

Heterotic-F-theory duality with Wilson line symmetry-breaking

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ABSTRACT: We begin with an $E_8 \times E_8$ Heterotic model broken to an $SU(5)_{\text{gauge}}$ and a mirror $SU(5)_{\text{gauge}}$, where one $SU(5)$ and its spectrum is identified as the visible sector while the other can be identified as a hidden mirror world. In both cases we obtain the minimal supersymmetric standard model spectrum after Wilson-line symmetry-breaking enhanced by a low energy R-parity enforced by a local (or global) $U(1)_X$ -symmetry. Using Heterotic/ F -theory duality, we show how to eliminate the vector-like exotics which were obtained in previous constructions. In these constructions, the Calabi-Yau [CY] four-fold was defined by an elliptic fibration with section over a base B_3 and a GUT surface given by $K3/\mathbb{Z}_2 = \text{Enriques surface}$. In the present paper we construct a quotient CY four-fold fibered by tori with two elliptic structures given by a pair of sections fibered over the Enriques surface. Using Heterotic/ F -theory duality we are able to define the cohomologies used to derive the massless spectrum.

Our model for the ‘correct’ F -theory dual of a Heterotic model with Wilson-line symmetry-breaking builds on prior literature but employs the stack-theoretic version of the dictionary between the Heterotic semi-stable E_8 -bundles with Yang-Mills connection and the dP_9 -fibrations used to construct the F -theory dual.

KEYWORDS: Differential and Algebraic Geometry, F-Theory, Supersymmetric Gauge Theory

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1 Introduction

1.1 The physics

Supersymmetric grand unified theories [SUSY GUTs] [17, 18, 29] have many nice properties. These include an explanation of the family structure of quarks and leptons with the requisite charge assignments under the Standard Model [SM] gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and a prediction of gauge coupling unification at a scale of order 10^{16} GeV. The latter is so far the only direct hint for the possible observation of supersymmetric particles at the LHC. UV completions of SUSY GUTs in string theory also provide a consistent quantum mechanical description of gravity. As a result of this golden confluence, many groups have searched for SUSY GUTs in string theory. In fact, it has been shown that by demanding SUSY GUTs in string constructions one can find many models with features much like that of the minimal supersymmetric Standard Model [MSSM] [2, 3, 8, 32, 36–38].

The past several years have seen significant attention devoted to the study of supersymmetric GUTs in F -theory [5, 6, 9, 20, 21, 24, 52]. Both local and global $SU(5)$ F -theory GUTs have been constructed where $SU(5)$ is spontaneously broken to the SM via

non-flat hypercharge flux. One problem with this approach for GUT breaking is that large threshold corrections are generated at the GUT scale due to the non-vanishing hypercharge flux [9, 10, 21, 44]. An alternative approach to breaking the GUT group is using a Wilson line in the hypercharge direction, i.e. a so-called flat hypercharge line bundle. In this case it is known that large threshold corrections are not generated at the GUT scale (or, in fact, leading to precise gauge coupling unification at the compactification scale in orbifold GUTs) [33, 49] and [1, 27, 50, 53].

In a previous paper, the present authors and collaborators constructed a global $SU(5)$ F -theory model with Wilson line breaking [43]. The model contained the vector multiplets for the MSSM gauge group, 3 families of quarks and leptons, 4 pairs of Higgs doublets, and in addition, a vector-like pair of chiral multiplets in the representation $(\mathbf{3}, \mathbf{2})_{-5/6} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{+5/6}$. In terms of the model defined on an elliptically fibered CY 4-fold with GUT surface defined as an Enriques surface, K_3/\mathbb{Z}_2 , the massless spectrum is given in terms of cohomologies of the flux line bundle (or twists of the flux line bundle) on the GUT surface. It was then shown that the holomorphic Euler character of any flat bundle on S_{GUT} is equivalent to its Todd genus [6, 20, 43], through

$$c_1(L_Y) = 0 \quad \implies \quad \chi(S_2, L_Y) = \int_{S_{\text{GUT}}} \text{Td}(TS_{\text{GUT}}), \quad (1.1)$$

so we have that

$$\chi(S_{\text{GUT}}, L_Y) = h^0(S_{\text{GUT}}, L_Y) - h^1(S_{\text{GUT}}, L_Y) + h^2(S_{\text{GUT}}, L_Y) = 1. \quad (1.2)$$

Since all $h^m(S_{\text{GUT}}, L_Y)$ cannot be vanishing, we are guaranteed to get some massless vector or chiral states, $(\mathbf{3}, \mathbf{2})_{-5/6}$'s and $(\bar{\mathbf{3}}, \mathbf{2})_{+5/6}$'s. We emphasize that the presence of some kind of vector-like exotic matter is not a specific issue with this Enriques model but rather a general property of any model that breaks $SU(5)_{\text{GUT}} \rightarrow SU(3) \times SU(2) \times U(1)_Y$ with a flat $U(1)_Y$ bundle on a holomorphic surface S_{GUT} . We note that this derivation is, however, only valid if the F -theory is compactified on a CY 4-fold with section. In the present paper we show how to eliminate the vector-like exotics and evade this theorem.

We need to understand how to modify the description of the exotics (and the matter content) in a situation where there is no single distinguished section, or, more precisely, where the torus fibration has two sections ‘on equal footing.’ The prescription we use gives exactly such a description. The purpose of this paper is to present a model for Heterotic/ F -theory duality in which $SU(5)$ symmetry is broken (on both sides) by Wilson lines. The work derives from the previous global F -theory model with Wilson line symmetry-breaking [43]. It modifies the previous model so as to allow the construction of a Heterotic dual. It adapts previously known Heterotic techniques for eliminating undesirable features of the model, such as vector-like exotics, by constructing the torus-fibration (on both sides) with two sections [22]. This allows us to use the Heterotic technique of translation by the difference of the two sections to form the requisite \mathbb{Z}_2 -action in order to evade the above-mentioned theorem and eliminate the vector-like exotics. The model has some very nice features. It contains the gauge group $SU(5)$ broken via a Wilson line in the hypercharge direction to $SU(3) \times SU(2) \times U(1)$. It contains three families of quarks

and leptons and one pair of Higgs doublets. Furthermore our F -theory model exhibits a \mathbb{Z}_4 \mathbf{R} -symmetry in the semi-stable limit of the F -theory model.

In addition it has a local $U(1)_X$ gauge symmetry, given by a global $(4 + 1)$ split spectral cover, where $U(1)_X$ is the $U(1)$ in $SO(10)$ commuting with $SU(5)_{\text{gauge}}$; Higgs pairs have charge, ± 2 , while $SU(5)$ $\mathbf{10}$'s and $\bar{\mathbf{5}}$'s have charge, $+1$ and -3 , respectively. This gauge symmetry preserves R -parity, eliminating baryon and lepton number violating dimension-4 operators [24]. $U(1)_X$ can also be used to identify possible right-handed neutrino states which are $SU(5)$ singlet matter states with $U(1)_X$ charge $+5$ [9, 52].

Some problems for this construction are as follows. The gauge symmetry $U(1)_X$ also forbids Majorana masses for the right-handed neutrinos. In addition, in subsection 8.7 we calculate the D -term for $U(1)_X$ (see eq. 3.30 in [24, 25]), requiring $D_{U(1)_X} = 0$ so that this symmetry is not spontaneously broken by fields derived from the adjoint representation of E_8 . In addition, $U(1)_X$ is not sufficient to prevent dimension-5 baryon and lepton number violating operators. It is possible that the local $U(1)_X$ symmetry is broken down to a global $U(1)_X$ symmetry via a Stueckelberg mechanism [24, 52], but this is beyond the scope of the present paper. The $U(1)_X$ symmetry may also be broken to a \mathbb{Z}_2 matter parity by a non-perturbative effect at the GUT scale. This would then allow for right-handed neutrino Majorana masses at the GUT scale.

Finally, a very novel feature of the model is that it contains a twin/mirror $SU(5)$ symmetry broken to a mirror SM with three families of mirror quarks and leptons and a pair of mirror Higgs multiplets.¹ This is a direct consequence of the fact that the GUT surface, $S_{\text{GUT}}^\vee = \text{Enriques}$, is a branched (therefore irreducible) double cover of the base B_2 . The gauge and Yukawa couplings in the visible and mirror worlds are determined by volume moduli which must still be stabilized and supersymmetry broken. As a result mirror matter does not necessarily have the same mass as visible matter or the same value of their gauge and Yukawa couplings. This mirror sector is a possible candidate for the dark matter in the universe.

1.2 The mathematics

We next give an idea of the mathematics of our model for Heterotic/ F -theory duality in which $SU(5)$ -symmetry is broken (on both sides) by a Wilson line construction. As mentioned, the mathematical model derives from a previous global F -theory model with Wilson line symmetry-breaking [43]. It modifies the previous model so as to allow the construction of a Heterotic dual. It also adapts previously known Heterotic techniques for eliminating undesirable features of the model, such as vector-like exotics, by constructing the torus-fibration (on both sides) that admits two sections. More properly, the two sections, taken together, should be thought of as determining an invariant \mathfrak{g}_2^1 linear series on the torus fiber.

¹Mirror world defined as the parity transform of the Standard Model has been reviewed in the paper by L.B. Okun [48]. This paper has many references which we refer to the reader. Some of these references however include another related definition of the mirror world given by a generalized \mathbb{Z}_2 -symmetry which takes the Standard Model into the twin or mirror sector with states having identical charges but different masses and couplings. Papers in this genre include, for example, [7, 13], and [4].

Our goal for this paper is to present a global phenomenologically consistent Heterotic model V_3^\vee/B_2^\vee and F -theory dual W_4^\vee/B_3^\vee . These will be constructed as equivariant \mathbb{Z}_2 -quotients of another set of dual Heterotic/ F -theory models V_3/B_2 and W_4/B_3 with respective involutions

$$\tilde{\beta}_3/\beta_2 : V_3/B_2 \rightarrow V_3/B_2 \tag{1.3}$$

and

$$\tilde{\beta}_4/\beta_3 : W_4/B_3 \rightarrow W_4/B_3 \tag{1.4}$$

whose quotients V_3^\vee/B_2^\vee and W_4^\vee/B_3^\vee are our ultimate goal. However much of the work will center on V_3/B_2 and W_4/B_3 and their respective involutions, passing to the quotients only late in the story. Furthermore much of the work has already been completed in our companion papers [14, 15], including in [15] the complete construction of

$$B_3 = \mathbb{P}^1 \times B_2$$

and the involution β_3 and calculation of their numerical invariants. The groundwork contained in those papers will be referred to as needed in what follows.

In short, the fundamental challenge is to construct compatible involutions (1.3) and (1.4) so that both leave their top-degree holomorphic forms invariant, that is, so that their respective quotients are Calabi-Yau manifolds. Because of other necessary characteristics of the Heterotic and F -theory models, we showed in [14] that (1.3) must act as

$$\frac{dx}{y} \mapsto \frac{dx}{y} \tag{1.5}$$

on the relative (Weierstrass) one-form on V_3/B_2 whereas (1.4) must act as

$$\frac{dx}{y} \mapsto -\frac{dx}{y} \tag{1.6}$$

on the relative (Weierstrass) one-form on W_4/B_3 . We showed this necessity on the F -theory side in [14] by tracing the Tate form

$$wy^2 = x^3 + a_5wxy + a_4zwx^2 + a_3z^2w^2y + a_2z^3w^2x + a_0z^5w^3 \tag{1.7}$$

back to its E_8 -origins, namely

$$wy^2 = x^3 + a_0z^5. \tag{1.8}$$

We showed in [15] that the coefficients a_j of the Tate form, as well as z and y/x must go to minus themselves under the \mathbb{Z}_2 -action. One identifies the configuration of exceptional curves in the crepant resolution of (1.8) with the configuration of the positive simple roots in the E_8 -Dynkin diagram. A consequence of this identification is that (1.6) sends each positive simple root to its negative. We preserve E_8 -symmetry by counteracting this reversal of roots by the operation of complex conjugation on the complex algebraic group $E_8^{\mathbb{C}}$, an operation that leaves untouched the compact real form E_8 . This last is reflected in the fact that tracing real roots back to E_8 requires that (1.7) and (1.8) be rescaled by dividing both sides by a_0^6 . This rescales both z and y/x by a_0^{-1} and y/a_0x indeed becomes invariant under the \mathbb{Z}_2 -action.

1.3 The organization of the paper

In section 2 we derive the Weierstrass equation for the Tate form. We introduce the construction of a second section τ of W_4/B_3 in addition to the tautological section ζ ‘at infinity.’ We discuss the action of the involution $\tilde{\beta}_4/\beta_3$ as reflecting complex conjugation on the complex algebraic groups $SL(5, \mathbb{C})$ and $E_8^{\mathbb{C}}$ whose compact real forms are $SU(5)$ and E_8 respectively.

In section 3 we derive the spectral variety from the Tate form and discuss its decomposition into components of degree four and one respectively.

In sections 4 and 5 we discuss the semi-stable limit and the relation to the Heterotic dual. In particular we show how to build the normal-crossing $K3$ from an elliptic curve with two flat E -bundles.

In section 6 we project the fourfold W_4/B_3 to the \mathbb{P}^1 -bundle Q/B_3 whose fiber is the degree-2 linear series determined by the two points $\tau(b_3)$ and $\zeta(b_3)$ where the two sections intersect the fiber. We do this by projecting each fiber of W_4/B_3 from the third point of intersection with the line joining $\tau(b_3)$ and $\zeta(b_3)$. This projection, that we name \overline{W}_4 blows up the singular locus of W_4 . In this way we obtain a commutative diagram

$$\begin{array}{ccc} \overline{W}_4 & \rightarrow & W_4 \\ \downarrow & & \downarrow \\ Q & \rightarrow & B_3 \end{array}$$

where the top horizontal map is crepant partial resolution, the left-hand vertical map is 2–1 and the bottom horizontal map is a \mathbb{P}^1 -fibration whose fibers correspond to the degree-2 linear system on the fibers of the right-hand vertical map determined by the lifts of the two sections (τ) and (ζ) to \overline{W}_4/B_3 . The Calabi-Yau fourfold \overline{W}_4 is a branched double cover of Q . The affine fiber coordinate ϑ_0 of Q/B_3 is a section of $K_{B_3}^{-1}$ as are the coefficients a_j in the Tate form and $t := y/x$.

We next construct a full crepant resolution of \overline{W}_4 to obtain a smooth model \tilde{W}_4 . This is accomplished in section 7. The first step is the partial resolution $W_4^{(1)}/B_3$ of \overline{W}_4/B_3 that will create an divisor D_0 connecting the divisors D_1 (the inherited component from W_4/B_3) and D_4 (the exceptional divisor of \overline{W}_4/W_4). Together these three divisors comprise

$$S_{\text{GUT}} \times_{B_3} W_4^{(1)}.$$

This will be followed by a third partial desingularization $W_4^{(2)}/B_3$ of $W_4^{(1)}/B_3$ extending over a general point of S_{GUT} . Its exceptional divisor will be reducible so that

$$S_{\text{GUT}} \times_{B_3} W_4^{(2)} = D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4$$

that configure themselves over a general point of S_{GUT} as an extended A_4 -Dynkin diagram. That will in turn be followed by a final codimension-2 desingularization \tilde{W}_4/B_3 of $W_4^{(2)}/B_3$ over the curve $\Sigma_{\mathbb{5}}^{(44)} \subseteq S_{\text{GUT}}$.

It should be noted that, in the process of putting (ζ) and (τ) on equal footing as the first step in the desingularization, neither can be given preference as the one passing

through the inherited component

$$S_{\text{GUT}} \times_{B_3} \overline{W}_4.$$

The ‘inherited’ role is assumed by D_0 while, over a general point of S_{GUT} , the proper transform $(\tilde{\zeta})$ of (ζ) intersects D_1 and the proper transform $(\tilde{\tau})$ of (τ) intersects D_4 .

In section 8 we calculate the Higgs line bundle that will govern the computation of the chiral spectrum. In subsections 8.5 and 8.6 we compute the G -flux. In subsection 8.7 we establish the vanishing of the D -term for a suitably chosen $\tilde{\beta}$ -symmetric Kähler metric.

In section 9 we discuss the symmetry-breaking induced by wrapping the Wilson line on the involution $\tilde{\beta}_4$ on \tilde{W}_4 . The Wilson line breaks $SU(5)$ -symmetry to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry of the Standard Model.

It is in section 10 that we calculate the complete spectrum of the theory. The desired invariants follow rather directly from the results of [15] where the toric presentation of B_3 and its involution β_3 were explored in detail. The present paper demonstrates a method for eliminating vector-like exotics when breaking the GUT symmetry with the Wilson line. In this section we discuss just one $SU(5)_{\text{gauge}} \times SU(5)_{\text{Higgs}}$ sector in the semi-stable limit of the F -theory model, however, the massless spectrum in the hidden $SU(5)_{\text{gauge}} \times SU(5)_{\text{Higgs}}$ is identical.

Finally in section 11 we discuss the \mathbb{Z}_4 R-symmetry on the semi-stable limit of our F -theory model.

Remark 1. Throughout this paper, we will employ the following notational convention. Projections with image space A will in general be denoted as π_A . This notation will be employed regardless of the domain of the map, which (hopefully) will be clear from the context.

Remark 2. Throughout this paper, we will let

$$\mathbb{P}_{[i_1, \dots, i_d]}^{d-1}$$

denote the weighted complex projective $(d - 1)$ -space with integer weights $[i_1, \dots, i_d]$ and we will let

$$\mathbb{P}_{[u_1, \dots, u_d]}$$

denote the (unweighted) complex projective space with homogeneous coordinates $[u_1, \dots, u_d]$.

2 W_4/B_3 , E_8 -unfolding and its symmetries

Our starting point is the Fano threefold

$$B_3 = \mathbb{P}_{[u_0, v_0]} \times B_2$$

where B_2 is the D_2 del Pezzo surface studied in section 6 of [15]. B_2 is a double cover of the projective plane

$$D_2 \rightarrow \mathbb{P}_{[n-1, m_i, m_{-i}]} \tag{2.1}$$

branched along a specific smooth quartic curve admitting a \mathbb{Z}_4 -action. Denote

$$N := c_1 \left(K_{B_3}^{-1} \right).$$

We form

$$P := \mathbb{P} \left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3} (2N) \oplus \mathcal{O}_{B_3} (3N) \right) \tag{2.2}$$

with homogeneous fiber coordinates $[w, x, y]$ and canonical bundle

$$\mathcal{O}_P (-3) \otimes \pi_{B_3}^* (-6N).$$

For

$$a_j, z, \frac{y}{x} = t \in H^0 \left(\mathcal{O}_{B_3} (N) \right), \tag{2.3}$$

we write the Tate form (1.7) for an elliptically fibered fourfold W_4/B_3 in determinantal form as

$$\begin{vmatrix} x^3 + a_4 z w x^2 + a_2 z^3 w^2 x + a_0 z^5 w^3 & 1 \\ w y^2 - (a_5 w x + a_3 z^2 w^2) y & 1 \end{vmatrix} = 0. \tag{2.4}$$

Since

$$K_P^{-1} = \mathcal{O}_P (3 (\{w = 0\} + 6N)),$$

W_4 is Calabi-Yau. Here as in [15] B_3 admits an involution β_3 with finite fixpoint set with respect to which

$$t := \frac{y}{x}, z, a_0, a_2, a_3, a_4, a_5 \in H^0 \left(\mathcal{O}_{B_3} (N) \right)^{[-1]},$$

the (-1) -eigenspace, and

$$\begin{aligned} w &\in H^0 \left(\mathcal{O}_P (1) \otimes \pi_{B_3}^* \mathcal{O}_{B_3} \right)^{[+1]} \\ x &\in H^0 \left(\mathcal{O}_P (1) \otimes \pi_{B_3}^* \mathcal{O}_{B_3} (2N) \right)^{[+1]} \\ y &\in H^0 \left(\mathcal{O}_P (1) \otimes \pi_{B_3}^* \mathcal{O}_{B_3} (3N) \right)^{[-1]}. \end{aligned}$$

with respect to the induced involution on W_4/B_3 . The equation $z = 0$ defines a smooth surface that, by the adjunction formula, must be a $K3$ -surface.

2.1 Weierstrass form with respect to ζ

Let

$$\zeta : B_3 \rightarrow W_4$$

be the standard section given by

$$\zeta (b_3) = \{ [w, x, y] = [0, 0, 1] \}.$$

Referring to (2.4) we change the equation of W_4/B_3 into Weierstrass form based at the section ζ in the standard way. Namely we complete the square with respect to y as follows.

$$\begin{aligned}
 wy^2 &= x^3 + a_5 wxy + a_4 zwx^2 + a_3 z^2 w^2 y + a_2 z^3 w^2 x + a_0 z^5 w^3 \\
 w \left(y^2 - (a_5 x + a_3 z^2 w) y + \frac{(a_5 x + a_3 z^2 w)^2}{4} \right) &= \\
 x^3 + a_4 zwx^2 + a_2 z^3 w^2 x + a_0 z^5 w^3 + w \frac{(a_5 x + a_3 z^2 w)^2}{4} & \\
 w \left(y - \frac{a_5 x + a_3 z^2 w}{2} \right)^2 &= x^3 + \left(a_4 z + \frac{a_5^2}{4} \right) wx^2 + z^2 \left(a_2 z + \frac{a_3 a_5}{2} \right) w^2 x + z^4 \left(a_0 z + \frac{a_3^2}{4} \right) w^3 \\
 &= x^3 + Awx^2 + Bz^2 w^2 x + Cz^4 w^3
 \end{aligned}$$

where

$$\begin{aligned}
 A &= a_4 z + \frac{a_5^2}{4} \\
 B &= a_2 z + \frac{a_3 a_5}{2} \\
 C &= a_0 z + \frac{a_3^2}{4}.
 \end{aligned}$$

Then we eliminate the x^2 -term by setting

$$\begin{aligned}
 \underline{x} &= x + \frac{a_5^2 + 4a_4 z}{12} w = x + \frac{A}{3} w \\
 \underline{y} &= y - \frac{a_5 x + a_3 z^2 w}{2}
 \end{aligned} \tag{2.5}$$

finally yielding the Weierstrass form

$$w\underline{y}^2 = \underline{x}^3 + \left(Bz^2 - \frac{A^2}{3} \right) w^2 \underline{x} + \left(Cz^4 - \frac{AB}{3} z^2 + \frac{2A^3}{27} \right) w^3. \tag{2.6}$$

for our Calabi-Yau fourfold W_4 .

The discriminant of (2.6) is given by

$$\begin{aligned}
 4 \left(Bz^2 - \frac{A^2}{3} \right)^3 + 27 \left(Cz^4 - \frac{AB}{3} z^2 + \frac{2A^3}{27} \right)^2 &= \\
 A^2 (4AC - B^2) z^4 + 2B (2B^2 - 9AC) z^6 + 27C^2 z^8. &
 \end{aligned}$$

Expanding the discriminant in powers of z the coefficient of z^4 becomes $\left(\frac{a_5^2}{4}\right)^2 \left(\frac{a_3^2 a_5^2}{4} - \left(\frac{a_3 a_5}{2}\right)^2\right) = 0$ and the coefficient of z^5 does not in general equal zero so that indeed, by Kodaira's classification, we have A_4 -singularities over $S_{\text{GUT}} = \{z = 0\}$.

