}_\alpha n_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B})$$
$$+ n_\alpha n_e(m_\alpha \bar{\nu}_{\alpha e}^+ \mathbf{u}_\alpha - m_e \bar{\nu}_{\alpha e}^- \mathbf{u}_e) + m_\alpha n_\alpha \bar{\nu}_{\alpha e}^M(\mathbf{u}_\alpha - \mathbf{u}_e), \tag{4.36}$$
$$\frac{\partial}{\partial t}m_e n_e \mathbf{u}_e + \nabla\cdot(m_e n_e \mathbf{u}_e \otimes \mathbf{u}_e + p_e \mathbf{I} + \Pi_e) = -en_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B})$$
$$+ n_e n_\alpha(m_e \bar{\nu}_{\alpha e}^- \mathbf{u}_e - m_\alpha \bar{\nu}_{\alpha e}^+ \mathbf{u}_\alpha) + m_e n_e \bar{\nu}_{e\alpha}^M(\mathbf{u}_e - \mathbf{u}_\alpha). \tag{4.37}$$

For the total energy system we will assume that each of the ion species is in thermal equilibrium, $T_i = T_\alpha$ for $0 \le i \le Z$. Summing over all of the ion energy equations we have

$$\frac{\partial}{\partial t}\sum_{i=0}^{Z}\mathscr{E}_i + \nabla\cdot\sum_{i=0}^{Z}((\mathscr{E}_i + p_i)\mathbf{u}_\alpha + \Pi_i\cdot\mathbf{u}_\alpha + \mathbf{h}_i) = \sum_{i=0}^{n} ein_i\mathbf{u}_\alpha\cdot\mathbf{E} + Q_i + \sum_{i=0}^{Z}\sum_{j\neq i} n_i\bar{\nu}_{ij}^+\mathscr{E}_j - n_j\bar{\nu}_{ij}^-\mathscr{E}_i$$
$$+ \sum_{i=0}^{Z} n_i\bar{\nu}_{ie}^+\mathscr{E}_e - n_e\bar{\nu}_{ie}^-\mathscr{E}_i + (T_e - T_\alpha)k_B\nu_{ie}^E + m_i n_i n_e \mathbf{u}_\alpha\cdot\nu_{ie}^M(\mathbf{u}_\alpha - \mathbf{u}_e). \tag{4.38}$$

First we will deal with the energy transfer between each ion species. Given that $\mathbf{u}_i = \mathbf{u}_\alpha$ for all species, we know that the total energy of each component, $\mathscr{E}_i$, differs only in the internal energy density, which obeys

$$\mathscr{E}_i - \frac{1}{2}m_i n_i|\mathbf{u}_\alpha|^2 = e_i \propto n_i T_i = n_i T_\alpha. \tag{4.39}$$

Assuming that the proportionality constant for each ionization state is identical we have

$$\sum_{i=0}^{Z}\sum_{j\neq i} n_i\bar{\nu}_{ij}^+\mathscr{E}_j - n_j\bar{\nu}_{ij}^-\mathscr{E}_i \propto T_\alpha\sum_{i=0}^{Z}\sum_{j\neq i} n_i n_j(\bar{\nu}_{ij}^+ - \bar{\nu}_{ij}^-) = 0. \tag{4.40}$$

The energy exchange due to ionization and recombination is harder to average, but given our assumptions thus far it is reasonable to conclude that

$$\mathscr{E}_i = \frac{n_i}{n_\alpha}\mathscr{E}_\alpha. \tag{4.41}$$

The thermal collision term, heat flux, and external heating can be averaged in the obvious way:

$$\bar{\nu}_{\alpha e}^{E} = \sum_{i=0}^{Z} \bar{\nu}_{ie}^{E}, \quad \mathbf{h}_\alpha = \sum_{i=0}^{Z} \mathbf{h}_i, \quad Q_\alpha = \sum_{i=0}^{Z} Q_i. \tag{4.42}$$

Conservation of energy for the atomic and electron systems is then expressed as the two equations

$$\frac{\partial}{\partial t}\mathscr{E}_\alpha + \nabla\cdot((\mathscr{E}_\alpha + p_\alpha)\mathbf{u}_\alpha + \Pi_\alpha\cdot\mathbf{u}_\alpha + \mathbf{h}_\alpha) = e\bar{Z}\mathbf{u}_\alpha\cdot\mathbf{E} + Q_\alpha$$
$$+ n_\alpha\bar{\nu}_{\alpha e}^{+}\mathscr{E}_e - n_e\bar{\nu}_{\alpha e}^{-}\mathscr{E}_\alpha + (T_e - T_\alpha)k_b\bar{\nu}_{\alpha e}^{T} + m_\alpha n_\alpha\mathbf{u}_\alpha\cdot(\mathbf{u}_\alpha - \mathbf{u}_e)\bar{\nu}_{\alpha e}^{M}, \tag{4.43}$$
$$\frac{\partial}{\partial t}\mathscr{E}_e + \nabla\cdot((\mathscr{E}_e + p_e)\mathbf{u}_e + \Pi_e\cdot\mathbf{u}_e + \mathbf{h}_\alpha) = -e\mathbf{u}_e\cdot\mathbf{E} + Q_e$$
$$- n_\alpha\bar{\nu}_{\alpha e}^{+}\mathscr{E}_e + n_e\bar{\nu}_{\alpha e}^{-}\mathscr{E}_\alpha + (T_\alpha - T_e)k_b\bar{\nu}_{\alpha e}^{T} + m_e n_e\mathbf{u}_e\cdot\bar{\nu}_{e\alpha}^{M}(\mathbf{u}_e - \mathbf{u}_\alpha). \tag{4.44}$$

Given this derivation, it appears clear that if constitutive laws are available to close the $(Z+2)$ fluid system, then a constitutive law is required to define the ratio

$$\frac{n_i}{n_\alpha} = \frac{n_i}{n_\alpha}(n_\alpha, e_\alpha) \tag{4.45}$$

at each point phase space. As we have assumed that the $Z+1$ ions are in thermal equilibrium, these statistics can be consistently computed at thermal equilibrium. Further, it may be reasonable for heavier atoms to assume that $m_i = m_0$, i.e., that mass changes due to the loss of electrons is negligible.

4.4. The Ideal Two-fluid Model

From an analytical and numerical standpoint, the collisional source terms are the most challenging aspect of the multi-fluid plasma equations. Most analysis of the multi-fluid plasma system therefore idealizes the fluids, and neglects all interaction source terms between the electron and ion fluids. This is reasonable if collision frequencies are sufficiently large that the fluids equilibrate on time scales shorter than the time scales of interest, for example. The ideal fluid assumption also drops the fluid stress tensor and heat flux terms, and so reduces the multi-fluid plasma model to an Euler-Maxwell system (see [21]). In this case, the electron and ion fluids couple only through Maxwell's equations, although momentum transfer terms may be retained if desired (not shown here). Note that the ideal two-fluid model remains considerably more general than the MHD approximation, as it tracks separate motion of ions and electrons.

By treating the fluids as ideal, i.e., inviscid and perfectly conducting, a pair of hyperbolic

Euler-like systems is obtained from (4.1)–(4.6) for the plasma model:

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = 0, \tag{4.46}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e) = 0, \tag{4.47}$$

$$\frac{\partial (\rho_i \mathbf{u}_i)}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i \otimes \mathbf{u}_i + p_i I) = \frac{\rho_i q_i}{m_i} (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}), \tag{4.48}$$

$$\frac{\partial (\rho_e \mathbf{u}_e)}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e \otimes \mathbf{u}_e + p_e I) = \frac{\rho_e q_e}{m_e} (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}), \tag{4.49}$$

$$\frac{\partial \varepsilon_i}{\partial t} + \nabla \cdot (\varepsilon_i \mathbf{u}_i + p_i \mathbf{u}_i) = \frac{\rho_i q_i}{m_i} \mathbf{E} \cdot \mathbf{u}_i, \tag{4.50}$$

$$\frac{\partial \varepsilon_e}{\partial t} + \nabla \cdot (\varepsilon_e \mathbf{u}_e + p_e \mathbf{u}_e) = \frac{\rho_e q_e}{m_e} \mathbf{E} \cdot \mathbf{u}_e. \tag{4.51}$$

Note that it is occasionally convenient to rewrite the momentum equations in terms of the partial current densities, $\mathbf{J}_s = q_s n_s \mathbf{u}_s$, to simplify the Maxwell coupling.

This idealized model (4.46)–(4.51) is useful for exploring the asymptotic limits of plasma behavior [21]. A representative length scale is the Larmor radius, defined as the radius of gyration of a charged particle in a magnetic field as it undergoes cyclotron motion, given by the ratio of the particle thermal velocity ($u_T = \sqrt{2p_s/\rho_s}$) to the cyclotron frequency ($\omega_c = \sqrt{q_s B/m_s}$), $r_L = u_T/\omega_c$. A nondimensionalization of the equations shows that the right-hand side terms are normalized by the inverse Larmor radius, so in the limit of $r_L \to \infty$, the neutral gas dynamic limit is recovered and the usual Euler equations are obtained for each fluid. In the limit of $r_L \to 0$, the MHD limit is recovered and the fluids become tightly coupled to the magnetic field (i.e., charge separation effects disappear). This relationship is also shown in Figure 1-1.

Other asymptotic approximations are also instructive. MHD models assume negligible electron inertia, and in the case of ideal MHD, an infinite speed of light [8, 22], and therefore treat the plasma as a single conducting fluid. The Hall-MHD model captures some charge separation phenomena by allowing for different bulk ion and electron velocities, which permits finite Larmor radii. A full discussion of the scalings for which the MHD, Hall-MHD, and two-fluid models are valid is presented in [8].

5. GENERALIZED OHM'S LAW

5.1. Recasting the Two-fluid Equations

To capture charge separation (displacement current) effects in a plasma, it is not always necessary to fully describe the ion and electron motion. In particular, the electron inertia may be negligible compared to the ion inertia, because of the physical difference in mass of each particle: even in a hydrogen proton-electron plasma, the mass ratio m_i/m_e is on the order of 1800. With this approximation, the two-fluid equations are rewritten in a center-of-mass frame to yield equations describing the bulk fluid motion and charge separation effects. In particular, a PDE is derived for the current density, $\mathbf{J}$, which is cast as a generalized Ohm's Law, hence the name for this system.

