

Nambu, A Foreteller of Modern Physics III

Conformal symmetry and its breaking in two-dimensional nearly anti-de Sitter space

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We study a two-dimensional dilaton gravity system, recently examined by Almheiri and Polchinski, which describes near-extremal black holes, or more generally, nearly AdS_2 spacetimes. The asymptotic symmetries of AdS_2 are all the time reparametrizations of the boundary. These symmetries are spontaneously broken by the AdS_2 geometry and they are explicitly broken by the small deformation away from AdS_2 . This pattern of spontaneous plus explicit symmetry breaking governs the gravitational backreaction of the system. It determines several gravitational properties such as the linear in temperature dependence of the near-extremal entropy as well as the gravitational corrections to correlation functions. These corrections include the ones determining the growth of out-of-time-order correlators that is indicative of chaos. These gravitational aspects can be described in terms of a Schwarzian derivative effective action for a reparametrization.

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Subject Index B00, B22

1. Introduction

In some respects AdS_2 is a bit harder to understand than its higher-dimensional siblings. The main reason is that pure gravity in AdS_2 is inconsistent with the existence of finite energy excitations above the AdS_2 vacuum [1–3]. Nevertheless, there is a sense in which *nearly* AdS_2 gravity is well defined. In nearly AdS_2 , or $NAdS_2$, one keeps the leading-order correction away from AdS_2 . These corrections have a universal form and a very nice analysis of this system was given by Almheiri and Polchinski [3].

The leading-order gravitational effects can be described by a particular dilaton gravity system first studied by Jackiw [4] and Teitelboim [5]. Dilaton gravity theories in two dimensions have no propagating degrees of freedom. In this case, the effective action is determined by the symmetries of the problem. This symmetry is a spontaneously and explicitly broken reparametrization symmetry. In the perfect AdS_2 limit the system develops a reparametrization symmetry, $t \rightarrow \tilde{t}(t)$. This arises as the asymptotic symmetry of AdS_2 [6–8]. Alternatively, we expect it from the tracelessness of the stress tensor of a putative boundary theory. Since that stress tensor has only one component, its tracelessness due to the scaling symmetry implies that it is zero. Thus in the perfect AdS_2 limit the system has a full reparametrization symmetry. Under such circumstances one would naively expect the full system to be topological. In some sense it is; the pure AdS_2 limit can only describe the ground states and their entropy [9,10].

In fact, this reparametrization symmetry is spontaneously broken. It is spontaneously broken because it is only an asymptotic symmetry. Only an $SL(2)$ subgroup is unbroken by the geometry of AdS_2 . A spontaneously broken symmetry has associated zero modes. If we think in terms of the Euclidean path integral, there is an infinite number of non-compact zero modes: all the Fourier modes of the reparametrization symmetry.

However, the symmetry is also explicitly broken because we have decided to keep the leading-order correction away from the conformal limit. This correction is given by the simplest local action that is invariant under a global $SL(2)$ symmetry, the Schwarzian derivative of the reparametrization. This action was originally found by Kitaev in his analysis of certain quantum mechanical fermion models with emergent reparametrization symmetry [11]. Those models have a similar realization of these symmetries (see also [12]).

The effective coefficient in front of this Schwarzian action becomes small at low energies or low temperatures. For this reason, gravitational corrections, though formally suppressed by the Newton constant, lead to important effects in the IR.

This action is responsible for the form of the near-extremal black hole entropy, which is linear in the temperature. It also leads to important corrections for correlation functions, such as the connected four-point function. In particular, it controls the chaos-related, exponentially growing, corrections to out-of-time-order correlators.

Here we show how this Schwarzian action emerges in detail and we study some of its consequences as well as some details regarding its physical interpretation. A similar perspective, but for AdS_3 , was discussed in [13].

While this paper was in preparation, similar results were presented in [14]. We have also learnt of others following similar ideas.¹

This paper is organized as follows. In Sect. 2 we review the asymptotic symmetries of AdS_2 , and we explain how they give rise to a family of geometries once we cut off the space. These give rise to an infinite family of zero modes. In Sect. 3 we include the leading perturbation away from the AdS_2 limit which corresponds to the near-extremal limit of black holes. This gives rise to a non-zero action for the above geometries. The leading-order action describes a near-extremal entropy that is linear in the temperature. In Sect. 4 we add matter and show how to couple it to the nearly zero modes described above. Using this coupling we perform some perturbative computations. These include the gravitational corrections to the four-point function, including the out-of-time-order correlator, a one-loop correction to the free energy, and a gravitational correction to the two-point function. In Sect. 5 we discuss some Lorentzian aspects of this action. We discuss its $SL(2)$ symmetries and why they cure a problem involving higher derivatives. In Sect. 6 we discuss how to resum all corrections in the chaos regime to describe the crossover at the scrambling time.

2. Pure AdS_2

2.1. Coordinate systems

On AdS_2 (with unit radius) it is convenient to use the following coordinate systems:

$$\text{Euclidean : } ds^2 = \frac{dt^2 + dz^2}{z^2}, \quad ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2; \quad (2.1)$$

¹ J. Engels y, T. Martens, and H. Verlinde, to appear.

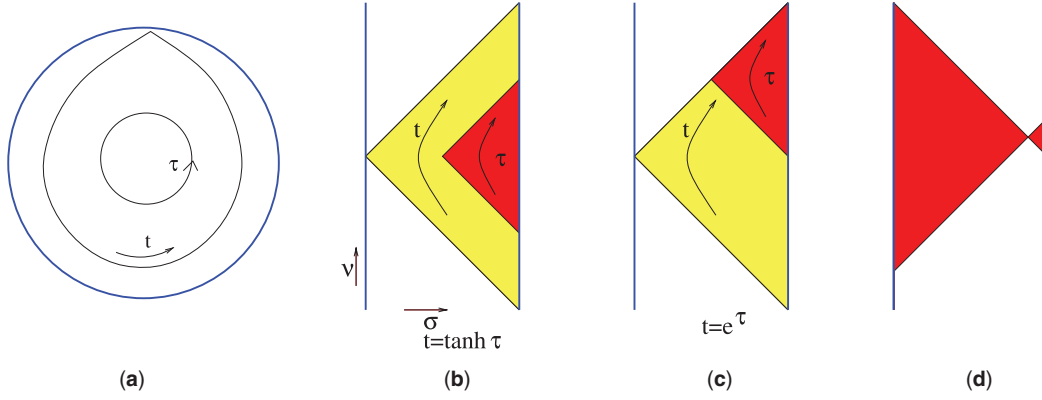


Fig. 1. (a) Hyperbolic space or Euclidean AdS_2 . The orbits of τ translations look like circles. Orbits of t are curves that touch the boundary at $t = \pm\infty$. (b) Lorentzian AdS_2 . The v, σ coordinates cover the whole strip. The $\hat{t}, z$ coordinates describe the Poincaré patch denoted here in yellow. The red region is covered by the $\hat{\tau}, \rho$ coordinates. There are different choices for how to place the $\hat{\tau}, \rho$ region that are generated by $SL(2)$ isometries. In (b) and (c) we show two choices and give the relation between the Poincaré time $\hat{t}$ and the $\hat{\tau}$ at the boundary of the space. In (d) we show a generic pair of Rindler wedges.

$$\begin{aligned} \text{Lorentzian : } ds^2 &= \frac{-d\hat{t}^2 + dz^2}{z^2}, \quad ds^2 = d\rho^2 - \sinh^2 \rho d\hat{\tau}^2, \quad ds^2 = \frac{-dv^2 + d\sigma^2}{\sin^2 \sigma}; \\ \text{Embedding : } -Y_{-1}^2 - Y_0^2 + Y_1^2 &= -1, \quad ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2. \end{aligned} \quad (2.2)$$

With Euclidean signature both coordinate choices cover all of hyperbolic space. In Lorentzian signature they cover different regions of the global space. The hatted versions of the times are Lorentzian, e.g. $t = i\hat{t}$. The causal structure of the global space is displayed clearly in the v, σ coordinates. The $\hat{t}, z$ and $\hat{\tau}, \rho$ coordinates cover different patches, as seen in Fig. 1. The $\hat{\tau}, \rho$ coordinates can be viewed as describing the exterior of a finite-temperature black hole. We can also view them as Rindler coordinates of AdS_2 . Note that the finite-temperature and zero-temperature solutions are just different coordinate patches of the same space.

2.2. Symmetries and a family of solutions

Let us imagine that we have a spacetime that is exactly AdS_2 , with a finite Newton constant. Then the gravitational action is

$$I = -\frac{\phi_0}{16\pi G} \left[\int d^2x \sqrt{g} R + 2 \int K \right] + I_m[g, \chi], \quad (2.3)$$

where I_m is the matter action and χ are the matter fields. Here, ϕ_0 is a constant, which sets the entropy $S_0 = \frac{\phi_0}{4G}$. In two dimensions, G is dimensionless.