2.2 The second section

Beside the standard section $\zeta : B_3 \rightarrow W_4$ we require a second section

$$\tau : B_3 \rightarrow W_4$$

defined by

$$b_3 \mapsto [w, x, y] = [1, z^2, z^3].$$

Substituting this section of P/B_3 into Tate form (1.7), as in [14, 15] one concludes that the condition that it lies in W_4 is

$$a_5 + a_4 + a_3 + a_2 + a_0 = 0. \tag{2.7}$$

This section allows a change of the group structure on the fibers of W_4/B_3 by a translation. Intertwining this translation with the action of $\tilde{\beta}_4$ allows us both to eliminate vector-like exotics from the F -theory model W_4^\vee/B_3^\vee and to introduce a $(4+1)$ -split in the spectral divisor giving the $U(1)_X$ discussed in the Introduction. Translation of fibers by this section of course leaves the Weierstrass form on smooth elliptic fibers and I_1 -fibers invariant, that is, all fibers over $(B_3 - S_{\text{GUT}})$.

Notice that, over a general point $b_3 \in B_3$, $\tau - \zeta$ is not of finite order on Pic^0 of the cuspidal curve $wy^2 = x^3$ since the parameter $t = \frac{y}{x} = z$ takes all values. So $\tau - \zeta$ is not of finite order in $\text{Pic}^0(\pi^{-1}(b_3))$ if the a_j are sufficiently small. Furthermore, if $a_5 = -a_0$ are small and $a_2 = a_3 = a_4 = 0$, the same argument gives

Lemma 1. *For a crepant resolution \tilde{W}_4/B_3 of W_4/B_3 , $\tau - \zeta$ is not of finite order in $\text{Pic}^0(\tilde{W}_4/B_3)$ over a general point of $S_{\text{GUT}} := \{z = 0\} \subseteq B_3$.*

So these same assertions hold for a general allowable choice of the coefficients a_j . This fact is essential to the proof in [15] of lemma 6ii) below. It says that our F -theory model has a single Higgs doublet.

Over $b_3 \in B_3$ the line in $\pi^{-1}(b_3)$ between $\zeta(b_3)$ and $\tau(b_3)$ is given by the equation $x - z^2w = 0$. Letting $a_{jkl} := a_j + a_k + a_l$ and using that $a_{420} = -a_{53}$ we obtain by (2.7) that the third point of intersection with W_4 is given by substituting $x = z^2w$ in (2.4) to obtain

$$w \begin{vmatrix} z^6w^2 + a_{420}z^5w^2 & 1 \\ y^2 + a_{420}z^2wy & 1 \end{vmatrix} = 0,$$

that is,

$$y^2 + a_{420}z^2wy - z^5w^2(z + a_{420}) = (y - z^3w)(y + (z + a_{420})z^2w) = 0.$$

We denote the third section as

$$v(b_3) = [w, z^2w, (-z - a_{420})z^2w].$$

Finally, we therefore have a fourth section

$$\mu : B_3 \rightarrow W_4$$

defined by the third point of intersection of the tangent line to

$$\begin{vmatrix} x^3 + a_4zwx^2 + a_2z^3w^2x + a_0z^5w^3 & 1 \\ wy^2 - (a_5wx + a_3z^2w^2)y & 1 \end{vmatrix} = 0 \tag{2.8}$$

at $v(b_3)$.

Taken together these calculations yield the following.

Lemma 2. *The sections ζ , τ , v and μ of W_4/B_3 satisfy the following relations in $\text{Pic}^3(W_4/B_3)$:*

$$\begin{aligned} 3 \cdot \zeta &\in |\mathcal{O}_{W_4/B_3}(1)| \\ \zeta + \tau + v &\in |\mathcal{O}_{W_4/B_3}(1)| \\ 2v + \mu &\in |\mathcal{O}_{W_4/B_3}(1)|. \end{aligned}$$

The last two relations imply that

$$\zeta + \tau \equiv v + \mu,$$

that is in classical language, the two divisors are members of the same distinguished g_2^1 on W_4/B_3 .

2.3 The quotient Calabi-Yau manifolds

In order to wrap a Wilson line, we must require that involution $\tilde{\beta}_4/\beta_3$ in (1.4) with quotient W_4^\vee/B_3^\vee be such that β_3 acts fixpoint-freely on the smooth anti-canonical divisor $S_{\text{GUT}} \subseteq B_3$ yielding an Enriques surface

$$S_{\text{GUT}}^\vee \subseteq B_3^\vee$$

and that the induced involution $\tilde{\beta}_3/\beta_2$ on the dual Heterotic with quotient V_3^\vee/B_2^\vee be such that $\tilde{\beta}_3$ acts freely. In order that W_4^\vee be Calabi-Yau, the involution β_3 on $B_3 = \mathbb{P}_{[u_0, v_0]} \times B_2$ must have only finite fixpoint-set since S_{GUT} is an ample divisor in the Fano manifold B_3 . This forces β_3 to act as (-1) on the meromorphic two-form on B_3 with pole on S_{GUT} . That in turn forces $\tilde{\beta}_4$ with quotient W_4^\vee/B_3^\vee to act as (-1) on the relative one-form dx/y on W_4/B_3 since otherwise W_4^\vee would not be Calabi-Yau. On the other hand, the induced involution $\tilde{\beta}_3/\beta_2$ on the Heterotic threefold V_3/B_2 must act as $(+1)$ on the relative one-form dx/y since otherwise V_3^\vee would not be Calabi-Yau. The possibility, even necessity, of the sign-reversal is explained in [14].

2.4 Three Calabi-Yau fourfolds related by quotienting

Following [15] we use root systems and toric geometry to actually define three base threefolds and associated Calabi-Yau fourfolds that we denote by

$$\begin{aligned} W_4^\wedge &\rightarrow B_3^\wedge \\ W_4 &\rightarrow B_3 = \mathbb{P}_{[u_0, v_0]} \times B_2 \\ W_4^\vee &\rightarrow B_3^\vee := B_3 / \{C_{u,v}\}. \end{aligned}$$

B_3^\wedge is defined to be the resolution of the graph of the Cremona involution on $\mathbb{P}(\mathfrak{h}_{\text{SU}(5)})$ with respect to a basis given by the choice of a system of simple roots of $\text{SU}(5)$ balanced between positive and negative Weyl chambers. We have reserved the least cumbersome notation for the intermediate one B_3 because, as we have already mentioned, it is computationally most convenient to work in that setting.

On the F -theory side our ultimate target is the (orbifold) Calabi-Yau fourfold W_4^\vee with smooth Heterotic dual Calabi-Yau threefold V_3^\vee having two bundles with Yang-Mills connections with structure group symmetry-breaking

$$E_8 \xrightarrow{\text{Tate}} \text{SU}(5)_{\text{gauge}} \times \text{SU}(5)_{\text{Higgs}} \xrightarrow{\text{Higgs}} \text{SU}(5)_{\text{gauge}} \xrightarrow{\text{Wilson-line}} \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y,$$

as described in [15]. Notice that the initial fourfolds and the involutions do not involve any choice of Weyl chamber. It is only the crepant resolutions of the Calabi-Yau fourfolds and threefolds that imply such choices. This is explained in [14].

In order to wrap a Wilson line, we require that W_4/B_3 given in (1.7) admit an equivariant involution $\tilde{\beta}_4/\beta_3$ with quotient W_4^\vee/B_3^\vee such that β_3 acts fixpoint-free on the smooth anti-canonical divisor $S_{\text{GUT}} \subseteq B_3$ yielding an Enriques surface

$$S_{\text{GUT}}^\vee \subseteq B_3^\vee.$$

In order that W_4^\vee be Calabi-Yau, the involution β_3 on $B_3 = \mathbb{P}_{[u_0, v_0]} \times B_2$ must have only finite fixpoint-set since S_{GUT} is an ample divisor in the Fano manifold B_3 . As explained in [14] this forces β_3 to act as (-1) on the meromorphic two-form on B_3 with pole on S_{GUT} .

2.5 Unfolding the E_8 -singularity

A basic principle in the mathematics of String Theory is that the geometric model (1.7) of F -theory must be considered as having evolved according to the unfolding of the E_8 -surface singularity

$$wy^2 = x^3 + a_0z^5.$$

In [14] we have observed that principle to the letter, tracing the equivariant crepant resolution implicit in the Tate form (1.7) back to the Brieskorn-Grothendieck equivariant crepant resolution [11, 51] of the semi-universal deformation of the rational double point singularity (1.8) by requiring that the section defining S_{GUT} be given by a formula

$$z = \sum_{j=2}^5 \kappa_j a_j$$

for generic κ_j .

The assumption (2.7) will, as we will see, reduce the maximal subgroup decomposition

$$\frac{\text{SU}(5)_{\text{gauge}} \times \text{SU}(5)_{\text{Higgs}}}{\mathbb{Z}_5} \subseteq E_8$$

of [14] to

$$\text{SU}(5)_{\text{gauge}} \times \text{U}(1)_X \times \text{SU}(4)_{\text{Higgs}} \subseteq E_8$$

so that, on the F -theory side, one begins with the identification of maximal tori compatible with the three-dimensional commutative diagram obtained by pasting the top and bottom morphisms of

$$\begin{array}{ccc}
 \dot{\text{S}}\text{L}(5; \mathbb{C})_{\text{gauge}} \times \dot{\text{S}}\text{L}(4; \mathbb{C})_{\text{Higgs}} \times \mathbb{C}^* & \xrightarrow{\dot{\kappa}} & \dot{E}_8^{\mathbb{C}} \\
 & \uparrow & \uparrow \\
 \text{SU}(5)_{\text{gauge}} \times \text{SU}(4)_{\text{Higgs}} \times \text{U}(1)_X & \hookrightarrow & E_8 \\
 & \downarrow & \downarrow \\
 \ddot{\text{S}}\text{L}(5; \mathbb{C})_{\text{gauge}} \times \ddot{\text{S}}\text{L}(4; \mathbb{C})_{\text{Higgs}} \times \mathbb{C}^* & \xrightarrow{\ddot{\kappa}} & \ddot{E}_8^{\mathbb{C}}
 \end{array} \tag{2.9}$$

to the top and bottom morphisms, respectively, of the commuting diagram

$$\begin{array}{ccc}
 \mathrm{SL}(5; \mathbb{C})_{\mathrm{gauge}} \times \mathrm{SL}(4; \mathbb{C})_{\mathrm{Higgs}} \times \mathbb{C}^* & \xrightarrow{\tilde{\kappa}} & \dot{E}_8^{\mathbb{C}} \\
 & \downarrow \iota & \downarrow \iota \\
 \ddot{\mathrm{SL}}(5; \mathbb{C})_{\mathrm{gauge}} \times \ddot{\mathrm{SL}}(4; \mathbb{C})_{\mathrm{Higgs}} \times \mathbb{C}^* & \xrightarrow{\ddot{\kappa}} & \ddot{E}_8^{\mathbb{C}}
 \end{array} \tag{2.10}$$

of real analytic outer complex conjugation involutions. [14] and [15] are built around the necessity of a choice of a positive-negative pair of Weyl chambers of E_8 with the requirement that every step of constructions must commute with the passage between these two chambers. In particular, as we have shown in [14], the equivariant crepant resolution \tilde{W}_4/B_3 of W_4/B_3 depends on the choice of Weyl chamber.

This means that we will have two copies of the crepantly resolved \tilde{W}_4/B_3 that we designate by letting \dot{W}_4/B_3 denote the F -theory model with a choice of positive chamber and \ddot{W}_4/B_3 with its negative as the choice of positive Weyl chamber. The action of $\tilde{\beta}_4/\beta_3$ on \dot{W}_4/B_3 will be a holomorphic involution that acts on roots as the longest element of the Weyl group $W \mathrm{SU}(5)$ yielding a Calabi-Yau quotient, and similarly for the quotient of the action on \ddot{W}_4/B_3 . However these quotients are only real-analytically equivalent, not complex-analytically equivalent. As explained in [15] the exceptional curves over the quotient S_{GUT}^{\vee} are ‘flopped’ when passing from one to the other. The flop is essentially invisible on the Heterotic side since it tracks only the real E_8 -bundles and the ‘flop’ becomes the passage between the two possible complexifications of the same real E_8 -bundle.

Identifying exceptional components over S_{GUT} with positive simple roots forces the involution $\tilde{\beta}_4$ to act as the non-trivial involution on the A_4 -Dynkin diagram, that is, by the longest element of the Weyl group $W \mathrm{SU}(5)$ on the exceptional divisors of the crepant resolution of W_4/B_3 . Again as shown in [14], it is the commutativity of the geometric involutions $\tilde{\beta}_4/\beta_3$ and $\tilde{\beta}_3/\beta_2$ with the complex conjugate involution ι in (2.10) that allows us to incorporate both in the simultaneous quotienting on both the Heterotic and F -theory models that preserves initial E_8 -symmetry and subsequent $\mathrm{SU}(5)$ -symmetry since (2.10) acts trivially on $\mathrm{SU}(5)$ and E_8 .

3 The spectral divisor

The crepant resolution \tilde{W}_4/B_3 of W_4/B_3 will have I_5 -type fibers over generic points of

$$S_{\mathrm{GUT}} := \{z = 0\} \subseteq B_3.$$

This I_5 -fibration over S_{GUT} carries the $\mathrm{SU}(5)_{\mathrm{gauge}}$ -symmetry. On the other hand, $\mathrm{SU}(5)_{\mathrm{Higgs}}$ -symmetry is broken on a five-sheeted branched covering of B_3 given by the lift of

$$\mathcal{C}_{\mathrm{Higgs}} := W_4 \cdot (\{wy^2 = x^3\} - \{w^4 = 0\}) \tag{3.1}$$

to a divisor $\tilde{\mathcal{C}}_{\mathrm{Higgs}} \subseteq \tilde{W}_4$. Its symmetry is broken by assigning non-trivial eigenvalues to the fundamental representation $\mathrm{SU}(5)_{\mathrm{Higgs}}$ using the spectral construction with respect to the push-forward to B_3 of a line bundle $\mathcal{L}_{\mathrm{Higgs}}$ on $\tilde{\mathcal{C}}_{\mathrm{Higgs}}$. We see this as follows.

We form

$$\hat{P} := \mathbb{P}(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(N) \oplus \mathcal{O}_{B_3}(2N) \oplus \mathcal{O}_{B_3}(3N))$$

using the fiber coordinate t for $\mathcal{O}_{B_3}(N)$. The natural projection

$$\hat{P} \dashrightarrow P \tag{3.2}$$

with center $\mathbb{P}(\mathcal{O}_{B_3}(N))$ is defined except along the section where $x = y = w = 0$. \hat{P} contains the smooth five-dimensional incidence hypersurface Y given by the equation

$$\begin{vmatrix} x & y \\ w & t \end{vmatrix} = 0,$$

thereby forming a smooth quadric hypersurface over B_3 with distinguished section given by $x = y = w = 0$. So the restriction of (3.2) to Y is defined except along the section where it spreads the exceptional locus over the linear locus $\{x = 0\}$ in P . The result is a birational morphism

$$\hat{Y} \rightarrow P$$

that maps the exceptional locus over the section isomorphically onto $\{x = 0\} \subseteq P$ and blows down the linear loci $\{x = y = 0\} \subseteq Y$ and $\{x = w = 0\} \subseteq Y$.

Ignoring what happens over $\{w = 0\}$, that is, setting $w = 1$ and using t as the affine fiber we obtain the equation

$$t^2x^2 = x^3 + a_5x^2t + a_4zx^2 + a_3z^2xt + a_2z^3x + a_0z^5$$

as the defining equation for the hypersurface $\hat{W}_4 - \{w = 0\} \subseteq \hat{Y} - \{w = 0\}$. Using

$$\begin{aligned} x &= wt^2 \\ y &= wt^3 \end{aligned}$$

the divisor given by the intersection with \hat{W}_4 then has equation

$$a_5t^5 + a_4t^4z + a_3t^3z^2 + a_2t^2z^3 + a_0z^5 = 0. \tag{3.3}$$

This is the equation on the affine set $w = 1$ of the intersection of the proper transform of W_4 with the locus given by

$$\{y^2 = x^3\} \tag{3.4}$$

and is called the *spectral divisor*. The spectral divisor, in particular, contains the singular locus of W_4 . The condition (2.7) implies that homogeneous form in (3.3) is divisible by $z - t$, that is, the spectral divisor admits a $(4 + 1)$ factorization. The involution $\tilde{\beta}_4/\beta_3$ takes

$$(z - t) \mapsto (t - z)$$

and leaves (3.3) invariant.

Said otherwise, since W_4 is smooth except over $\{z = 0\} \subseteq B_3$, the proper transform $\hat{W}_4 \subseteq \hat{Y}$ of W_4 blows up the codimension-2 subvariety

$$\{z = t = 0\}$$

by forming the incidence fivefold

$$\tilde{Y} = \left\{ \left| \begin{array}{cc} z & t \\ \tilde{z} & \tilde{t} \end{array} \right| = 0 \right\} \subseteq Y \times \mathbb{P}_{[\tilde{t}, \tilde{z}]}$$

with exceptional locus $\mathbb{P}_{[\tilde{t}, \tilde{z}]} \times S_{\text{GUT}}$. The proper transform \hat{W}_4 of W_4 intersects this exceptional locus in the hypersurface

$$\mathcal{D} \subseteq \mathbb{P}_{[\tilde{t}, \tilde{z}]} \times B_3.$$

given by the equation

$$0 = a_5 \tilde{t}^5 + a_4 \tilde{z} \tilde{t}^4 + a_3 \tilde{z}^2 \tilde{t}^3 + a_2 \tilde{z}^3 \tilde{t}^2 + a_0 \tilde{z}^5. \tag{3.5}$$

(Compare (3.3) with (3.5).)

It is immediate to check that the sections ζ and τ both lie in W_4/B_3 and both lie in

$$SP := \left\{ \left| \begin{array}{cc} x^3 & 1 \\ wy^2 & 1 \end{array} \right| = 0 \right\} \subseteq P. \tag{3.6}$$

For fiber coordinate t for SP with

$$wt^2 := x$$

$$wt^3 := y$$

the value $t = z$ gives the section $\tau = \{[w, x, y] = [1, z^2, z^3]\}$ of $W_4 \cap SP$. The cohomology class of

$$SP \cap W_4$$

is given by

$$c_1(\mathcal{O}_P(3))^2.$$

One sees easily that W_4 and (3.4) have contact of order 4 along (ζ) and order 1 along (τ) . Given $b_3 \in (B_3 - S_{\text{GUT}})$, we denote by $SP_{(4)}(b_3)$ the other four points in which $W_4|_{\pi^{-1}(b_3)}$ intersects SP . We denote the closure in P of the locus $\cup_{b_3 \in (B_3 - S_{\text{GUT}})} SP_{(4)}(b_3)$ as

$$SP_{(4)} \subseteq SP \cap W_4 \tag{3.7}$$

that since $0 = a_0 + a_2 + a_3 + a_4 + a_5$ has equation

$$\begin{aligned} & a_5 \tilde{t}^5 + a_4 \tilde{t}^4 \tilde{z} + a_3 \tilde{t}^3 \tilde{z}^2 + a_2 \tilde{t}^2 \tilde{z}^3 + a_0 \tilde{z}^5 = \\ & (a_5 \tilde{t}^4 + a_{54} \tilde{t}^3 \tilde{z} - a_{20} \tilde{t}^2 \tilde{z}^2 - a_0 z^3 (\tilde{t} + \tilde{z})) (\tilde{t} - \tilde{z}) = 0 \end{aligned} \tag{3.8}$$

where $a_{jk} = a_j + a_k$, etc.²

Thus the spectral divisor

$$\mathcal{C}_{\text{Higgs}} = \mathcal{C}_{\text{Higgs}}^{(4)} + \mathcal{C}_{\text{Higgs}}^{(1)} \tag{3.9}$$

²Often this equation is written in terms of the variable $s = \tilde{t}/\tilde{z}$ so that $(\tilde{t} - \tilde{z}) = 0$ becomes $s - 1 = 0$, thereby eliminating $\mathbf{10}_{\{-4\}}$ representations, as in (70) of [9].

is the image of $\mathcal{D} = \mathcal{D}^{(4)} + \mathcal{D}^{(1)}$ in \hat{W}_4 .³ Thus the involution $\tilde{\beta}_4/\beta_3$ preserves (3.9). Using that $a_{54320} = 0$, the Higgs curve, that will be important through this paper, is defined as the image in S_{GUT} of the common solutions to the two equations

$$\begin{aligned} a_5 \tilde{t}^4 - a_{20} \tilde{t}^2 \tilde{z}^2 - a_0 \tilde{z}^4 &= 0 \\ a_{54} \tilde{t}^2 - a_0 \tilde{z}^2 &= 0. \end{aligned} \tag{3.10}$$

Writing (3.10) as two equations in the variable \tilde{t}^2/\tilde{z}^2 , the solution set doubly covers the surface in B_3 defined by the resultant equation obtained by substituting

$$\frac{\tilde{t}^2}{\tilde{z}^2} = \frac{a_0}{a_{54}}$$

in the first equation to obtain

$$a_5 \left(\frac{a_0}{a_{54}} \right)^2 - a_{20} \left(\frac{a_0}{a_{54}} \right) - a_0 = 0$$

that, again using $a_{54320} = 0$, reduces to

$$a_0 \cdot \begin{vmatrix} a_4 & -a_5 \\ a_3 + a_0 & a_3 \end{vmatrix} = 0 \tag{3.11}$$

with branch locus defined by the restriction of the divisor class N .

3.1 Adjusting the E_8 -evolution

Because the image (τ) of our second section τ actually generates with $\mathcal{C}_{\text{Higgs}}^{(1)}$ in our F -theory model $\text{SU}(5)_{\text{gauge}} \times \text{SU}(5)_{\text{Higgs}}$ will be actually replaced by

$$\text{SU}(5)_{\text{gauge}} \times (\text{SU}(4)_{\text{Higgs}} \times \text{U}(1)_X) \tag{3.12}$$

with maximal torus comprising a maximal torus in E_8 and so a vector space isomorphism of Cartan subalgebras

$$\mathfrak{h}_{su(5)} \oplus \mathfrak{h}_{su(4)} \oplus \mathfrak{h}_{u(1)} \xrightarrow{\cong} \mathfrak{h}_{e_8}$$

and associated commutative diagram

$$\begin{array}{ccc} \text{SU}(5) \times \text{SU}(4) \times \text{U}(1) & \xrightarrow{Ad_{\text{SU}(5)} \times Ad_{\text{SU}(4)} \times Ad_{\text{U}(1)}}} & \text{GL}(su(5) \times su(4) \times u(1)) \\ \downarrow & & \downarrow \\ E_8 & \xrightarrow{Ad_{E_8}} & \text{GL}(e_8). \end{array}$$

³By (7.16) below, the $\mathcal{D}^{(1)}$ -component projects to the invariant global section v of W_4/B_3 given in lemma 2, therefore yielding a global $\text{U}(1)_X$.