The model begins with the ideal two-fluid model, (4.46)–(4.51) without momentum transfer terms. We note that this derivation largely follows [12] in which momentum transfer source terms are retained, however. First, we define quantities in the center-of-mass frame: the density,

$$\rho = \rho_i + \rho_e = m_i n_i + m_e n_e, \tag{5.1}$$

the charge density,

$$q = q_i n_i + q_e n_e, \tag{5.2}$$

the center-of-mass velocity,

$$\mathbf{u} = \frac{\rho_i \mathbf{u}_i + \rho_e \mathbf{u}_e}{\rho}, \tag{5.3}$$

and the current density,

$$\mathbf{J} = q_i n_i \mathbf{u}_i + q_e n_e \mathbf{u}_e. \tag{5.4}$$

The separate fluid velocities may now be rewritten in terms of the new variables:

$$\mathbf{u}_i = \frac{q_e \rho \mathbf{u} - m_e \mathbf{J}}{n_i (q_e m_i - q_i m_e)} = \mathbf{u} + \frac{\rho_e}{\rho}\Gamma, \tag{5.5}$$

$$\mathbf{u}_e = \frac{-q_i \rho \mathbf{u} + m_i \mathbf{J}}{n_e (q_e m_i - q_i m_e)} = \mathbf{u} - \frac{\rho_i}{\rho}\Gamma, \tag{5.6}$$

where the Hall velocity, Γ is defined as

$$\Gamma \equiv \mathbf{u}_i - \mathbf{u}_e = \frac{\rho (q\mathbf{u} - \mathbf{J})}{n_i n_e (q_e m_i - q_i m_e)}. \tag{5.7}$$

Casting the continuity equations into the new variables is trivial. The mass conservation equation is simply the sum of the single fluid equations, and the charge conservation equation is the sum of the ion equation multiplied by q_i/m_i and the electron equation multiplied by q_e/m_e:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Note that these equations are exact, i.e., the negligible mass approximation was not needed to derive them.

The momentum equation is derived by summing the two-fluid equations and substituting the velocity expressions (5.5)–(5.6). This yields

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left[\rho \mathbf{u} \otimes \mathbf{u} + pI + \frac{\rho m_i m_e}{n_i n_e (q_e m_i - q_i m_e)^2} \left(q^2 \mathbf{u} \otimes \mathbf{u} - q(\mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u}) + \mathbf{J} \otimes \mathbf{J} \right) \right] = q\mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (5.8)$$

Multiplying the ion and electron momentum equations by q_i/m_i and q_e/m_e and summing gives a long expression for the current density:

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \left[\frac{q_i q_e q \rho^2 \mathbf{u} \otimes \mathbf{u} - q_i q_e \rho^2 (\mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u}) + (q_i n_e m_e^2 + q_e n_i m_i^2) \mathbf{J} \otimes \mathbf{J}}{n_i n_e (q_e m_i - q_i m_e)^2} + \left(\frac{q_i}{m_i} p_i + \frac{q_e}{m_e} p_e \right) I \right] = \frac{q_i^2 n_i}{m_i} (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \frac{q_e^2 n_e}{m_e} (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) \quad (5.9)$$

These equations are somewhat unwieldy in this form until the approximation of $m_e \ll m_i$ is applied. By inspection, the momentum correction term is $O(m_e/m_i)$, and the simplified equation can be written down:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + pI) = q\mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (5.10)$$

Note that this is the same form for the fluid momentum equation as is seen in MHD models; the generalized Ohm's Law differs by including the PDE for the current density alongside. To simplify (5.9), note that $\rho \sim m_i n_i$, and

$$\mathbf{u}_i \sim \mathbf{u},$$
$$\mathbf{u}_e \sim \frac{1}{q_e n_e} (\mathbf{J} - q_i n_i \mathbf{u}).$$

With some further manipulation, the generalized Ohm's Law equation for negligible electron inertia is derived:

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \left[\frac{\mathbf{J} \otimes \mathbf{J}}{q_e n_e} + \left(\frac{q}{q_e n_e} - 1 \right) (q \mathbf{u} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{J} - \mathbf{J} \otimes \mathbf{u}) + \frac{q_e}{m_e} p_e I \right] = \frac{q_e}{m_e} \left[q_e n_e \mathbf{E} + \left(\mathbf{J} + (1 - \frac{q}{q_e n_e}) \mathbf{u} \right) \times \mathbf{B} \right]. \quad (5.11)$$

The energy equations are derived for the total energy density, $\mathscr{E}$, and the electron energy density, $\mathscr{E}_e$. The definition of $\mathscr{E}_e$ is unchanged, but is now rewritten in terms of the new variables:

$$\mathscr{E}_e = \frac{p_e}{\gamma-1} + \frac{1}{2}\rho_e u_e^2 = \frac{p_e}{\gamma-1} + \frac{1}{2}\frac{m_e}{n_e(q_e m_i - q_i m_e)^2}\left(m_i \mathbf{J} - q_i \rho \mathbf{u}\right)^2. \tag{5.12}$$

The total energy density $\mathscr{E} = \mathscr{E}_i + \mathscr{E}_e$, is given by

$$\mathscr{E} = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2 + \frac{1}{2}\frac{m_i m_e \rho}{n_i n_e (q_e m_i - q_i m_e)^2}\left(\rho\mathbf{u} - \mathbf{J}\right)^2. \tag{5.13}$$

Writing the electron energy equation requires only substitution of the appropriate quantities to obtain

$$\frac{\partial \mathscr{E}_e}{\partial t} + \nabla\cdot\left[\frac{m_i\mathbf{J} - q_i\rho\mathbf{u}}{n_e(q_e m_i - q_i m_e)}\left(\mathscr{E}_e + p_e\right)\right] = q_e \mathbf{E}\cdot\frac{m_i\mathbf{J} - q_i\rho\mathbf{u}}{q_e m_i - q_i m_e}. \tag{5.14}$$

The total energy equation is the sum of the ion and electron equations,

$$\begin{aligned}\frac{\partial \mathscr{E}}{\partial t} + \nabla\cdot\left[\frac{q_e\rho\mathbf{u} - m_e\mathbf{J}}{n_i(q_e m_i - q_i m_e)}\left(\mathscr{E} + p - \mathscr{E}_e - p_e\right) + \frac{m_i\mathbf{J} - q_i\rho\mathbf{u}}{n_e(q_e m_i - q_i m_e)}\left(\mathscr{E}_e + p_e\right)\right] \\ = q_i\mathbf{E}\cdot\frac{q_e\rho\mathbf{u} - m_e\mathbf{J}}{q_e m_i - q_i m_e} + q_e\mathbf{E}\cdot\frac{m_i\mathbf{J} - q_i\rho\mathbf{u}}{q_e m_i - q_i m_e}.\end{aligned} \tag{5.15}$$

Now applying $m_e \ll m_i$, with a little algebra we obtain for the total energy,

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla\cdot\left[(\mathscr{E} + p)\mathbf{u} + (\mathscr{E}_e + p_e)\frac{\mathbf{J} - q\mathbf{u}}{q_e n_e}\right] = \mathbf{E}\cdot\mathbf{J}, \tag{5.16}$$

and for the electron energy equation,

$$\frac{\partial \mathscr{E}_e}{\partial t} + \nabla\cdot\left[(\mathscr{E}_e + p_e)\left(\mathbf{u} + \frac{\mathbf{J} - q\mathbf{u}}{q_e n_e}\right)\right] = \mathbf{E}\cdot\left(\mathbf{J} + \left(q - q_e n_e\right)\mathbf{u}\right), \tag{5.17}$$

where

$$\mathscr{E} = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2 + \frac{1}{2}\frac{m_e}{q_e^2 n_e}\left(\mathbf{J} - q\mathbf{u}\right)^2, \tag{5.18}$$

$$\mathscr{E}_e = \frac{p_e}{\gamma-1} + \frac{1}{2}m_e n_e\left(\mathbf{u} + \frac{\mathbf{J} - q\mathbf{u}}{q_e n_e}\right)^2. \tag{5.19}$$

5.2. Nondimensionalization

In order to derive the generalized Ohm's Law presented in [12], a nondimensionalization of these equations is required in order to make scaling arguments for terms that can be neglected. First, the complete dimensional system:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{5.20}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + pI) = q\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{5.21}$$

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \left[(\mathscr{E} + p)\mathbf{u} + (\mathscr{E}_e + p_e)\frac{\mathbf{J} - q\mathbf{u}}{q_e n_e} \right] = \mathbf{E} \cdot \mathbf{J}, \tag{5.22}$$

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{5.23}$$

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \left[\frac{\mathbf{J} \otimes \mathbf{J}}{q_e n_e} + \left(\frac{q}{q_e n_e} - 1 \right) (q\mathbf{u} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{J} - \mathbf{J} \otimes \mathbf{u}) + \frac{q_e}{m_e} p_e I \right]$$
$$= \frac{q_e}{m_e} \left[q_e n_e \mathbf{E} + \left(\mathbf{J} - (\frac{q}{q_e n_e} - 1)\mathbf{u} \right) \times \mathbf{B} \right] \tag{5.24}$$

$$\frac{\partial \mathscr{E}_e}{\partial t} + \nabla \cdot \left[(\mathscr{E}_e + p_e) \left(\mathbf{u} + \frac{\mathbf{J} - q\mathbf{u}}{q_e n_e} \right) \right] = \mathbf{E} \cdot \left(\mathbf{J} - \left(q - q_e n_e \right) \mathbf{u} \right). \tag{5.25}$$

Next, we define the reference scales for the nondimensionalization, listed in Table 5-1. Two important reference velocity scales are defined: first, the Alfvén speed,

$$V_{a_0} = \frac{B_0}{\sqrt{\rho_0 \mu_0}}, \tag{5.26}$$

which is the reference speed of the fluid center-of-mass flow, and second, the Hall speed,

$$V_{h_0} = \frac{B_0}{q_e n_{e_0} \mu_0 L_0}, \tag{5.27}$$

which is the reference speed for the charge flow. Note that the reference charge density, q_0, has an unexpected definition that is a function of both Alfvén and Hall velocity scales to be consistent with [12].