We now want to imagine a situation where this spacetime arises as a low-energy limit of a well-defined UV theory. For this purpose we imagine that we cut off the spacetime. The UV theory has some time coordinate u . Throughout the paper, we denote the time in the boundary theory by u . Let us say that the (Euclidean) AdS_2 spacetime has the metric in (2.1). We want to cut off the space along a trajectory given by $(t(u), z(u))$. We expect to fix the proper length of the boundary curve

$$g|_{\text{bdy}} = \frac{1}{\epsilon^2}, \quad \frac{1}{\epsilon^2} = g_{uu} = \sqrt{\frac{t'^2 + z'^2}{z^2}} \longrightarrow z = \epsilon t' + O(\epsilon^3), \quad (2.4)$$

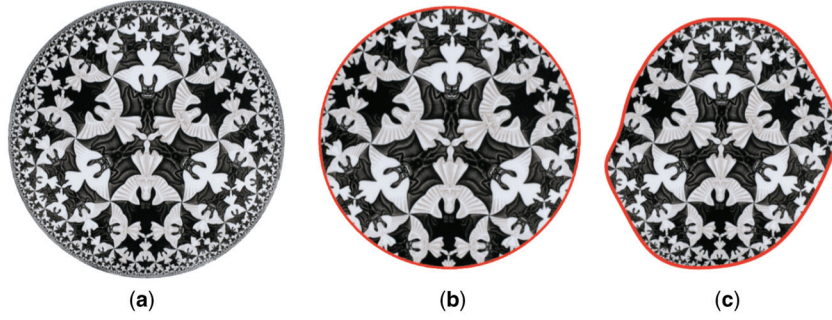


Fig. 2. In (a) we see the full AdS_2 space. In (b) we cut it off at the location of a boundary curve. In (c) we choose a more general boundary curve. The full geometry of the cutout space does depend on the choice of the boundary curve. On the other hand, the geometry of this cutout region remains the same if we displace it or rotate it by an $SL(2)$ transformation of the original AdS_2 space.

where primes are u derivatives. Note that, given an arbitrary $t(u)$, we can choose $z(u) = \epsilon t'(u)$ in order to obey the above equations. Since all other fields are constant on the AdS_2 vacuum, when we set the boundary conditions for all the fields to be such constants, we will obey all other boundary conditions. Therefore we find that we have a family of solutions to the problem, given by $t(u)$.

Let us clarify in what sense these are different solutions. The main point is that we are cutting out a region of AdS_2 , with different shapes that depend on the function $t(u)$; see Fig. 2. Though the interior AdS_2 space is locally the same, the full cutout shape does depend on $t(u)$. For example, correlation functions of matter fields will depend on the shape chosen by the function $t(u)$. Note, however, that overall translations or rotations of the whole shape in hyperbolic space do not change the physics. These are described by the action of an $SL(2)$ symmetry group on AdS_2 . It acts by sending

$$t(u) \rightarrow \tilde{t}(u) = \frac{at(u) + b}{ct(u) + d}, \quad \text{with } ad - cb = 1. \quad (2.5)$$

We see that $t(u)$ or $\tilde{t}(u)$ produce exactly the same cutout shape. Therefore the full set of different interior geometries is given by the set of all functions $t(u)$ up to the above $SL(2)$ transformations [or modded out by these $SL(2)$ transformations (2.5)].

It is worth noting that we can also look at the asymptotic symmetries of AdS_2 . They are generated by reparametrizations of the asymptotic form

$$\zeta^t = \varepsilon(t), \quad \zeta^z = z\varepsilon'(t). \quad (2.6)$$

These will map one boundary curve into another. In fact, (2.6) sends the curve $t(u) = u$ to $t(u) = u + \varepsilon(u)$.

If we insert these geometries into the action (2.3), the Gauss–Bonnet theorem implies that we always get the same action, namely the extremal entropy. Thus we have a set of exact zero modes parametrized by $t(u)$ [up to the $SL(2)$ identification (2.5)].

Notice that, near the boundary, the geometries are indistinguishable; we need to go through the bulk in order to distinguish them. In fact, this is the realization of the full reparametrization symmetry that we expect in this problem. In other words, we expect that $SL(2)$ is enhanced to a full Virasoro-like symmetry; in this case, just the reparametrization symmetries. However, the reparametrization symmetry is spontaneously broken by AdS_2 . It is broken to $SL(2, R)$. The zero modes are characterized by the functions $t(u)$. These can be viewed as Goldstone bosons, except that here we consider them

in the Euclidean problem. We can call these zero modes “boundary gravitons.” They are similar to the ones that appear in three dimensions. An important difference with the three-dimensional case is that, here, these modes have precisely zero action in the conformal limit: there is no local conformal invariant action we can write down for them.

3. $NAdS_2$, or nearly AdS_2 spacetimes

The pure AdS_2 gravity theory discussed above is not consistent with any configuration with non-zero energy, since the variation of the metric imposes that the stress tensor of matter is identically zero. The Einstein term is topological and does not contribute to the equation of motion for the metric. If one is only interested in understanding the ground state entropy, this can be enough [9,10].

In order to obtain a reasonable gravity theory it is important to consider a nearly AdS_2 geometry. In other words, we need to keep track of the leading effects that break the conformal symmetry. This is a configuration that still remembers that the conformal symmetry is slightly broken. A model that correctly captures a large number of situations where AdS_2 arises from a higher-dimensional system (or from some otherwise well-defined UV theory) is the following [3]:

$$I = -\frac{\phi_0}{16\pi G} \left[\int \sqrt{g} R + 2 \int_{\text{bdy}} K \right] - \frac{1}{16\pi G} \left[\int d^2x \phi \sqrt{g} (R + 2) + 2 \int_{\text{bdy}} \phi_b K \right] + I_M[g, \chi] + \dots \quad (3.1)$$

Here we imagine that $\phi_0 \gg \phi$ and the dots denote higher-order terms in ϕ . We will neglect all such higher-order terms here. ϕ_b is the boundary value of ϕ . If AdS_2 is arising from the near-horizon geometry of a near-extremal black hole, then $\phi_0 + \phi$ is the area of the two-sphere, and ϕ_0 is the area of the extremal black hole, with ϕ denoting the deviations from this extremal value. The middle term in the action is the Jackiw–Teitelboim two-dimensional gravity theory [4,5]. The first term is purely topological and its only role is to give the extremal entropy. We have included the extrinsic curvature terms at the boundary to make the metric variational problem well defined. From now on, we will ignore the dots in (3.1). Since the first term in the action is topological we will also ignore it.

A thorough analysis of this model was presented in an article by Almheiri and Polchinski [3]. Here we simply emphasize how the pattern of breaking of the reparametrization symmetry determines many aspects of the theory. Now, let us analyze the equations of motion of the Jackiw–Teitelboim theory:

$$I_{JT} = -\frac{1}{16\pi G} \left[\int d^2x \phi \sqrt{g} (R + 2) + 2 \int_{\text{bdy}} \phi_b K \right]. \quad (3.2)$$

The equations of motion for ϕ imply that the metric has constant negative curvature or is AdS_2 . This is also the case if we include the matter term in (3.1), since it is independent of the dilaton ϕ . The equations of motion for the metric are

$$T_{\mu\nu}^\phi \equiv \frac{1}{8\pi G} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi + g_{\mu\nu} \phi) = 0. \quad (3.3)$$

Due to the Bianchi identity, this implies that $T_{\mu\nu}^\phi$ is automatically conserved. It turns out that the general solution is

$$\phi = \frac{\alpha + \gamma t + \delta(t^2 + z^2)}{z} = Z.Y, \quad (3.4)$$

where we also rewrote the expression in embedding coordinates (2.2), where Z is an arbitrary vector.²

The solution breaks the $SL(2)$ isometries to $U(1)$. In fact, the vector $\zeta^\mu = \epsilon^{\mu\nu} \partial_\nu \phi$ is always a Killing vector of the metric thanks to the equations (3.3) [15]. Thus, the combined dilaton gravity theory always preserves this isometry.

Since ϕ is diverging near the boundary, we now have a new dimensionful coupling constant which is the strength of that divergence. In other words, beyond the condition (2.4) we also need to impose the condition

$$\phi_b = \phi|_{\text{bdy}} = \frac{\phi_r(u)}{\epsilon}, \quad (3.5)$$

where $\phi_r(u)$ is an arbitrary dimension -1 coupling. The r stands for “renormalized,” in the sense that it remains finite in the $\epsilon \rightarrow 0$ limit. For generality we have chosen it to depend on u , but we could also choose it to be independent of u . When we choose it to be constant we will denote it as $\bar{\phi}_r$.

