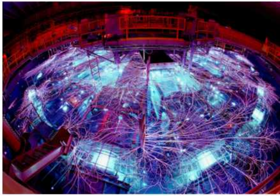
is paper describes objective technical results and analysis. Any subjective views or opinions that might be express
the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Governm

Laboratories

SAND2019-1825C





Black Box Multigrid for MHD

Ari Rappaport, Ray Tuminaro, John Shadid, Luc Berger-Vergiat
Center for Computing Research, Sandia National Laboratories

Jehanzeb Chaudhry
Department of Mathematics and Statistics, University of New Mexico

Wednesday 20th February, 2019

oratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia,
neywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract

Context

1. Black Box Multigrid (BoxMG)[1,2] structured algebraic algorithm to build hierarchy
2. Test extension to coupled PDE systems[3]
3. Problems motivated by magnetohydrodynamics (MHD)
4. Use Drekar (Sandia MHD/CFD code) to assemble linear systems, solve with Matlab BoxMG code

[1] J. E. Dendy. "Black Box Multigrid". In: *Journal of Computational Physics* (1982).

[2] J. E. Dendy Jr. and J.D. Moulton. "Black box multigrid with coarsening by a factor of three". In: *Numerical Linear Algebra with Applications* (2010). DOI: 10.1002/nla.705.

[3] J.E. Dendy Jr. "Black Box Multigrid for Systems". In: *Applied Mathematics and Computation* 19 (1986).

Outline

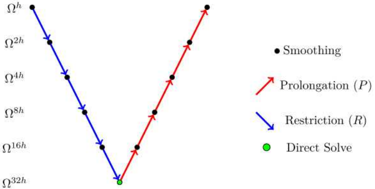
Black Box Multigrid

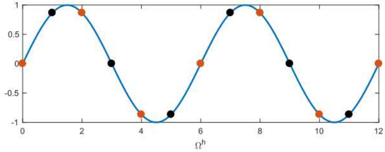
- Scalar BoxMG
- Vector BoxMG

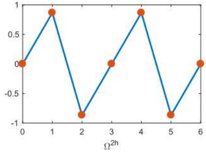
Magnetohydrodynamics (MHD)

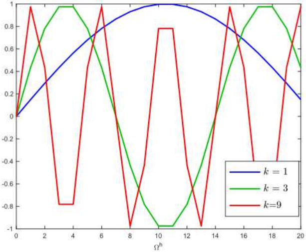
Numerical Results

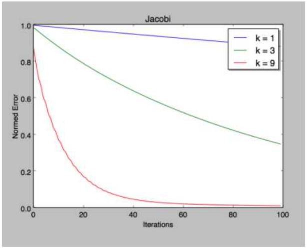
Multigrid Overview



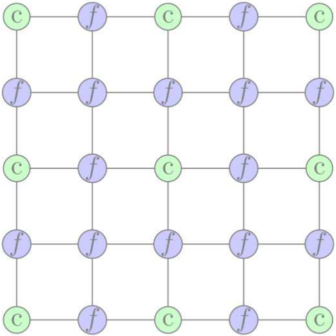








Abstract fine and coarse points



$$A = \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}$$

'Perfect" interpolation

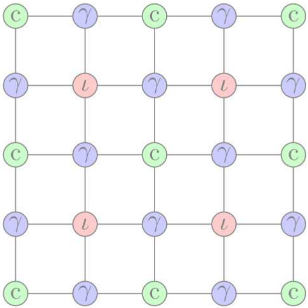
$$P_{fc} = -A_{ff}^{-1} A_{fc},\ P_{cc} = I, P = \begin{bmatrix} P_{fc} \\ P_{cc} \end{bmatrix},\ R = P^T$$

Galerkin projection produces Schur complement:

$$RAP = A_{cc} - A_{cf} A_{ff}^{-1} A_{fc}$$

Infeasible because A_{ff}^{-1} is dense!