Replacing the dimensional variables with dimensionless equivalents, denoted by a tilde, the continuity and momentum equations are:

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{u}}) = 0,$$

$$\frac{\partial \tilde{\rho} \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left(\tilde{\rho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} + \frac{p_0}{\rho_0 V_{a_0}^2} \tilde{p} I \right) = \frac{V_{a_0}^2}{c_0^2} \tilde{q} \tilde{\mathbf{E}} + \tilde{\mathbf{J}} \times \tilde{\mathbf{B}}$$

$$\frac{V_{a_0}^2}{c_0^2} \frac{\partial \tilde{q}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \tilde{\mathbf{J}} = 0,$$

$$\frac{\partial \tilde{\mathbf{J}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left[\frac{V_{h_0}}{V_{a_0} \tilde{n}_e} \tilde{\mathbf{J}} \otimes \tilde{\mathbf{J}} + \left(\frac{V_{a_0} V_{h_0}}{\tilde{n}_e c_0^2} \tilde{q} - 1 \right) \left(\frac{V_{a_0}^2}{c_0^2} \tilde{q} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} - \tilde{\mathbf{u}} \otimes \tilde{\mathbf{J}} - \tilde{\mathbf{J}} \otimes \tilde{\mathbf{u}} \right) + \frac{p_{e_0}}{m_e n_{e_0} V_{h_0} V_{a_0}} \tilde{p}_e I \right]$$
$$= \frac{q_e \tilde{n}_e B_0 L_0}{m_e V_{h_0}} \left[\tilde{\mathbf{E}} + \left(\frac{V_{h_0}}{V_{a_0} \tilde{n}_e} \tilde{\mathbf{J}} - (\frac{V_{a_0} V_{h_0}}{\tilde{n}_e c_0^2} \tilde{q} - 1) \tilde{\mathbf{u}} \right) \times \tilde{\mathbf{B}} \right]$$

Length	$\mathbf{x}$	L_0
Time	t	$t_0 = L_0/u_0$
Density	ρ	ρ_0
Velocity	$\mathbf{u}$	V_{a_0}
Pressure	p	p_0
Electron number density	n_e	n_{e_0}
Charge density	q	$q_0 = q_e n_{e_0} V_{a_0} V_{h_0}/c_0^2$
Current density	$\mathbf{J}$	$J_0 = q_e n_{e_0} V_{h_0}$
Electric field	$\mathbf{E}$	$E_0 = B_0 u_0$
Magnetic field	$\mathbf{B}$	B_0
Energy	$\mathscr{E}$	$\mathscr{E}_0 = \rho_0 u_0^2$
Electron energy	$\mathscr{E}_e$	$\mathscr{E}_{e_0} = m_e n_{e_0} u_0^2$

Table 5-1. Reference quantities for nondimensionalization of the generalized Ohm's Law system (5.20)–(5.25). **The reference Alfvén speed,** V_{a_0}**, and the reference Hall speed,** V_{h_0}**, are defined in** (5.26) **and** (5.27)**, respectively.**

To simplify some of these terms, we introduce the following definitions:

$$\beta = \frac{p_0}{\rho_0 V_{a_0}^2}\tilde{p} = \frac{p_0 \mu_0}{B_0^2}\tilde{p}, \tag{5.28}$$

$$q' = \frac{V_{a_0}^2}{c_0^2}\tilde{q}, \tag{5.29}$$

$$V_h = \frac{V_{h_0}}{V_{a_0}\tilde{n}_e}, \tag{5.30}$$

$$\beta_e = \frac{p_{e_0}}{m_e n_{e_0} V_{a_0}^2}\tilde{p}_e = \frac{p_{e_0}\mu_0\rho_0}{\rho_{e_0} B_0^2}\tilde{p}_e, \tag{5.31}$$

$$\lambda_e = \sqrt{\frac{m_e V_{h_0}}{q_e \tilde{n}_e B_0 L_0}} = \sqrt{\frac{c_0^2}{\tilde{n}_e L_0^2}\frac{q_e^2 n_{e_0}}{m_e \varepsilon_0}} = \lambda_{e_0}/\sqrt{\tilde{n}_e L_0^2}. \tag{5.32}$$

This last quantity is a normalization of the reference electron inertial length, $\lambda_{e_0} = c_0/\omega_{pe_0}$, which is defined in terms of the plasma frequency

$$\omega_{pe_0} = \sqrt{\frac{q_e^2 n_{e_0}}{m_e \varepsilon_0}}. \tag{5.33}$$

For the energy equations, the nondimensional forms of (5.18) and (5.19) are

$$\tilde{\mathscr{E}} = \frac{\beta}{\gamma-1} + \frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{2}\frac{\rho_{e_0}}{\rho_0}\tilde{n}_e V_h^2\left(\tilde{\mathbf{J}} - q'\tilde{\mathbf{u}}\right)^2, \tag{5.34}$$

$$\tilde{\mathscr{E}}_e = \frac{\beta_e}{\gamma-1} + \frac{1}{2}\tilde{n}_e\left(\tilde{\mathbf{u}} + V_h(\tilde{\mathbf{J}} - q'\tilde{\mathbf{u}})\right)^2. \tag{5.35}$$

With a little manipulation, and using the quantities identified above, the complete nondimensional system is as follows:

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{u}}) = 0, \tag{5.36}$$

$$\frac{\partial \tilde{\rho} \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} + \beta I) = q' \tilde{\mathbf{E}} + \tilde{\mathbf{J}} \times \tilde{\mathbf{B}}, \tag{5.37}$$

$$\frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left[(\tilde{\mathscr{E}} + \beta) \tilde{\mathbf{u}} + \frac{\rho_{e_0}}{\rho_0} (\tilde{\mathscr{E}}_e + \beta_e) V_h \left(\tilde{\mathbf{J}} - q' \tilde{\mathbf{u}} \right) \right] = \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}, \tag{5.38}$$

$$\frac{\partial q'}{\partial \tilde{t}} + \tilde{\nabla} \cdot \tilde{\mathbf{J}} = 0, \tag{5.39}$$

$$\frac{\partial \tilde{\mathbf{J}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left[V_h \tilde{\mathbf{J}} \otimes \tilde{\mathbf{J}} + \left(V_h q' - 1 \right) \left(q' \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} - \tilde{\mathbf{u}} \otimes \tilde{\mathbf{J}} - \tilde{\mathbf{J}} \otimes \tilde{\mathbf{u}} \right) + \frac{V_{a_0}}{V_{h_0}} \beta_e I \right]$$
$$= \frac{1}{\lambda_e^2} \left[\tilde{\mathbf{E}} + \left(V_h \tilde{\mathbf{J}} - \left(V_h q' - 1 \right) \tilde{\mathbf{u}} \right) \times \tilde{\mathbf{B}} \right], \tag{5.40}$$

$$\frac{\partial \tilde{\mathscr{E}}_e}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left[\left(\tilde{\mathscr{E}}_e + \beta_e \right) \left(\tilde{\mathbf{u}} + V_h (\tilde{\mathbf{J}} - q' \tilde{\mathbf{u}}) \right) \right] = \frac{1}{\lambda_e^2} \frac{V_{h_0}}{V_{a_0}} \left(V_h \tilde{\mathbf{J}} - (V_h q' - 1) \tilde{\mathbf{u}} \right) \cdot \tilde{\mathbf{E}}. \tag{5.41}$$

From this system, scaling arguments can be made to simplify the equations (and the equation for $\tilde{\mathbf{J}}$ in particular) to derive a models of intermediate complexity between the full two-fluid system and MHD.

5.3. Simplified Generalized Ohm's Law Models

We can simplify towards an MHD model by neglecting charge separation effects due to q', the flow of free charge relative to the current density, $q' \tilde{\mathbf{u}} \ll \tilde{\mathbf{J}}$, and electron energy/pressure terms. Thus

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{u}}) = 0, \tag{5.42}$$

$$\frac{\partial \tilde{\rho} \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} + \beta I) = \tilde{\mathbf{J}} \times \tilde{\mathbf{B}}, \tag{5.43}$$

$$\frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left(\tilde{\mathbf{u}} (\tilde{\mathscr{E}} + \beta) \right) = \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}, \tag{5.44}$$

$$\frac{\partial q'}{\partial \tilde{t}} + \tilde{\nabla} \cdot \tilde{\mathbf{J}} = 0, \tag{5.45}$$

$$\frac{\partial \tilde{\mathbf{J}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot \left(\tilde{\mathbf{u}} \otimes \tilde{\mathbf{J}} + \tilde{\mathbf{J}} \otimes \tilde{\mathbf{u}} + V_h \tilde{\mathbf{J}} \otimes \tilde{\mathbf{J}} + \frac{V_{a_0}}{V_{h_0}} \beta_e I \right)$$
$$= \frac{1}{\lambda_e^2} \left[\tilde{\mathbf{E}} + \left(V_h \tilde{\mathbf{J}} + \tilde{\mathbf{u}} \right) \times \tilde{\mathbf{B}} \right]. \tag{5.46}$$

Note that the energy, $\tilde{\mathscr{E}}$, is now defined as

$$\tilde{\mathscr{E}} = \frac{\beta}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{2}\frac{\rho_{e_0}}{\rho_0}\tilde{n}_e V_h^2 \tilde{\mathbf{J}}^2. \tag{5.47}$$

Further simplification can be made by assuming that the electron inertial length goes to zero to yield the Hall MHD approximation,

$$\tilde{\mathbf{E}} + \left(V_h \tilde{\mathbf{J}} + \tilde{\mathbf{u}}\right) \times \tilde{\mathbf{B}} = 0. \tag{5.48}$$

Note that a conductivity term would appear in this equation if collision terms had been kept in the ion and electron equations.