When we embed this into a full higher-dimensional picture, we have in mind situations where $\phi_b \propto 1/\epsilon$ is large, but $\phi_b \ll \phi_0$ so that we are still in the near-extremal region.³ In other words, we cut off the space before ϕ becomes too large. Note that the notion of “too large” is really external to the theory (3.2).

Now, once we impose both (2.4) and (3.5) we determine completely the shape of the curve or reparametrization $t(u)$. It is simply given by computing $z(u)$ from (2.4), substituting in (3.4), and then using (3.5) to obtain

$$\frac{\alpha + \gamma t(u) + \delta t(u)^2}{t'(u)} = \phi_r(u). \quad (3.6)$$

It is interesting that this condition can also be obtained from an effective action for $t(u)$. A simple way to obtain the effective action is the following. Starting from (3.2) we impose the equation of motion for ϕ , which implies that we have an AdS_2 space. Inserting that into the action (3.2) we find that the first term vanishes and we only get the boundary term, which involves the boundary value of ϕ (3.5),

$$I_{TJ} \rightarrow -\frac{1}{8\pi G} \int \frac{du}{\epsilon} \frac{\phi_r(u)}{\epsilon} K, \quad (3.7)$$

where we also used that the induced metric is given by du/ϵ (2.4). The extrinsic curvature is given by

$$K = \frac{t'(t'^2 + z'^2 + zz'') - zz't''}{(t'^2 + z'^2)^{\frac{3}{2}}} = 1 + \epsilon^2 \text{Sch}(t, u),$$

$$\text{Sch}(t, u) \equiv -\frac{1}{2} \frac{t''^2}{t'^2} + \left(\frac{t''}{t'} \right)'. \quad (3.8)$$

Inserting this into I_{TJ} we get

$$I = -\frac{1}{8\pi G} \int du \phi_r(u) \text{Sch}(t, u). \quad (3.9)$$

² More precisely, in (3.4) we use the Euclidean version of the embedding coordinates.

³ This type of expansion is somewhat analogous to the slow roll expansion for inflationary universes.

We see that the zero modes get an action determined by the Schwarzian. Here, $\phi_r(u)$ is an external coupling and $t(u)$ is the field variable.

It is interesting to contemplate why we obtained this. We expect that the breaking of conformal symmetry should be local along the boundary, and proportional to $\phi_r(u)$. In addition, we expect to obtain a local action which involves the pseudo-Nambu–Goldstone modes. Since these are specified by $t(u)$ up to global $SL(2)$ transformations, we conclude that the simplest term is the Schwarzian action, which is indeed $SL(2)$ invariant; $Sch(t, u) = Sch\left(\frac{at+b}{ct+d}, u\right)$.

Finally, it is easy to check that by varying (3.9) with respect to $t(u)$ we obtain the equation

$$\left[\frac{1}{t'} \left(\frac{(t' \phi_r)'}{t'} \right)' \right]' = 0, \quad (3.10)$$

which can be easily integrated to (3.6), where α , γ , and δ are integration constants.⁴ Thus we see that the action (3.9), which is defined purely on the boundary, captures the same information as the bulk expression for the dilaton ϕ . Notice that this also implies that the equations of motion of the action (3.9) are equivalent to imposing the equations of motion that result from varying the metric, which were not imposed in deriving (3.9). The time dependence of $\phi_r(u)$ allows us to pick an arbitrary $t(u)$ as the saddle point geometry. On the other hand, we can also remove it by picking a new time coordinate via $d\tilde{u} = \bar{\phi}_r du / \phi_r(u)$. When $\phi_r(u)$ is constant, (3.10) becomes $\bar{\phi}_r \frac{[Sch(t, u)]'}{t'} = 0$.

The Schwarzian action summarizes many gravitational effects of the model. As we have explained, it follows from the symmetries of the problem and its applicability can go beyond systems that are described by a local gravity theory. In fact, this Schwarzian action was introduced, for these reasons, by Kitaev in his analysis of certain interacting fermion models [11] (see [16] for a description).

3.1. The near-extremal entropy

It is convenient to make a change of field variable in the Schwarzian action from t to τ of the form

$$t = \tan \frac{\tau}{2}. \quad (3.11)$$

We can then use the general transformation rule for the Schwarzian to find

$$\begin{aligned} I &= -C \int du Sch(t, u) = -C \int du \left[Sch(\tau, u) + \tau'^2 Sch(t, \tau) \right] \\ &= -C \int du \left[Sch(\tau, u) + \frac{1}{2} \tau'^2 \right], \quad C \equiv \frac{\bar{\phi}_r}{8\pi G}. \end{aligned} \quad (3.12)$$

We could have derived this form of the action by starting with AdS_2 in terms of the coordinates $ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2$ and viewing the boundary as parametrized by $\tau(u)$, with $\rho(u)$ determined by the analog of (2.4).

This is an interesting action, whose solutions are $\tau = \frac{2\pi}{\beta} u$ (up to $SL(2)$ transformations). Note that $\tau \sim \tau + 2\pi$. For these solutions, only the term involving τ'^2 in (3.12) is important. On such solutions the action gives

$$\log Z = -I = 2\pi^2 \frac{C}{\beta} = 2\pi^2 CT, \quad (3.13)$$

⁴ A fourth integration constant arises by integrating (3.6).

which leads to a near-extremal entropy $S = S_0 + 4\pi^2 CT$ that is linear in the temperature. Note that $4\pi^2 CT$ is also the specific heat. This linear in T behavior is a simple consequence of the reparametrization symmetry and its breaking.

This gives us only the near-extremal entropy. The extremal entropy, S_0 , can be obtained by adding a purely topological term to the above action of the form

$$-I_{\text{top}} = \frac{\phi_0}{8\pi G_N} \int du \tau'. \quad (3.14)$$

It might seem unusual that we reproduce the entropy from a classical action. This is familiar from the bulk point of view, but it seems unusual to reproduce it from a boundary-looking action. However, this is common in discussions of hydrodynamics. In that case, the free energy is reproduced from a classical action. Here the crucial feature is that the solution depends on the temperature through the condition that τ winds once as we go from $u = 0$ to $u = \beta$.

We could wonder whether we should consider solutions where τ winds n times, $\tau = n \frac{2\pi}{\beta} u$. It appears that this effective action makes sense only for the case with winding number one.⁵

Note that this is not a microscopic derivation of the entropy. This is simply phrasing the computation of the entropy as a consequence of a symmetry. We have not given an explicit description of the black hole microstates. If one had a microscopic system which displayed this symmetry breaking pattern, then we would microscopically explain the form of the entropy.

It is also possible to compute the Arnowitt–Deser–Misner (ADM) energy of the system. This is given in terms of the boundary values of the fields. In this case, we get [3]

$$M = \frac{1}{8\pi G} \frac{1}{\epsilon} [\phi_b - \partial_n \phi] = \frac{\bar{\phi}_r}{8\pi G} \text{Sch}(t, u) = C \text{Sch}(t, u) = -C \text{Sch}(\hat{t}, \hat{u}). \quad (3.15)$$

The second expression gives the mass terms of the Schwarzian action. This can be obtained by either solving the equations for ϕ or by deriving the conserved-quantity-associated u translations for (3.9). It is valid in the absence of boundary sources for massive fields. The general formula is given in (A.11).

4. Adding matter

We can now add matter as in (3.1). Since ϕ does not appear in the matter action, the metric is still fixed to the AdS_2 metric by the ϕ equation of motion, and the matter fields move on this fixed AdS_2 geometry. The gravitational backreaction is completely contained in the equations obeyed by the dilaton, which are simply (3.3) but with the matter stress tensor on the right-hand side.⁶ These are three equations for a single variable ϕ , but the conservation of the matter stress tensor implies that the equations are consistent. The boundary is located by finding the curve where $\phi = \phi_b$. This can be done by first solving for ϕ in the bulk as described above and then finding the trajectory of the boundary curve. Alternatively, one can show that the final equation for the trajectory is given by an equation where we add a new term on the right-hand side of (3.10). For the case of massless matter

⁵ For $n = 0$ the τ' terms in the numerator are a problem. For $n > 1$ the small fluctuations around the solution have negative modes.

⁶ This structure is similar to other models of dilaton gravity where the metric is forced to be flat, instead of AdS_2 ; see [17] for a review.

fields we obtain

$$C \frac{(\text{Sch}(t, u))'}{t'} = -t' T_{tz}. \quad (4.1)$$

A simple derivation is obtained by equating the change in energy (3.15) to the flux of energy, $-t'^2 T_{tz}$, into the space. A factor of t' comes from redshifting the energy from t to u time and another factor from going from energy per unit t to energy per unit u .⁷ Solving (4.1) we find $t'(u)$ and solve directly for the trajectory of the boundary curve. The correction to (4.1) when we have sources for massive fields is given in Appendix A.