BoxMG Splitting and Reduction



$$A = \left[\begin{array}{cc|c} A_{\iota\iota} & A_{\iota\gamma} & A_{\iota c} \\ A_{\gamma\iota} & A_{\gamma\gamma} & A_{\gamma c} \\ \hline A_{c\iota} & A_{c\gamma} & A_{cc} \end{array}\right] \xrightarrow{\text{collapse}} \left[\begin{array}{cc|c} A_{\iota\iota} & A_{\iota\gamma} & A_{\iota c} \\ 0 & \widehat{A}_{\gamma\gamma} & \widehat{A}_{\gamma c} \\ \hline A_{\iota c}^T & A_{\gamma c}^T & A_{cc} \end{array}\right].$$

if $\widehat{A}_{\gamma\gamma}$ diagonal

$$A_{ff} = \begin{bmatrix} A_{\iota\iota} & A_{\iota\gamma} \\ 0 & \widehat{A}_{\gamma\gamma} \end{bmatrix}$$

upper triangular $\rightarrow$ easily inverted

Stencil Collapse/Nullspace Preservation

Sandia National Laboratories

Row for interior γ node with 9-point Laplacian:

$$[\cdots -1 \quad -1 \quad -1 \cdots -1 \quad 8 \quad -1 \cdots -1 \quad -1 \quad -1 \ldots]$$

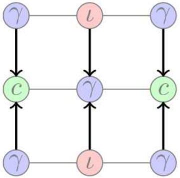


Figure: γ point on x coarse line

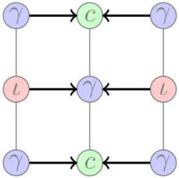


Figure: γ point on y coarse line

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{\text{collapse } y} \begin{bmatrix} 0 & 0 & 0 \\ -3 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{\text{collapse } x} \begin{bmatrix} 0 & -3 & 0 \\ 0 & 6 & 0 \\ 0 & -3 & 0 \end{bmatrix}$$

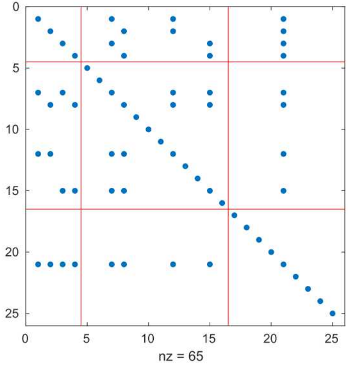
Accomplish two things:

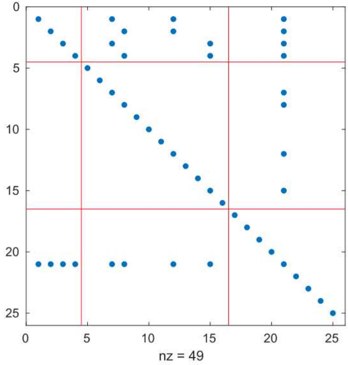
- Decoupling: $\widehat{A}_{\gamma\gamma}$ now diagonal, $A_{\gamma\iota} = 0$
- Preserve action of matrix on constant vector: for Laplacian preserves nullspace

Example

Take ∇^2 with 9-point stencil on a 5x5 mesh in 2d

$$\left[\begin{array}{cc|c} A_{\iota\iota} & A_{\iota\gamma} & A_{\iota c} \\ A_{\iota\gamma}^T & A_{\gamma\gamma} & A_{\gamma c} \\ \hline A_{\iota c}^T & A_{\gamma c}^T & A_{cc} \end{array}\right] \qquad \left[\begin{array}{cc|c} A_{\iota\iota} & A_{\iota\gamma} & A_{\iota c} \\ 0 & \widehat{A}_{\gamma\gamma} & \widehat{A}_{\gamma c} \\ \hline A_{\iota c}^T & A_{\gamma c}^T & A_{cc} \end{array}\right]$$





System Extension

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Block prolongators,

- $P_1 = \text{BoxMG}(A_{11})$, $R_1 = P_1^T$
- $P_2 = \text{BoxMG}(A_{22})$, $R_2 = P_2^T$

monolithic operators

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

So

$$RAP = \begin{bmatrix} R_1 A_{11} P_1 & R_1 A_{12} P_2 \\ R_2 A_{21} P_1 & R_2 A_{22} P_2 \end{bmatrix}$$

Diagonal operators are being projected well

Outline

Black Box Multigrid

Scalar BoxMG

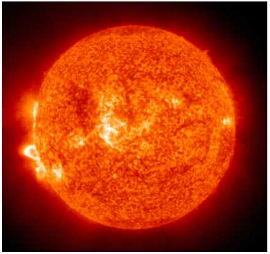
Vector BoxMG

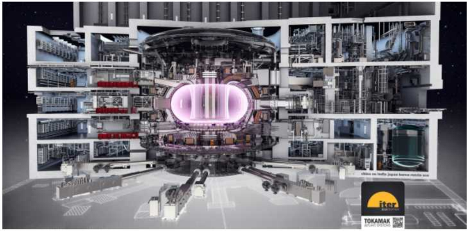
Magnetohydrodynamics (MHD)

Numerical Results

What is MHD?