6. TWO-FLUID VERIFICATION USING LINEAR PLASMA WAVE SOLUTIONS

The general two-fluid plasma equations have a rich and detailed mathematical structure even in the limits of low temperature and limited collisionality. This can be understood by a simple counting argument. In three spatial dimensions there are five equations for each species (mass, momentum and energy) and three equations each for Faraday's and Ampere's laws. Thus the basic two-fluid system has sixteen degrees of freedom and is hyperbolic with interacting source terms. A useful starting point for building understanding of the system is a dispersive linear wave analysis about a reference base state. This will be used here, along with some additional approximations, to provide a verification approach for the two-fluid system.

For cold plasmas, a comprehensive picture of the possible wave modes is possible using the Clemmow-Mullaly-Allis (CMA) diagram (e.g., in [4]) to classify waves at different frequencies. This diagram provides a visualization in density-magnetic field space of the waves supported in different frequency regions and their character. The waves are generally anisotropic with respect to the magnetic field direction. Capturing both phase and group velocity (ray velocity, wave front) surfaces as a function of wave-normal direction are of interest for numerical verification purposes. Since plasma waves are highly dispersive, the wave analysis is much more complicated than simpler non-dispersive systems.

A very large class of analytical and semi-analytical solutions to propagating small-amplitude waves in plasmas exists [4]. These solutions are obtained by assuming a traveling wave form $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ for the solution fields and appropriately linearizing the governing equations. To maintain tractability of the plasma wave solutions developed here, we will restrict our attention to single-fluid (electron) and two-fluid (electron-ion) cases. For the latter, denoting the electron fluid with subscript e and the singly-ionized fluid with subscript i, the continuity and momentum equations from the two-fluid model are given by

$$\frac{\partial \rho_e}{\partial t} + \nabla\cdot(\rho_e \mathbf{u}_e) = 0, \tag{6.1}$$

$$\frac{\partial \rho_i}{\partial t} + \nabla\cdot(\rho_i \mathbf{u}_i) = 0, \tag{6.2}$$

$$\frac{\partial(\rho_e \mathbf{u}_e)}{\partial t} + \nabla\cdot(\rho_e \mathbf{u}_e \otimes \mathbf{u}_e + p_e I) = -\frac{\rho_e e}{m_e}(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \mathbf{R}^{en}, \tag{6.3}$$

$$\frac{\partial(\rho_i \mathbf{u}_i)}{\partial t} + \nabla\cdot(\rho_i \mathbf{u}_i \otimes \mathbf{u}_i + p_i I) = \frac{\rho_i e}{m_i}(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \mathbf{R}^{in}. \tag{6.4}$$

We are neglecting ionization/recombination reactions and the viscous stress tensor, but retaining collisions with a background neutral species. The pressure terms are modeled by an adiabatic equation of state for species a where $p_a n_a^{-\gamma}$ is constant, which results in $\nabla p_a = \gamma k_B T_a \nabla n_a$. Here, the adiabatic constant is for one-dimensional (single degree-of-freedom) compressive waves, which is set to $\gamma = 3$. For the collision terms, we assume a stationary neutral species ($\mathbf{u}_n = 0$) yielding $\mathbf{R}^{an} = -m_a n_a \mathbf{u}_a \nu_{an}$.

Maxwell's equations for free space are repeated here for completeness,

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}, \tag{6.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6.6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{6.7}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{6.8}$$

where the charge density is $q = e(n_i - n_e)$ and the current is $\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$.

For harmonic traveling waves in the **k** direction, the solution fields can be represented as

$$n_e = \rho_e/m_e = n_0 + \tilde{n}_e \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{6.9}$$

$$n_i = \rho_i/m_i = n_0 + \tilde{n}_i \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{6.10}$$

$$\mathbf{u}_e = \tilde{\mathbf{u}}_e \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{6.11}$$

$$\mathbf{u}_i = \tilde{\mathbf{u}}_i \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{6.12}$$

$$\mathbf{E} = \tilde{\mathbf{E}} \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{6.13}$$

$$\mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}} \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t). \tag{6.14}$$

Here, the tilde versions of the variables refer to small amplitude perturbations with respect to plasma number density n_0 and external magnetic field $\mathbf{B}_0$.

As shown in [4], the sole requirement for linearizing the system above is that the particle velocities are small compared to the wave phase velocity, i.e. $||\mathbf{u}|| \ll \omega/k$. Under this assumption, the nonlinear terms in the continuity, momentum, and Ampere's Law equations are approximated as

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}_a) \approx m_a \left(\frac{\partial n_a}{\partial t} + n_0 \nabla \cdot \mathbf{u}_a \right), \tag{6.15}$$

$$\frac{\partial (\rho_a \mathbf{u}_a)}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}_a \otimes \mathbf{u}_a) \approx m_a n_0 \frac{\partial \mathbf{u}_a}{\partial t}, \tag{6.16}$$

$$\mathbf{E} + \mathbf{u}_a \times \mathbf{B} \approx \mathbf{E} + \mathbf{u}_a \times \mathbf{B}_0, \tag{6.17}$$

$$\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e) \approx n_0 e (\mathbf{u}_i - \mathbf{u}_e). \tag{6.18}$$

The overall system of equations in harmonic form thus reduces to

$$\omega \tilde{n}_e = n_0 \mathbf{k} \cdot \tilde{\mathbf{u}}_e, \tag{6.19}$$

$$\omega \tilde{n}_i = n_0 \mathbf{k} \cdot \tilde{\mathbf{u}}_i, \tag{6.20}$$

$$-i\omega \tilde{\mathbf{u}}_e = -\frac{e}{m_e}(\tilde{\mathbf{E}} + \tilde{\mathbf{u}}_e \times \mathbf{B}_0) - i\frac{kV_{se}^2}{n_0}\tilde{n}_e - \nu_{en}\tilde{\mathbf{u}}_e, \tag{6.21}$$

$$-i\omega \tilde{\mathbf{u}}_i = \frac{e}{m_e}(\tilde{\mathbf{E}} + \tilde{\mathbf{u}}_i \times \mathbf{B}_0) - i\frac{kV_{si}^2}{n_0}\tilde{n}_i - \nu_{in}\tilde{\mathbf{u}}_i, \tag{6.22}$$

$$i\varepsilon_0 \mathbf{k} \cdot \tilde{\mathbf{E}} = e(\tilde{n}_i - \tilde{n}_e), \tag{6.23}$$

$$\mathbf{k} \cdot \tilde{\mathbf{B}} = 0, \tag{6.24}$$

$$\mathbf{k} \times \tilde{\mathbf{E}} = \omega \tilde{\mathbf{B}}, \tag{6.25}$$

$$i\varepsilon_0 c^2 \mathbf{k} \times \tilde{\mathbf{B}} = n_0 e(\tilde{\mathbf{u}}_i - \tilde{\mathbf{u}}_e) - i\varepsilon_0 \omega \tilde{\mathbf{E}}. \tag{6.26}$$

Here, the adiabatic sound frequencies are $V_{sa} = \sqrt{\gamma k_B T_a / m_a}$. These equations can be solved to obtain the fundamental mode shapes and corresponding dispersion relations $\omega(k)$, where the number and character of the modes depend strongly on many factors, including the existence and alignment of $\mathbf{B}_0$ relative to $\mathbf{k}$, species temperatures, and collisionality.

The dispersion relations for these modes are described in detail in [4], while the full expressions for each verification problem are given the following sections. For convenience, we will assume without loss of generality that the wavevector $\mathbf{k} = (k, 0, 0)$ is pointing in the x_1 direction, and consider external magnetic fields which are zero ($B_0 = 0$), aligned with the direction of propagation ($\mathbf{B}_0 = (B_0, 0, 0)$), or normal to the direction of propagation ($\mathbf{B}_0 = (0, B_0, 0)$). For consistency in treating these linearized waves, we introduce a dimensionless perturbation parameter

$$\delta = \frac{k}{\omega} \max\{||\mathbf{u}_e||, ||\mathbf{u}_i||\} \ll 1$$

to ensure that the linearization condition is always satisfied.

Since we do not discuss any particular numerical method in this report, the dots (labeled "sim") on some of the plots that follow in the sections below give representative results to show how specific frequencies can be computed at a given wavenumber and cross-plotted on the exact dispersion curves. Each dot represents a single computation in which numerical initial conditions are chosen to match the desired mode and thus isolates the wave mode of interest.

6.1. Cold Unmagnetized Electron Plasma Waves

The cold unmagnetized collisionless electron plasma waves are the simplest version of the general formulation above. Here, the ion species is assumed to be immobile and represented exclusively as a constant charge density $n_i = n_0$ used to balance Gauss' Law for $\mathbf{E}$ via $\varepsilon_0 \nabla \cdot \mathbf{E}(\mathbf{x},t) = e[n_i(\mathbf{x}) - n_e(\mathbf{x},t)]$, $\mathbf{B} = 0$, and $T_e = 0$. Under these assumptions, both longitudinal electron plasma (LEP) and transverse electromagnetic (TEM) waves propagate. The longitudinal electron plasma wave has a complex dispersion relation that is independent of k and is simply:

$$\omega = \omega_{pe} = \sqrt{\frac{e^2 n_0}{\varepsilon_0 m_e}}. \tag{6.27}$$

The full LEP wave mode solution is:

$$\rho_e(\mathbf{x},t) = m_e n_0 [1 + \delta \sin(kx_1 - \omega t)], \tag{6.28}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k} \delta \Big(\sin\left(kx_1 - \omega t\right), 0, 0 \Big), \tag{6.29}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{e n_0}{\varepsilon_0 k} \delta \Big(\cos(kx_1 - \omega t), 0, 0 \Big), \tag{6.30}$$

$$\mathbf{B}(\mathbf{x},t) = \big(0,0,0\big). \tag{6.31}$$

Note that this is fundamentally an electrostatic mode since $\nabla \times \mathbf{E} = 0$.