For correlation function computations it is useful to calculate the effective action as a function of the boundary conditions for the matter fields, $\chi_r(u)$, which can be functions of the boundary time.

It is convenient to solve first an auxiliary problem, which consists of finding the effective action for the matter fields in AdS , with boundary conditions $\tilde{\chi}_r(t)$ that are functions of the AdS_2 boundary time. For a free field in AdS_2 this is simple to compute, and we obtain

$$-I_{eff} = D \int dt dt' \frac{\tilde{\chi}_r(t) \tilde{\chi}_r(t')}{|t - t'|^{2\Delta}}, \quad (4.2)$$

where D is a constant.⁸ Once we specify the trajectory of the actual boundary curve via $t(u)$ we can transform this to the desired boundary conditions:

$$\chi \sim z^{1-\Delta} \tilde{\chi}_r(t) = \epsilon^{1-\Delta} t'^{1-\Delta} \tilde{\chi}_r(t) = \epsilon^{1-\Delta} \chi_r(u) \longrightarrow \chi_r(u) = [t'(u)]^{1-\Delta} \tilde{\chi}_r(t(u)). \quad (4.3)$$

The first expression defines $\tilde{\chi}_r(t)$; then we used the expression for z from (2.4), and finally we compared it to the expression for χ that defines $\chi_r(u)$. Using (4.3), we rewrite (4.2) as

$$-I_{eff} = D \int du du' \left[\frac{t'(u) t'(u')}{[t(u) - t(u')]^2} \right]^\Delta \chi_r(u) \chi_r(u'). \quad (4.4)$$

Though we did this for the two-point function, the same is true for any n -point function. If we had a self-interacting matter theory in AdS_2 and we computed the AdS_2 n -point correlation function, then the correct physical one in $NAdS_2$, after coupling to gravity, would be obtained by writing them using $t(u)$, and rescaling by a factor of $t'(u_i)^{\Delta_i}$ at the insertion of each operator.⁹ Even if we had free fields in AdS_2 this coupling to gravity makes them interact with each other. What is remarkable is the simplicity of this coupling.

In this way we have found a coupling between $t(u)$ and the matter action. We see that the coupling proceeds by a reparametrization of the original two-point function. The full correlation functions are obtained by integrating over $t(u)$, after we add the Schwarzian action (3.9). These are the same formulas derived in [3]. The classical equations for $t(u)$ that follow from the variation of the Schwarzian action (3.9) plus (4.4) are the same as the ones in [3].

⁷ In Lorentzian signature we get a minus sign in (4.1) from the minus in (3.15).

⁸ $D = \frac{(\Delta - \frac{1}{2})\Gamma(\Delta)}{\sqrt{\pi}\Gamma(\Delta - \frac{1}{2})}$, or $D = 1/2\pi$ for $\Delta = 1$.

⁹ We can say that if $Z_M[\tilde{\chi}_r(t), z(t)]$ is the partition function of the pure matter theory with boundary conditions at $z = z(t)$, then the one in the theory coupled to dilaton gravity is $Z_{\text{Dressed}}[\chi_r(u), \epsilon] = Z_M[t'^{\Delta-1} \chi(u), z(t(u))]$, where $z = \epsilon t' + \dots$.

4.1. Perturbative expansion of the Schwarzian action

Since the Schwarzian action is of order $1/G$ we can evaluate its effects using perturbation theory around a solution. To avoid carrying unnecessary factors of the temperature, we set $\beta = 2\pi$. The factors of temperature can be reinstated by dimensional analysis. We then set

$$\tau = u + \varepsilon(u) \quad (4.5)$$

in (3.12) and expand to second order in ε to obtain

$$I_\epsilon = \frac{C}{2} \int du [\varepsilon''^2 - \varepsilon'^2], \quad \text{for } \beta = 2\pi. \quad (4.6)$$

We would like to compute the propagator for this action. A problem is that the action has three zero modes, going like $\epsilon = 1, e^{iu}, e^{-iu}$. These zero modes arise from $SL(2)$ transformations of the background solution $\tau = u$. Recall that these $SL(2)$ transformations did not generate new geometries. Therefore we should not be integrating over them in the first place, since the integral over ε is only over distinct geometries. This is equivalent to viewing the $SL(2)$ symmetry as a gauge symmetry, so that we can gauge fix those three zero modes to zero and invert the propagator. The answer is

$$\langle \epsilon(u) \epsilon(0) \rangle = \frac{1}{2\pi C} \left[-\frac{(|u| - \pi)^2}{2} + (|u| - \pi) \sin |u| + a + b \cos u \right]. \quad (4.7)$$

The last two terms are proportional to $SL(2)$ zero modes and cancel in any gauge-invariant computation.¹⁰ Note that the propagator is formally suppressed by G , but it is enhanced as $\bar{\phi}_r$ becomes small. We will now use this propagator for some computations. The effective coupling is β/C , which is the same as the inverse of the near-extremal entropy.¹¹

4.2. Gravitational contributions to the four-point function

Suppose that we have operators V, W , which are dual to two different fields that are free in AdS_2 before coupling to gravity. The gravitational contribution to the four-point function can be computed as follows. (Some four-point functions were also considered in [3]. These steps are identical to the ones discussed in [16], since the effective action is the same.) We start from the factorized expression for the four-point function, $\langle V(t_1)V(t_2)W(t_3)W(t_4) \rangle = \frac{1}{t_{12}^{2\Delta}} \frac{1}{t_{34}^{2\Delta}}$. We then insert the reparametrizations (3.11) and (4.5) into (4.4) and expand to linear order in ε to obtain

$$\frac{1}{t_{12}^{2\Delta}} \longrightarrow \mathcal{B}(u_1, u_2) \frac{\Delta}{[2 \sin \frac{u_{12}}{2}]^{2\Delta}}, \quad \mathcal{B}(u_1, u_2) \equiv \left[\varepsilon'(u_1) + \varepsilon'(u_2) - \frac{\varepsilon(u_1) - \varepsilon(u_2)}{\tan \frac{u_{12}}{2}} \right]. \quad (4.8)$$

We make a similar replacement for $t_{34}^{-2\Delta}$, and then contract the factors of ε using the propagator (4.7). This gives the $O(1/C) = O(G)$ contribution to the four-point function. Note that the bilocal operator $\mathcal{B}$ is $SL(2)$ invariant.¹² The final expression depends on the relative ordering of the four points. When $u_4 < u_3 < u_2 < u_1$ we obtain the factorized expression

$$\frac{\langle V_1 V_2 W_3 W_4 \rangle_{\text{grav}}}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \Delta^2 \langle \mathcal{B}(u_1, u_2) \mathcal{B}(u_3, u_4) \rangle = \frac{\Delta^2}{2\pi C} \left(-2 + \frac{u_{12}}{\tan \frac{u_{12}}{2}} \right) \left(-2 + \frac{u_{34}}{\tan \frac{u_{34}}{2}} \right). \quad (4.9)$$

¹⁰ A direct inversion of the operator gives $a = 1 + \pi^2/6$ and $b = 5/2$ [16].

¹¹ This is the reason that there is trouble with naive black hole thermodynamics at $C/\beta \sim 1$ [1].

¹² Indeed, a and b in (4.7) disappear if we consider $\langle \epsilon(u) \mathcal{B}(u_1, u_2) \rangle$.