Therefore the restriction of $E_8 \xrightarrow{Ad_{E_8}} \text{Aut}(e_8)$ to $\text{SU}(5)_{\text{gauge}} \times (\text{SU}(4)_{\text{Higgs}} \times \text{U}(1)_X)$ decomposes as in (88) of [9] into

$$\begin{aligned}
 & (\mathbf{15}, \mathbf{1})_0 \oplus \\
 & (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{10})_{-4} \oplus (\mathbf{1}, \overline{\mathbf{10}})_4 \oplus (\mathbf{1}, \mathbf{24})_0 \\
 & \oplus (\mathbf{4}, \mathbf{1})_5 \oplus (\mathbf{4}, \overline{\mathbf{5}})_{-3} \oplus (\mathbf{4}, \mathbf{10})_1 \\
 & \oplus (\overline{\mathbf{4}}, \mathbf{1})_{-5} \oplus (\overline{\mathbf{4}}, \mathbf{5})_3 \oplus (\mathbf{4}, \overline{\mathbf{10}})_{-1} \\
 & \oplus (\mathbf{6}, \mathbf{5})_{-2} \oplus (\mathbf{6}, \overline{\mathbf{5}})_2 .
 \end{aligned} \tag{3.13}$$

Here $\mathbf{5}$ denotes the standard matrix representation

$$\text{SU}(5) \rightarrow \text{GL}(\mathbb{C}^5),$$

$\mathbf{10}$ denotes the induced representation

$$\text{SU}(5) \rightarrow \text{GL}(\wedge^2 \mathbb{C}^5)$$

and the ‘bar’ indicates the conjugate representation obtained by composing with the inverse map on $\text{SU}(5)$. Analogously for $\text{SU}(4)$.

4 Semi-stable degeneration and V_3/B_2

To link a crepant resolution \tilde{W}_4/B_3 to its Heterotic dual V_3/B_2 , recall that $B_3 = B_2 \times \mathbb{P}_{[u_0, v_0]}$ so that one can realize \tilde{W}_4/B_3 as $W_{4,1}/B_{3,1}$ in a family of elliptic Calabi-Yau 4-folds $W_{4,\delta}/B_{3,\delta}$ constructed over

$$B_{3,\delta} = B_2 \times \{a'b' = \delta \cdot a''b''\} \subseteq B_2 \times \mathbb{P}_{[a', a'']} \times \mathbb{P}_{[b', b'']}$$

as follows. Using the affine parameters

$$\begin{aligned}
 [a', a''] &= [a, 1] \\
 [b', b''] &= [b, 1]
 \end{aligned} \tag{4.1}$$

we define the affine family

$$B_2 \times \{ab = \delta : 0 \leq \delta \leq 1\} \subseteq B_2 \times \mathbb{C}^2 \tag{4.2}$$

and identify $\mathbb{P}_{[u_0, v_0]}$ with the closure $\{a'b' = a''b''\}$ of $\{ab = 1\}$ by the rule

$$\frac{u_0 - v_0}{u_0 + v_0} = a = b^{-1}. \tag{4.3}$$

That is

$$\begin{aligned}
 & \begin{vmatrix} u_0 - v_0 & u_0 + v_0 \\ a' & a'' \end{vmatrix} = 0 \\
 & \begin{vmatrix} u_0 - v_0 & u_0 + v_0 \\ b'' & b' \end{vmatrix} = 0.
 \end{aligned} \tag{4.4}$$

In this way we identify

$$\tilde{W}_4/B_3/B_2$$

as an elliptic fibration over the closure $B_2 \times \{a'b' = a''b''\}$ of $B_2 \times \{ab = 1\}$. Also as in [15] as coefficients in the Tate form (1.7) for W_4/B_3 we require that

$$a_2, a_3, a_4, a_5, z, t = \frac{y}{x} \in H^0\left(K_{B_3}^{-1}\right)^{[-1]}, \tag{4.5}$$

(that is, anti-invariant under the involution β_3), in other words they must be linear combinations of the forms listed in table 2 in [15]. Furthermore also as described via table 2 in [15] the four-dimensional family of forms spanned by $a_2, a_3, a_4, a_5 \in H^0\left(K_{B_3}^{-1}\right)^{[-1]}$ should deform as

$$a_{j,\delta} = \delta \cdot a_j + a_{j,0} \tag{4.6}$$

where the $a_{j,0}$ lie in the three-dimensional $(-i)$ -eigenspace of the \mathbb{Z}_4 \mathbf{R} -symmetry $T_{u,v}$, again as given in table 2 in [15]. Finally

$$z_\delta := \delta z + z_0$$

where $z_0 := q \cdot (u_0^2 - v_0^2)$ is skew-invariant under β_3 so that $q \in H^0\left(K_{B_2}^{-1}\right)$ must be invariant under the involution

$$B_2 \xrightarrow{\beta_2} B_2.$$

As we again showed in [15],

$$B_2 = D_2$$

is a particular degree-2 del Pezzo surface on which the involution β_2 on B_2 acts with four fixpoints.

Thus $B_{2,\delta} = B_2$ for all δ and

$$\begin{aligned} B_{3,1} &= B_3 \\ B_{3,0} &= (B_2 \times \mathbb{P}_{[a',a'']}) \cup (B_2 \times \mathbb{P}_{[b',b'']}) \end{aligned}$$

with

$$(B_2 \times \mathbb{P}_{[a',a'']}) \cap (B_2 \times \mathbb{P}_{[b',b'']}) = (B_2 \times \{[0, 1]\}) \cup (B_2 \times \{[1, 0]\})$$

and applying the relations (4.4). a_2, a_3, a_4, a_5 are required to have no common zeros on $B_{3,\delta}$ for δ on the half-open real interval $(0, 1]$.

The Tate form (1.7) and the action β_3 on B_3 as described in the tables in section 4 of [15] then determine equivariant involutions $\tilde{\beta}_4/\beta_3$ on W_4/B_3 . For generic choice of z , S_{GUT} will not contain any of the eight fixpoints of β_3 acting on B_3 so β_3 will act freely on S_{GUT} yielding a smooth Enriques surface as quotient. The sections $a_{j,\delta}, z_\delta \in H^0\left(K_{B_3}^{-1}\right)$ in the Tate form will be allowed to vary under the contraction (4.6) and in particular the discriminant component $S_{\text{GUT},\delta} \subseteq B_{3,\delta}$ varies as defined by

$$z_\delta := \delta \cdot z + (1 - \delta) q \cdot (u_0^2 - v_0^2) = 0$$

with β_2 -invariant $q \in H^0\left(K_{B_2}^{-1}\right)$.

4.1 Degeneration of a single $K3$ -surface

Over each point $(a, b) \in \mathbb{C}^2$ associate the Weierstrass form

$$y^2 = x^3 + g_2(a, b)x + g_3(a, b) \tag{4.7}$$

where g_2 is homogeneous of total degree 4 and g_3 homogeneous of total degree 6. Restrict the Weierstrass form to the locus

$$\Gamma_\delta := \{(a, b) : a \cdot b = \delta\} \tag{4.8}$$

so that for $\delta \neq 0$, the discriminant has degree 24. As shown below compactifying at infinity yields a $K3$ -surface elliptically fibered over the closure of Γ_δ and setting $\delta = 0$ yields the union of two dP_9 's meeting over $\{a = b = 0\}$. For fixed value $b_2 \in B_2$ the fiber $\pi_{B_2}^{-1}(b_2)$ has homogeneous linear coordinates $[u_0, v_0]$ and $\pi_{B_3}^{-1}(\pi_{B_2}^{-1}(b_2))$ is a $K3$ -surface in W_4 . Since g_2 and g_3 in (4.12) must be homogeneous forms of degree 4 and 6 respectively in $[u_0, v_0]$, dividing (4.12) by $(u_0 + v_0)^6$ gives the affine equation

$$y^2 = x^3 + g_2(a)x_0 + g_3(a) \tag{4.9}$$

where

$$\begin{aligned} a &:= \frac{u_0 - v_0}{u_0 + v_0} \\ x &:= \frac{x_{nf}}{(u_0 + v_0)^2} \\ y &:= \frac{y_{nf}}{(u_0 + v_0)^3}. \end{aligned}$$

On the \mathbb{P}^1 -fiber over $b_2 \in B_2$ we obtain the Weierstrass form

$$y^2 = x^3 + g_2(b_2, a)x + g_3(b_2, a) \tag{4.10}$$

where $g_2(b_2, a)$ is a rational function of a with denominator of degree 4 and $g_3(b_2, a)$ can be expressed as a rational function of a with denominator of degree 6.

Thus one can write a global decomposition of (4.10) with

$$\begin{aligned} g_2(b_2, a) &= \sum_{j=1}^4 g_{2,j}^-(b_2) a^{-j} + g_{2,0}(b_2) + \sum_{j=1}^4 g_{2,j}^+ a^j \\ g_3(b_2, a) &= \sum_{j=1}^6 g_{3,j}^-(b_2) a^{-j} + g_{3,0}(b_2) + \sum_{j=1}^6 g_{3,j}^+ a^j. \end{aligned}$$

Then, letting $b = a^{-1}$, we can equivalently write

$$\begin{aligned} g_2(b_2; a, b) &= \sum_{j=1}^4 g_{2,j}^-(b_2) b^j + g_{2,0}(b_2) + \sum_{j=1}^4 g_{2,j}^+ a^j \\ g_3(b_2; a, b) &= \sum_{j=1}^6 g_{3,j}^-(b_2) b^j + g_{3,0}(b_2) + \sum_{j=1}^6 g_{3,j}^+ a^j. \end{aligned}$$

where the g_2 and g_3 are the functions on the curve $\{ab = 1\}$ over the point b_2 in B_2 .

Then for any $b_2 \in B_2$ and any $(a, b) \in \mathbb{C}^2$ we can write the Weierstrass form

$$y^2 = \frac{1}{2}x^3 + \left(\frac{g_{2,0}(b_2)}{2} + \sum_{j=1}^4 g_{2,j}^+ a^j \right) x + \left(\frac{g_{3,0}(b_2)}{2} + \sum_{j=1}^6 g_{3,j}^+ b^j \right) + \frac{1}{2}x^3 + \left(\frac{g_{2,0}(b_2)}{2} \sum_{j=1}^4 g_{2,j}^-(b_2) b^j \right) x + \left(\frac{g_{3,0}(b_2)}{2} + \sum_{j=1}^6 g_{3,j}^-(b_2) b^j \right).$$

4.2 dP_9 -bundles over B_2 giving a singular Calabi-Yau fourfold

As in section 4.1 we consider $(\mathbb{P}^1)^2$ with coordinates $([a', a''], [b', b''])$ as a fiber of the total space

$$\mathbb{P} \left(K_{B_2}^{-1} \oplus K_{B_2}^{-1} \right) \times_{B_2} \mathbb{P} \left(K_{B_2}^{-1} \oplus K_{B_2}^{-1} \right).$$

The union of our two dP_9 -bundles is then given as the subspace of defined by (5.11) and the equation

$$a'b' = 0.$$

Returning to our fibration

$$\pi_{B_2} : B_3 \rightarrow B_2,$$

for each fiber $\pi_{B_3}^{-1}(b_3)$ of \tilde{W}_4/B_3 we will associate *two* copies of the Weierstrass equation, namely the one distinguished by designating $\tilde{\zeta}(b_3)$ as the identity element of the group structure and the other distinguished by designating $\tilde{\tau}(b_3)$ as the identity element of the group structure. The two fibers $\pi_{B_3}^{-1}(b_3)$ and $\pi_{B_3}^{-1}(\beta(b_3))$ are identified under the isomorphism induced by the involution β_4 induced by the involution β_3 on B_3 . However, as will become clear in section 6 the line bundle

$$\mathcal{O}_{\pi_{B_3}^{-1}(b_3)} \left(\tilde{\zeta}(b_3) - \tilde{\tau}(b_3) \right) \tag{4.11}$$

is not trivial over any $b_3 \in S_{\text{GUT}}$. As we have seen in (2.6), the identification acts on sheaves \mathcal{F} on the elliptic curve in Weierstrass form

$$wy_{nf}^2 = x_{nf}^3 + g_2 w^2 x_{nf} + w^3 \tag{4.12}$$

where $g_2(b_3) = \left(Bz^2 - \frac{A^2}{3} \right)$ and $g_3(b_3) = \left(Cz^4 - \frac{AB}{3}z^2 + \frac{2A^3}{27} \right)$. The action on \mathcal{F} is given on $\pi_{B_3}^{-1}(b_3)$ by

$$\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_{\pi_{B_3}^{-1}(b_3)} \left(\tilde{\zeta}(b_3) - \tilde{\tau}(b_3) \right) \tag{4.13}$$

so that for each $b_2 \in B_2$ the $K3$ -surface $(\pi_{B_2} \circ \pi_{B_3})^{-1}(b_2)$ can be obtained as the smoothing the union of two dP_9 's described in subsection 4.1. That is, we realize \tilde{W}_4/B_2 as the smoothing of two dP_9 -bundles

$$dP_a \cup dP_b \tag{4.14}$$

over B_2 .

Simultaneously, via

$$S_{\text{GUT},\delta} := \{\delta z + (1 - \delta) z_0 = 0\} \subseteq B_2,$$

we move S_{GUT} to the reducible quadric z_0 given by

$$z_0 = (u_0^2 - v_0^2) \cdot q(u_1, v_1, u_2, v_2) \tag{4.15}$$

where $q(u_1, v_1, u_2, v_2)$ is invariant under the action of the involution β_2 on B_2 . So at $\delta = 0$ S_{GUT} splits into two ‘horizontal’ components given by $u_0 - v_0 = 0$ and $u_0 + v_0 = 0$ and a ‘vertical’ component given by $\pi_{B_2}^{-1}(\{q(u_1, v_1, u_2, v_2) = 0\})$. Thus the intersection

$$S_{\text{GUT},\delta} \cap \pi_{B_2}^{-1}(\text{fixpoint set of } \beta_2)$$

is independent of δ .

From the Weierstrass forms just above, we read off that \tilde{W}_4 is the smoothing over $ab = 1$ of the union of the two dP_9 -bundles given over each point $b_2 \in B_2$ by

$$\frac{1}{2}y^2 = \frac{1}{2}x^3 + \left(\frac{g_{2,0}(b_2)}{2} + \sum_{j=1}^8 g_{2,j}^+ a^j \right) x + \left(\frac{g_{3,0}(b_2)}{2} + \sum_{j=1}^{12} g_{3,j}^+ a^j \right) \tag{4.16}$$

and

$$\frac{1}{2}y^2 = \frac{1}{2}x^3 + \left(\frac{g_{2,0}(b_2)}{2} \sum_{j=1}^8 g_{2,j}^-(b_2) b^j \right) x_0 + \left(\frac{g_{3,0}(b_2)}{2} + \sum_{j=1}^{12} g_{3,j}^-(b_2) b^j \right). \tag{4.17}$$

These two dP_9 's over b_2 contain the common fiber of V_3/B_2 whose Weierstrass form is

$$wy^2 = x^3 + w^2 g_{2,0}(b_2) x + w^3 g_{3,0}(b_2).$$

The spectral data on the Heterotic side, namely the two E_8 -bundles on the fiber of V_3/B_2 over b_2 , are given via the Friedman-Morgan-Witten classification [23] by the two dP_9 -bundles. Namely in section 4.2 of [23] Friedman-Morgan-Witten give a classifying space for imbeddings of an elliptic fiber E_{b_2} of V_3/B_2 into a rational elliptic surface $dP_9(b_2)$, each such corresponding canonically by a theorem of E. Looijenga [41] to an isomorphism class of flat $E_8^{\mathbb{C}}$ -bundles F over E_{b_2} . Considered as fibrations over B_2 , fibers are so-called dP_9 -surfaces. Setting $[s, t] = [a', a'']$, respectively $[s, t] = [b', b'']$ for s, t as in [23], fibers are given uniquely in $\mathbb{P}_{1,1,2,3}^3$ by an equation

$$y^2 = 4x^3 - (g_2 t^4 - \beta_1 s t^3 - \dots - \beta_4 s^4) x - (g_3 t^6 - \alpha_2 s^2 t^4 - \dots - \alpha_6 s^6) \tag{4.18}$$

of weighted homogeneous degree 6 in the variables x, y, s, t with respective weights 2, 3, 1, 1 over the weighted projective space with coordinates

$$[\alpha_2, \dots, \alpha_6, \beta_1, \dots, \beta_4] \in \mathbb{P}_{[2,3,4,5,6,1,2,3,4]}^8$$

weighted as per their respective indices.

So to realize the semi-stable degeneration geometrically, the crepant resolution \tilde{W}_4/B_3 of W_4/B_3 is given over the locus

$$\Gamma_1 = \{(a, b) : ab = 1\} \subseteq B_3 \times \mathbb{C}^2.$$

The deformation in this section is given by restricting the Weierstrass form to the locus

$$\Gamma_\delta = \{(a, b) : ab = \delta\}$$

as δ goes to 0. Furthermore S_{GUT} deforms with δ via the formula

$$z_\delta = \delta \cdot z + z_0. \tag{4.19}$$

We thereby obtain the family $\{\tilde{W}_{4,\delta}\}$ of fourfolds over the affine line \mathbb{C}_δ^* as in subsection 4.1.

Then the Heterotic model (V_3, F_a, F_b) over B_2 canonically corresponds to a normal crossing Calabi-Yau 4-fourfold with two components dP_a and dP_b obtained by making the construction described just above equivariantly over B_2 .

Thus the family $W_{4,\delta}/B_{3,\delta}$ defined by the Tate form on $B_{3,\delta}$ degenerates as δ approaches zero to a reducible Calabi-Yau 4-fold

$$W_{4,0} = dP_a \cup dP_b$$

over

$$(B_2 \times \mathbb{P}_{[a,1]}) \cup (B_2 \times \mathbb{P}_{[b,1]})$$

with

$$V_3 = dP_a \cap dP_b$$

giving the Heterotic model V_3/B_2 defined by $a = b = 0$. Finally there are common distinguished sections on every $W_{4,\delta}/B_{3,\delta}$. defined equivariantly by

$$\begin{aligned} \zeta_\delta &= \{w = x = 0\} \\ \tau_\delta &= \{[w, x, y] = [w, wz_\delta^2, wz_\delta^3]\}. \end{aligned}$$

The Calabi-Yau threefold

$$V_3 = dP_a \cap dP_b$$

is elliptically fibered over $B_2 \times \{a = b = 0\}$, the general fiber of dP_a/B_2 has an I_5 -fiber at $a = \infty$ and the general fiber of dP_b/B_2 has an A_4 -fiber at $b = \infty$.

4.3 The involution

Then by construction the involution $\tilde{\beta}_4/\beta_3$ induces involutions $\tilde{\beta}_{4,\delta}/\beta_{3,\delta}$ on $W_{4,\delta}/B_{3,\delta}$ for all $\delta \geq 0$. Furthermore the above assumptions force

$$\begin{array}{ccc} W_{4,0}/B_{3,0} & \xrightarrow{\tilde{\beta}_{4,0}} & W_{4,0}/B_{3,0} \\ \downarrow & & \downarrow \\ B_2 & \xrightarrow{\beta_2} & B_2 \end{array}$$

to be given by the elliptic fibrations with involutions

$$\begin{aligned} dP_a / (B_2 \times \mathbb{P}_{[a,1]}) &\xrightarrow{\tilde{\beta}_a} dP_{9,a} / (B_2 \times \mathbb{P}_{[a,1]}) \\ ([w, x, y], (b_2, [a, 1])) &\mapsto ([w, x, -y], (\beta(b_2), [-a, 1])) \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} dP_{9,b} / (B_2 \times \mathbb{P}_{[b,1]}) &\xrightarrow{\tilde{\beta}_b} dP_{9,b} / (B_2 \times \mathbb{P}_{[b,1]}) \\ ([w, x, y], (b_2, [b, 1])) &\mapsto ([w, x, -y], (\beta(b_2), [-b, 1])). \end{aligned} \tag{4.21}$$

The action of these involutions over the fixpoints of the action of β_2 on B_2 is treated in detail in section 2.4 of [14].

5 Passing from Heterotic theory to F-theory

Essentially one passes from the Heterotic model to the F -theory model by reading the subsections of the previous section in reverse order and from bottom to top. We can paste one of these dP_9 along E at $s = 0$ and the other along E at $s = 0$ to obtain a normal crossing elliptic $K3$ -surface with unobstructed deformation space thereby joining dP_a^\vee to dP_b^\vee to form a normal crossing Calabi-Yau fourfold. Having that, a theorem of Kawamata-Namikawa [31] guarantees that the normal crossing elliptic Calabi-Yau fourfold has an unobstructed deformation theory. $B_3 = \mathbb{P}_{[u_0, v_0]} \times B_2$ so that our only choices are the section of $K_{B_2}^{-1}$ in the definition of z_0 and the smooth section z of $K_{B_3}^{-1}$ in (4.19).

5.1 Initial data

In Heterotic theory, we begin with the smooth degree-2 del Pezzo del Pezzo (B_2, β_2) with involution β_2 with 4 fixpoints constructed in [15]. Letting B_2^\vee denote the quotient, we are given a smooth, elliptically fibered Calabi-Yau threefold

$$V_3^\vee \rightarrow B_2^\vee$$

with Weierstrass form

$$y^2 = x^3 + g_2(b_2)x + g_3(b_2) \tag{5.1}$$

for each $b_2 \in B_2^\vee$. This Weierstrass equation lives at each point b_2 of a $\mathbb{P}_{[w,x,y]}$ -bundle over B_2 with

$$\begin{aligned} x &\in H^0(K_{B_2}^{-2}) \\ y &\in H^0(K_{B_2}^{-3}) \\ g_2 &\in H^0(K_{B_2}^{-4}) \\ g_3 &\in H^0(K_{B_2}^{-6}) \end{aligned}$$

in order that V_3^\vee have trivial canonical bundle. Has fundamental group \mathbb{Z}_2 .

Furthermore V_3^\vee supports two principal E_8 -bundles

$$F_a^\vee \oplus F_b^\vee \tag{5.2}$$

with Yang-Mills connections. Pulling back via

$$\begin{array}{ccc} V_3 = B_2 \times_{B_2^\vee} V_3^\vee & \rightarrow & B_2 \\ & \downarrow & \downarrow \\ & V_3^\vee & \rightarrow & B_2^\vee \end{array}$$

we have two β_2 -invariant E_8 -bundles, F_a and F_b each with a Yang-Mills connection with fiber over $b_2 \in B_2$.

The smooth, torus-fibered Calabi-Yau threefold V_3^\vee/B_2^\vee with fundamental group \mathbb{Z}_2 is also endowed with two disjoint sections

$$\zeta, \tau : B_2^\vee \rightarrow V_3^\vee.$$

The pull-back of the two sections under

$$\pi_{B_2^\vee} : B_2 \rightarrow B_2^\vee$$

becomes the union of two disjoint sections

$$\zeta, \tau : B_2 \rightarrow V_3 := B_2 \times_{B_2^\vee} V_3^\vee.$$

Also, following section 2.4 of [14], the fiber of the smooth threefold V_3^\vee/B_2^\vee over the four orbifold points of B_2^\vee must be acquired with multiplicity two, being doubly covered by the elliptic fibers E_{b_2} of V_3/B_2 over the β_2 -fixpoints $b_2 \in B_2$. The covering must be unbranched via translation by a distinguished half-period. Given the prior conditions imposed on our model that half-period must be $\mathcal{O}_{E_{b_2}}(\zeta(b_2) - \tau(b_2))$. The role of the logarithmic transform over a neighborhood of each β_2 -fixpoint, especially how it induces the action

$$(b_2, (x, y)) \mapsto (\beta_2(b_2), (x, y))$$

of $\tilde{\beta}_3/\beta_2$ on (Weierstrass) fibers of V_3/B_2 , as well as the transition from the β_2 -pull-backs

$$F_a \oplus F_b \tag{5.3}$$

is explained in [14].