The transverse electromagnetic wave has a dispersion relation given by:

$$\omega = \sqrt{\omega_{pe}^2 + c^2 k^2}. \tag{6.32}$$

This asymptotes for small k to $\omega = \omega_{pe}$, and for large k to $\omega = ck$. This mode is linearly polarized and has the following form:

$$\rho_e(\mathbf{x},t) = m_e n_0, \tag{6.33}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k} \delta \Big(0, \sin\left(kx_1 - \omega t\right), 0 \Big), \tag{6.34}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{e n_0}{\varepsilon_0 k} \frac{\delta}{1 - \eta^2} \Big(0, \cos(kx_1 - \omega t), 0 \Big), \tag{6.35}$$

$$\mathbf{B}(\mathbf{x},t) = \frac{e n_0}{\varepsilon_0 \omega} \frac{\delta}{1 - \eta^2} \Big(0, 0, \cos\left(kx_1 - \omega t\right) \Big), \tag{6.36}$$

where the index of refraction is $\eta = ck/\omega$.

The dispersion relation for these two waves is shown in Figure 6-1.

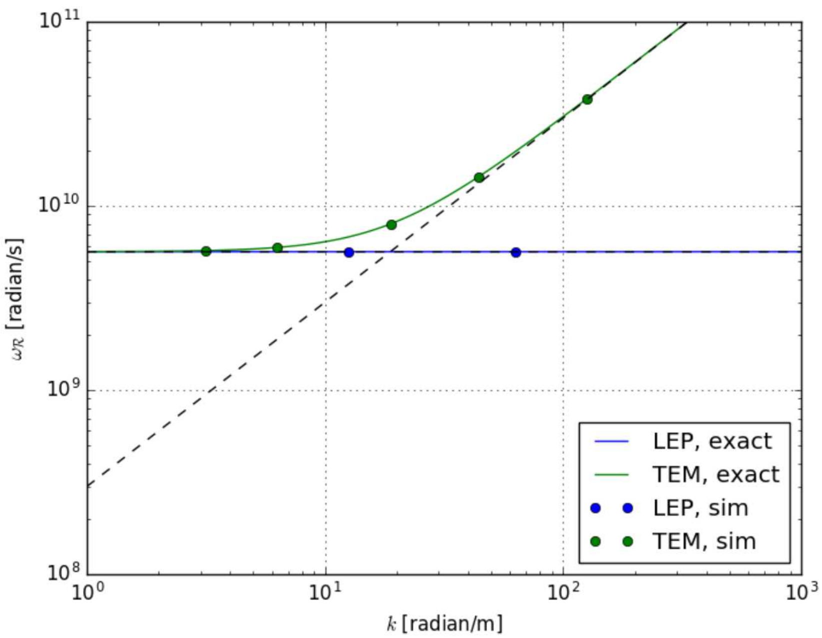

Figure 6-1. Dispersion plot for collisionless LEP (blue) and TEM (green) unmagnetized cold plasma waves, with $n_0 = 10^{16}$ m^{-3}.

6.2. Cold Magnetized Electron Plasma Waves

The magnetized plasma waves considered here differ from the previous section due to the presence of an external magnetic field $\mathbf{B}_0$, which results in a richer set of wave modes. These cases are useful to test the interaction between magnetic fields and charged particle currents, which was not represented in the unmagnetized wave solutions. For tractability, we only consider the collisionless ($\nu_{en} = 0$) case. In addition to the plasma frequency, the electron cyclotron frequency $\omega_{ce} = eB_0/m_e$ now appears.

Fundamentally different classes of waves propagate in directions along and normal to $\mathbf{B}_0$; we consider each class separately by considering $\mathbf{B}_0 = (B_0, 0, 0)$ and $\mathbf{B}_0 = (0, B_0, 0)$ cases while maintaining a common wave propagation direction $\mathbf{k} = (k, 0, 0)$. The modes that propagate along the magnetic field are the LEP wave, right-hand circularly polarized (RCP), and left-hand circularly polarized (LCP) waves, for which the dispersion relations are shown in Figure 6-2. The modes that propagate normal to the magnetic field are the ordinary (O) and extraordinary (X) waves, for which the dispersion relations are shown in Figure 6-3.

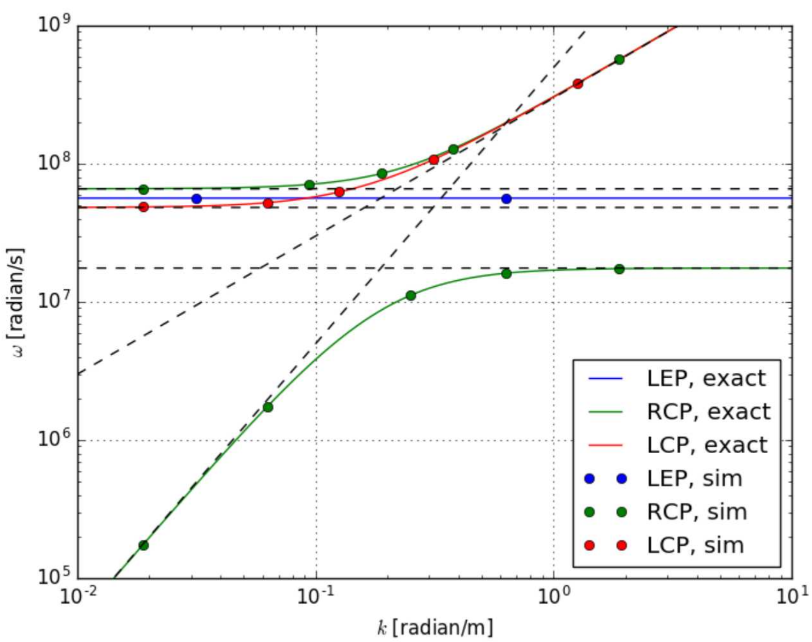

Figure 6-2. Dispersion plot for magnetized cold plasma waves along the magnetic field, with $n_0 = 10^{12}$ m^{-3} and $B_0 = 10^{-4}$ T.

The LEP wave appears in the magnetized case completely decoupled from the magnetic field. Thus the dispersion relation is again

$$\omega = \omega_{pe}, \tag{6.37}$$

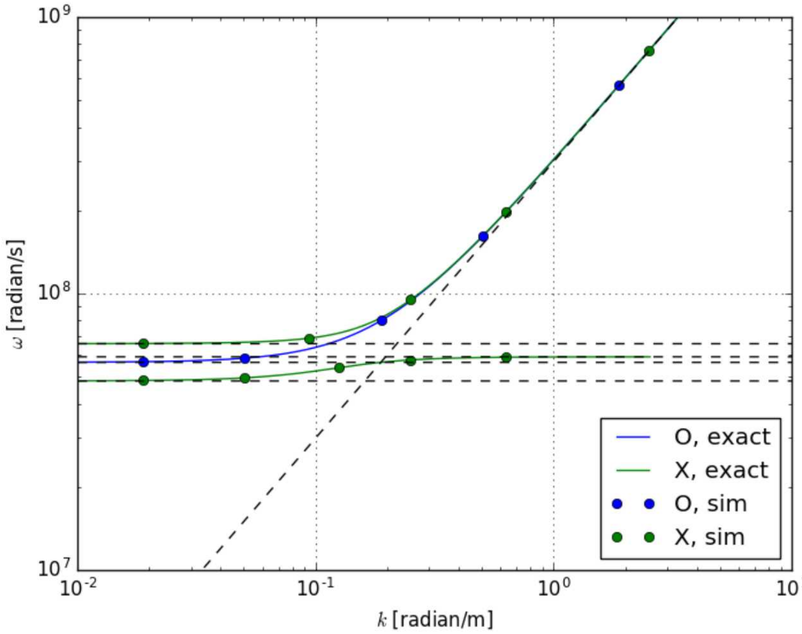

Figure 6-3. Dispersion plot for magnetized cold plasma waves normal to the magnetic field, with $n_0 = 10^{12}$ m^{-3} and $B_0 = 10^{-4}$ T.

and the wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0[1+\delta\sin(kx_1 - \omega t)], \tag{6.38}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\delta\Big(\sin(kx_1-\omega t),0,0\Big), \tag{6.39}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 k}\delta\Big(\cos(kx_1-\omega t),0,0\Big), \tag{6.40}$$

$$\mathbf{B}(\mathbf{x},t) = \Big(B_0,0,0\Big). \tag{6.41}$$

For a magnetic field aligned with the wave vector, the RCP wave has an implicitly-defined dispersion relation given by

$$c^2k^2 = \omega^2 - \frac{\omega_{pe}^2\omega}{\omega - \Omega_{ce}}. \tag{6.42}$$

This equation has two solutions for $\omega(k)$, resulting in two distinct branches. The lower branch propagates for $\omega \leq \omega_{ce}$, where the asymptotes for small and large k are $\omega = \omega_{ce}c^2k^2/\omega_{pe}$ and $\omega = \omega_{ce}$, respectively. The upper branch propagates for $\omega \geq \omega_{02} = \sqrt{\omega_{pe}^2 + \frac{1}{4}\omega_{ce}^2} + \frac{1}{2}\omega_{ce}$, where the asymptotes for small and large k are $\omega = \omega_{02}$ and $\omega = ck$, respectively. The wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0, \tag{6.43}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\frac{\delta}{\sqrt{2}}\Big(0, \sin(kx_1 - \omega t), \cos(kx_1 - \omega t)\Big), \tag{6.44}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\sqrt{2}\varepsilon_0 k}\frac{\delta}{1-\eta^2}\Big(0, \cos(kx_1 - \omega t), -\sin(kx_1 - \omega t), 0\Big), \tag{6.45}$$

$$\mathbf{B}(\mathbf{x},t) = \Big(B_0, 0, 0\Big) + \frac{en_0}{\sqrt{2}\varepsilon_0 \omega}\frac{\delta}{1-\eta^2}\Big(0, \sin(kx_1 - \omega t), \cos(kx_1 - \omega t)\Big). \tag{6.46}$$

The LCP wave admits only a single solution branch, with the dispersion relation

$$c^2k^2 = \omega^2 - \frac{\omega_{pe}^2\omega}{\omega + \Omega_{ce}}. \tag{6.47}$$