As discussed in [16], this expression can be viewed as arising from energy fluctuations. Each two-point function generates an energy fluctuation which then affects the other. Since energy is conserved, the result does not depend on the relative distance between the pair of points. In other words, we can think of it as

$$\langle V_1 V_2 W_3 W_4 \rangle_{\text{grav}} = \partial_M \langle V_1 V_2 \rangle \partial_M \langle W_3 W_4 \rangle \frac{1}{-\partial_M^2 S(M)} = \partial_\beta \langle V_1 V_2 \rangle \partial_\beta \langle W_3 W_4 \rangle \frac{1}{\partial_\beta^2 \log Z(\beta)}, \quad (4.10)$$

where M is the mass of the black hole background, or β its temperature, and $S(M)$ or $\log Z$ are its entropy or partition function.¹³ Both expressions give the same answer, thanks to thermodynamic identities between entropy and mass.¹⁴ If one expands as $u_{12} \rightarrow 0$ we get a leading term going like u_{12}^2 , which one would identify with an operator of dimension two. In this case this is the Schwarzian itself, which is also the energy, and it is conserved (3.15). Its two-point functions are constant.¹⁵

It is also interesting to evaluate the correlator in the other ordering $u_4 < u_2 < u_3 < u_1$. We get

$$\begin{aligned} \frac{\langle V_1 W_3 V_2 W_4 \rangle_{\text{grav}}}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} &= \frac{\Delta^2}{2\pi C} \left[\left(-2 + \frac{u_{12}}{\tan \frac{u_{12}}{2}} \right) \left(-2 + \frac{u_{34}}{\tan \frac{u_{34}}{2}} \right) + \right. \\ &\quad \left. + \frac{2\pi [\sin(\frac{u_1 - u_2 + u_3 - u_4}{2}) - \sin(\frac{u_1 + u_2 - u_3 - u_4}{2})]}{\sin \frac{u_{12}}{2} \sin \frac{u_{34}}{2}} + \frac{2\pi u_{23}}{\tan \frac{u_{12}}{2} \tan \frac{u_{34}}{2}} \right]. \quad (4.11) \end{aligned}$$

This expression interpolates between (4.9) when $u_3 = u_2$ and an expression like (4.9), but with $u_{34} \rightarrow -2\pi + u_{34}$, when $u_3 = u_1$. Note that now the answer depends on the overall separation of the two pairs. This dependence, which involves the second sine term in the numerator as well as the u_{23} factor, looks like we are exciting the various zero modes of the Schwarzian action, including the exponential ones. It is interesting to continue (4.11) to Lorentzian time and into the chaos region, which involves the correlator in the out-of-time-order form

$$\langle V(a) W_3(b + \hat{u}) V(0) W(\hat{u}) \rangle \sim \frac{\beta \Delta^2}{C} e^{\frac{2\pi \hat{u}}{\beta}}, \quad \frac{\beta}{2\pi} \ll \hat{u} \ll \frac{\beta}{2\pi} \log \frac{C}{\beta}, \quad (4.12)$$

where $a, b \sim \beta$. Here we restored the temperature dependence in (4.11) by multiplying by an overall factor of $\frac{\beta}{2\pi}$ and sending $u_i \rightarrow \frac{2\pi}{\beta} u_i$.

We can also connect (4.12) to a scattering process. It is peculiar that in this setup the two particles do not scatter since they behave like free fields on a fixed AdS_2 background. On the other hand, they create a dilaton profile which gives rise to a nontrivial interaction once we relate the AdS_2 time to the boundary time. The net result is the same as what is usually produced by the scattering of shock waves; see Appendix B. Here we see that the gravitational effects are very delocalized; we can remove them from the bulk and take them into account in terms of the boundary degree of freedom $t(u)$.

4.3. Loop corrections

We can use the Schwarzian action as a full quantum theory and we can compute loop corrections. The simplest example corresponds to the one-loop correction to the free energy.¹⁶ This arises from

¹³ The correlator at finite β is $\langle VV \rangle = \left[\frac{\beta}{\pi} \sin \frac{\pi u_{12}}{\beta} \right]^{-2\Delta}$.

¹⁴ (4.10) is valid for a general spherically symmetric reduction of general relativity to two dimensions.

¹⁵ Note that this is different than in 1+1 dimensions, where the stress tensor correlators go like $1/z^4$.

¹⁶ J. Polchinski and A. Streicher, private communication.

computing the functional determinant of the quadratic operator in (5.1). This was done in detail in [16] and we will not repeat the details. The important point is simply that it gives a temperature-dependent correction to the free energy going like

$$\log Z|_{\text{one loop}} = -\frac{3}{2} \log \left(\frac{\beta}{C} \right). \quad (4.13)$$

This is a correction to the leading classical expression (3.13). The determinants of all matter fields in AdS_2 are conformally invariant and should not give rise to a temperature-dependent contribution, but they can and do contribute to the extremal entropy [18,19]. The correction (4.13) is such that there is no logarithmic correction to the entropy as a function of energy. This is good, since there are cases, such as BPS black holes, where we do not expect corrections that diverge in the IR.

As a second example, we can consider a loop correction to the two-point function. We expand the reparametrization

$$\frac{1}{t_{12}^{2\Delta}} \rightarrow \frac{(1 + \varepsilon'_1)^\Delta (1 + \varepsilon'_2)^\Delta}{(\sin \frac{u_{12} + \varepsilon_1 - \varepsilon_2}{2})^{2\Delta}} \quad (4.14)$$

to quadratic order in the ε , and then contract using the propagator (4.7):

$$\begin{aligned} \frac{\langle V_1 V_2 \rangle_{\text{one loop}}}{\langle V_1 V_2 \rangle_{\text{tree}}} &= \Delta \left\langle \frac{(\varepsilon_1 - \varepsilon_2)^2}{4 \sin^2 \frac{u}{2}} - \frac{1}{2} (\varepsilon'_1{}^2 + \varepsilon'_2{}^2) \right\rangle + \frac{\Delta^2}{2} \left\langle \left(\varepsilon'_1 + \varepsilon'_2 - \frac{(\varepsilon_1 - \varepsilon_2)}{\tan \frac{u}{2}} \right)^2 \right\rangle \\ &= \frac{1}{2\pi C} \left[\Delta \frac{(u^2 - 2\pi u + 2 - 2 \cos u + 2(\pi - u) \sin u)}{4 \sin^2 \frac{u}{2}} + \right. \\ &\quad \left. + \frac{\Delta^2}{2} \left(-2 + \frac{u}{\tan \frac{u}{2}} \right) \left(-2 + \frac{(u - 2\pi)}{\tan \frac{u}{2}} \right) \right], \quad u = u_1 - u_2 > 0. \end{aligned} \quad (4.15)$$

It is interesting to continue these formulas to Lorentzian signature $u \rightarrow i\hat{u}$ and then expand them for large Lorentzian times. The largest term, which goes at $\hat{u}^2$ for large Lorentzian time, can arise from energy fluctuations in a manner analogous to (4.10). The tree-level correlator includes a quasinormal decay as $e^{-\Delta \hat{t}} \sim e^{-\Delta 2\pi \hat{u}/\beta}$. But the energy fluctuations cause a temperature fluctuation which would then lead to a correction for the ratio of the one loop to tree level as

$$\frac{\Delta^2}{2} \frac{4\pi^2 \hat{u}^2}{\beta^4} \frac{1}{\partial_\beta^2 \log Z} = \frac{\Delta^2 \hat{u}^2}{2\beta C}, \quad (4.16)$$

which agrees with the $\hat{u}^2$ piece from (4.15).¹⁷

5. Lorentzian picture and the $SL(2)$ symmetry

5.1. $SL(2)$ symmetry of the Schwarzian action

We have seen that gravitational effects are summarized by the Schwarzian action (3.9). This action seems problematic when viewed as an action in Lorentzian signature since it involves higher-derivative terms. These usually lead to ghosts. We can see this more explicitly by starting with

¹⁷ In fact, (4.15) can be written exactly as $\langle V_1 V_2 \rangle_{\text{one loop}} = \frac{1}{2\partial_\beta^2 \log Z} \partial_\beta \left\{ \left[\partial_\beta \langle V_1 V_2 \rangle_{\text{tree}} \right]_{u \rightarrow \beta - u} \right\}$. The derivatives seem to be a way of varying the temperature in a way that maintains periodicity.

the Lorentzian action for small fluctuations,

$$iI_L = -iC \int d\hat{u} \text{Sch}(\hat{t}, \hat{u}) = i\frac{C}{2} \int d\hat{u}(\varepsilon''^2 + \varepsilon'^2), \quad \text{for } \beta = 2\pi, \quad C \equiv \frac{\bar{\phi}_r}{8\pi G}. \quad (5.1)$$

It is possible to rewrite this higher-derivative action in terms of a two-derivative action for two fields by introducing a new field η ,

$$\int_{\varepsilon = r+q, \quad \eta = r-q} d\hat{u}[\varepsilon''^2 + \varepsilon'^2] \rightarrow \int d\hat{u} \left[\eta(\varepsilon'' - \frac{\varepsilon}{2}) - \frac{\eta^2}{4} - \frac{\varepsilon^2}{4} \right] = \int d\hat{u}[-r'^2 - r^2 + q'^2] \quad (5.2)$$

Integrating out η we get $\eta = 2\varepsilon'' - \varepsilon$ and recover the original action, (5.1). We can also use this expression for η to express r and q in terms of ε , which gives

$$r = \varepsilon'', \quad q = -\varepsilon'' + \varepsilon. \quad (5.3)$$

The full set of solutions of the original Lorentzian action (5.1) is given by

$$\varepsilon = (\alpha e^{\hat{u}} + \beta e^{-\hat{u}}) + (\gamma \hat{u} + \delta). \quad (5.4)$$

We see that the first parenthesis corresponds to the ghost-like mode r and the last one to the mode q . Note that in Euclidean space we started out with a full function worth of nearly zero modes, but in the Lorentzian theory these only give rise to the two degrees of freedom r and q .