The Yang-Mills connections on the bundles (5.3) restrict to sums of eight flat line bundles on each fiber of V_3/B_2 and are completely determined by that family of restrictions.

5.2 Building a normal-crossing K3 from an elliptic curve with two flat E_8 -bundles

As in (4.18) the two flat E_8 -bundles on the elliptic fiber E_{b_2} of V_3/B_2 over b_2 determine two dP_9 -bundles

$$dP_a(b_2) \cup dP_b(b_2),$$

with E_{b_2} given in each by

$$s = 0.$$

That is, over B_2 we have the union

$$dP_a \cup dP_b$$

of two dP_9 -bundles with

$$dP_a \cap dP_b = V_3.$$

Since the canonical bundle of this normal-crossing variety is trivial, the theorem of Kawamata-Namikawa cited above establishes that its deformation space is unobstructed. In fact we employ the very specific smoothing determined by (4.2) and (4.6).

Finally for the homogeneous coordinates $[u_0, v_0]$ in [15] and the curve (4.3), the equations

$$\begin{vmatrix} u_0 - v_0 & u_0 + v_0 \\ a' & a'' \end{vmatrix} = 0$$

$$\begin{vmatrix} u_0 + v_0 & u_0 - v_0 \\ b' & b'' \end{vmatrix} = 0$$

allow us as in (4.2) to form the family of hypersurfaces

$$B_{3,\delta} \subseteq B_2 \times \left(\mathbb{P}_{[a',a'']}^1 \times \mathbb{P}_{[b',b'']}^1 \right) \rightarrow B_2$$

with fibers given by

$$\{a' \cdot b' = \delta \cdot a'' \cdot b''\} \tag{5.4}$$

with

$$K_{B_{3,\delta}}^{-1} = K_{B_2}^{-1} \boxtimes (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)) \Big|_{\{a' \cdot b' = \delta \cdot a'' \cdot b''\}} =: N_\delta.$$

The forms (4.6) then yield our F -theory model W_4/B_3 at $\delta = 1$.

5.3 The action of the involution on the Heterotic model

In section 4.2 of [23] Friedman-Morgan-Witten gives a classifying space for imbeddings of an elliptic curve E in Weierstrass form into dP_9 's, each corresponding canonically to an isomorphism class of flat E_8 -bundles

$$F \rightarrow E.$$

The flat E_8 -bundle F is given as the sum of eight flat line bundles, each given by the divisor of the form

$$p - e$$

where $e = (\tilde{\zeta}) \cap E$ is the identity element of E considered as an abelian group and p is a geometrically given point on the torus E . Again we follow Friedman-Morgan-Witten in [14] and consider the family of dP_9 -hypersurfaces

$$y^2 = 4x^3 - (g_2t^4 - \beta_1st^3 - \dots - \beta_4s^4)x - (g_3t^6 - \alpha_2s^2t^4 - \dots - \alpha_6s^6) \tag{5.5}$$

in $\mathbb{P}_{1,1,2,3}^3$ where the α_j and β_j are homogeneous forms of weight j in a $\mathbb{P}_{1,2,2,3,3,4,4,5,6}^8$. Fixing the values of α_j and β_j we think of the solution set of (5.5) as a rational hypersurface in $\mathbb{P}_{1,1,2,3}^3$ with distinguished pencil

$$\gamma s + \delta t = 0. \tag{5.6}$$

The given elliptic curve E sits in each dP_9 in (5.5) as the solution set to the equation

$$s = 0.$$

If we change the basepoint of the elliptic curve E_{b_3} from $\tilde{\zeta}(b_3)$ to $\tilde{\tau}(b_3)$, then the trivial bundle goes to itself so that translation by $(\tilde{\tau}(b_3) - \tilde{\zeta}(b_3))$ acts trivially on $\text{Pic}^0(E_{b_3})$ and therefore acts trivially on flat E_8 -bundles on E_{b_3} . However changing the identity element on the torus E from $(\tilde{\zeta}) \cap E$ to $(\tilde{\tau}) \cap E$ and then applying the Friedman-Morgan-Witten dictionary gives a different sum of eight flat line bundles, namely those obtained by divisors

$$p - e + ((\tilde{\zeta}) \cap E) - ((\tilde{\tau}) \cap E)$$

that is, all eight flat line bundles are tensored with the non-trivial flat line bundle

$$\mathcal{O}_{\tilde{W}_4}((\tilde{\zeta}) - (\tilde{\tau})).$$

Lemma 3.

- i) The induced action of the involution $\tilde{\beta}_{4,0}$ on the union of the two dP_9 -bundles at $\delta = z_0 = 0$ takes each of the two to itself.
- ii) The action of the involution $\tilde{\beta}_3$ on the intersection V_3 of the two dP_9 -bundles is given by the map

$$(b_2; x, y) \mapsto (\beta_2(b_2); x, y) + (\tilde{\zeta}(\beta_2(b_2)) - \tilde{\tau}(\beta_2(b_2)))$$

where addition is with respect to the addition law on the elliptic curve and the (identical) Weierstass forms

$$\begin{aligned} y^2 &= x^3 + g_2(b_2)x + g_3(b_2) \\ y^2 &= x^3 + g_2(\beta(b_2))x + g_3(\beta(b_2)). \end{aligned} \tag{5.7}$$

- iii) Over fixpoints b_2 of the action of β_2 on B_2 , $\tau(b_2) - \zeta(b_2)$ is a non-trivial half-period on the fiber E_{b_2} of V_3/B_2 but the action of $\tilde{\beta}_3$ on the intersection cycle

$$4 \cdot \zeta(b_2) + 5 \cdot \tau(b_2)$$

of length nine determining the E_8 -bundles on the fiber is given by tensoring with the canonical section of the line bundle associated to the divisor $\zeta(b_2) - \tau(b_2)$.

Proof.

- i) This assertion is immediate from the definition of $\tilde{\beta}_4$.

ii) First of all the involution is given by

$$\begin{aligned} (x, y) &\mapsto (x, -y) \\ a &\mapsto -a \end{aligned} \tag{5.8}$$

but the fiber of

$$V_3/B_2$$

is given in (4.16) and (4.17) as

$$y^2 = x^3 + g_2^0 x + g_3^0$$

so that it does not reflect the sign change of a and we showed in [14] how the sign change of y is absorbed in the process of taking residue. Secondly the equations of the two $K3$ -surfaces fibers

$$\tilde{W}_4 \times_{B_2} \{b_2\} \cup \tilde{W}_4 \times_{B_2} \{\beta(b_2)\}$$

are invariant under $\tilde{\beta}_4$ and so are identical. Furthermore by (5.8) the globally defined relative holomorphic 2-form on \tilde{W}_4/B_2 is invariant under the action of $\tilde{\beta}_4$. Thus, for any point $b_3 \in B_3$ lying over $b_2 \in B_2$, the relative one-form on \tilde{W}_4/B_2 obtained from the contraction of the relative holomorphic 2-form on \tilde{W}_4/B_2 against a section of the relative tangent bundle of B_3/B_2 is invariant under the action of $\tilde{\beta}_4$. So the induced relative one-form on V_3/B_2 is invariant under the action of $\tilde{\beta}_3$. So as we showed in [14] $\tilde{\beta}_3$ must act as a translation of distinguished basepoint with respect to the common Weierstrass form (5.7). Since

$$\begin{aligned} \tilde{\zeta}(b_2) &\mapsto \tilde{\tau}(b_2) \\ \tilde{\zeta}(\beta(b_2)) &\mapsto \tilde{\tau}(\beta(b_2)) \end{aligned}$$

respectively, ii) is proved.

iii) Choosing $a_5 \neq 0$ at the fixpoints b_3 of the action of $\tilde{\beta}$ on B_3 separates $\zeta(b_3)$ from $\tau(b_3)$. The assertion is forced by the fact that, in the canonical identification of an elliptic curve E with $\text{Pic}^0(E)$, the trivial bundle is associated to $\zeta(b_3)$ for the Weierstrass form based at $\zeta(b_3)$ and is associated to $\tau(b_2)$ for the Weierstrass form based at $\tau(b_3)$.

Said otherwise the involution on dP_9 -structures (5.5) induced by

$$\begin{aligned} y &\mapsto -y \\ (s, t) &\mapsto (-s, t) \end{aligned} \tag{5.9}$$

induces the involution

$$\begin{aligned} \text{Pic}^0(E) &\rightarrow \text{Pic}^0(E) \\ L &\mapsto L^{-1} \end{aligned}$$

so that there is a unique translation

$$\begin{aligned} E &\rightarrow E \\ x &\mapsto x + (e' - e) \end{aligned} \tag{5.10}$$

such that the composition takes the trivial bundle to itself under the change of basepoint of the torus E from e to e' . Using the decomposition

$$\text{Pic}^0(E)^{\oplus 8}$$

of the moduli space of semi-stable E_8 -bundles on E , the functorial diagram

$$\begin{array}{ccc} F_a \xrightarrow{\tilde{\beta}_3^*} & & F_a \\ \downarrow & & \downarrow \\ \text{Pic}^0(V_3/B_2)^{\oplus 8} & \xrightarrow{(\otimes_{\mathcal{O}_{\tilde{W}_0}}(\tilde{\zeta}-\tilde{\tau})) \circ ()^{-1}} & \text{Pic}^0(V_3/B_2)^{\oplus 8} \end{array} \tag{5.11}$$

is such that the bottom horizontal map is the identity map. This is what allows us to induce a semi-stable E_8 -bundle on the quotient

$$B_2^\vee = \frac{B_2}{\{\beta_2\}}.$$

See [14] for a more detailed analysis.

Remark 3. The classifying space for flat E_8 -structures on the elliptic curve E is acted on by the semi-direct product

$$\text{Pic}^0(E) \rtimes \mathbb{Z}_2$$

where \mathbb{Z}_2 is generated by

$$([s, t], (x, y)) \mapsto ([-s, t], (x, -y)).$$

5.4 $SU(5)_{\text{gauge}}$ -roots and the semi-stable limit

The action of $\tilde{\beta}_3$ on

$$V_3/B_2 = dP_a \cap dP_b$$

is explained in [14] through the lens of its local action over a neighborhood of β_2 -fixpoints, as is the compatibility of this action with the action induced by $\tilde{\beta}_4$ on the semi-stable degeneration

$$\tilde{W}_4/B_2 \Rightarrow dP_a \cup dP_b.$$

One checks directly that this action is compatible with the semi-stable degeneration of S_{GUT} to

$$\{z_0 = q \cdot (u_0^2 - v_0^2) = 0\}.$$

z_0 deposits an I_5 -fiber at the point $\{a'' = 0\}$ on each fiber of dP_a/B_2 and at the point $\{b'' = 0\}$ on each fiber of dP_b/B_2 . By construction, $\tilde{\zeta}$ and $\tilde{\tau}$ can only meet over $\{z_\delta = 0\}$. If

$$b_2 \in \{u_1^2 - v_1^2 = u_2^2 - v_2^2 = 0\} \subseteq B_2$$

is any of the 4 fixpoints of the involution β_2 , then when $\delta = 0$ the fixpoints of the action of β_3 on the fiber $\mathbb{P}_{[u_0, v_0]} \times \{b_2\}$ occur at $a = 0, \infty$ since β takes a to $-a$. We choose the form $q \in H^0(K_{B_2}^{-1})$ in the definition of z_0 so as not to vanish on any of the fixpoints of the action of β_2 on B_2 .

6 Geometric model-double cover form

6.1 Putting the sections ζ and τ of W_4/B_3 on equal footing

As mentioned earlier, the \mathbb{Z}_2 -action will ultimately incorporate a translation in the fiber direction that will interchange the two distinguished sections ζ and τ , and so will interchange the two distinguished divisors, (ζ) and (τ) , to which they correspond. In particular, the Weierstrass forms for the elliptic fibration determined by either of the two sections must be ‘on equal footing with the other one’ throughout. So we will have to begin from another birational model for W_4/B_3 that achieves the desired ‘equal footing’. On each fiber of W_4/B_3 , the intersection with the sections $(\zeta) + (\tau)$ distinguishes a g_2^1 (that is, a linear series of projective dimension one and degree two). Taken together the g_2^1 ’s yield a \mathbb{P}^1 -bundle

$$Q := \mathbb{P}((\pi_{B_3})_* \mathcal{O}_{W_4}((\zeta) + (\tau)))$$

over B_3 .

Since we are eventually going to change the elliptic group structure on the torus fibers of W_4/B_3 from the one given by (ζ) that does not pass through the set of singular points of W_4 to the one given by (τ) that actually contains the set of singular points of W_4 , we first want to define a crepant partial resolution of our F -theory model W_4/B_3 that becomes a branched double cover of Q/B_3 and has the property that the proper transform of (τ) misses the singular set of the resulting fourfold entirely. We achieve this by projecting each torus fiber from the third point $v(b_3)$ of intersection of the line between $\zeta(b_3)$ and $\tau(b_3)$ with the fiber of W_4/B_3 over b_3 .

6.1.1 Line between the sections

Over $b_3 \in (B_3 - S_{\text{GUT}})$, the line through $\zeta(b_3)$ and $\tau(b_3)$ in $\{b_3\} \times_{B_3} P$ is given by

$$x - z^2 w = 0.$$

We have denoted the third point of intersection of this line with $\{b_3\} \times_{B_3} W_4$ as

$$v(b_3) = [w, z^2 w, (-z - a_{420}) z^2 w].$$

Thus we can modify the defining equation

$$\begin{vmatrix} x^3 + a_4 z w x^2 + a_2 z^3 w^2 x + a_0 z^5 w^3 & 1 \\ w y (y - (a_5 x + a_3 z^2 w)) & 1 \end{vmatrix} = 0$$

of W_4 by projecting $\{b_3\} \times_{B_3} W_4$ from the point $v(b_3)$. That is we write

$$\begin{aligned} x - z^2 w &= X \\ y + (z + a_{420}) z^2 w &= Y \end{aligned} \tag{6.1}$$

in order that the section (v) correspond to the point $[w, X, Y] = [1, 0, 0]$. The change of fiber coordinates in P/B_3 is given by

$$\begin{pmatrix} w \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -z^2 & 1 & 0 \\ (z + a_{420}) z^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \end{pmatrix}. \tag{6.2}$$

(Notice that this is the identity matrix over a first-order neighborhood of S_{GUT} .) Recalling again that $-a_{53} = a_{420}$, we obtain

$$\begin{vmatrix} (X^3 + 3z^2 w X^2 + 3z^4 w^2 X) + a_4 z w (X^2 + 2z^2 w X) + a_2 z^3 w^2 X & 1 \\ w Y (Y - z^3 w - a_5 X) - (z + a_{420}) z^2 w^2 (Y - a_5 X) & 1 \end{vmatrix} = 0 \tag{6.3}$$

as the equation for W_4 in the projective coordinates $[w, X, Y]$.

6.2 Fundamental projection and modified Weierstrass form

By considering (6.3) as a quadratic equation in w , the Tate form then yields a birationally equivalent double-cover

$$\overline{W}_4/Q$$

where the branched double cover \overline{W}_4 of Q is given by the equation

$$\begin{aligned} w^2 &= (3z^2 X^2 + a_4 z X^2 - Y^2 + a_5 XY)^2 \\ &\quad - 4 \left((3z^4 + (2a_4 + a_2 - a_5) z^3 - a_5 a_{420} z^2) X^4 + (2z^3 + a_{420} z^2) X^3 Y \right) \end{aligned} \tag{6.4}$$

where the fiber coordinates of Q/B_3 are given by

$$\begin{aligned} X &= x - z^2 w \\ Y &= y + (z + a_{420}) z^2. \end{aligned}$$

(6.4) also allows us to see that W_4 is birationally a double cover

$$\pi_Q : \overline{W}_4 \rightarrow Q$$

with equation

$$\begin{aligned} w^2 &= (3z^2 X^2 + a_4 z X^2 - Y^2 + a_5 XY)^2 \\ &\quad - 4 \left((3z^4 + (2a_4 + a_2 - a_5) z^3 - a_5 a_{420} z^2) X^4 + (2z^3 + a_{420} z^2) X^3 Y \right) \end{aligned} \tag{6.5}$$

where

$$\overline{W}_4 \subseteq \overline{P} := \mathbb{P}(\mathcal{O}_{B_3}(6N) \oplus \mathcal{O}_{B_3}(2N) \oplus \mathcal{O}_{B_3}(3N)) = \mathbb{P}_{[w, X, Y]}$$

and

$$Q = \mathbb{P}_{[X,Y]} := \mathbb{P}(\mathcal{O}_{B_3}(2N) \oplus \mathcal{O}_{B_3}(3N))$$

with branch locus Δ given by the zeros of the homogeneous polynomial (6.4). Notice that (6.5) is invariant under the transformations

$$[(a_0, a_2, a_3, a_4, a_5), z, X, Y, w] \mapsto [(-a_0, -a_2, -a_3, -a_4, -a_5), -z, X, -Y, \pm w]$$

so both are possible but only one will leave the holomorphic four-form on \overline{W}_4 invariant, namely the one that is compatible with the action of $\tilde{\beta}_4/\beta_3$ that transforms the relative one-form by

$$\frac{dx}{y} \mapsto -\frac{dx}{y}.$$

To see which one, we divide (6.5) by X^4 and define

$$\vartheta_0 := \frac{Y}{X}.$$

We have the affine equation

$$w_0^2 = (3z^2 + a_4z - \vartheta_0^2 + a_5\vartheta_0)^2 - 4((3z^4 + (2a_4 + a_2 - a_5)z^3 - a_5a_{420}z^2) + (2z^3 + a_{420}z^2)\vartheta_0). \quad (6.6)$$

Therefore, since $\vartheta_0 \mapsto -\vartheta_0$, $\tilde{\beta}_4/\beta_3$ is only compatible with the transformation

$$\frac{dw_0}{\vartheta_0} \mapsto -\frac{dw_0}{\vartheta_0}$$

on the relative one-form so we conclude that w_0 must be invariant. Furthermore, since the sections ζ and τ are now given by $\{X = 0\}$, so, referring to (6.5), their equation becomes

$$0 = w^2 - Y^4 = (w + Y^2)(w - Y^2)$$

and so each of the two sections must be taken to itself under $\tilde{\beta}_4/\beta_3$.

The canonical bundle of Q is

$$-2(X_0) - 2N$$

where (X_0) denotes the divisor $\{X_0 = 0\}$ whereas the branch locus Δ has divisor class

$$4(X_0) + 4N.$$

We therefore replace W_4 with the birationally equivalent double cover

$$\pi_Q : \overline{W}_4 \rightarrow Q$$

branched over Δ and \overline{W}_4 is Calabi-Yau. Furthermore

$$\overline{W}_4 \cap \{X = 0\} = (\bar{\zeta}) \cup (\bar{\tau})$$

is the union of the proper transforms of the two original sections of W_4/B_3 .

6.3 The branch locus

The branch locus Δ is defined by the equation

$$0 = \left((3z^2 + a_4z) - \vartheta_0(\vartheta_0 - a_5) \right)^2 - 4 \left((3z^4 + (2a_4 + a_2 - a_5)z^3 - a_5a_{420}z^2) + (2z^3 + a_{420}z^2)\vartheta_0 \right) \quad (6.7)$$

on the space

$$Q - \{X = 0\} = |\mathcal{O}_{B_3}(N)|,$$

the total space of the line bundle $\mathcal{O}_{B_3}(N)$. Then the equation for

$$(\overline{W}_4 - \{X = 0\}) \subseteq (\overline{P} - \{X = 0\})$$

can be rewritten as

$$w_0^2 = \left((3z^2 + a_4z) - \vartheta_0(\vartheta_0 - a_5) \right)^2 - 4a_{420}z^2(\vartheta_0 - a_5) - 4z^3(3z + 2a_4 + a_2 - (\vartheta_0 + (\vartheta_0 - a_5))). \quad (6.8)$$

Again rescaling the a_j and appealing to Bertini's theorem, we will have for general choices of the $a_j \in \mathcal{A}$ that singularities of Δ are supported on the locus

$$\{\vartheta_0(\vartheta_0 - a_5) = z = 0\}.$$

Thus $S_{\text{GUT}} \times_{B_3} \overline{W}_4$ has two components, each isomorphic to $S_{\text{GUT}} \times_{B_3} Q$. They are given by

$$w_0 = z = \pm \vartheta_0(\vartheta_0 - a_5).$$

One of these components, that we will denote as D_1 , intersects (ζ) , while the other, that we will denote as D_4 , intersects (τ) . We write $D_1 =: \{G_1 = 0\}$ and $D_4 =: \{G_4 = 0\}$ so that

$$G_1 G_4 \quad (6.9)$$

on \overline{W}_4 .

Over $\{z = 0\}$ the two components coincide as the component $\{\vartheta(\vartheta_0 - a_5) = 0\}$ locus of the reducible double cover

$$\left\{ w_0^2 = \vartheta_0^2 (\vartheta_0 - a_5)^2 \right\} \quad (6.10)$$

of

$$Q \times_{B_3} \{z = 0\}.$$

6.4 The standard \mathbb{P}^{112} -formulation

To relate the presentation (6.4) to the more standard \mathbb{P}^{112} -notation used for this type of model, let

$$\check{P} := \mathbb{P}(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(2) \oplus \mathcal{O}_{B_3}(4))$$

with coordinates (w, x, y) and write

$$\begin{aligned} X &\rightarrow w \\ Y &\rightarrow x \\ w &\rightarrow y + x^2 \end{aligned}$$

so that

$$w^2 = (3z^2X^2 + a_4X^2z - Y(Y - a_5X))^2 - 4((3z^4 + (2a_4 + a_2 - a_5)z^3 - a_5a_{420}z^2)X^4 + (2z^3 + a_{420}z^2)X^3Y)$$

becomes

$$(y + x^2)^2 = (w^2(3z^2 + a_4z) - x(x - a_5w))^2 - 4((3z^4 + (2a_4 + a_2 - a_5)z^3 - a_5a_{420}z^2)w^4 + (2z^3 + a_{420}z^2)xw^3)$$

that can be rewritten as

$$\begin{aligned} y^2 + 2x^2y &= -2a_5x^3w \\ &+ (a_5^2 - 2a_4z - 6z^2)x^2w^2 \\ &+ (2a_5a_4z + (6a_5 - 4a_{420})z^2 - 8z^3)xw^3 \\ &- ((-a_4^2 - 4a_5a_{420})z^2 + (2a_4 + 4a_2 - 4a_5)z^3 + 3z^4)w^4. \end{aligned} \tag{6.11}$$

We include appendix A.1 by Sakura Schäfer-Nameki containing a brief overview of elliptically-fibered Calabi-Yau manifolds symmetric with respect to two sections.

7 Desingularization of \overline{W}_4

7.1 Localizing at the singularities of \overline{W}_4

We first rearrange terms of (6.7) in increasing order of total degree in the variables (z, ϑ_0) so that the branch locus Δ is given by

$$\begin{aligned} &(a_5\vartheta_0 + a_4z)^2 + 4a_5a_{420}z^2 \\ &- 2(a_5\vartheta_0^3 + a_4\vartheta_0^2z) + (6a_5 - 4a_{420})z^2\vartheta_0 + (-2a_4 - 4a_2 + 4a_5)z^3 \\ &+ \vartheta_0^4 - 6\vartheta_0^2z^2 - 8z^3\vartheta_0 - 3z^4. \end{aligned} \tag{7.1}$$

Δ will have ordinary nodal singularities along $\{z = \vartheta_0 = 0\}$ except where the quadratic normal cone

$$(a_5\vartheta_0 + a_4z)^2 + 4a_5a_{420}z^2 = 0$$

is not of maximal rank, namely where

$$a_5a_{420} = 0.$$

If $a_5a_{420} = 0$, the reduced quadratic cone is given by

$$[\vartheta_0, z] = [a_4, -a_5].$$

on which the cubic cone evaluates as

$$a_5^2(a_{420}a_4 + (2a_4 + a_2 + a_5)a_5) = 0.$$

So the quadratic and cubic cone both vanish identically over $\{a_5 = 0\}$ but both only vanish over $\{a_{420} = a_{53} = 0\}$ when additionally

$$a_5^3 (a_4 - a_0 + a_5) = 0, \tag{7.2}$$

a locus that has only finite intersection with S_{GUT} .