This mode propagates for $\omega \geq \omega_{01} = \sqrt{\omega_{pe}^2 + \frac{1}{4}\Omega_{ce}^2} - \frac{1}{2}\Omega_{ce}$. The asymptotes for small and large k are $\omega = \omega_{01}$ and $\omega = ck$, respectively. The wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0, \tag{6.48}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\frac{\delta}{\sqrt{2}}\Big(0, \sin(kx_1 - \omega t), -\cos(kx_1 - \omega t)\Big), \tag{6.49}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\sqrt{2}\varepsilon_0 k}\frac{\delta}{1-\eta^2}\Big(0, \cos(kx_1 - \omega t), \sin(kx_1 - \omega t), 0\Big), \tag{6.50}$$

$$\mathbf{B}(\mathbf{x},t) = \Big(B_0, 0, 0\Big) + \frac{en_0}{\sqrt{2}\varepsilon_0 \omega}\frac{\delta}{1-\eta^2}\Big(0, -\sin(kx_1 - \omega t), \cos(kx_1 - \omega t)\Big). \tag{6.51}$$

In a transverse external magnetic field, $\mathbf{B}_0 = (0, B_0, 0)$, the ordinary (O) wave has a dispersion relation given by

$$\omega = \sqrt{\omega_{pe}^2 + c^2k^2}. \tag{6.52}$$

Asymptotes for small and large k are $\omega = \omega_{pe}$ and $\omega = ck$, respectively. The wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0, \tag{6.53}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\delta\Big(0, \sin(kx_1 - \omega_{\mathscr{R}} t), 0\Big), \tag{6.54}$$

$$\mathbf{E}_(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 k}\frac{\delta}{1-\eta^2}\Big(0, \cos(kx_1 - \omega_{\mathscr{R}} t), 0\Big), \tag{6.55}$$

$$\mathbf{B}(\mathbf{x},t) = \Big(0, B_0, 0\Big) + \frac{en_0}{\varepsilon_0 \omega}\frac{\delta}{1-\eta^2}\Big(0, 0, \cos\left(kx_1 - \omega_{\mathscr{R}} t\right)\Big). \tag{6.56}$$

The elliptically polarized extraordinary (X) wave has dispersion relation

$$c^2k^2 = \frac{(\omega^2 - \omega_{01}^2)(\omega^2 - \omega_{02}^2)}{\omega^2 - \omega_{uh}^2}, \tag{6.57}$$

where $\omega_{uh} = \sqrt{\omega_{pe}^2 + \Omega_{ce}^2}$ is the upper hybrid frequency. This mode has two distinct branches: a lower branch that propagates for $\omega_{01} \leq \omega \leq \omega_{uh}$, and an upper branch that propagates for $\omega > \omega_{02}$. The lower branch has asymptotes for small and large k of $\omega = \omega_{01}$ and $\omega = \omega_{uh}$, respectively, while the corresponding asymptotes for the upper branch are $\omega = \omega_{02}$ and $\omega = ck$. The wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0 [1 + \delta' \sin(kx_1 - \omega t)], \tag{6.58}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\delta' \left(\sin(kx_1 - \omega_{\mathscr{R}} t), 0, -(1-\eta^2)\frac{\omega^2 - \omega_{uh}^2}{\omega_{pe}^2}\frac{\omega}{\omega_{ce}}\cos(kx_1 - \omega t) \right), \tag{6.59}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 k}\delta' \left(\cos(kx_1 - \omega_{\mathscr{R}} t), 0, \frac{\omega^2 - \omega_{uh}^2}{\omega_{pe}^2}\frac{\omega}{\omega_{ce}}\sin(kx_1 - \omega t) \right), \tag{6.60}$$

$$\mathbf{B}(\mathbf{x},t) = B_0 \left(0, 1, -\frac{\omega^2 - \omega_{uh}^2}{\omega_{ce}^2}\delta' \sin(kx_1 - \omega t) \right), \tag{6.61}$$

where the scaled perturbation strength δ' is related to δ by

$$\frac{\delta}{\delta'} = \sqrt{1 + \left[(1-\eta^2)\frac{\omega^2 - \omega_{uh}^2}{\omega_{pe}^2}\frac{\omega}{\omega_{ce}} \right]^2}. \tag{6.62}$$

6.3. Cold Collisional Electron Plasma Waves

Analytic progress can be made for the simple collisional case of a cold unmagnetized electron plasma interacting with a stationary neutral fluid, where a single collision frequency ν_{en} describes the interaction. It serves as a damping term for the waves described in section 6.1. A dispersion plot of the real part of the frequency for varying collisionality is shown in Figure 6-4.

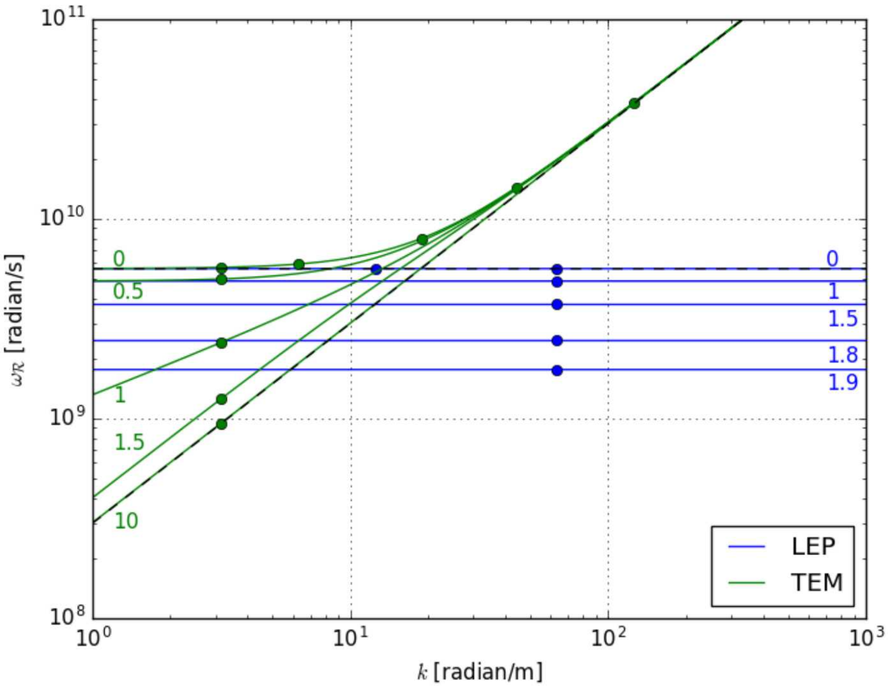

Figure 6-4. Dispersion plot for LEP (blue) and TEM (green) unmagnetized cold plasma waves with various levels of collisionality and $n_0 = 10^{16}\ \mathrm{m}^{-3}$. Here, the numbers for each curve are values of ν_{en}/ω_{pe}.

The collisional LEP wave has the complex-valued dispersion relation

$$\omega = \sqrt{\omega_{pe}^2 - (\nu_{en}/2)^2} - i\nu_{en}/2. \tag{6.63}$$

Here, we split ω into real and imaginary parts via $\omega = \omega_{\mathcal{R}} - i\omega_{\mathcal{I}}$, where $\omega_{\mathcal{R}}$ and $\omega_{\mathcal{I}}$ are both positive real numbers. The real part $\omega_{\mathcal{R}}$ is interpreted as the frequency of oscillation while $\omega_{\mathcal{I}} \geq 0$ is the damping factor; note that $\omega_{\mathcal{I}} < 0$ would imply a growing (or unstable) mode, which is not expected for linear plasma wave theory. The corresponding wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0[1+\delta e^{-\omega_{\mathscr{I}} t}\sin(kx_1 - \omega_{\mathscr{R}} t)], \tag{6.64}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{|\omega|}{k}\delta e^{-\omega_{\mathscr{I}} t}\Big(\sin\big(kx_1 - \omega_{\mathscr{R}} t - \arctan(\omega_{\mathscr{I}}/\omega_{\mathscr{R}})\big),0,0\Big), \tag{6.65}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 k}\delta e^{-\omega_{\mathscr{I}} t}\Big(\cos(kx_1 - \omega_{\mathscr{R}} t),0,0\Big), \tag{6.66}$$

$$\mathbf{B}(\mathbf{x},t) = \big(0,0,0\big), \tag{6.67}$$

where $|\omega| = \sqrt{\omega_{\mathscr{R}}^2 + \omega_{\mathscr{I}}^2}$.

In the collisional case, the transverse electromagnetic wave has the implicitly defined complex dispersion relation

$$c^2k^2 = \omega^2 - \frac{\omega_{pe}^2}{1+(\nu_{en}/\omega)^2} + i\frac{\omega_{pe}^2(\nu_{en}/\omega)}{1+(\nu_{en}/\omega)^2}. \tag{6.68}$$

This mode is linearly polarized and has the form

$$\rho_e(\mathbf{x},t) = m_e n_0, \tag{6.69}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{|\omega|}{k}\delta e^{-\omega_{\mathscr{I}} t}\Big(0,\sin\big(kx_1 - \omega_{\mathscr{R}} t - \phi_{\text{TEM}}\big),0\Big), \tag{6.70}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 k}\frac{|\omega|}{\omega_{\text{TEM}}}\delta e^{-\omega_{\mathscr{I}} t}\Big(0,\cos(kx_1 - \omega_{\mathscr{R}} t),0\Big), \tag{6.71}$$

$$\mathbf{B}(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 \omega_{\text{TEM}}}\delta e^{-\omega_{\mathscr{I}} t}\Big(0,0,\cos\big(kx_1 - \omega_{\mathscr{R}} t + \arctan(\omega_{\mathscr{I}}/\omega_{\mathscr{R}})\big)\Big), \tag{6.72}$$

where the the frequency ω_{TEM} and phase shift ϕ_{TEM} are

$$\omega_{\text{TEM}} = \frac{1}{|\omega|^2}\sqrt{\omega_{\mathscr{R}}^2\left(|\omega|^2 - c^2k^2\right)^2 + \omega_{\mathscr{I}}^2\left(|\omega|^2 + c^2k^2\right)^2}, \tag{6.73}$$

$$\tan\left(\phi_{\text{TEM}}\right) = \frac{\omega_{\mathscr{I}}\left(|\omega|^2 + c^2k^2\right)}{\omega_{\mathscr{R}}\left(|\omega|^2 - c^2k^2\right)}. \tag{6.74}$$

(6.75)