Should we be worried by the appearance of the ghost-like mode that has a negative sign in its kinetic term in (5.2)? Should we view the exponentially growing solutions in (5.4) as an instability? To answer these questions we need to recall that the original metric had an unbroken $\text{SL}(2)$ symmetry. Such $\text{SL}(2)$ diffeomorphisms do not generate a new cutout geometry. Thus, we should not include them in our integration over pseudo-Goldstone modes. One way to remove them is to treat such diffeomorphisms as a gauge symmetry. More precisely, the Schwarzian action has an $\text{SL}(2)$ global symmetry. This global symmetry has its associated conserved charges

$$\begin{aligned} Q^- &= C \left[\frac{\hat{t}'''}{\hat{t}'^2} - \frac{\hat{t}''^2}{\hat{t}'^3} \right] = C e^{-\hat{t}} \left[\frac{\hat{t}'''}{\hat{t}'^2} - \frac{\hat{t}''^2}{\hat{t}'^3} + \frac{\hat{t}''}{\hat{t}'} \right], \\ Q^0 &= C \left[\frac{\hat{t}'''\hat{t}}{\hat{t}'^2} - \frac{\hat{t}''^2\hat{t}}{\hat{t}'^3} - \frac{\hat{t}''}{\hat{t}'} \right] = C \left[\frac{\hat{t}'''}{\hat{t}'^2} - \frac{\hat{t}''^2}{\hat{t}'^3} - \hat{t}' \right], \\ Q^+ &= C \left[\frac{\hat{t}'''\hat{t}^2}{\hat{t}'^2} - \frac{\hat{t}''^2\hat{t}^2}{\hat{t}'^3} - \frac{2\hat{t}\hat{t}''}{\hat{t}'} + 2\hat{t}' \right] = C e^{\hat{t}} \left[\frac{\hat{t}'''}{\hat{t}'^2} - \frac{\hat{t}''^2}{\hat{t}'^3} - \frac{\hat{t}''}{\hat{t}'} \right], \end{aligned} \quad (5.5)$$

written after setting $\hat{t} = e^{\hat{t}}$, which is appropriate for Lorentzian finite-temperature computations. Treating them as gauge symmetry amounts to saying that the full state should be invariant under these symmetries. However, we see that a solution with nonzero $\hat{t}'$ cannot have zero charges! Recall, though, that in the bulk this $\text{SL}(2)$ symmetry acts on the full AdS_2 spacetime. This means that it is a symmetry of the thermofield double. In the quantum mechanical description of the thermofield double we have two sides and the charges are equal and opposite to the two sides, $Q_L^a = -Q_R^a$, so that the total charge can be zero. Therefore, purely on one side the charges can be anything. We can view the charges Q^a as proportional to the vector Z^a in (3.4) that determines the location of the bifurcation point. The $\text{SL}(2)$ transformations move this point in AdS_2 . This motion has no physical

consequence because the location of the boundary is determined by the value of the dilaton and thus the boundary curve moves together with the bifurcation point as we perform an $SL(2)$ transformation.

For the simplest solution $\hat{\tau} = \frac{2\pi}{\beta}u$, the charges are

$$Q^\pm = 0, \quad Q^0 = -C \frac{2\pi}{\beta}. \quad (5.6)$$

This value of Q^0 (when $Q^\pm = 0$) can be viewed as (minus) the near-extremal entropy of the black hole. More precisely, $S = -2\pi Q$. As Wald has pointed out [20], we can view black hole entropy as a Noether charge associated to the translation generated by the horizon-generating Killing vector.

There is an additional conserved quantity of the Schwarzian action, which is simply associated to u time translations. This is the Hamiltonian discussed in (3.15). It is interesting to note the relation

$$H = \frac{1}{2C} [-Q^+ Q^- + (Q^0)^2] \quad (5.7)$$

between the energy and the charges. Here, the Q are the charges of only the t field on one side, as in (5.5).

It is also interesting to evaluate the charges and the Hamiltonian for a first-order perturbation around the thermal solution, $\hat{\tau} = u + \varepsilon(u)$:

$$\begin{aligned} Q^\pm &\sim C e^{\pm u} [\varepsilon''' \mp \varepsilon''], & Q^0 &\sim C [-1 + \varepsilon''' - \varepsilon'], \\ H &\sim C \left[\frac{1}{2} - (\varepsilon''' - \varepsilon') \right]. \end{aligned} \quad (5.8)$$

With these expressions we see that the zero mode $\varepsilon = e^u$ only contributes to Q^- and $\varepsilon = e^{-u}$ only to Q^+ . Two-point functions of H are constant, as expected for a conserved quantity.¹⁸ Saying that we treat the $SL(2)$ symmetry as a gauge symmetry implies that we are not free to excite these modes. These modes are excited in an amount that is set by the value of the charges.

At this linear order in ε , we can also show that the Hamiltonian has the expected commutation relation with operators. From (5.8) and (5.3) we have $H \sim C \left(\frac{1}{2} + q' \right)$. Assuming a canonical quantization of the non-ghost mode q , we conclude that $[H, \varepsilon] = C[q', q] = -i$ and $[H, \varepsilon'] = 0$. To evaluate the commutator with V , we include the reparametrization dressing $V \rightarrow (1 + \varepsilon')^\Delta V(u + \varepsilon)$ and then expand to linear order in ε . This immediately gives $[H, V] = -iV'$.

5.2. Adding matter

If we have matter in AdS_2 then the matter can also carry $SL(2)$ charges. The total $SL(2)$ charge is the sum of the matter one plus the one carried by the field $\hat{\tau}(\hat{u})$ that appears in the Schwarzian action. For massless matter we simply have

$$Q_T^a = Q^a(\hat{\tau}) + q_M^a, \quad (5.9)$$

where q_M^a are the standard charges associated to the AdS_2 isometries for the matter fields. The $SL(2)$ gauge symmetry is saying that Q_T will remain constant as we add matter. This is compatible with

¹⁸ Two-point functions of the Q^a , such as $\langle Q^a(u) Q^b(0) \rangle$, are *not* constant, despite their classical conservation law. This is due to the fact that we needed to break the $SL(2)$ symmetry to compute the propagator for ε (4.7). This is not a problem because these are not gauge-invariant quantities.

the equations of motion (4.1), and the fact that the $SL(2)$ charges change by a flux of energy; for massless matter we have simply $\partial_{\hat{t}}(Q^-, Q^0, Q^+) = T_{\hat{t}z} \hat{t}'(1, \hat{t}, \hat{t}^2)$. When sources for massive fields are turned on, one has to add an extra stress tensor term to q_M to define matter charges that satisfy the correct conservation conditions; see (A.13). This $SL(2)$ gauge symmetry implies that we cannot purely excite one of the ghost modes, we have to excite them together with some matter fields.

The total energy is still given by the ADM expression (3.15) and it is written purely in terms of the $\hat{t}$ variable. In particular, (5.7) continues to be true where Q^a in (5.7) are the $SL(2)$ charges of the τ system only, they are *not* the total $SL(2)$ charge appearing in (5.9). So we see that the matter inside AdS_2 only carries $SL(2)$ charge and their contribution to the mass only appears through the $SL(2)$ constraints that relate these charges to the $SL(2)$ charge of the $\hat{t}$ variable.

Suppose that we start from the thermofield double state and then we add matter on the right part. Then from the right part point of view we can view the charges and masses of the left part of the thermofield double as being carried at the horizon. Then the condition that the total charge vanishes becomes simply the condition that

$$Q_h^a = Q^a + q_M^a = Q_R^a, \quad (5.10)$$

where Q_R^a are the total charges of the right system, q_M^a are the $SL(2)$ charge of possible matter falling into the black hole, and $Q_h^a = -Q_L^a$ is the value of the charge at the bifurcation point, also equal to minus the charges of the left side. Here we assumed that there is no matter on the left side of the spacetime. We have seen that for the simple solution $\tau = u$ the charge $Q^0 < 0$, and it is related to the energy (5.7). On the other hand, with the same conventions, the matter charge q_M^0 of a matter particle would be *positive*. That makes (5.10) compatible with the energy conservation condition for small fluctuations which says that the mass of the black hole plus the energy of matter should be the same as the energy measured at the boundary.