Sandia National Laboratories





- Continuum model of plasma states
- Mathematical Model: Continuum representation of collisional plamsa systems.
- Structure of simple resistive MHD is Navier-Stokes + Lorentz force coupled with low frequency Maxwell equations.

MHD System

Stationary incompressible MHD system

$$-\eta\Delta\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = \mathbf{f},$$

$$\nabla\cdot\mathbf{u} = 0,$$

$$\nabla\times\left(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right) - \nabla\times(\mathbf{u}\times\mathbf{B}) = \mathbf{0},$$

$$\nabla\cdot\mathbf{B} = 0.$$

- Unknowns: magnetic field $\mathbf{B}$, velocity $\mathbf{u}$, pressure p
- Physical parameters: resistivity η, density ρ, magnetic permeability of free space μ_0.

Take induction equation as PDE system

$$\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

- $\nabla \times \nabla \times$ has large nullspace (gradients of all scalars)
- Introduce penalty term in weak form

$$\int_\Omega \frac{\eta}{\mu_0} (\nabla \cdot \mathbf{B})(\nabla \cdot \mathbf{C}) - \frac{\eta}{\mu_0} (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{C}) + \mathbf{C} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B})] \, \mathrm{d}x = 0$$

- Well posedness proved in [Gunzburger et. al][4]
- Augmented strong form,

$$\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) - \frac{\eta}{\mu_0} \nabla (\nabla \cdot \mathbf{B}) - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

[4] Max D. Gunzburger, Ammon J. Meir, and Janet S. Peterson. "On the Existence, Uniqueness, and Finite Element Approximation of Solutions of the Equations of Stationary, Incompressible Magnetohydrodynamics". In: *Mathematics of Computation* 56 (1991).

Exact Penalty Method Cont.

$$\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) - \frac{\eta}{\mu_0} \nabla \left(\nabla \cdot \mathbf{B} \right) - \nabla \times \left(\mathbf{u} \times \mathbf{B} \right) = \mathbf{0}$$

- What have we gained?
- Vector identity $\nabla \times (\nabla \times \mathbf{B}) - \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}$
- We have "completed" vector Laplacian
- Nullspace reduction to constants: good for multigrid

Outline

Sandia National Laboratories

Black Box Multigrid
- Scalar BoxMG
- Vector BoxMG

Magnetohydrodynamics (MHD)

Numerical Results

Poisson Equation

Let $\Omega = (0,1)^2$, $\Gamma = \partial\Omega$,
$f(x,y) = 4\pi^2 \sin(2\pi x)\sin(2\pi y)$.

$$\begin{cases} \nabla^2 u = f, & x \in \Omega \\ u = 0, & x \in \Gamma \end{cases}$$

True solution
$u(x,y) = -\sin(2\pi x)\sin(2\pi y)$.

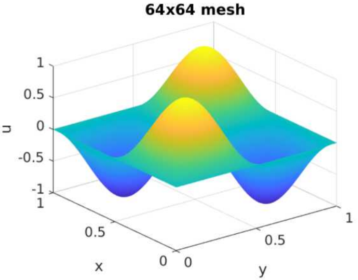


Figure: Solution plot

BoxMG for Scalar Poisson (tol = 1e-7)				
# Elements	Total DOFs	V Cycles	Jacobi Sweeps	Levels
32^2	1089	7	(3,3)	5
64^2	4225	8	(3,3)	6
128^2	16641	8	(3,3)	7
256^2	66049	8	(3,3)	8
512^2	264169	8	(3,3)	9