6.4. Warm Electron Plasma Waves

The introduction of temperature to the electron fluid modifies the linear dispersion wave behavior of the system by changing the asymptotes of longitudinal oscillations in the large-wavenumber limit. It is convenient to define a set of normalized parameters to describe the relative strength of thermal and electromagnetic effects. A normalized representation of electron temperature is given by the ratio of the electron sound speed to the speed of light:

$$\frac{v_{se}^2}{c^2} = \frac{k_B T_e}{m_e c^2}. \tag{6.76}$$

A normalized electron density is given by the ratio of the wave frequency to the plasma frequency:

$$\frac{\omega^2}{\omega_{pe}^2} = \frac{m_e \varepsilon_0 \omega^2}{e^2 n_e}. \tag{6.77}$$

The background magnetization can also be normalized by the electron plasma frequency:

$$\frac{\omega_{ce}^2}{\omega_{pe}^2} = \frac{B_0^2}{\mu_0 m_e n_e} \frac{1}{c^2} = \frac{v_{Ae}^2}{c^2}, \tag{6.78}$$

which is effectively a normalization of the electron Alfven velocity,

$$v_{Ae} = \frac{B_0}{\sqrt{\mu_0 \rho_e}}, \tag{6.79}$$

by the speed of light. Finally, the wavenumber can be normalized by the electron skin depth, δ_e:

$$\delta_e^2 k_x^2 = \frac{c^2}{\omega_{pe}^2} k_x^2. \tag{6.80}$$

The dispersion relations for the warm electron fluid case are plotted in Figures 6-5 and 6-6 for the cases of the applied magnetic field being parallel to the wave vector and normal, respectively. These show the thermal effect in asymptotic behavior compared to the cold case shown in Figures 6-2 and 6-3.

For the warm LEP wave, the dispersion relation now depends on the electron temperature:

$$\frac{\omega^2}{\omega_{pe}^2} = 1 + \frac{v_{se}^2}{c^2} d_e^2 k_x^2, \tag{6.81}$$

with the resulting wave mode solution, now including the pressure term:

$$\rho_e(\mathbf{x},t) = m_e n_0 \left(1 + \delta \sin(k_x - \omega t)\right), \tag{6.82}$$

$$\mathbf{u}_e(\mathbf{x},t) = \delta \frac{\omega}{k} \sin\left(k_x - \omega t\right) \hat{\mathbf{x}}, \tag{6.83}$$

$$P_e(\mathbf{x},t) = P_0 \left(1 + \gamma \delta \sin(k_x - \omega t)\right), \tag{6.84}$$

$$\mathbf{E}(\mathbf{x},t) = \delta \frac{e n_0}{\varepsilon_0 k} \cos(k_x - \omega t) \hat{\mathbf{y}}, \tag{6.85}$$

$$\mathbf{B}(\mathbf{x},t) = 0, \tag{6.86}$$

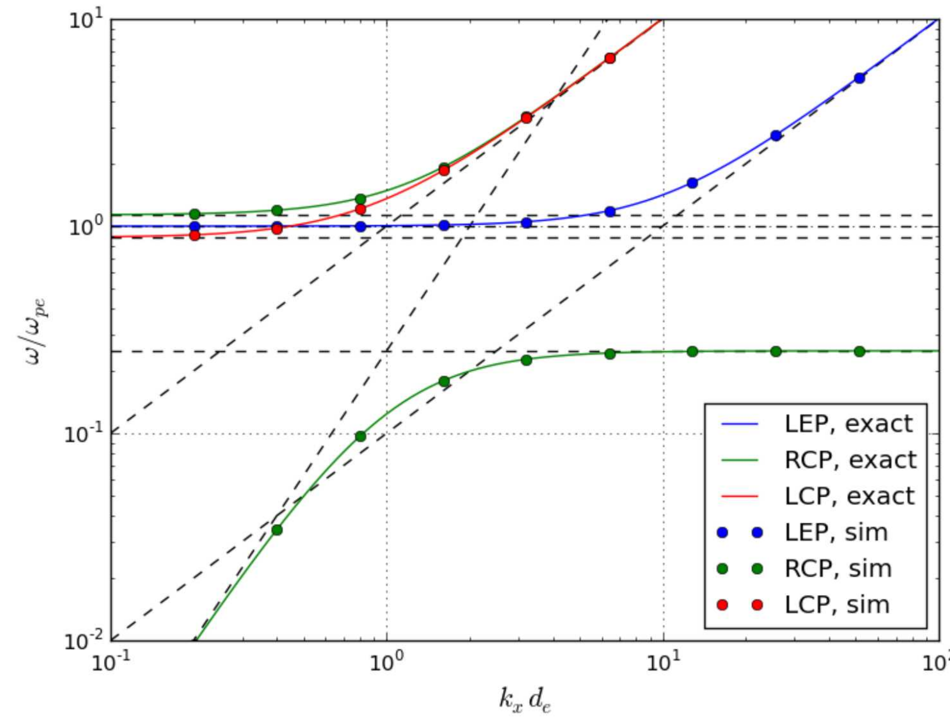

Figure 6-5. Dispersion plot for warm plasma waves parallel to an applied magnetic field. For RCP and LCP, the ratio of cyclotron frequency to plasma frequency is a constant $\omega_{ce}/\omega_{pe} = 1/4$. The leftmost diagonal dashed line represents the speed of light, while the rightmost one represents the speed of sound, $v_{se} = 0.1c$.

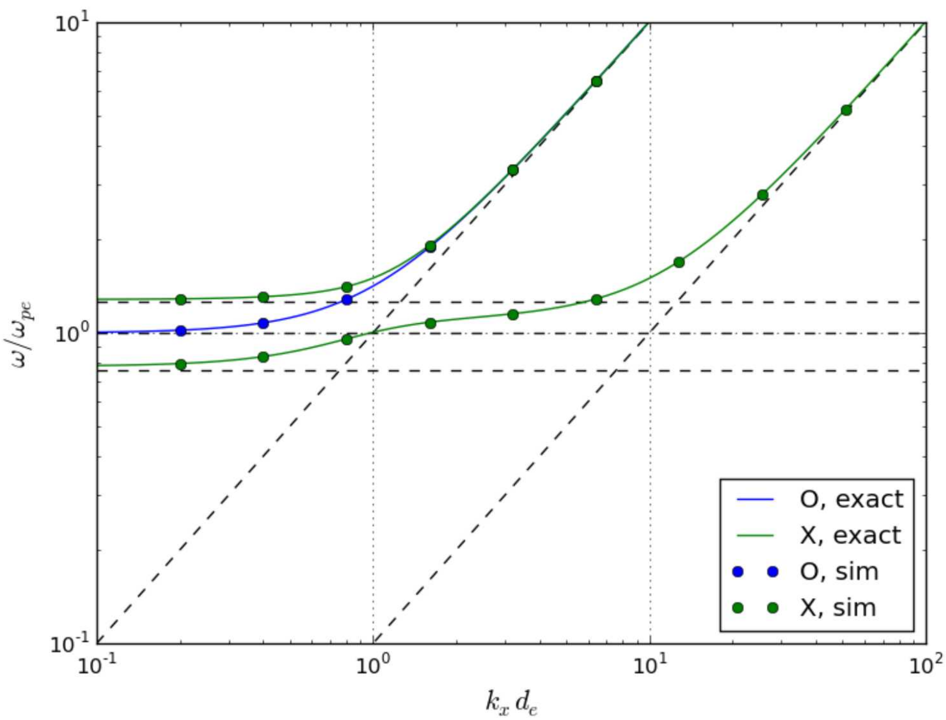

Figure 6-6. Dispersion plot for warm plasma waves normal to an applied magnetic field. The left diagonal dashed line represents the speed of light, while the right one represents the speed of sound, $v_{se} = 0.1c$.

where the reference electron pressure is

$$P_0 = \frac{m_e n_0 c^2}{\gamma} \frac{v_{se}^2}{c^2}. \tag{6.87}$$

Note that the solution remains independent of the magnetic field along the direction of propagation.

Solutions to the other wave modes can be derived by the same procedure as before.

6.5. Warm Unmagnetized Electron-Ion Plasma Waves

The warm ion-electron plasma waves introduce two fundamental complexities to the cold electron waves described to this point. First, the introduction of the much slower ion species results in significant stiffness in the numerical system. Second, the adiabatic pressure gradient term appears when a finite temperature is specified for each species. In the unmagnetized case, LEP, longitudinal ion plasma (LIP), and TEM waves propagate, which are described below. The dispersion relations for the LEP, LIP, and TEM waves are shown for an ion mass of $100m_e$ (chosen for computational expediency) in Figure 6-7.

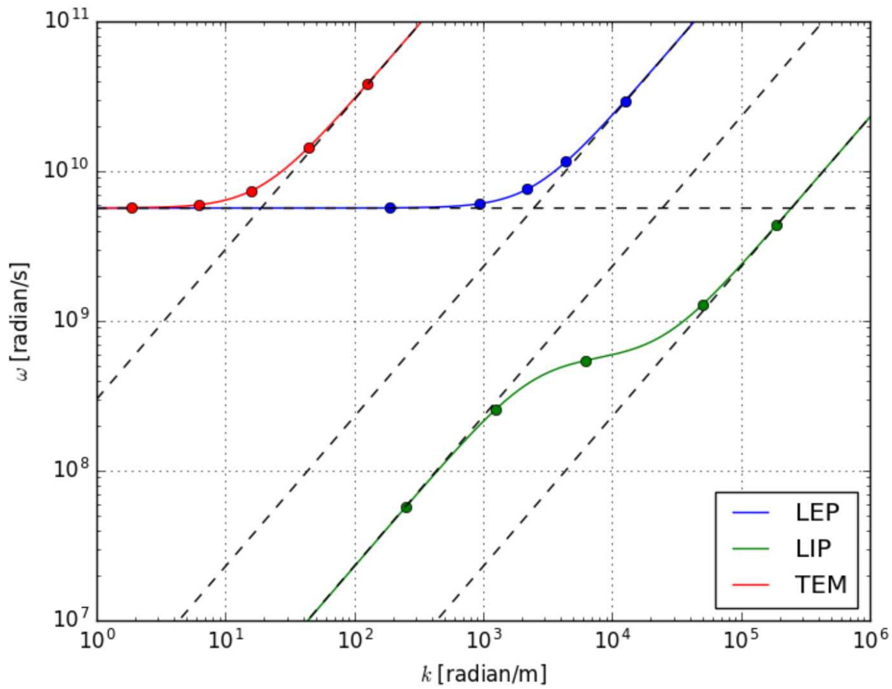

Figure 6-7. Dispersion plot for unmagnetized warm electron-ion plasma waves with $100m_e$ ions and $n_0 = 10^{16}$ m^{-3}.