When we throw matter into the black hole, the values of these charges change, but always in agreement with the conservation law. As we send in matter from the boundary, its additional mass is immediately recorded in the new value for the on-shell Schwarzian. The $SL(2)$ charges of the matter, together with the boundary charges, are constrained to add to the same value that the boundary system had before we threw in the matter. For an initial configuration with $Q^\pm = 0$, the changes of the Q^0 charges demanded by (5.10) can be viewed as a consequence of the first law, once we remember that Q^0 is related to the entropy.

Let us add a classical massive particle following a geodesic in the background AdS_2 spacetime. The equations for this geodesic are given by $A.Y = 0$ in embedding coordinates. If we choose $A^2 = \pm m^2$, then $\vec{A}$ is also proportional to the $SL(2)$ charges. Then the dilaton on the other side of the geodesic is given by $\Phi = (Z + A).Y$, where $Z + A$ reflect the $SL(2)$ charges on the other side of the geodesic.

An important point to note is that the Schwarzian action breaks conformal symmetry for the u time, so that general $SL(2)$ transformation of u , such as $u \rightarrow (au + b)/(cu + d)$, is *not* a symmetry. (Only $u \rightarrow u + \text{constant}$ is a symmetry.) Nevertheless, the $SL(2)$ charges acting on t are still conserved, since they are gauge symmetries. In other words, we should not confuse the $SL(2)$ symmetry acting on t , which is gauged and thus unbroken, with the $SL(2)$ symmetry acting on u , which is not gauged and is broken by the Schwarzian action.

These charges are analogous to the edge modes of the electromagnetic field discussed in [21,22], or the “center” in [23], or horizon symmetries in [24].

It is likely that there is a more elegant way to think about these $SL(2)$ charges using the $SL(2)$ gauge theory formulation of (3.2) [25,26].

6. Higher orders in the chaos region

The order $G \sim 1/C$ term in the four-point function (4.12) is exponentially growing in the time separation $\hat{u}$ of the V and W operators. However, we actually expect the full correlator to become small at large $\hat{u}$. This is due to higher-order effects in powers of $1/C$. As an application of the Schwarzian action, we will show how to sum powers of $(e^{\hat{u}}/C)$, which will be enough to capture the late-time decay. More precisely, we work in a limit $C \rightarrow \infty$, $\hat{u} \rightarrow \infty$ with $e^{\hat{u}}/C$ fixed.

6.1. Bulk inspiration

The procedure is equivalent to a bulk analysis where the tree-level amplitude is upgraded to an eikonal S matrix. We will briefly review this analysis (see [27], using [28–30]), which is very simple in AdS_2 . The V and W operators are represented by bulk quanta with momenta p_- and q_+ , respectively. To capture all powers of $e^{\hat{u}}/C$, one can replace the bulk metric by two effective “shock wave” modes. These are parametrized by shifts X^+ and X^- on the future and past horizons of the black hole. The bulk metric perturbations and stress tensor are

$$h_{++} = 4\delta(x^+)X^-, \quad h_{--} = 4\delta(x^-)X^+, \quad T_{--} = -p_- \delta(x^-), \quad T_{++} = -q_+ \delta(x^+). \quad (6.1)$$

Here we are using Kruskal coordinates x^+, x^- to describe the region near the horizon; see (B.1). The bulk action $I_{JT} + S_M$ (3.1) for these quantities to quadratic order is

$$I_L = -2CX^+X^- - X^+q_+ - X^-p_-. \quad (6.2)$$

The integral over X^+, X^- of e^{iI_L} gives the scattering matrix $S = e^{ip_-q_+/2C}$. The four-point function is an in–out overlap with this S matrix. The essential feature is that at late time the product p_-q_+ is large, and the S matrix implements a large translation of the wave packets, making the overlap small.

6.2. Full resummation from the Schwarzian action

We will now do the calculation for real, using the boundary formulation of the theory. The metric is always exactly AdS_2 ; the only variable is the reparametrization $t(u)$. We have to identify X^+, X^- with certain modes of $t(u)$ and then evaluate the Schwarzian action and the coupling to V, W . The main subtlety is that for the out-of-time-order correlator, we have to think about the function $t(u)$ on a folded time contour.

We start with a single shock, so just X^+ is nonzero. The bulk solution consists of two black holes glued together with a shift along the horizon. In terms of $t(u)$, we glue two solutions together at $\hat{u} = \infty$ with an $SL(2)$ transformation. In an $SL(2)$ frame where the original black hole solution is $\hat{t} = e^{\hat{u}}$, the transformation is a simple translation, and the shock wave solution is

$$\hat{t} = e^{\hat{u}} \quad (\text{sheet 1}), \quad \hat{t} = e^{\hat{u}} + X^+ \quad (\text{sheet 2}). \quad (6.3)$$

Notice that these can be glued at $\hat{u} = \infty$. We will find it more convenient to work in a different $SL(2)$ frame, where the original solution is $\hat{t} = \tanh(\frac{\hat{u}}{2})$; see Fig. 1. Then the one-shock solution is

$$\hat{t} = x \quad (\text{sheet 1}), \quad \hat{t} = x + \frac{(1-x)^2 X^+}{2 + (1-x)X^+} \quad (\text{sheet 2}), \quad x \equiv \tanh\left(\frac{\hat{u}}{2}\right). \quad (6.4)$$

This new $SL(2)$ frame makes it clear that if X^+ is small then we have a small perturbation to $\hat{t}$ for all values of $\hat{u}$.

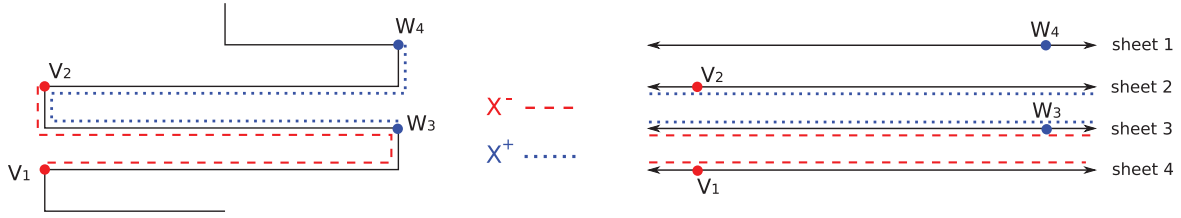


Fig. 3. The black contour at left is the minimal time contour needed to compute $\langle V_1 W_3 V_2 W_4 \rangle$. Horizontal is real time, vertical is imaginary time; the ends are identified. It is convenient to extend the folds to infinity, as shown at right. The blue and red indicate which sheets have the X^- and X^+ perturbations turned on.

To compute the $\langle V_1 W_3 V_2 W_4 \rangle$ correlator, we need a contour with four folds, as in Fig. 3. We would like to superpose the (6.4) solution and a similar expression for X^- in the right way to capture the important part of the functional integral over $t(u)$. We can guess the answer based on the bulk picture, or from the discussion of $SL(2)$ charges in the previous section, which suggests that each pair of operators is associated to a relative $SL(2)$ transformation between the portion of the contour inside and outside the pair. So we consider the configuration

$$\hat{t} = x + \frac{(1-x)^2 X^+}{2 + (1-x)X^+} \theta(2, 3) - \frac{(1+x)^2 X^-}{2 + (1+x)X^-} \theta(3, 4), \quad x \equiv \tanh\left(\frac{\hat{u}}{2}\right). \quad (6.5)$$

The θ symbols are defined to be equal to one on the sheets indicated and zero elsewhere; see Fig. 3. This is not a solution to the equations of motion, it is an off-shell configuration of $t(u)$. The idea is that by integrating over X^+, X^- , we are capturing the part of the integral over $t(u)$ that gives powers of $e^{\hat{u}}/C$. The entire dependence of the action (3.9) on X^+, X^- comes from the fold where both terms are nonzero. The product $X^+ X^-$ is small, of order $1/C$ in the limit we are taking, so it is enough to compute the action to quadratic order:

$$iI_L \supset -iC \int_{\text{sheet 3}} d\hat{u} \text{Sch}(\hat{t}, \hat{u}) = -2iC X^- X^+ + (X\text{-independent}). \quad (6.6)$$

To compute the four-point function, we also have to consider the reparametrized two-point functions. If the V operators are at early time, then only the X^+ part of the reparametrization is important. It acts on sheets two and three, and therefore affects only V_2 (see Fig. 3). Similarly, for the W operators we only have to consider the X^- part acting on W_3 :

$$G_V(\hat{u}_1, \hat{u}_2) = \left[\frac{-\hat{t}'(\hat{u}_1)\hat{t}'(\hat{u}_2)}{(\hat{t}(\hat{u}_1) - \hat{t}(\hat{u}_2))^2} \right]^\Delta \approx \left[\frac{-i}{2 \sinh \frac{\hat{u}_{12}}{2} - X^+ e^{-(\hat{u}_1 + \hat{u}_2)/2}} \right]^{2\Delta}, \quad (6.7)$$

$$G_W(\hat{u}_3, \hat{u}_4) = \left[\frac{-\hat{t}'(\hat{u}_3)\hat{t}'(\hat{u}_4)}{(\hat{t}(\hat{u}_3) - \hat{t}(\hat{u}_4))^2} \right]^\Delta \approx \left[\frac{-i}{2 \sinh \frac{\hat{u}_{34}}{2} - X^- e^{(\hat{u}_3 + \hat{u}_4)/2}} \right]^{2\Delta}. \quad (6.8)$$