Magnetic Field Diffusion Test Problem

Let $\Omega = (0,1)^2$, $\Gamma = \Gamma_D \cap \Gamma_N$, where $\Gamma_D = \{x = 0 \text{ or } x = 1\}$, and $\Gamma_P = \{y = 0 \text{ or } y = 1\}$. $\mathbf{B} = (B_1, B_2)^T$, $\frac{\eta}{\mu_0} = 1$,

$$
\begin{cases}
\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) - \frac{\eta}{\mu_0} \nabla (\nabla \cdot \mathbf{B}) = \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \Omega \\
B_1 \equiv 0, & \mathbf{x} \in \Gamma \\
B_2 \equiv 0, & \mathbf{x} \in \Gamma_D \\
B_2(0, y) = B_2(1, y), & \mathbf{x} \in \Gamma_P
\end{cases}
$$

$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0, & -2 \end{bmatrix}^T$ to produce analytic solution
$\mathbf{B} = \begin{bmatrix} 0, & x(x-1) \end{bmatrix}^T$.
$\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0} \implies$ no off diagonal coupling because
$\nabla \times (\nabla \times \mathbf{B}) - \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}$ so block structure

$$
A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}
$$

2D Magnetic Field Diffusion Results

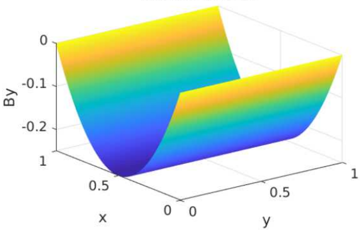


Figure: Solution plot

BoxMG for Uncoupled Problem (tol = 1e-7)				
# Elements	Total DOFs	V Cycles	Jacobi Sweeps	Levels
32^2	2178	8	(3,3)	5
64^2	8450	9	(3,3)	6
128^2	33282	9	(3,3)	7
256^2	132098	9	(3,3)	8
512^2	526338	9	(3,3)	9

2D Fully Coupled Magnetic Field Convection-Diffusion

$\Omega = (0,1)^2, \Gamma = \partial\Omega, \ \frac{\eta}{\mu_0} = 1.$

Fixed velocity field $\mathbf{u} = (1,1)^T$ in 2D produces

$\nabla \times (\mathbf{u} \times \mathbf{B}) = \left[\frac{\partial(B_2 - B_1)}{\partial y}, -\frac{\partial(B_2 - B_1)}{\partial x}\right]^T.$

Take

$$\mathbf{g}(\mathbf{x}) = \left[4\pi^2 \sin(2\pi y) - 2\pi \cos(2\pi x), \quad 4\pi^2 \sin(2\pi x) - 2\pi \cos(2\pi y)\right]^T$$

chosen to manufacture analytic solution:

$$\mathbf{B}_{analytic} = \left[\sin(2\pi y), B_2 = \sin(2\pi x)\right]^T$$

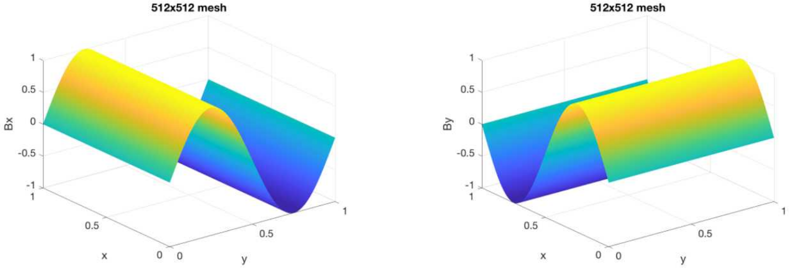
$$\begin{cases} \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B}\right) - \frac{\eta}{\mu_0} \nabla(\nabla \cdot \mathbf{B}) - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \Omega \\ \mathbf{B} = \mathbf{B}_{analytic}, & \mathbf{x} \in \partial\Omega \end{cases}$$

Block structure,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

2D Coupled Magnetic Field Convection-Diffusion Results

Sandia National Laboratories



BoxMG for Uncoupled Problem (tol = 1e-7)				
# Elements	Total DOFs	V Cycles	Jacobi Sweeps	Levels
32^2	2178	8	(3,3)	5
64^2	8450	8	(3,3)	6
128^2	33282	8	(3,3)	7
256^2	132098	8	(3,3)	8
512^2	526338	8	(3,3)	9

Concluding Remarks

- Encouraging result for coupled system
- Like to perform experiment for full MHD system
- Also explore other PDE systems (elasticity)
- Current solver written in Matlab; developing system version in Trilinos/MueLu
- Possibly explore extension to Nédélec elements