For the LEP wave, the dispersion relation is given by

$$\omega = \sqrt{\frac{A + \sqrt{A^2 - 4C}}{2}}, \tag{6.88}$$

where the constants A and C are

$$A = \omega_{pe}^2 + \omega_{pi}^2 + (V_{se}^2 + V_{si}^2)k^2, \tag{6.89}$$

$$C = (\omega_{pe}^2 + \omega_{pi}^2 + V_{se}^2 + V_{si}^2)k^2 + V_{se}^2 V_{si}^2 k^4. \tag{6.90}$$

This asymptotes to $\omega = \omega_p = \sqrt{\omega_{pe}^2 + \omega_{pi}^2}$ and $\omega = V_{se}k$ for small and large values of k, respectively. The LEP wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0[1+\delta' \sin(kx_1 - \omega t)], \tag{6.91}$$

$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\delta'\Big(\sin\big(kx_1 - \omega t\big),0,0\Big), \tag{6.92}$$

$$\rho_i(\mathbf{x},t) = m_i n_0\left[1 - \frac{\omega^2 - \omega_{pe}^2 - k^2V_{se}^2}{\omega_{pe}^2}\delta' \sin(kx_1 - \omega t)\right], \tag{6.93}$$

$$\mathbf{u}_i(\mathbf{x},t) = -\frac{\omega}{k}\frac{\omega^2 - \omega_{pe}^2 - k^2V_{se}^2}{\omega_{pe}^2}\delta'\Big(\sin\big(kx_1 - \omega t\big),0,0\Big), \tag{6.94}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{en_0}{\varepsilon_0 k}\frac{\omega^2 - k^2V_{se}^2}{\omega_{pe}^2}\delta'\Big(\cos(kx_1 - \omega t),0,0\Big), \tag{6.95}$$

$$\mathbf{B}(\mathbf{x},t) = \big(0,0,0\big), \tag{6.96}$$

where the scaled perturbation parameter is given by

$$\frac{\delta}{\delta'} = \max\left\{1, \left|\frac{\omega^2 - \omega_{pe}^2 - k^2V_{se}^2}{\omega_{pe}^2}\right|\right\}. \tag{6.97}$$

For the equivalent ion wave, the LIP, the dispersion relation is

$$\omega = \sqrt{\frac{A - \sqrt{A^2 - 4C}}{2}}, \tag{6.98}$$

with the same constants as for the LEP wave. This asymptotes to $\omega = V_{sp}k$ and $\omega = V_{si}k$, where $V_{sp} = \sqrt{V_{si}^2 + (m_e/m_i)^2V_{se}^2}$, for small and large values of k, respectively. The wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0\left[1 - \frac{\omega^2 - \omega_{pi}^2 - k^2V_{si}^2}{\omega_{pi}^2}\delta' \sin(kx_1 - \omega t)\right], \tag{6.99}$$

$$\mathbf{u}_e(\mathbf{x},t) = -\frac{\omega}{k}\frac{\omega^2 - \omega_{pi}^2 - k^2V_{si}^2}{\omega_{pi}^2}\delta'\Big(\sin\big(kx_1 - \omega t\big),0,0\Big), \tag{6.100}$$

$$\rho_i(\mathbf{x},t) = m_i n_0[1+\delta' \sin(kx_1 - \omega t)], \tag{6.101}$$

$$\mathbf{u}_i(\mathbf{x},t) = \frac{\omega}{k}\delta'\Big(\sin\big(kx_1 - \omega t\big),0,0\Big), \tag{6.102}$$

$$\mathbf{E}(\mathbf{x},t) = -\frac{en_0}{\varepsilon_0 k}\frac{\omega^2 - k^2V_{si}^2}{\omega_{pi}^2}\delta'\Big(\cos(kx_1 - \omega t),0,0\Big), \tag{6.103}$$

$$\mathbf{B}(\mathbf{x},t) = \big(0,0,0\big), \tag{6.104}$$

where

$$\frac{\delta}{\delta'} = \max\left\{1, \left|\frac{\omega^2 - \omega_{pi}^2 - k^2V_{si}^2}{\omega_{pi}^2}\right|\right\}. \tag{6.105}$$

The warm electron-ion TEM wave has dispersion relation

$$\omega = \sqrt{\omega_p^2 + c^2k^2}, \tag{6.106}$$

where $\omega_p = \sqrt{\omega_{pe}^2 + \omega_{pi}^2}$. This mode asymptotes to $\omega = \omega_p$ and $\omega = ck$ for small and large values of k, respectively. The wave mode solution is

$$\rho_e(\mathbf{x},t) = m_e n_0, \tag{6.107}$$
$$\mathbf{u}_e(\mathbf{x},t) = \frac{\omega}{k}\delta'\Big(0, \sin\left(kx_1 - \omega t\right), 0\Big), \tag{6.108}$$
$$\rho_i(\mathbf{x},t) = m_i n_0, \tag{6.109}$$
$$\mathbf{u}_i(\mathbf{x},t) = -\frac{\omega}{k}\frac{\omega^2 - \omega_{pe}^2 - k^2c^2}{\omega_{pe}^2}\delta'\Big(0, \sin\left(kx_1 - \omega t\right), 0\Big), \tag{6.110}$$
$$\mathbf{E}(\mathbf{x},t) = \frac{m_e\omega^2}{ek}\delta'\Big(\cos(kx_1 - \omega t), 0, 0\Big), \tag{6.111}$$
$$\mathbf{B}(\mathbf{x},t) = \frac{m_e\omega}{e}\delta'\left(0, 0, \cos\left(kx_1 - \omega t\right)\right), \tag{6.112}$$

where

$$\frac{\delta}{\delta'} = \max\left\{1, \left|\frac{\omega^2 - \omega_{pe}^2 - k^2c^2}{\omega_{pe}^2}\right|\right\}. \tag{6.113}$$

6.6. Warm Magnetized Electron-Ion Plasma Waves

The magnetized versions of the warm electron-ion plasma waves are significantly more complex than for the cold electron waves. As in the cold magnetized case, different waves propagate parallel to and normal to an imposed magnetic field. For $\mathbf{B}_0$ parallel to the wave vector, LEP, LIP, RCP, and LCP modes propagate, and these are shown in Figure 6-8. For $\mathbf{B}_0$ normal to the wave vector, LEP, LIP, transverse O, and partially transverse X waves propagate. Note that the LEP and LIP waves propagating normal in the $\mathbf{B}_0 = (0, B_0, 0)$ case are very different from those propagating along $\mathbf{B}_0$, while the O wave is the same as the TEM mode from the warm two-fluid unmagnetized case.

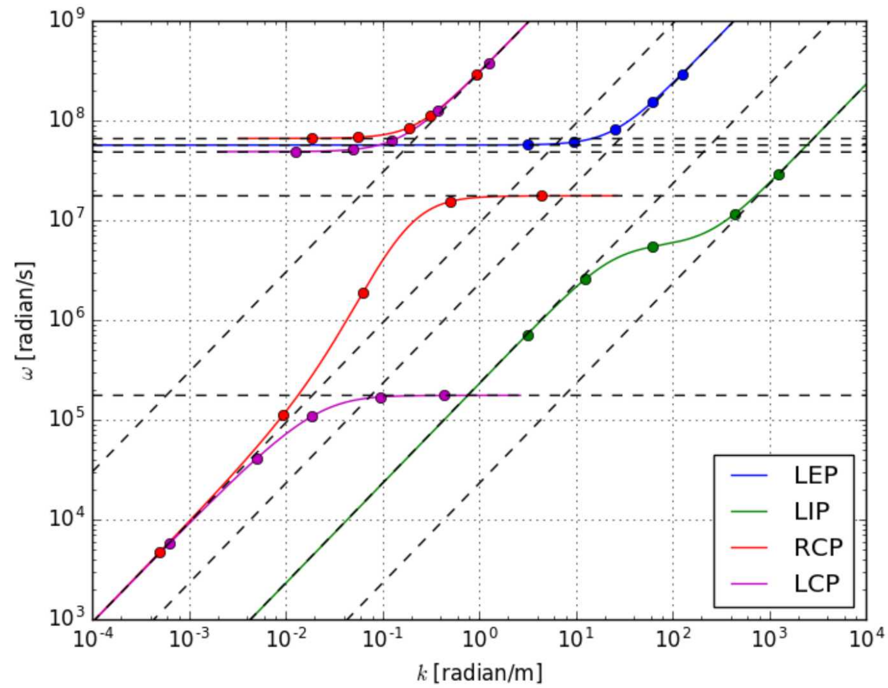

Figure 6-8. Dispersion plot for magnetized warm electron-ion plasma waves parallel to an applied magnetic field with $100m_e$ ions and $n_0 = 10^{12}\ \mathrm{m}^{-3}$.

The expressions for the dispersion relations in each of these cases are significantly more complex and tedious to extract, but can still useful for code verification since there are few other analytic solutions in this regime.

7. CONCLUSION

We have summarized in this report numerous details associated with a hierarchy of plasma models that are appropriate for modeling plasmas from very low to high density and varying levels of charge separation. The models include the multi-species Boltzmann equation and various moment models leading to useful fluid plasma approximations. Focusing on five-moment models, we discussed a three-fluid model, a two-fluid model, and an associated Generalized Ohm's Law model including additional possible simplifications. Each of the plasma fluid models describes a set of transport equations coupled to Maxwell's equations. These systems are primarily hyperbolic in nature, with complex dispersion relations associated with multiple propagating wave types. We demonstrated a linear-wave verification approach for verifying the dispersive wave characteristics of various forms of the two-fluid equations. A similar approach can be used to help verify the numerical implementation of any of the equation sets discussed in this report.

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