We next rearrange terms of (6.7) in increasing order of total degree in the variables $(z, a_5 - \vartheta_0)$. The branch locus Δ is then given by

$$\begin{aligned} 0 = & (a_5 (a_5 - \vartheta_0) + a_4 z)^2 \\ & - 2a_5 (a_5 - \vartheta_0)^3 - 2a_4 z (a_5 - \vartheta_0)^2 + (4a_{420} + 6a_5) (a_5 - \vartheta_0) z^2 - (2a_4 - 4a_2 - 4a_5) z^3 \\ & + (a_5 - \vartheta_0)^4 - 6(a_5 - \vartheta_0)^2 z^2 + 8z^3 (a_5 - \vartheta_0) - 3z^4. \end{aligned} \tag{7.3}$$

The quadratic normal cone along $\{z = a_5 - \vartheta_0 = 0\}$ is of rank one except where $a_5 = a_4 = 0$. It is spanned by the vector $(a_5 - \vartheta_0, z) = (a_4, -a_5)$. Evaluating the cubic normal cone along this vector and recalling that $a_{420} = -a_{53}$ gives

$$(-a_4 a_3 + a_5 (a_0 + a_3)) a_5^2 = 0.$$

Thus a simple modification will resolve the singularity of \overline{W}_4 over $\{a_5 - \vartheta_0 = z = 0\}$ unless $a_5 = 0$ or

$$\begin{vmatrix} a_4 & -a_5 \\ a_0 + a_3 & a_3 \end{vmatrix} = 0. \tag{7.4}$$

7.2 Resolving the singularities of \overline{W}_4

7.2.1 First modification of Δ

We first make the modification of Δ and \overline{W}_4 at $\{\vartheta = z = 0\}$. Namely, inside

$$Q^{(1)} := \left\{ \begin{vmatrix} \vartheta_0 & z \\ \vartheta_1 & Z_{14} \end{vmatrix} \right\} \subseteq Q \times \mathbb{P}_{[\vartheta_1, Z_{14}]}$$

we write

$$Z_0 := \frac{\vartheta_0}{\vartheta_1} = \frac{z}{Z_{14}}.$$

The proper transform $\Delta^{(1)}$ of Δ in $Q^{(1)}$ is given by the equation

$$\begin{aligned} 0 = & (a_5 \vartheta_1 + a_4 Z_{14})^2 + 4a_5 a_{420} Z_{14}^2 \\ & + (-2a_5 \vartheta_1^3 - 2a_4 \vartheta_1^2 Z_{14} + (6a_5 - 4a_{420}) \vartheta_1 Z_{14}^2 - (2a_4 + 4a_2 - 4a_5) Z_{14}^3) Z_0 \\ & + (\vartheta_1^4 - 6\vartheta_1^2 Z_{14}^2 - 8Z_{14}^3 \vartheta_1 - 3Z_{14}^4) Z_0^2. \end{aligned} \tag{7.5}$$

Over $z = Z_{14} Z_0 = 0$ we have two possible singular loci in $\Delta^{(1)}$. These loci are given respectively by

$$\left\{ Z_{14} = (\vartheta_1 Z_0 - a_5)^2 = 0 \right\}$$

where $Z_{14} = 0$ and singular points of

$$\left\{ Z_0 = (a_5\vartheta_1 + a_4Z_{14})^2 + 4a_5a_{420}Z_{14}^2 = 0 \right\}$$

at which $Z_0 = 0$ and $Z_{14} \neq 0$. In this last case we write

$$Z_0 = \left(a_5 \frac{\vartheta_1}{Z_{14}} + a_4 \right)^2 + 4a_5a_{420} = 0$$

to conclude that singular points can only occur where the discriminant

$$\{4a_5a_{420} = 0\}$$

of the quadratic equation in $\frac{\vartheta_1}{Z_{14}}$ is singular, that is, where $Z_0 = a_5 = a_{420} = 0$. Substituting in (7.5) we conclude that, in addition, $a_4 = 0$. Since $\frac{\partial}{\partial Z_0}$ applied to (7.5) and evaluated at such singular points would also have to vanish, we would also have $a_2 = 0$ and so $a_0 = 0$ contradicting the assumption of a generic allowable selection of the a_j and z in the linear system $|\mathcal{A}|$.

Lemma 4. *Singularities of the branch locus $\Delta^{(1)}$ lie on locus*

$$\left\{ Z_{14} = (\vartheta_1 Z_0 - a_5)^2 = 0 \right\}.$$

The assumption that the curve $\{a_5 = z = 0\} \subseteq B_3$ is smooth can be weakened to allow nodal singularities. Since $a_4 = 0$ there, the potential nodal singularities in $\Delta^{(1)}$ thereby created are in fact already resolved by the fact that the entire fiber of Z_0 already lies in $\Delta^{(1)}$ over any point where $a_5 = a_4 = 0$. In fact, in section 6.3 of [15] we impose a normal-crossing assumption on $\{a_5 = z = 0\} \subseteq B_3$.

The proper transform $\Delta^{(1)}$ of Δ in $Q^{(1)}$ is the total transform minus twice the exceptional divisor, so that the proper transform is again twice the anticanonical divisor of $Q^{(1)}$. So the branched double cover $W_4^{(1)}$ given by the equation

$$\begin{aligned} w_1^2 &= (a_5\vartheta_1 + a_4Z_{14})^2 + 4a_5a_{420}Z_{14}^2 \\ &+ (-2a_5\vartheta_1^3 - 2a_4\vartheta_1^2Z_{14} + (6a_5 + 4a_{420})\vartheta_1Z_{14}^2 - (2a_4 + 4a_2 - 4a_5)Z_{14}^3)Z_0 \\ &+ (\vartheta_1^4 - 6\vartheta_1^2Z_{14}^2 - 8Z_{14}^3\vartheta_1 - 3Z_{14}^4)Z_0^2. \end{aligned} \quad (7.6)$$

is again Calabi-Yau. We retain the notation D_1 and D_4 for the proper transforms in $W_4^{(1)}$ of those respective divisors in \overline{W}_4 , namely the two components of $W_4^{(1)} \times_{\Delta^{(1)}} \{Z_{14} = 0\}$. We denote the irreducible branched double cover $W_4^{(1)} \times_{Q^{(1)}} \{Z_0 = 0\}$ as D_0 . Over $\{z = a_5a_{420} = 0\}$, D_0 splits into two components, $\{w_1 = \pm (a_5\vartheta_1 + a_4Z_{14})\}$, and each component projects isomorphically to $\{Z_0 = 0\} \subseteq Q^{(1)}$.

7.2.2 Second modification of Δ

Recalling that $z = Z_0Z_{14}$ we must next attend to the singularities of $\Delta^{(1)}$ lying in $\{Z_{14} = 0\}$. As we have seen in the last subsection, these lie on the locus $\{a_5 - \vartheta_1Z_0 = Z_{14} = 0\}$ that only intersects D_0 over $\{a_5 = Z_0 = 0\}$.

First, referring to (7.5), we rewrite the equation for $\Delta^{(1)}$ in terms of the variables $((a_5 - \vartheta_1 Z_0), Z_{14})$ as

$$\begin{aligned}
 0 &= (\vartheta_1 (a_5 - \vartheta_1 Z_0) + a_4 Z_{14})^2 \\
 &\quad (6a_5 + 4a_{420}) (a_5 - \vartheta_1 Z_0) Z_{14}^2 - (4a_5 + 2a_4 + 4a_2) Z_{14}^3 Z_0 \\
 &\quad - 6 (a_5 - \vartheta_1 Z_0)^2 Z_{14}^2 + 8 (\vartheta_1 Z_0 - a_5) Z_{14}^3 Z_0 - 3 Z_{14}^4 Z_0^2.
 \end{aligned} \tag{7.7}$$

Next define

$$Q^{(2)} := \left\{ \begin{array}{c} a_5 - \vartheta_1 Z_0 \quad Z_{14} \\ \vartheta_2 \quad \tilde{Z}_{14} \end{array} \middle| = 0 \right\} \subseteq Q^{(1)} \times \mathbb{P}_{[\vartheta_2, \tilde{Z}_{14}]}$$

and

$$Z_{23} := \frac{a_5 - \vartheta_1 Z_0}{\vartheta_2} = \frac{Z_{14}}{\tilde{Z}_{14}}$$

for which we have

$$\pi^{(2)} : Q^{(2)} \rightarrow B_3.$$

The equation for the proper transform $\Delta^{(2)}$ of $\Delta^{(1)}$ becomes

$$\begin{aligned}
 0 &= \left(\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14} \right)^2 \\
 &\quad + \left((6a_5 + 4a_{420}) \vartheta_2 - (4a_5 + 2a_4 + 4a_2) Z_0 \tilde{Z}_{14} \right) \tilde{Z}_{14}^2 Z_{23} \\
 &\quad - \left(6\vartheta_2^2 - 8\vartheta_2 Z_0 \tilde{Z}_{14} + 3\tilde{Z}_{14}^2 Z_0^2 \right) \tilde{Z}_{14}^2 Z_{23}^2.
 \end{aligned} \tag{7.8}$$

The proper transform $\Delta^{(2)}$ in $Q^{(2)}$ is the total transform minus twice the exceptional divisor, so that the proper transform is again twice the anticanonical divisor of $Q^{(2)}$. Therefore we have a Calabi-Yau fourfold $W_4^{(2)}$ given by the completion over $\{X = 0\}$ of the solution set of

$$\begin{aligned}
 w_2^2 &= \left(\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14} \right)^2 \\
 &\quad + \left((6a_5 + 4a_{420}) \vartheta_2 - (4a_5 + 2a_4 + 4a_2) Z_0 \tilde{Z}_{14} \right) \tilde{Z}_{14}^2 Z_{23} \\
 &\quad - \left(6\vartheta_2^2 - 8\vartheta_2 Z_0 \tilde{Z}_{14} + 3\tilde{Z}_{14}^2 Z_0^2 \right) \tilde{Z}_{14}^2 Z_{23}^2.
 \end{aligned} \tag{7.9}$$

Over the exceptional locus, given by $\{Z_{23} = 0\} \subseteq Q^{(2)}$, the equation for $\Delta^{(2)}$ is a perfect square so that $W_4^{(2)} \times_{B_3} S_{\text{GUT}}$ splits into five components. Two of these are new components lying over $\{Z_{23} = 0\} \subseteq Q^{(2)}$ that we call D_2 and D_3 . In addition we have lifted components, the two components that we continue to call D_1 and D_4 lying over $\{\tilde{Z}_{14} = 0\} \subseteq Q^{(2)}$, and finally the lifted component over $\{Z_0 = 0\}$ that we continue to call D_0 . We number things so that, over a general point of S_{GUT} , D_2 intersects D_1 and D_3 intersects D_4 . The incidence of the five components over a general point of S_{GUT} is that of the extended Dynkin diagram \tilde{A}_4 .

By lemma 4 singular points of $\Delta^{(2)}$ can only occur where $Z_{23} = 0$, so $\tilde{Z}_{14} \neq 0$ there and these points can be singular points of $\Delta^{(2)}$ only if $\frac{\vartheta_1 \vartheta_2}{\tilde{Z}_{14}} = -a_4$ and

$$\frac{\vartheta_2}{\tilde{Z}_{14}} = \frac{(2a_4 + 4a_2 + 4a_5) Z_0}{6a_5 + 4a_{420}}.$$

Multiplying both sides of this last equation by ϑ_1 and substituting, then recalling that at these points $a_5 - \vartheta_1 Z_0 = 0$ and that $a_{54320} = 0$, we obtain

$$a_5 a_0 + a_3 (a_5 + a_4) = 0$$

that can be rewritten as the relation

$$Z_0 \cdot \begin{vmatrix} a_4 & -a_5 \\ a_0 + a_3 & a_3 \end{vmatrix} = 0$$

given in (7.4).

Lemma 5. *Singularities of the branch locus $\Delta^{(2)}$ only occur over*

$$z = \begin{vmatrix} a_4 & -a_5 \\ a_0 + a_3 & a_3 \end{vmatrix} = 0$$

and then only where

$$\frac{\vartheta_2}{\tilde{Z}_{14}} = \frac{(a_4 + 2a_2 + 2a_5) Z_0}{a_5 - 2a_3}.$$

Over a general point of $\{z = a_5 = 0\}$, D_0 splits into the two components $D_{01} + D_{04}$ that reconfigure with the specialization of $D_2 + D_3$ over $\{z = a_5 = 0\}$ to form the extended Dynkin diagram for D_5 as is shown by the relation

$$\vartheta_1 Z_0 + \vartheta_2 Z_{23} = a_5.$$

Over a general point of $\{z = a_{420} = a_{53} = 0\}$ again D_0 splits into two components, augmenting the extended Dynkin diagram \tilde{A}_4 to the extended Dynkin diagram \tilde{A}_5 .

7.2.3 Singularities of higher codimension

We have seen in lemma 5 that, at singular points of $\Delta^{(2)}$, $Z_{23} = 0$ and $\tilde{Z}_{14} \neq 0$ and they lie over

$$\left\{ z = \begin{vmatrix} a_4 & -a_5 \\ a_0 + a_3 & a_3 \end{vmatrix} = 0 \right\} \subseteq B_3. \tag{7.10}$$

In fact, if one rewrites (7.9) in the form

$$\begin{vmatrix} w_2 - (\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14}) & \tilde{Z}_{14}^2 Z_{23} \\ -A & w_2 + (\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14}) \end{vmatrix} = 0 \tag{7.11}$$

where

$$\begin{aligned} -A = & \left((6a_5 + 4a_{420}) \vartheta_2 - (4a_5 + 2a_4 + 4a_2) Z_0 \tilde{Z}_{14} \right) \\ & - \left(6\vartheta_2^2 - 8\vartheta_2 Z_0 \tilde{Z}_{14} + 3\tilde{Z}_{14}^2 Z_0^2 \right) Z_{23} \end{aligned}$$

over $\{\tilde{Z}_{14} = 0\}$ the equation of $W_4^{(2)}$ becomes

$$(w_2 - \vartheta_1 \vartheta_2)(w_2 + \vartheta_1 \vartheta_2) = 0.$$

Recalling that \tilde{Z}_{14} and $\vartheta_1 \vartheta_2$ cannot vanish simultaneously, one sees that

$$C_{\mathfrak{S}}^{(44)} \cap \{\tilde{Z}_{14} = 0\} = \emptyset \tag{7.12}$$

where $C_{\mathfrak{S}}^{(44)}$ is given by the vanishing

$$\begin{aligned} w_2 &= \vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14} = Z_{23} = \\ (a_5 - 2a_3) \vartheta_2 - (a_4 + 2a_2 + 2a_5) Z_0 \tilde{Z}_{14} &= 0 \end{aligned} \tag{7.13}$$

of all four entries in the 2×2 matrix (7.11) is a simple nodal locus of $W_4^{(2)}$. Since \tilde{Z}_{14} is never zero along $C_{\mathfrak{S}}^{(44)}$ we conclude that

$$(D_1 \cup D_4) \cap C_{\mathfrak{S}}^{(44)} = \emptyset.$$

The entries in (7.11) are all invariant under the action of $\tilde{\beta}_4/\beta_3$. Therefore we can take either small blow-up to complete the crepant resolution.

7.3 The smooth model \tilde{W}_4

After taking the small resolution of the nodal curve $C_{\mathfrak{S}}^{(44)}$ in $W_4^{(2)}$ as described just above, we obtain the inclusion of proper transform

$$\tilde{\Delta} \subseteq \tilde{W}_4$$

where $\tilde{\Delta}$ is non-singular.

We retain the notations D_j , $j = 0, \dots, 4$, for the proper transforms in \tilde{W}_4 of the corresponding divisors in $W_4^{(2)}$. We will let G_j denote the canonical section of the line bundle determined by the divisor D_j , that is

$$D_j = \{G_j = 0\}.$$

Remark 4. The exceptional divisors D_j are identified with the positive simple roots of $SU(5)_{\text{gauge}}$ in such a way that the involution β_3 on B_3 and the action

$$\frac{w_0}{\vartheta_0} \mapsto -\frac{w_0}{\vartheta_0}$$

on the fibers of \tilde{W}_4/B_3 induces the non-trivial geometric action

$$\begin{aligned} D_0 &= -(D_1 + D_2 + D_3 + D_4) \mapsto -D_0 \\ D_1 &\mapsto -D_4 \\ D_2 &\mapsto -D_3 \\ D_3 &\mapsto -D_2 \\ D_4 &\mapsto -D_1 \end{aligned} \tag{7.14}$$

on the exceptional fibers of \tilde{W}_4 . As explained in [14], the action of $\tilde{\beta}_4$ on the roots D_1, \dots, D_4 reverses the choice of positive Weyl chamber used in making the dictionary between exceptional divisors of \tilde{W}_4 and the $SU(5)_{\text{gauge}}$ -roots. This reversal of positive Weyl chamber exactly reverses the non-trivial involution (7.14) thereby preserving the $SU(5)_{\text{gauge}}$ -symmetry of the quotient W_4^\vee .⁴

The identity

$$a_5 = Z_0\vartheta_1 + Z_{23}\vartheta_2$$

intertwines D_0 and $D_2 + D_3$ above $\{z = a_5 = 0\}$. After the first modification, the equation of the spectral variety on $\{Z_0 = 0\}$ is

$$(a_5\vartheta_1 + a_4Z_{14})^2 - 4a_5a_{420}Z_{14}^2 = 0$$

whereas the equations of the center of the second modification are

$$a_5 - \vartheta_1 = Z_{14} = 0.$$

Since ϑ_1 and Z_{14} cannot vanish simultaneously, therefore after the second modification, the proper transform of $\{Z_0 = 0\}$ will only intersect $\{Z_{23} = 0\}$ along the fiber of $\{Z_{23} = 0\}$ over $\{z = a_5 = 0\}$. If the modifications were done in the opposite order, the divisors $\{Z_0 = 0\}$ and $\{Z_{23} = 0\}$ in $\tilde{\Delta}$ would have been flopped.

7.4 The spectral divisor in \tilde{W}_4

Referring to (3.1), the spectral divisor is obviously invariant under the involution $(x, y) \mapsto (x, -y)$. To compute the image

$$\mathcal{C}_0^{(4)} + \mathcal{C}_0^{(1)} \subseteq Q$$

of $\mathcal{D}^{(4)} + \mathcal{D}^{(1)}$ we write

$$t = \frac{y}{x} = \frac{Y - (z + a_{420})z^2w}{X + z^2w}$$

so that

$$t - z = \frac{Y - zX - (2z + a_{420})z^2w}{X + z^2w}$$

and

$$t + z = \frac{Y + zX - a_{420}z^2w}{X + z^2w}.$$

So, setting $\frac{X}{z^2w} = 1$ and $\frac{Y}{z^2w} = \vartheta$, the affine equations

$$\begin{aligned} (\vartheta - (z + a_{420}))^2 \left(a_5 (\vartheta - (z + a_{420}))^2 + a_{54} (z (\vartheta - 2(z + a_{420}))) - 4a_{20}z^2 \right) \\ - 8a_0z^3 (z (\vartheta + z - a_{420})) = 0 \end{aligned} \tag{7.15}$$

and

$$\vartheta - (3z + a_{420}) = 0$$

⁴This assertion is reflected in the fact that the action of $\tilde{\beta}_4$ leaves all entries of equation (4.10) of [14] invariant.

are the respective equations for the image $\mathcal{C}_0^{(4)} + \mathcal{C}_0^{(1)} \subseteq Q$ of the components of the spectral divisor. Setting $z = 0$ we obtain

$$\begin{aligned} a_5 (\vartheta_0 - a_{420})^4 &= 0 \\ \vartheta_0 - a_{420} &= 0 \end{aligned} \tag{7.16}$$

consistent with the fact that the inverse image of both $\mathcal{C}_0^{(4)}$ and $\mathcal{C}_0^{(1)}$ in \overline{W}_4 are reducible, only one of its two components correspond to the image

$$\overline{\mathcal{C}}_{\text{Higgs}} = \overline{\mathcal{C}}_{\text{Higgs}}^{(4)} + \overline{\mathcal{C}}_{\text{Higgs}}^{(1)} \subseteq \overline{W}_4 \tag{7.17}$$

of $\mathcal{D}^{(4)} + \mathcal{D}^{(1)}$ and only $\mathcal{C}_0^{(4)}$ intersects the proper transform $\{X = 0\}$ of $(\zeta) + (\tau)$ and it only intersects simply along one of the two sections. We let

$$\mathcal{C}_{\text{Higgs}} = \mathcal{C}_{\text{Higgs}}^{(4)} + \mathcal{C}_{\text{Higgs}}^{(1)} \subseteq \tilde{Q} = Q^{(2)}$$

denote the proper (also the total) transform of (7.17) and

$$\tilde{\mathcal{C}}_{\text{Higgs}} \subseteq \tilde{W}_4$$

denote the proper (also the total) transform of $\overline{\mathcal{C}}_{\text{Higgs}}$. Then

$$\mathcal{C}_{\text{Higgs}} := \left(\pi_{\tilde{Q}} \right)_* \left(\tilde{\mathcal{C}}_{\text{Higgs}} \right).$$

The equations

$$\begin{aligned} z &= Z_0 \tilde{Z}_{14} Z_{23} \\ \vartheta - a_{420} &= \vartheta_1 Z_0 - a_{420} \end{aligned}$$

imply that $\mathcal{C}_{\text{Higgs}}$ is the total transform of \mathcal{C}_0 so that

$$\mathcal{C}_{\text{Higgs}} \times_{B_3} S_{\text{GUT}} \subseteq \left\{ \tilde{Z}_{14} = 0 \right\}. \tag{7.18}$$

Therefore $\vartheta_1 \vartheta_2$ never vanishes on $\mathcal{C}_{\text{Higgs}} \times_{B_3} S_{\text{GUT}}$, and this in turn implies that neither component of $\tilde{\mathcal{C}}_{\text{Higgs}} \times_{B_3} S_{\text{GUT}}$ can lie in the inherited component $D_1 \subseteq \tilde{W}_4$ containing the proper transform of (ζ) . Thus

$$\tilde{\mathcal{C}}_{\text{Higgs}} \times_{B_3} S_{\text{GUT}} \subseteq \{D_4 = 0\}. \tag{7.19}$$

Finally, by (7.15), $\mathcal{C}_{\text{Higgs}}^{(4)} + \mathcal{C}_{\text{Higgs}}^{(1)}$ has homology class is

$$(4(X_2) + 5N) + ((X_2) + N). \tag{7.20}$$

7.4.1 Locating matter curves and the Higgs curve

Referring to (7.16) $S_{\text{GUT}} \subseteq B_3$ is lifted into \tilde{W}_4 via

$$\left\{ \vartheta_1 Z_0 - a_{420} = \tilde{Z}_{14} = 0 \right\} \subseteq \tilde{Q}$$

with the property that ϑ_1 is never zero on the lifting. The matter and Higgs curves will all lie in this lifting.

Therefore, assuming that no component lies in $\{a_{420} = z = 0\}$,

$$\Sigma_{\mathbf{10}}^{(4)} = \left\{ a_5 = \tilde{Z}_{14} = \vartheta_1 Z_0 - a_{420} = 0 \right\} \subseteq \mathcal{C}_{\text{Higgs}}. \quad (7.21)$$

Since ϑ_1 cannot vanish when $\tilde{Z}_{14} = 0$,

$$\Sigma_{\mathbf{5}}^{(41)} := \left\{ Z_0 = a_{420} = \tilde{Z}_{14} = 0 \right\} \subseteq \mathcal{C}_{\text{Higgs}}^{(4)}. \quad (7.22)$$

By (7.12) and (7.13) the singular curve $C_{\mathbf{5}}^{(44)}$ determining the Higgs curve must also lie on

$$\begin{aligned} \vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14} &= Z_{23} = \\ (a_5 - 2a_3) \vartheta_2 - (a_4 + 2a_2 - 2a_5) Z_0 \tilde{Z}_{14} &= 0 \end{aligned}$$

as well as

$$\left\{ \begin{vmatrix} a_4 & -a_5 \\ a_0 + a_3 & a_3 \end{vmatrix} = 0 \right\} \cap \{Z_{23} = 0\}$$

so that $\vartheta_1 \vartheta_2 \neq 0$ and, so by lemma 5

$$\frac{\vartheta_2}{\tilde{Z}_{14}} = \frac{(a_4 + 2a_2 + 2a_5) Z_0}{a_5 - 2a_3}$$

and therefore

$$\Sigma_{\mathbf{5}}^{(44)} := \left\{ \begin{vmatrix} a_4 & a_5 \\ a_0 + a_3 & -a_3 \end{vmatrix} = \tilde{Z}_{14} = \vartheta_1 Z_0 - a_{420} = 0 \right\} \subseteq \mathcal{C}_{\text{Higgs}}^{(4)}. \quad (7.23)$$

Furthermore the two values

$$\frac{\tilde{t}}{\tilde{z}} = \pm \sqrt{\frac{a_0}{a_{54}}}$$

in (3.10) lie in $\mathcal{C}_{\text{Higgs}}^{(4)}$ does not meet the singular curve $C_{44} \subseteq \tilde{\Delta}$. The lifted spectral curves $\tilde{\Sigma}_{\mathbf{10}}^{(4)}, \tilde{\Sigma}_{\mathbf{5}}^{(41)}, \tilde{\Sigma}_{\mathbf{5}}^{(44)} \subseteq \tilde{\mathcal{C}}_{\text{Higgs}}$ and by (7.19) have associated spectral surfaces given by

$$\begin{aligned} \tilde{E}_{\mathbf{10}}^{(4)} &:= \tilde{\Sigma}_{\mathbf{10}}^{(4)} \times_{S_{\text{GUT}}} D_4 \cong E_{\mathbf{10}}^{(4)} := \Sigma_{\mathbf{10}}^{(4)} \times_{S_{\text{GUT}}} \left\{ \tilde{Z}_{14} = 0 \right\}. \\ \tilde{E}_{\mathbf{5}}^{(41)} &:= \tilde{\Sigma}_{\mathbf{5}}^{(41)} \times_{S_{\text{GUT}}} D_4 \cong \Sigma_{\mathbf{5}}^{(41)} := \Sigma_{\mathbf{5}}^{(41)} \times_{S_{\text{GUT}}} \left\{ \tilde{Z}_{14} = 0 \right\} \\ \tilde{E}_{\mathbf{5}}^{(44)} &:= \tilde{\Sigma}_{\mathbf{5}}^{(44)} \times_{S_{\text{GUT}}} D_4 \cong \Sigma_{\mathbf{5}}^{(44)} := \Sigma_{\mathbf{5}}^{(44)} \times_{S_{\text{GUT}}} \left\{ \tilde{Z}_{14} = 0 \right\}. \end{aligned} \quad (7.24)$$

7.5 Topology of \tilde{W}_4

7.5.1 Picard groups

The Picard group of \tilde{Q} is freely generated by generators

$$\begin{aligned} &\pi_{B_3}^* (\text{Pic}(B_3)) \\ &(X), (Z_0), (Z_{23}) \end{aligned}$$

where (X) denotes the divisor given by $X_0 = 0$, etc. The Picard group of \tilde{W}_4 is generated by

$$\pi_{\tilde{Q}}^* \text{Pic}(\tilde{Q}), (\tilde{\zeta}), (\tilde{\tau}), D_0, \dots, D_4$$

with relations

$$\begin{aligned} \pi_{\tilde{Q}}^*((X_0)) &= (\tilde{\zeta}) + (\tilde{\tau}) \\ \pi_{\tilde{Q}}^*((Z_{34})) &= D_2 + D_3 \\ \pi_{\tilde{Q}}^*((Z_1)) &= D_0 \\ \tilde{\pi}_{B_3}^*(S_{\text{GUT}}) &= \sum_{j=0}^4 D_j. \end{aligned} \tag{7.25}$$

The cohomology class of $\tilde{\Delta} \subseteq \tilde{Q}$ is given by

$$4((X_2) + N) - 2((Z_0) + (Z_{23})) \tag{7.26}$$

and

$$K_{\tilde{Q}} = - (2(X_2) + 2N) + ((Z_0) + (Z_{23}))$$

where, as before, $N = \tilde{\pi}^*(c_1(\mathcal{N}))$ and \mathcal{N} is the line bundle on B_3 whose sections include z and the a_j . Thus the Picard group of $\tilde{\Delta}$ admits the effective square root

$$2((X_2) + N) - ((Z_0) + (Z_{23}))$$

of the branch locus of $\pi_{\tilde{Q}}$. Also

$$\begin{aligned} \mathcal{C}_{\text{Higgs}} &= \mathcal{C}_{\text{Higgs}}^{(4)} + \mathcal{C}_{\text{Higgs}}^{(1)} = \mathcal{C}_{\text{Higgs}}^{(4)} + \text{image}(\tilde{\tau}) \\ &\equiv 5(X_2) + 5N \in \text{Pic}(\tilde{Q}). \end{aligned} \tag{7.27}$$

We have the following linear equivalences on \tilde{Q} :

$$\begin{aligned} (\vartheta_0) &= (Y_0) - (X_0) \equiv N \\ (\vartheta_1) &= (Y_1) - (X_1) = ((Y_0) - (Z_0)) - (X_0) \\ (\vartheta_2) &= (Y_2) - (X_2) = ((Y_0) - (Z_{23})) - (X_0) \\ (\vartheta_1) + (Z_0) &= (\vartheta_2) + (Z_{23}) \equiv N. \end{aligned} \tag{7.28}$$

Finally, since none of the blow-ups in the resolution over B_3 touch (X_0) , from now on we will simply identify

$$(X) := (X_0) = (X_1) = (X_2).$$

7.5.2 Intersections in $Q^{(2)}$

We compute push-forwards to B_3 of intersections in \tilde{Q} as follows. From the fact that $(\pi_{B_3})_*((Z_0) \cdot (Z_{23}))$ is supported on the curve $\{z = a_5 = 0\}$ we conclude that

$$(\pi_{B_3})_*((Z_0) \cdot (Z_{23})) \equiv 0.$$

Appealing to (7.25) and (7.26), we compute

$$\begin{aligned}
 (\pi_{B_3})_* \left((X) \cdot_{\tilde{Q}} (X) \right) &\equiv -N \\
 (\pi_{B_3})_* \left((X) \cdot_{\tilde{Q}} (Z_0) \right) &\equiv (\pi_{B_3})_* \left((X) \cdot_{\tilde{Q}} (Z_{23}) \right) \equiv 0 \\
 (\pi_{B_3})_* \left((X) \cdot_{\tilde{Q}} \mathcal{C}_{\text{Higgs}} \right) &\equiv 0 \\
 (\pi_{B_3})_* \left((Z_0)^2 \right) &\equiv -N \\
 (\pi_{B_3})_* \left((Z_{23})^2 \right) &\equiv -N.
 \end{aligned}
 \tag{7.29}$$

8 Higgs line bundle and the G -flux

8.1 Physical interpretation

We begin with the projection map

$$\mathcal{D} \subseteq B_3 \times \mathbb{P}_{[\tilde{t}, \tilde{z}]} \xrightarrow{\psi} B_3$$

Recall that E_8 has the subgroup $SU(5)_{\text{gauge}} \times SU(4)_{\text{Higgs}} \times U(1)_X$ and the Higgs operator is a non-trivial element of the center of the enveloping algebra of $SU(4)_{\text{Higgs}}$. We use it to break E_8 -symmetry of the subgroup $SU(4)_{\text{Higgs}}$. To accomplish this we must identify a line bundle

$$\mathcal{L}_{\text{Higgs}}$$

on \mathcal{D} such that the first Chern class of the rank-4 vector bundle

$$\mathcal{E} := \psi_* (\mathcal{L}_{\text{Higgs}})$$

is zero. We then geometrically specify the eigenvalues of the Higgs operator acting on the complexified Cartan subalgebra of $SU(4)_{\text{Higgs}}$. We identify the fibers of \mathcal{E} with the representation space for the fundamental representation of $SU(4)$ so that the roots

$$\frac{t}{z} = \tilde{t}_1, \dots, \tilde{t}_4$$

of

$$a_5 t^4 + a_{54} t^3 z - a_{20} t^2 z^2 - a_0 z^3 (t + z) = 0$$

in (3.8) become the eigenvalues of the Higgs operator on the eigenvectors of the standard representation of $SU(4)_{\text{Higgs}}$. The condition

$$c_1 (\mathcal{E}) = 0$$

becomes the condition that the sum of these eigenvalues is zero.

8.2 The Higgs bundle

So we begin with any line bundle \mathcal{L} on $\mathcal{D}^{(4)}$. By the Grothendieck Riemann-Roch theorem,

$$c_1(\psi_*(\mathcal{L})) = \psi_*(c_1(\mathcal{L})) - \psi_*\left(c_1\left(\mathcal{O}_{\mathcal{D}}\left(\frac{R}{2}\right)\right)\right)$$

where R is the ramification divisor of ψ .

To construct $\mathcal{L}_{\text{Higgs}}$ we proceed as follows. Applying the formula for the discriminant of a fourth-degree equation (7.1) we obtain that the discriminant has class $6 \cdot N$ on B_3 . Thus we must have

$$\psi_*\left(c_1\left(\mathcal{O}_{\mathcal{D}}\left(\frac{R}{2}\right)\right)\right) = c_1(\mathcal{N}^3)$$

so that we must choose an effective divisor class on $\mathcal{D}^{(4)}$ whose push-forward to B_3 has class $c_1(\mathcal{N}^3)$.

One obvious line bundle to use is

$$\mathcal{O}_{B_3}(N) \boxtimes \mathcal{O}_{\mathbb{P}_{[\bar{t}, \bar{z}]}}(-1)|_{\mathcal{D}} \tag{8.1}$$

that pushes forward on B_3 to

$$3N.$$

We denote the line bundle on $\mathcal{D}^{(4)}$ given by this divisor as

$$\mathcal{L}_{\text{Higgs}}^{(0)} = \mathcal{O}_{B_3}(N) \boxtimes \mathcal{O}_{\mathbb{P}_{[U, V]}}(-1)|_{\mathcal{D}^{(4)}}.$$

From subsection 7.4 the matter curves are given by

$$\begin{aligned} \Sigma_{\mathbf{10}}^{(4)} &= \{t/z = \infty\} \cdot \mathcal{D}^{(4)} \\ \Sigma_{\bar{\mathbf{5}}}^{(41)} &= \{t/z = 1\} \cdot \mathcal{D}^{(4)} \end{aligned}$$

and, as in [15] the Higgs curve is given by a branched double cover $\hat{\Sigma}_{\bar{\mathbf{5}}}^{(44)} \subseteq \mathcal{D}^{(4)}$ of the curve

$$\Sigma_{\bar{\mathbf{5}}}^{(44)} = \left\{ z = \left| \begin{array}{cc} a_4 & -a_5 \\ a_3 + a_0 & a_3 \end{array} \right| = 0 \right\} \subseteq B_3.$$

We will need the skew component of the push-forward to $\Sigma_{\bar{\mathbf{5}}}^{(44)}$ of $\mathcal{L}_{\text{Higgs}}^{(0)}$. Since the skew component of the restriction of $\mathcal{O}_{\mathbb{P}_{[\bar{t}, \bar{z}]}}(-1)$ is trivial, we need only compute the skew component of the push-forward of the pull-back of $\mathcal{O}_{B_3}(N)$. Again by the Grothendieck Riemann-Roch theorem, this is a line bundle on $\Sigma_{\bar{\mathbf{5}}}^{(44)}$ whose square is the restriction of $\mathcal{O}_{B_3}(2N)$. We will denote this line bundle as $\mathcal{L}_{\text{Higgs}}^{(0,-)}$. Since the restriction of $\mathcal{O}_{B_3}(3N)$ to $\Sigma_{\bar{\mathbf{5}}}^{(44)}$ is its canonical bundle, the restriction of $\mathcal{L}_{\text{Higgs}}^{(0,-)}$ will be a theta-characteristic of $\Sigma_{\bar{\mathbf{5}}}^{(44)}$.

8.3 Restrictions to the matter and Higgs curves

From (4.1) of [15] we have

$$c_1 \left(K_{B_3}^{-1} \right)^3 = 12$$

and so

$$\begin{aligned} \deg \left(K_{\Sigma_{10}^{(4)}} \right) &= N^3 = 12 \\ \deg \left(K_{\Sigma_{\bar{5}}^{(41)}} \right) &= N^3 = 12 \\ \deg \left(K_{\Sigma_{\bar{5}}^{(44)}} \right) &= 4N^3 = 48. \end{aligned}$$

Since the degree of the canonical bundle of a curve is $2g - 2$, the genus of $\Sigma_{\bar{5}}^{(44)}$ is 25 and the genus of both $\Sigma_{10}^{(4)}$ and $\Sigma_{\bar{5}}^{(41)}$ is 7.

To compute with $\mathcal{L}_{\text{Higgs}}^{(0)}$ we work on the preimage \mathcal{D} of $\mathcal{C}_{\text{Higgs}}$ in $B_3 \times \mathbb{P}_{[U,V]}$ where

$$\mathcal{L}_{\text{Higgs}}^{(0)} = K_{B_3}^{-1} \boxtimes \mathcal{O}_{\mathbb{P}_{[U,V]}}(-1).$$

Thus

$$\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{10}^{(4)}} = \mathcal{N} \Big|_{\Sigma_{10}^{(4)}} = K_{B_3}^{-1} \Big|_{\Sigma_{10}^{(4)}} = K_{\Sigma_{10}^{(4)}}$$

and

$$\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{\bar{5}}^{(41)}} = \mathcal{N} \Big|_{\Sigma_{\bar{5}}^{(41)}} = K_{B_3}^{-1} \Big|_{\Sigma_{\bar{5}}^{(41)}} = K_{\Sigma_{\bar{5}}^{(41)}}.$$

As we showed just above

$$\mathcal{L}_{\text{Higgs}}^{(0,-)} \Big|_{\Sigma_{\bar{5}}^{(44)}} = \mathcal{N} \Big|_{\Sigma_{\bar{5}}^{(44)}}$$

so that, in particular

$$\left(\mathcal{L}_{\text{Higgs}}^{(0,-)} \Big|_{\Sigma_{\bar{5}}^{(44)}} \right)^2 = K_{\Sigma_{\bar{5}}^{(44)}}.$$

In fact, $\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{\bar{5}}^{(44)}}$ is nothing more than the theta-characteristic $Z_2 \cdot Z$ in section 5.1 of [15] for $Z_2 = \Sigma_{\bar{5}}^{(44)}$.

Thus

$$\begin{aligned} h^0 \left(\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{10}^{(4)}} \right) - h^1 \left(\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{10}^{(4)}} \right) &= 7 - 1 = 6 \\ h^0 \left(\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{\bar{5}}^{(41)}} \right) - h^1 \left(\mathcal{L}_{\text{Higgs}}^{(0)} \Big|_{\Sigma_{\bar{5}}^{(41)}} \right) &= 7 - 1 = 6 \\ h^0 \left(\mathcal{L}_{\text{Higgs}}^{(0,-)} \Big|_{\Sigma_{\bar{5}}^{(44)}} \right) - h^1 \left(\mathcal{L}_{\text{Higgs}}^{(0,-)} \Big|_{\Sigma_{\bar{5}}^{(44)}} \right) &= 0. \end{aligned}$$

We therefore have the desired Euler characteristics for the Higgs bundle on the matter and Higgs curves. However the ranks of the relevant spaces of sections are not quite right.

We will rectify the undesired outcome by modifying $\mathcal{L}_{\text{Higgs}}^{(0)}$ by recalling from section 7.1 of [15] that

$$\left((S_{\text{GUT}} \cap (F_{\dot{x}\dot{z}j\dot{y}\dot{w}} \cup F_{j\dot{w}\dot{x}\dot{z}})) \times \mathbb{P}_{[U,V]} \right) \subseteq \mathcal{D}.$$

This allows us to define

$$\mathcal{L}_{\text{Higgs}} = \mathcal{L}_{\text{Higgs}}^{(0)} \otimes \mathcal{O}_{\mathcal{D}} \left(m \cdot \left((S_{\text{GUT}} \cap (F_{\dot{x}\dot{z}\dot{y}\dot{w}} - F_{\dot{y}\dot{w}\dot{x}\dot{z}})) \times \mathbb{P}_{[U,V]} \right) \right)$$

for $m \neq 0$ as in lemma 6 of [15] and define

$$\begin{aligned} \mathcal{L}_{\mathbf{10}}^{(4)} &:= \mathcal{L}_{\text{Higgs}}|_{\Sigma_{\mathbf{10}}^{(4)}} \\ \mathcal{L}_{\bar{\mathbf{5}}}^{(41)} &:= \mathcal{L}_{\text{Higgs}}|_{\Sigma_{\bar{\mathbf{5}}}^{(41)}} \\ \mathcal{L}_{\bar{\mathbf{5}}}^{(44)} &:= \mathcal{L}_{\text{Higgs}}^{(0,-)}|_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}. \end{aligned}$$

Since the restriction of $\mathcal{O}_{\mathcal{D}} \left(m \cdot \left((S_{\text{GUT}} \cap (F_{\dot{x}\dot{z}\dot{y}\dot{w}} - F_{\dot{y}\dot{w}\dot{x}\dot{z}})) \times \mathbb{P}_{[U,V]} \right) \right)$ to the respective curves is not the trivial bundle, we conclude that

$$\begin{aligned} h^0 \left(\mathcal{L}_{\mathbf{10}}^{(4)} \right) &= 6 \\ h^1 \left(\mathcal{L}_{\mathbf{10}}^{(4)} \right) &= 0 \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} h^0 \left(\mathcal{L}_{\bar{\mathbf{5}}}^{(41)} \right) &= 6 \\ h^1 \left(\mathcal{L}_{\bar{\mathbf{5}}}^{(41)} \right) &= 0. \end{aligned} \tag{8.3}$$

Furthermore by lemma 7 of [15] we have with respect to the eigenbundles of the involution $C_{u,v}$ that

$$\begin{aligned} h^0 \left(\left(\pi_{\Sigma_{\bar{\mathbf{5}}}^{(44)}} \right)_* \left(\mathcal{L}_{\bar{\mathbf{5}}}^{(44)} \right)^{[+1]} \right) &= 0 \\ h^1 \left(\left(\pi_{\Sigma_{\bar{\mathbf{5}}}^{(44)}} \right)_* \left(\mathcal{L}_{\text{Higgs}}|_{\hat{\Sigma}_{\bar{\mathbf{5}}}^{(44)}}^{[-1]} \right)^{[+1]} \right) &= 0 \\ h^0 \left(\left(\pi_{\Sigma_{\bar{\mathbf{5}}}^{(44)}} \right)_* \left(\mathcal{L}_{\text{Higgs}}|_{\hat{\Sigma}_{\bar{\mathbf{5}}}^{(44)}}^{[-1]} \right)^{[-1]} \right) &= 1 \\ h^1 \left(\left(\pi_{\Sigma_{\bar{\mathbf{5}}}^{(44)}} \right)_* \left(\mathcal{L}_{\text{Higgs}}|_{\hat{\Sigma}_{\bar{\mathbf{5}}}^{(44)}}^{[-1]} \right)^{[-1]} \right) &= 1. \end{aligned} \tag{8.4}$$

This gives three-generations

$$\begin{aligned} h^0 \left(\Sigma_{\mathbf{10}}^{(4)\vee}; \mathcal{L}_{\mathbf{10}}^{(4)[\pm 1]} \right) &= h^0 \left(\Sigma_{\bar{\mathbf{5}}}^{(41)\vee}; \mathcal{L}_{\bar{\mathbf{5}}}^{(41)[\pm 1]} \right) = 3 \\ h^1 \left(\Sigma_{\mathbf{10}}^{(4)\vee}; \mathcal{L}_{\mathbf{10}}^{(4)[\pm 1]} \right) &= h^1 \left(\Sigma_{\bar{\mathbf{5}}}^{(41)\vee}; \mathcal{L}_{\bar{\mathbf{5}}}^{(41)[\pm 1]} \right) = 0 \end{aligned} \tag{8.5}$$

as well as

$$\begin{aligned} h^0 \left(\Sigma_{\bar{\mathbf{5}}}^{(44)\vee}; \mathcal{L}_{\bar{\mathbf{5}}}^{(44)[+1]} \right) &= h^1 \left(\Sigma_{\bar{\mathbf{5}}}^{(44)\vee}; \mathcal{L}_{\bar{\mathbf{5}}}^{(44)[+1]} \right) = 0 \\ h^0 \left(\Sigma_{\bar{\mathbf{5}}}^{(44)\vee}; \mathcal{L}_{\bar{\mathbf{5}}}^{(44)[-1]} \right) &= h^1 \left(\Sigma_{\bar{\mathbf{5}}}^{(44)\vee}; \mathcal{L}_{\bar{\mathbf{5}}}^{(44)[-1]} \right) = 1 \end{aligned} \tag{8.6}$$

where the sign indicates the $C_{u,v}$ -eigen-line-bundle.

8.4 Action of $\tilde{\beta}_4$ on $\mathcal{L}_{\text{Higgs}}$

The isomorphisms

$$\left(\tilde{\zeta}\right) \leftrightarrow B_3 \leftrightarrow (\tilde{\tau}) \tag{8.7}$$

induced by $\tilde{\beta}_4$ and π_{B_3} identifies sections of $\mathcal{O}_{B_3}(N)$ with sections of the co-normal bundles $\mathcal{N}_{(\tilde{\zeta})|\tilde{W}_4}^\vee$ and $\mathcal{N}_{(\tilde{\tau})|\tilde{W}_4}^\vee$ respectively. A section of $\mathcal{O}_{B_3}(N)$ pulls back to sections of $\mathcal{N}_{(\tilde{\zeta})|\tilde{W}_4}^\vee$ and $\mathcal{N}_{(\tilde{\tau})|\tilde{W}_4}^\vee$ respectively that are interchanged under the action of $\tilde{\beta}_4$ thereby identifying $\mathcal{N}_{(\tilde{\zeta})|\tilde{W}_4}^\vee$ with $\tilde{\beta}_4^* \mathcal{N}_{(\tilde{\tau})|\tilde{W}_4}^\vee$. In (8.4) we have used that identification to define an involution on the rank-2 vector bundle $\left(\pi_{\mathcal{C}_{\text{Higgs}}^\vee}\right)_* \mathcal{L}_{\text{Higgs}}$ that decomposes into the direct sum of line bundles consisting of sections that respectively symmetric and skew-symmetric. It is the skew-symmetric line bundle on $\mathcal{C}_{\text{Higgs}}^\vee$ that we call $\mathcal{L}_{\text{Higgs}}^\vee$.

Let

$$\check{\mathcal{L}}_{10}^{(4)}, \check{\mathcal{L}}_{\bar{5}}^{(41)}, \check{\mathcal{L}}_{\bar{5}}^{(44)} \tag{8.8}$$

denote the restriction of the line bundle $\mathcal{L}_{\text{Higgs}}^\vee$ to the respective quotient curves.

8.5 The G -flux in \tilde{W}_4

The G -flux $\mathcal{G}_{\tilde{W}_4}$ is defined as the push-forward into \tilde{W}_4 of the Chern class of $\mathcal{L}_{\text{Higgs}}$ modified by a $\tilde{\beta}_4$ -invariant two cycle in (X) so that it has intersection number zero with $\pi_{B_3}^*(C)$ for any curve C in S_{GUT} . Since we can essentially identify $\mathcal{C}_{\text{Higgs}} \subseteq \tilde{Q}$ with its lifting $\tilde{\mathcal{C}}_{\text{Higgs}} \subseteq \tilde{W}_4$ we will first work to compute the class of the G -flux $\mathcal{G}_{\tilde{Q}} \subseteq \tilde{Q}$. By (7.27)

$$\mathcal{C}_{\text{Higgs}}^{(4)} \equiv 4(X) + 5N \in \text{Pic}(\tilde{Q}).$$

Therefore the class of the push-forward of $c_1(\mathcal{L}_{\text{Higgs}})$ to \tilde{Q} is

$$(4(X) + 5N) \cdot_{\tilde{Q}} (N + mF). \tag{8.9}$$

Since $F = F_+ - F_-$ projects into the curve $\{z = \vartheta_0 + a_{420} = 0\} \subseteq B_3$

$$(\pi_{B_3})_*(mF) = 0$$

and we have by the projection formula that the intersection of (8.9) with $\pi_{B_3}^*(C)$ projects to $C \cdot_{B_3} (4N)$, we must have

$$\mathcal{G}_{\tilde{Q}} \equiv (4N + 5N) \cdot_{\tilde{Q}} (N + mF).$$

So

$$(\pi_{B_3})_* (\mathcal{G}_{\tilde{Q}}^2) = 0$$

and therefore, recalling that $\mathcal{G}_{\tilde{W}_4}$ projects birationally to $\mathcal{G}_{\tilde{Q}}$,

$$\mathcal{G}_{\tilde{Q}}^2 = \mathcal{G}_{\tilde{W}_4}^2 = 0. \tag{8.10}$$

8.6 Numerical conditions on the square of the G -flux

The standard conditions that $\mathcal{G}_{W_4^\vee}$ will have to satisfy are that

$$\begin{aligned} \mathcal{G}_{W_4^\vee} \cdot_{W_4^\vee} \mathcal{G}_{W_4^\vee} &\geq 0 \\ \mathcal{G}_{W_4^\vee} \cdot_{W_4^\vee} \mathcal{G}_{W_4^\vee} - \frac{\chi(W_4^\vee)}{24} &\leq 0 \end{aligned} \tag{8.11}$$

where the subscript on the intersection dot indicates the space in which the intersection is taking place. But by (8.10),

$$\mathcal{G}_{W_4^\vee} \cdot_{W_4^\vee} \mathcal{G}_{W_4^\vee} = \mathcal{G}_{\tilde{W}_4}^2 = 0$$

so we must only check that $\frac{\chi(W_4^\vee)}{24}$ is a non-negative integer.

Now the involution $\tilde{\beta}_4$ on W_4 has four fixpoints over each of the eight fixpoints of the action of the involution β_3 on B_3 . As in formula (5.3) of [43] each orbifold point will affect the computation of $\chi(W_4^\vee)$ in (8.11). That is, since the (string-theoretic) orbifold Euler characteristic of the fiber of \tilde{W}_4/B_3 over each fixpoint of β_3 in B_3 is 6,

$$\chi(W_4^\vee) = \frac{1}{2}\chi(\tilde{W}_4) + 8 \cdot 6.$$

Since \tilde{W}_4 is smooth with a $(4+1)$ -split of $\tilde{\mathcal{C}}_{\text{Higgs}}$ we can apply the formula

$$\chi(\tilde{W}_4) = 6 \int_{B_3^\vee} (24c_1^3 - 44c_1^2 \cdot N + 27c_1 \cdot N^2 - 5N^3 + 2c_1c_2)$$

given in (5.19) of [43] where c_j denotes the j -th Chern class of B_3 . Now in our case $N = c_1$ and, since $\chi(\mathcal{O}_{B_3}) = 1$, the Riemann-Roch theorem gives that $\int_{B_3} c_1 \cdot c_2 = 24$. Furthermore, by (4.1) of [15] $c_1^3 = 12$. So

$$\chi(\tilde{W}_4) = 12 \int_{B_3^\vee} (c_1^3 + c_1c_2) = 12(12 + 24)$$

and

$$\chi(W_4^\vee) = 6 \cdot (36 + 8) = 24 \cdot 11.$$

8.7 The D -term

We next consider the $U(1)$ factor of the $(4+1)$ -decomposition of the spectral variety. Following [24] this D term must vanish, otherwise we might break R -parity in the low energy theory.

After resolution,

$$\tilde{W}_4 - E_{\tilde{5}}^{(44)} \rightarrow \tilde{Q} - C_{\tilde{5}}^{(44)}$$

is a branched double cover, so that \tilde{W}_4 will inherit ample divisors from sufficiently ample divisors on \tilde{Q} . So, to compute the D -term we must first adjust the divisor

$$\tilde{\Delta} - 4(X) \equiv 4N - 2((Z_0) + (Z_{23}))$$

so that its intersections with all curves of the form $(X) \cdot_{\tilde{Q}} D$ and $\pi_{B_3}^*$ (*point*) are zero. Now

$$(X) \cdot_{\tilde{Q}} D \cdot_{\tilde{Q}} (4N - 2((Z_0) + (Z_{23}))) = \pi_{B_3} \left((X) \cdot_{\tilde{Q}} D \right) \cdot_{B_3} 4N$$

so that we must adjust by $-4N$. But the D -term is computed by identifying an ample divisor on \tilde{Q} that has zero intersection number with the two-class

$$\mathcal{G}_{\tilde{Q}} \cdot_{\tilde{Q}} \left(\mathcal{C}_{\text{Higgs}} - 4 \left(\hat{X} \right) - \hat{S}_{\text{GUT}} \right). \tag{8.12}$$

But

$$\mathcal{G}_{\tilde{Q}} \cdot_{\tilde{Q}} \left(\mathcal{C}_{\text{Higgs}} - 4 \left(\hat{X} \right) - \hat{S}_{\text{GUT}} \right) = \mathcal{G}_{\tilde{Q}} \cdot_{\tilde{Q}} 0 = 0$$

so any ample divisor trivially has zero intersection with (8.12).

9 Wilson line: symmetry-breaking to the standard model

We write

$$\left(\pi_{B_3^\vee} \right)_* \mathcal{O}_{B_3} = \mathcal{O}_{B_3^\vee} \oplus \mathcal{O}_{B_3^\vee} (\varepsilon_{u,v})$$

with connection isomorphism

$$\varepsilon_{u,v} : \pi_1 \left(B_3^\vee \right) \rightarrow \{ \pm 1, \cdot \}.$$

Let

$$Y = \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{pmatrix} \in \mathfrak{su}(5)$$

denote the hypercharge direction when acting on the fundamental representation. The Wilson line is the flat rank-5 vector bundle L_Y given by the homomorphism

$$\pi_1 \left(S_{\text{GUT}}^\vee \right) \rightarrow \text{SU}(5)_{\text{gauge}}$$

that takes the generator to $\exp(6\pi i \cdot Y)$ when viewed as acting on the fundamental representation of $\text{SU}(5)_{\text{gauge}}$. This has the effect of breaking $\text{SU}(5)_{\text{gauge}}$ -representations over S_{GUT} to $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$ -representations over S_{GUT}^\vee . Via tensor and exterior product, these representations yield eigenspaces respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{ \pm 1 \} \times \{ \exp(6\pi i \cdot Y) \}$ holonomy group. Only eigenspaces with eigenvalue pairs in the diagonal of $\mathbb{Z}_2 \times \mathbb{Z}_2$ remain in the theory. The flat $\text{U}(1)_Y$ -bundle gives mass to the gauge states in $\text{SU}(5)/[\text{SU}(3) \times \text{SU}(2) \times \text{U}_Y(1)]$ at a scale given by the cycle $\propto 1/R_{\text{cycle}}$ on the GUT surface.⁵ It endows the quotient

$$W_4^\vee / B_3^\vee$$

with the Standard Model gauge symmetry $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$.

⁵Referring to lemma 2 the flat $\text{U}(1)_Y$ -bundle wraps the non-contractible cycle (v) while (ζ) and (τ) are interchanged.

We are about to incorporate the very particular translation by $\tau(b_3) - \zeta(b_3)$ in the fiber direction into the definition of the \mathbb{Z}_2 -action on \tilde{W}_4/B_3 . This translation commutes with all the properties we have attributed to $\tilde{\beta}_4/\beta_3$ up to this point. The purpose of including the change of basepoint into the involution $\tilde{\beta}_4/\beta_3$ is to eliminate vector-like exotics from the bulk spectrum. If we were *not* to incorporate the translation $\tau(b_3) - \zeta(b_3)$ into $\tilde{\beta}_4/\beta_3$, the bulk spectrum data for $SU(3) \times SU(2) \times U(1)_Y$ would be detailed in the following table, in which the non-vanishing of the cohomology groups in the last two rows indicate the existence of vector-like exotics [6, 20, 43]:

Representation	Type of multiplet	Cohomology group dimension
$(\mathbf{8}, \mathbf{1})_0$	Vector	$h^2(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) = 1$
$(\mathbf{1}, \mathbf{3})_0$	Vector	$h^2(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) = 1$
$(\mathbf{1}, \mathbf{1})_0$	Vector	$h^2(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) = 1$
$(\mathbf{8}, \mathbf{1})_0$	Chiral	$h^0(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) \oplus h^1(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) = 0$
$(\mathbf{1}, \mathbf{3})_0$	Chiral	$h^0(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) \oplus h^1(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) = 0$
$(\mathbf{1}, \mathbf{1})_0$	Chiral	$h^0(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) \oplus h^1(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}}) = 0$
$(\mathbf{3}, \mathbf{2})_{-5/6}$	Vector	$h^0(S_{\text{GUT}}^{\vee}, \mathcal{O}_{S_{\text{GUT}}^{\vee}}(\varepsilon_{u,v})) = 0$
$(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$	Vector	$h^0(S_{\text{GUT}}^{\vee}, \mathcal{O}_{S_{\text{GUT}}^{\vee}}(\varepsilon_{u,v})) = 0$
$(\mathbf{3}, \mathbf{2})_{-5/6}$	Chiral	$h^1(S_{\text{GUT}}^{\vee}, \mathcal{O}_{S_{\text{GUT}}^{\vee}}(\varepsilon_{u,v})) \oplus h^2(S_{\text{GUT}}^{\vee}, \mathcal{O}_{S_{\text{GUT}}^{\vee}}(\varepsilon_{u,v})) = 1$
$(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$	Chiral	$h^1(S_{\text{GUT}}^{\vee}, \mathcal{O}_{S_{\text{GUT}}^{\vee}}(\varepsilon_{u,v})) \oplus h^2(S_{\text{GUT}}^{\vee}, \mathcal{O}_{S_{\text{GUT}}^{\vee}}(\varepsilon_{u,v})) = 1$

Reviewing the construction of the ‘semi-stable degeneration’ in section 5 one sees that all the modification steps

$$\begin{array}{ccccc}
 \bar{W}_4^{\wedge}/B_3^{\wedge} & \rightarrow & \bar{V}_3^{\wedge}/B_2 & \subseteq & \bar{dP}_a \cup \bar{dP}_b \\
 \downarrow & & \downarrow & & \downarrow \\
 \bar{W}_4/B_3 & \rightarrow & \bar{V}_3/B_2 & \subseteq & \bar{dP}_a \cup \bar{dP}_b \\
 \uparrow & & \uparrow & & \uparrow \\
 W_4^{(1)}/B_3 & \rightarrow & V_3^{(1)}/B_2 & \subseteq & dP_a^{(1)} \cup dP_b^{(1)} \\
 \uparrow & & \uparrow & & \uparrow \\
 W_4^{(2)}/B_3 & \rightarrow & V_3^{(2)}/B_2 & \subseteq & dP_a^{(2)} \cup dP_b^{(2)} \\
 \uparrow & & \uparrow & & \uparrow \\
 \tilde{W}_4/B_3 & \rightarrow & V_3/B_2 & \subseteq & dP_a \cup dP_b
 \end{array}$$

are equivariant with respect to the action of the automorphism group of B_3 given by the

permutation group $S_4 \subseteq W \text{SU}(5)$ as in [15]. Now on the Heterotic side $\overline{dP}_a \cup \overline{dP}_b$ is simply the blow-up of $W_{4,0}$ with center (v) .

Since

$$(x, y) \mapsto (x, -y)$$

before we introduce the translation into the definition of $\tilde{\beta}_4$ we have that

$$\begin{aligned} y(\zeta(\beta_3(b_3))) &= -y(\zeta(b_3)) \\ y(\tau(\beta_3(b_3))) &= -y(\tau(b_3)) \end{aligned}$$

whereas composing with the translation we have that

$$\begin{aligned} y(\zeta(\beta_3(b_3))) &= -y(\tau(b_3)) \\ y(\tau(\beta_3(b_3))) &= -y(\zeta(b_3)) \end{aligned}$$

so that

$$y(\tau(\beta_3(b_3))) - y(\zeta(\beta_3(b_3))) = -y(\zeta(b_3)) - (-y(\tau(b_3)))$$

allowing the translation to descend to the quotient.

Previous construction utilized a single section that was invariant under the involution, thereby forcing the existence of vector-like exotics. The existence of two sections and the incorporation of the translation between them into the \mathbb{Z}_2 -action allows the elimination of the vector-like exotics.

It is (v) on which the non-contractible cycle of $dP_a \cup dP_b$ is wrapped, while (ζ) and (τ) are interchanged. Thus it is (v) that endows the quotient of the action by $\tilde{\beta}_4$ with a $U(1)_X$ -symmetry.

10 Bulk and chiral spectra with Wilson Line on W_4^\vee

10.1 Bulk spectrum with Wilson line

Since the canonical bundle of S_{GUT} is trivial, the introduction of the translation into the definition of $\tilde{\beta}_4$ will allow us to replace the trivial line bundle by

$$\mathcal{O}_{\tilde{W}_4} \left((\tilde{\tau}) - (\tilde{\zeta}) \right) \in \text{Pic} \left(\tilde{W}_4 \right) \tag{10.1}$$

in the computation of the bulk spectrum. More specifically, the semi-stable degeneration is the geometric bridge between Heterotic theory and F -theory. The introduction of the translation into the definition of $\tilde{\beta}_{4,0}$ affects the dictionary (4.18) that is given in terms of the Weierstrass form on the dP_9 -bundles coming from the F -theory side and flat line bundles on the elliptic fibers on the Heterotic side. Change of basepoint from the one given by ζ to the one given by τ does not change the uniquely given Weierstrass form of the elliptic fiber however on the other side it does change the sum of eight flat line bundles given by the restriction of the E_8 -bundle since they are given by the differences between each of the eight points marked by the (asymptotic) Tate form and the identity element of the elliptic fiber as a group and the involution changes this last from the intersection of the torus fiber with (ζ) to the intersection of the torus fiber with (τ) .

Therefore instead of pushing forward the canonical bundle $K_{S_{\text{GUT}}} = \mathcal{O}_{S_{\text{GUT}}}$ to S_{GUT}^{\vee} we must incorporate the twist by the bundle (10.1) in order to compute the bulk spectrum for W_4^{\vee}/B_3^{\vee} , that is, we must compute the (derived) cohomology of the push-forward of the twisted structure sheaf

$$\left(\mathcal{O}_{S_{\text{GUT}} \times_{\tilde{B}_3} \tilde{W}_4} \right) \otimes \mathcal{O}_{\tilde{W}_4} \left(\left(\tilde{\zeta} \right) - (\tilde{\tau}) \right)$$

that goes to minus itself under the isomorphism $\tilde{\beta}_4$. The push-forward of the line bundle (10.1) to $S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}$ breaks into the sum

$$\left(\pi^* K_{S_{\text{GUT}}^{\vee}} \otimes \mathcal{O}_{S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}} (\varepsilon_{u,v}) \right) \oplus \left(\mathcal{O}_{S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}} (\varepsilon_{u,v} \cdot \tilde{\tau} - \varepsilon_{u,v} \cdot \tilde{\zeta}) \right) \quad (10.2)$$

of symmetric and a skew-symmetric line bundle summands where $\varepsilon_{u,v}$ is the flat orbifold line bundle induced by the involution $\tilde{\beta}_4$ with finite fixpoint set.

Thus the derived push-forwards of the summands in (10.2) are respectively

$$\mathcal{O}_{S_{\text{GUT}}^{\vee}} \oplus \{0\}$$

are all zero since the divisor $\left((\tilde{\tau}) - (\tilde{\zeta}) \right)$ is not linearly equivalent to zero anywhere, in particular nowhere over S_{GUT} . Therefore referring to the table in section 9 but now with the incorporation of the translation $(\tilde{\tau}) - (\tilde{\zeta})$ bulk spectrum data is detailed in the following table:

Representation	Type of multiplet	Cohomology group dimension
$(\mathbf{8}, \mathbf{1})_0$	Vector	$h^2 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) = h^0 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee}} \right) = 1$
$(\mathbf{1}, \mathbf{3})_0$	Vector	$h^2 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) = h^0 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee}} \right) = 1$
$(\mathbf{1}, \mathbf{1})_0$	Vector	$h^2 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) = h^0 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee}} \right) = 1$
$(\mathbf{8}, \mathbf{1})_0$	Chiral	$h^0 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) \oplus h^1 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) = 0$
$(\mathbf{1}, \mathbf{3})_0$	Chiral	$h^0 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) \oplus h^1 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) = 0$
$(\mathbf{1}, \mathbf{1})_0$	Chiral	$h^0 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) \oplus h^1 \left(S_{\text{GUT}}^{\vee}, K_{S_{\text{GUT}}^{\vee}} \right) = 0$
$(\mathbf{3}, \mathbf{2})_{-5/6}$	Vector	$h^0 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}} (\varepsilon_{u,v} \cdot \tilde{\tau} - \varepsilon_{u,v} \cdot \tilde{\zeta}) \right) = 0$
$(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$	Vector	$h^0 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}} (\varepsilon_{u,v} \cdot \tilde{\tau} - \varepsilon_{u,v} \cdot \tilde{\zeta}) \right) = 0$
$(\mathbf{3}, \mathbf{2})_{-5/6}$	Chiral	$h^1 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}} (\varepsilon_{u,v} \cdot \tilde{\tau} - \varepsilon_{u,v} \cdot \tilde{\zeta}) \right) \oplus h^2 (\dots) = 0$
$(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$	Chiral	$h^1 \left(\mathcal{O}_{S_{\text{GUT}}^{\vee} \times_{B_3^{\vee}} W_4^{\vee}} (\varepsilon_{u,v} \cdot \tilde{\tau} - \varepsilon_{u,v} \cdot \tilde{\zeta}) \right) \oplus h^2 (\dots) = 0$

Thus there are no vector-like exotics!

10.2 Matter spectrum

In fact we have already computed the correct chiral spectra in (8.2), (8.3), and (8.4) of subsection 8.3.

Lemma 6. *With respect to the $C_{u,v}$ -eigen-line-bundle decomposition of the push-forwards to B_3^\vee of the (restrictions of the) Higgs bundle to the matter and Higgs curves in B_3 ,*

i)

$$\begin{aligned} h^0 \left(\check{\mathcal{L}}_{\mathbf{10}}^{(4)[\pm 1]} \right) &= h^0 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(41)[\pm 1]} \right) = 3 \\ h^1 \left(\check{\mathcal{L}}_{\mathbf{10}}^{(4)[\pm 1]} \right) &= h^1 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(41)[\pm 1]} \right) = 0. \end{aligned}$$

ii)

$$\begin{aligned} h^0 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[+1]} \right) &= h^1 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[+1]} \right) = 0 \\ h^0 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[-1]} \right) &= h^1 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[-1]} \right) = 1. \end{aligned}$$

Therefore referring to tables 1 and 2 in section 7 of [15], the flux distribution associated to the \mathbb{Z}_2 -action given by the involution $\tilde{\beta}_4$ and the Wilson line that is wrapped by that involution is presented in the tables below. Namely the distribution of MSSM matter fields is as follows:

$\Sigma_{\mathbf{10}}^{(4)} = \{a_5 = z = 0\}$	$C_{u,v}$	L_Y	$\mathcal{L}_{\text{Higgs}}$	$\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$
$h^0 \left(\check{\mathcal{L}}_{\mathbf{10}}^{(4)[\pm 1]} \right)$	+1	+1	3	$(\mathbf{1}, \mathbf{1})_{+1}$
	-1	-1		$(\mathbf{3}, \mathbf{2})_{+1/6}$
	+1	+1		$(\bar{\mathbf{3}}, \mathbf{1})_{-2/3}$
$h^1 \left(\check{\mathcal{L}}_{\mathbf{10}}^{(4)[\pm 1]} \right)$	+1	+1	0	$(\mathbf{1}, \mathbf{1})_{+1}$
	-1	-1		$(\bar{\mathbf{3}}, \mathbf{2})_{+1/6}$
	+1	+1		$(\mathbf{3}, \mathbf{1})_{+2/3}$

$\Sigma_{\bar{\mathbf{5}}}^{(41)} = \{a_{420} = z = 0\}$	$C_{u,v}$	L_Y	$\mathcal{L}_{\text{Higgs}}$	$\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$
$h^0 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(41)[\pm 1]} \right)$	+1	+1	3	$(\bar{\mathbf{3}}, \mathbf{1})_{+1/3}$
	-1	-1		$(\mathbf{1}, \mathbf{2})_{-1/2}$
$h^1 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(41)[\pm 1]} \right)$	+1	+1	0	$(\mathbf{3}, \mathbf{1})_{-1/3}$
	-1	-1		$(\mathbf{1}, \mathbf{2})_{+1/2}$

As for the Higgs fields we have the following:

$\Sigma_{\bar{\mathbf{5}}}^{(44)} = \{a_4 a_3 + a_5 (a_0 - a_3) = z = 0\}$	$C_{u,v}$	L_Y	$\mathcal{L}_{\text{Higgs}}$	$\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$
$h^0 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[+1]} \right)$	+1	+1	0	$(\bar{\mathbf{3}}, \mathbf{1})_{+1/3}$
$h^0 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[-1]} \right)$	-1	-1	1	$(\mathbf{1}, \mathbf{2})_{-1/2}$
$h^1 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[+1]} \right)$	+1	+1	0	$(\mathbf{3}, \mathbf{1})_{-1/3}$
$h^1 \left(\check{\mathcal{L}}_{\bar{\mathbf{5}}}^{(44)[-1]} \right)$	-1	-1	1	$(\mathbf{1}, \mathbf{2})_{+1/2}$

In conclusion we have reproduced the spectrum of the minimal supersymmetric Standard Model.

11 Asymptotic \mathbb{Z}_4 R-symmetry

The asymptotic \mathbb{Z}_4 -symmetry $T_{u,v}$ of the semi-stable F -theory limit constructed on B_3^\vee in section 6 of [15] acts on sections and twisted sections of the anti-canonical bundle B_3^\vee . Referring to the results and notation in sections 5 and 6 of [15], the asymptotic \mathbb{Z}_4 R-symmetry $T_{u,v}$ constructed there on

$$\lim_{\delta \rightarrow 0} \tilde{W}_{4,\delta}$$

is compatible with the following table from subsection 8.2 of [15]:⁶

TABLE 3: $T_{u,v}$	$T_{u,v}$ -charge	space
matter fields on $\frac{\Sigma_{\mathbf{10}}^{(4)}}{\{C_{u,v}\}}$	-1	$H^0 \left(\frac{\Sigma_{\mathbf{10}}^{(4)}}{\{C_{u,v}\}}; \mathcal{L}_{\text{Higgs}}^{\vee, [\pm 1]} \right)$
matter fields on $\frac{\Sigma_{\bar{\mathbf{5}}}^{(41)}}{\{C_{u,v}\}}$	-1	$H^0 \left(\frac{\Sigma_{\bar{\mathbf{5}}}^{(41)}}{\{C_{u,v}\}}; \mathcal{L}_{\text{Higgs}}^{\vee, [\pm 1]} \right)$
Higgs fields on $\frac{\Sigma_{\bar{\mathbf{5}}}^{(44)}}{\{C_{u,v}\}}$	+i	$H^0 \left(\frac{\Sigma_{\bar{\mathbf{5}}}^{(44)}}{\{C_{u,v}\}}; \mathcal{L}_{\text{Higgs}}^{\vee, [-1]} \right) / H^1 \left(\frac{\Sigma_{\bar{\mathbf{5}}}^{(44)}}{\{C_{u,v}\}}; \mathcal{L}_{\text{Higgs}}^{\vee, [-1]} \right)$
bulk matter on $\frac{S_{\text{GUT}}}{\{C_{u,v}\}}$	-i	$H^2 \left(K_{\frac{S_{\text{GUT}}}{\{C_{u,v}\}}} \right)$

The \mathbb{Z}_4 R symmetry forbids the Higgs μ -term and dimension 4 and 5 baryon and lepton number violating operators.

⁶Given the \mathbb{Z}_4 R-charges, i^{q_f+1} , for the fermionic components of $N = 1$ superfields, then the bosonic components of the chiral superfields are given by i^{q_f} . This is in accord with the anti-commuting superspace coordinate transforming as $\theta' = i^{-1}\theta$. Otherwise, these are equivalent to the \mathbb{Z}_4 R-charges of [39, 40].

12 Conclusions

In this paper we have constructed an $SU(5)$ GUT F -theory model. We have shown how to break the GUT group with a non-local Wilson line. Thus we are able to identify the GUT scale with the compactification scale of the GUT surface. Our model includes three families of quarks and leptons and one pair of Higgs doublets. The price to pay for this result is that we have a mirror world where the mass scales and couplings of the mirror states may be different than for the MSSM. This mirror world can, in principle, be the dark matter of the universe. There may or may not also be direct couplings of the mirror and MSSM sectors of the theory. These give interesting physics as in [4, 7, 13]. There are no vector-like exotics in the bulk spectrum or on the matter curves, neither are there chiral exotics.

The existence of the \mathbb{Z}_4 \mathbf{R} -symmetry generated by the automorphism $T_{u,v}$ that reverses the sign on the holomorphic 4-form addresses the issue of dimension-5 proton decay operators and forbids a μ -term [39]. The charges of the matter states under the \mathbb{Z}_4 \mathbf{R} -symmetry are given in table 3. Finally, Wilson line symmetry-breaking is addressed in sections 9 and 10.

However there are several issues which are not resolved in this paper. Moduli stabilization is not addressed. We have not generated a μ -term. In principle the \mathbb{Z}_4 \mathbf{R} -symmetry can be broken by non-perturbative physics down to matter parity which then allows for a μ -term of order the weak scale and severely suppressed dimension-5 proton decay operators. We have not discussed the possible Yukawa interactions needed to give quarks and leptons mass. We may or may not also have right-handed neutrinos which would be useful for a see-saw mechanism of neutrino masses. Finally, the $U(1)_X$ gauge symmetry may be broken to \mathbb{Z}_2 matter parity via non-perturbative effects at the GUT scale or by a Stueckelberg mechanism. This would then allow for right-handed neutrino Majorana masses near the GUT scale.

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A \mathbb{P}^{112} Formulations by Sakura Schäfer-Nameki

A.1 Realizing \overline{W}_4 in \mathbb{P}^{112}

An alternative formulation of the elliptic fibrations \overline{W}_4 can be given in terms of the \mathbb{P}^{112} -fibration in [45]. Let

$$\mathbb{P}^{112} := \mathbb{P}(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(2) \oplus \mathcal{O}_{B_3}(4))$$

with projective coordinates $[w, x, y]$. For $c_i \in H^0(\mathcal{O}_{B_3}(8-2i))$ define

$$\begin{aligned} & c_0 w^4 + c_1 w^3 x + c_2 w^2 x^2 + c_3 w x^3 \\ & = y^2 + (b_0 x^2 + b_1 w x + b_2 w^2) y. \end{aligned} \tag{A.1}$$

This elliptic fibration has two rational sections

$$\begin{aligned} w = y = 0 \\ w = y + b_0 x^2 = 0 \end{aligned}$$

that can be made symmetric by the shift

$$y \mapsto y - \frac{b_0 x^2}{2}. \tag{A.2}$$

$$\begin{aligned} & \left| \begin{array}{cc} h_0 & -\hat{t}(\hat{h}_1 \hat{z} - \hat{h}_2 \hat{t}) \\ \hat{t}(d_2 \hat{z} + d_3 \hat{t}) & \hat{c}_4 \hat{t}^2(\hat{t}^2 + \hat{z}^2) + \hat{c}_3 \hat{t}^3 \hat{z} + \hat{c}_1 z^3(\hat{t} - \hat{z}) \end{array} \right| = 0. \\ & \left| \begin{array}{cc} h_0 & -\hat{t}(\hat{h}_1 \hat{z} - \hat{h}_2 \hat{t}) \\ \hat{t}(d_2 \hat{z} + d_3 \hat{t}) & \hat{c}_4 \hat{t}^2(\hat{t}^2 + \hat{z}^2) + \hat{c}_3 \hat{t}^3 \hat{z} + \hat{c}_1 z^3(\hat{t} - \hat{z}) \end{array} \right| = 0. \end{aligned}$$

Denote by

$$\begin{aligned} \gamma_i &= \text{ord}_z(c_i) \\ \beta_j &= \text{ord}_z(b_j) \end{aligned}$$

and let

$$Q(\gamma_0, \gamma_1, \gamma_3, \gamma_0, \beta_0, \beta_1, \beta_2)$$

denote the quartic (A.1) for the given values. Using Tate’s algorithm, there are various ways to degenerate this to an I_5 fiber about the locus $\{z = 0\}$ [34], both by specifying the vanishing orders in (A.1) without further relations among the leading order coefficients c_i and b_j (‘canonical Tate models’), or by imposing relations among the coefficients (‘non-canonical models’).

In particular, the fibration \overline{W}_4 in this language is a non-canonical model with coordinates (w, x, y) given by

$$\begin{aligned} X_0 &\leftrightarrow w \\ Y_0 &= x \\ W_0 &\leftrightarrow y + x^2 \end{aligned}$$

so that (6.8) becomes the $Q(2, 1, 0, 0, 0, \infty, \infty)$ given in (6.11) by

$$\begin{aligned} y^2 + 2x^2 y &= -2a_5 x^3 w \\ &+ (a_5^2 - 2a_4 z - 6z^2) x^2 w^2 \\ &+ (2a_5 a_4 z + (6a_5 - 4a_{420}) z^2 - 8z^3) x w^3 \\ &- ((-a_4^2 - 4a_5 a_{420}) z^2 + (2a_4 + 4a_2 - 4a_5) z^3 + 3z^4) w^4. \end{aligned}$$

Matter loci	Rep and U(1) charge	Codim 2 fiber
σ_3	$\mathbf{10}_1 + \overline{\mathbf{10}}_{-1}$	$I_1^{*(0 1)}$
σ_1	$\mathbf{10}_{-4} + \overline{\mathbf{10}}_4$	$I_1^{*(01)}$
σ_2	$\mathbf{5}_{-7} + \overline{\mathbf{5}}_7$	$I_6^{(0 1)}$
P_1	$\mathbf{5}_{-2} + \overline{\mathbf{5}}_2$	$I_6^{(0 1)}$
P_2	$\mathbf{5}_3 + \overline{\mathbf{5}}_{-3}$	$I_6^{(0 1)}$

Table 1. Matter loci and U(1)-charges for the $I^{(0||1)}$ fiber.

The I_5 singular fiber enhances to I_6 along $-a_{53} = a_{420} = 0$ and $\left| \begin{smallmatrix} a_4 & -a_5 \\ a_0+a_3 & a_3 \end{smallmatrix} \right| = 0$, and we have I_1^* enhancement along $a_5 = 0$.

The sections intersect the I_5 fiber along D_0 and D_2 , where, as above, the rational curves in the I_5 associated to the simple roots α_i are denoted by D_i , with the extended node corresponding to D_0 . This type of I_5 fiber with two rational sections was denoted by $I_5^{(0||1)}$, where the separation of the sections is $\#|-1$. [34, 35]

A.2 Other models with similar fiber type

A different set of $I_5^{(0||1)}$ non-canonical models were determined in [34]

This model is based on the I_4 fiber $Q(3, 2, 1, 0, 0, 0, 1)$ given by

$$\begin{aligned} c_0 z^3 w^4 + c_1 z^2 w^3 x + c_2 z w^2 x^2 + c_3 z w x^3 \\ = y^2 + b_0 x^2 y + b_1 y w x + b_2 z w^2 y. \end{aligned}$$

where in addition we impose

$$P_0 = \begin{vmatrix} b_0 & b_2 \\ c_3 & c_2 \end{vmatrix} = 0.$$

This last is solved as follows:

$$\begin{vmatrix} b_0 & b_2 \\ c_3 & c_2 \end{vmatrix} = \begin{vmatrix} \sigma_1 \\ \sigma_4 \end{vmatrix} \begin{vmatrix} \sigma_2 & \sigma_3 \end{vmatrix} = 0.$$

This fibration $Q(3, 2, 1, 1, 0, 0, 1)|_{P_0=0}$ has generically I_5 fiber, with same intersection pattern with the two rational sections as W_4 , but it provides more matter loci, shown in the table, with the following expressions for the matter curves

$$\begin{aligned} P_1 &= \sigma_4 b_{2,1}^2 + \sigma_1^2 \sigma_3 c_{0,3} - \sigma_1 b_{2,1} c_{1,2} \\ P_2 &= \sigma_1 \sigma_2^2 \begin{vmatrix} \sigma_1 & b_{2,1} \\ \sigma_4 & c_{1,2} \end{vmatrix} + \sigma_1 \sigma_3^2 \begin{vmatrix} \sigma_1 & b_{0,1} \\ \sigma_4 & c_{3,2} \end{vmatrix} \\ &\quad + \sigma_3 \sigma_2 (\sigma_4 \sigma_1 b_{1,1} - \sigma_1^2 c_{2,2} + \sigma_4^2). \end{aligned}$$

The sections are not symmetric in the form above, but this can again be remedied by shifting as in (A.2).

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References

- [1] A. Anandakrishnan and S. Raby, *SU(6) GUT Breaking on a Projective Plane*, *Nucl. Phys. B* **868** (2013) 627 [[arXiv:1205.1228](https://arxiv.org/abs/1205.1228)] [[INSPIRE](#)].
- [2] L.B. Anderson, J. Gray, A. Lukas and E. Palti, *Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds*, *Phys. Rev. D* **84** (2011) 106005 [[arXiv:1106.4804](https://arxiv.org/abs/1106.4804)] [[INSPIRE](#)].
- [3] L.B. Anderson, J. Gray, A. Lukas and E. Palti, *Heterotic Line Bundle Standard Models*, *JHEP* **06** (2012) 113 [[arXiv:1202.1757](https://arxiv.org/abs/1202.1757)] [[INSPIRE](#)].
- [4] R. Barbieri, T. Gregoire and L.J. Hall, *Mirror world at the large hadron collider*, [hep-ph/0509242](https://arxiv.org/abs/hep-ph/0509242) [[INSPIRE](#)].
- [5] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory — I*, *JHEP* **01** (2009) 058 [[arXiv:0802.3391](https://arxiv.org/abs/0802.3391)] [[INSPIRE](#)].
- [6] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory — II: Experimental Predictions*, *JHEP* **01** (2009) 059 [[arXiv:0806.0102](https://arxiv.org/abs/0806.0102)] [[INSPIRE](#)].
- [7] Z.G. Berezhiani, *Astrophysical implications of the mirror world with broken mirror parity*, *Acta Phys. Polon. B* **27** (1996) 1503 [[hep-ph/9602326](https://arxiv.org/abs/hep-ph/9602326)] [[INSPIRE](#)].
- [8] R. Blumenhagen, V. Braun, T.W. Grimm and T. Weigand, *GUTs in Type IIB Orientifold Compactifications*, *Nucl. Phys. B* **815** (2009) 1 [[arXiv:0811.2936](https://arxiv.org/abs/0811.2936)] [[INSPIRE](#)].
- [9] R. Blumenhagen, T.W. Grimm, B. Jurke and T. Weigand, *Global F-theory GUTs*, *Nucl. Phys. B* **829** (2010) 325 [[arXiv:0908.1784](https://arxiv.org/abs/0908.1784)] [[INSPIRE](#)].
- [10] R. Blumenhagen, *Gauge Coupling Unification in F-theory Grand Unified Theories*, *Phys. Rev. Lett.* **102** (2009) 071601 [[arXiv:0812.0248](https://arxiv.org/abs/0812.0248)] [[INSPIRE](#)].
- [11] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, in proceedings of the *Congrès International des Mathématiciens*, Nice, France, 1–10 Septembre 1970, in *Actes du Congrès International des Mathématiciens. Tome 2*, Gauthier-Villars, Paris France (1971), pp. 279–284.
- [12] F. Catanese, *Automorphisms of rational double points and moduli spaces of surfaces of general type*, *Compos. Math.* **61** (1987) 81.
- [13] Z. Chacko, H.-S. Goh and R. Harnik, *The Twin Higgs: Natural electroweak breaking from mirror symmetry*, *Phys. Rev. Lett.* **96** (2006) 231802 [[hep-ph/0506256](https://arxiv.org/abs/hep-ph/0506256)] [[INSPIRE](#)].
- [14] H. Clemens and S. Raby, *Heterotic/F-theory Duality and Narasimhan-Seshadri Equivalence*, [arXiv:1906.07238](https://arxiv.org/abs/1906.07238) [[INSPIRE](#)].
- [15] H. Clemens and S. Raby, *F-theory over a Fano threefold built from A_4 -roots*, [arXiv:1908.01110](https://arxiv.org/abs/1908.01110) [[INSPIRE](#)].
- [16] R. Davies, *Dirac gauginos and unification in F-theory*, *JHEP* **10** (2012) 010 [[arXiv:1205.1942v3](https://arxiv.org/abs/1205.1942v3)] [[INSPIRE](#)].

- [17] S. Dimopoulos, S. Raby and F. Wilczek, *Supersymmetry and the Scale of Unification*, *Phys. Rev. D* **24** (1981) 1681 [INSPIRE].
- [18] S. Dimopoulos and H. Georgi, *Softly Broken Supersymmetry and SU(5)*, *Nucl. Phys. B* **193** (1981) 150 [INSPIRE].
- [19] M.J. Dolan, J. Marsano and S. Schäfer-Nameki, *Unification and Phenomenology of F-theory GUTs with U(1)_{PQ}*, *JHEP* **12** (2011) 032 [arXiv:1109.4958] [INSPIRE].
- [20] R. Donagi and M. Wijnholt, *Model Building with F-theory*, *Adv. Theor. Math. Phys.* **15** (2011) 1237 [arXiv:0802.2969] [INSPIRE].
- [21] R. Donagi and M. Wijnholt, *Breaking GUT Groups in F-theory*, *Adv. Theor. Math. Phys.* **15** (2011) 1523 [arXiv:0808.2223v2] [INSPIRE].
- [22] R. Donagi, Y.-H. He, B.A. Ovrut and R. Reinbacher, *The Spectra of heterotic standard model vacua*, *JHEP* **06** (2005) 070 [hep-th/0411156] [INSPIRE].
- [23] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [hep-th/9701162] [INSPIRE].
- [24] T.W. Grimm and T. Weigand, *On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs*, *Phys. Rev. D* **82** (2010) 086009 [arXiv:1006.0226] [INSPIRE].
- [25] T.W. Grimm, M. Kerstan, E. Palti and T. Weigand, *Massive Abelian Gauge Symmetries and Fluxes in F-theory*, *JHEP* **12** (2011) 004 [arXiv:1107.3842] [INSPIRE].
- [26] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, *New Aspects of Heterotic-F Theory Duality*, *Nucl. Phys. B* **806** (2009) 224 [arXiv:0805.1057] [INSPIRE].
- [27] A. Hebecker and M. Trapletti, *Gauge unification in highly anisotropic string compactifications*, *Nucl. Phys. B* **713** (2005) 173 [hep-th/0411131] [INSPIRE].
- [28] A. Hebecker and J. Unwin, *Precision Unification and Proton Decay in F-theory GUTs with High Scale Supersymmetry*, *JHEP* **09** (2014) 125 [arXiv:1405.2930] [INSPIRE].
- [29] L.E. Ibáñez and G.G. Ross, *Low-Energy Predictions in Supersymmetric Grand Unified Theories*, *Phys. Lett. B* **105** (1981) 439 [INSPIRE].
- [30] V. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge U.K. (1990).
- [31] Y. Kawamata and Y. Namikawa, *Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties*, *Invent. Math.* **118** (1994) 395.
- [32] J.E. Kim, J.-H. Kim and B. Kyae, *Superstring standard model from \mathbb{Z}_{12-I} orbifold compactification with and without exotics and effective R-parity*, *JHEP* **06** (2007) 034 [hep-ph/0702278] [INSPIRE].
- [33] S. Krippendorf, H.P. Nilles, M. Ratz and M.W. Winkler, *Hidden SUSY from precision gauge unification*, *Phys. Rev. D* **88** (2013) 035022 [arXiv:1306.0574] [INSPIRE].
- [34] M. Kuntzler and S. Schäfer-Nameki, *Tate Trees for Elliptic Fibrations with Rank one Mordell-Weil group*, arXiv:1406.5174 [INSPIRE].
- [35] C. Lawrie, S. Schäfer-Nameki and J.-M. Wong, *F-theory and All Things Rational: Surveying U(1) Symmetries with Rational Sections*, *JHEP* **09** (2015) 144 [arXiv:1504.05593] [INSPIRE].

- [36] O. Lebedev et al., *A Mini-landscape of exact MSSM spectra in heterotic orbifolds*, *Phys. Lett. B* **645** (2007) 88 [[hep-th/0611095](#)] [[INSPIRE](#)].
- [37] O. Lebedev, H.P. Nilles, S. Ramos-Sanchez, M. Ratz and P.K.S. Vaudrevange, *Heterotic mini-landscape (II): Completing the search for MSSM vacua in a \mathbb{Z}_6 orbifold*, *Phys. Lett. B* **668** (2008) 331 [[arXiv:0807.4384](#)] [[INSPIRE](#)].
- [38] O. Lebedev et al., *The Heterotic Road to the MSSM with R parity*, *Phys. Rev. D* **77** (2008) 046013 [[arXiv:0708.2691](#)] [[INSPIRE](#)].
- [39] H.M. Lee et al., *A unique \mathbb{Z}_4 R-symmetry for the MSSM*, *Phys. Lett. B* **694** (2011) 491 [[arXiv:1009.0905](#)] [[INSPIRE](#)].
- [40] H.M. Lee et al., *Discrete R symmetries for the MSSM and its singlet extensions*, *Nucl. Phys. B* **850** (2011) 1 [[arXiv:1102.3595](#)] [[INSPIRE](#)].
- [41] E. Looijenga, *Root Systems And Elliptic Curves*, *Invent. Math.* **38** (1977) 17.
- [42] J. Marsano and S. Schäfer-Nameki, *Yukawas, G-flux and Spectral Covers from Resolved Calabi-Yau's*, *JHEP* **11** (2011) 098 [[arXiv:1108.1794](#)] [[INSPIRE](#)].
- [43] J. Marsano, H. Clemens, T. Pantev, S. Raby and H.-H. Tseng, *A Global SU(5) F-theory model with Wilson line breaking*, *JHEP* **01** (2013) 150 [[arXiv:1206.6132](#)] [[INSPIRE](#)].
- [44] C. Mayrhofer, E. Palti and T. Weigand, *Hypercharge Flux in IIB and F-theory: Anomalies and Gauge Coupling Unification*, *JHEP* **09** (2013) 082 [[arXiv:1303.3589](#)] [[INSPIRE](#)].
- [45] D.R. Morrison and D.S. Park, *F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds*, *JHEP* **10** (2012) 128 [[arXiv:1208.2695](#)] [[INSPIRE](#)].
- [46] D. Mumford, *Theta characteristics of an algebraic curve*, *Ann. Sci. Éc. Norm. Supér.* **4** (1971) 181.
- [47] D. Mumford, *Prym Varieties I*, in *Contributions to Analysis: A collection of papers dedicated to Lipman Bers*, Academic Press (1974), pp. 325–350.
- [48] L.B. Okun, *Mirror particles and mirror matter: 50 years of speculations and search*, *Phys. Usp.* **50** (2007) 380 [[hep-ph/0606202](#)] [[INSPIRE](#)].
- [49] S. Raby, M. Ratz and K. Schmidt-Hoberg, *Precision gauge unification in the MSSM*, *Phys. Lett. B* **687** (2010) 342 [[arXiv:0911.4249](#)] [[INSPIRE](#)].
- [50] G.G. Ross, *Wilson line breaking and gauge coupling unification*, [hep-ph/0411057](#) [[INSPIRE](#)].
- [51] P. Slodowy, *Simple singularities and simple algebraic groups*, Springer-Verlag, *Lect. Notes Math.* **815** (1980) 1.
- [52] R. Tatar, Y. Tsuchiya and T. Watari, *Right-handed Neutrinos in F-theory Compactifications*, *Nucl. Phys. B* **823** (2009) 1 [[arXiv:0905.2289](#)] [[INSPIRE](#)].
- [53] M. Trapletti, *Gauge symmetry breaking in orbifold model building*, *Mod. Phys. Lett. A* **21** (2006) 2251 [[hep-th/0611030](#)] [[INSPIRE](#)].