Now, to compute the four-point function, we simply integrate these expressions over $X^\pm$ with the weighting given by (6.6). Up to measure factors, we have

$$\langle V_1 W_3 V_2 W_4 \rangle \propto \int dX^+ dX^- e^{-2iC X^+ X^-} G_V(\hat{u}_1, \hat{u}_2) G_W(\hat{u}_3, \hat{u}_4). \quad (6.9)$$

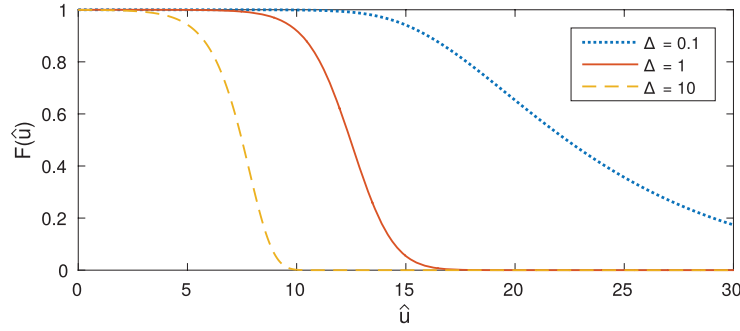


Fig. 4. The correlation function (6.10) in the configuration (6.11), with $8C = 10^6$. Scaling up C simply translates all of the curves to the right without changing their shape. The initial descent from the plateau is characterized by the $e^{\frac{2\pi}{\beta}\hat{u}}$ behavior. The final approach to zero is determined by quasinormal decay.

One of the integrals can be done simply, and the second can be expressed using the confluent hypergeometric function $U(a, 1, x) = \Gamma(a)^{-1} \int_0^\infty ds e^{-sx} \frac{s^{a-1}}{(1+s)^a}$. The answer is

$$\frac{\langle V_1 W_3 V_2 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \frac{U(2\Delta, 1, \frac{1}{z})}{z^{2\Delta}}, \quad z = \frac{i}{8C} \frac{e^{(\hat{u}_3 + \hat{u}_4 - \hat{u}_1 - \hat{u}_2)/2}}{\sinh \frac{\hat{u}_{12}}{2} \sinh \frac{\hat{u}_{34}}{2}}. \quad (6.10)$$

The real part of z is positive for the ordering of operators we have assumed. A simple configuration to keep in mind is the one where the V, W operators are equally spaced around the Euclidean circle, for example

$$\hat{u}_1 = -\frac{\hat{u}}{2} - i\pi, \quad \hat{u}_2 = -\frac{\hat{u}}{2}, \quad \hat{u}_3 = \frac{\hat{u}}{2} - i\frac{\pi}{2}, \quad \hat{u}_4 = \frac{\hat{u}}{2} + i\frac{\pi}{2} \quad \Rightarrow \quad z = \frac{e^{\hat{u}}}{8C}. \quad (6.11)$$

Here, $\hat{u}$ is the separation of the early V operators and the late W operators. Notice that the z variable is real and positive in this type of configuration. We give a plot of (6.10) in Fig. 4. For early and late times, we have the limiting behaviors

$$\frac{U(2\Delta, 1, \frac{1}{z})}{z^{2\Delta}} \approx 1 - 4\Delta^2 z \quad (z \ll 1), \quad \frac{U(2\Delta, 1, \frac{1}{z})}{z^{2\Delta}} \approx \frac{\log z}{\Gamma(2\Delta) z^{2\Delta}} \quad (z \gg 1). \quad (6.12)$$

The small- z expression reproduces the initial exponential growth of the connected correlator (4.12). The large- z behavior gives exponential decay of the full correlator at late time, where it is dominated by the decay of the quasinormal modes.

6.3. Role of the $SL(2)$ charges

To understand the charges of the matter, it is helpful represent (6.7) and (6.8) in a basis that diagonalizes certain $SL(2)$ generators. It is convenient to return to the $SL(2)$ frame where the background solution is $\hat{t} = e^{\hat{u}}$. One can write

$$G_V(\hat{u}_1, \hat{u}_2) = \frac{1}{\Gamma(2\Delta)} \int_{-\infty}^0 \frac{dq_+}{-q_+} \Psi_1(q_+) \Psi_2(q_+) e^{-iX^+ q_+}, \quad (6.13)$$

$$G_W(\hat{u}_3, \hat{u}_4) = \frac{1}{\Gamma(2\Delta)} \int_{-\infty}^0 \frac{dp_-}{-p_-} \Psi_3(p_-) \Psi_4(p_-) e^{-iX^- p_-}, \quad (6.14)$$

where the wave functions are given by

$$\Psi_j = x_j^\Delta e^{-x_j}, \quad x_1 = -iq_+ e^{\hat{u}_1}, \quad x_2 = iq_+ e^{\hat{u}_2}, \quad x_3 = ip_- e^{-\hat{u}_3}, \quad x_4 = -ip_- e^{-\hat{u}_4}. \quad (6.15)$$

Equations (6.13) and (6.14) decompose the bilocals into pieces with definite charge, e.g. $q_M^- = -q_+$. Based on the discussion in Sect. 5.2, we expect these charges to be related to the charges of the shocks in $t(u)$. For example, when we pass the V operator, the X^- shock turns on, and one can check from (5.5) that the charge changes by $Q_{after}^- - Q_{before}^- = -2CX^-$. This gets related to q_+ as follows: when we insert (6.13) and (6.14) into (6.9) and integrate over X^+ , we will get a delta function setting $-2CX^- - q_+ = 0$, so that

$$Q_{after}^- - Q_{before}^- = -q_M^-. \quad (6.16)$$

This means that the total charge Q_T indeed remains constant.

The charges give a new perspective on the exponential growth of the four-point function. Acting with V at an early time changes Q^- of the $\hat{\tau}$ sector. Because the charge is conserved, and because of the explicit factor of $e^{-\hat{\tau}}$ in (5.5), this has an exponentially growing effect on $\hat{\tau}(u)$ as we move toward the future. We can always make an $SL(2)$ gauge transformation to remove this effect on either the portion of the contour before V or the portion after, but not both. In the out-of-time-order four-point function we have W operators probing both sides, so the exponential effect is physical.

7. Discussion

We have pointed out how the asymptotic symmetries of AdS_2 can be used to determine many aspects of the gravitational dynamics of nearly AdS_2 spacetimes, or $NAdS_2$. The essential feature is the emergence of a reparametrization symmetry which is both spontaneously and explicitly broken. The corresponding pseudo-Goldstone bosons are described by a reparametrization $t(u)$ that expresses AdS time t in terms of the physical boundary time u . The explicit breaking leads to a Schwarzian action for $t(u)$ (3.9). In addition, we also have a simple coupling to bulk fields (4.4). These together give rise to several features of $NAdS_2$ or near-extremal black holes. These include the computation of the near-extremal free energy as well as several gravitational effects involving correlation functions. These include gravitational corrections to the four-point function (4.9) and (4.12) as well as corrections to the two-point function (4.15).

For all these features it was important to assume that the Schwarzian action was the leading effect that breaks the reparametrization symmetry. This is the case in many interesting physical situations. However, one can imagine cases involving AdS_2 spaces with particularly light fields, dual to operators with dimensions $1 < \Delta < 3/2$. In these cases, if these fields are excited, then we have larger irrelevant perturbations and the infrared dynamics is different. See Appendix D for a detailed discussion.

The Schwarzian action involves higher-derivative terms, which raise ghost fears. The ghosts are made invisible by treating the $SL(2)$ symmetry of the Schwarzian action as a gauge symmetry. This reflects the fact that the whole configuration, including the boundary, can be shifted around in AdS_2 space with no physical consequence. This is distinct from the physical $SL(2)$ symmetry acting on u , which is broken by the Schwarzian action. The ghost-like degrees of freedom lead to exponentially growing corrections in the out-of-time-order configuration.

Note that the Schwarzian action has the flavor of a hydrodynamical theory. Namely, it reproduces the thermodynamics of the system. The fact that the entropy is the conserved charge associated to the τ circle translations also resonates with recent discussions of a $U(1)_T$ symmetry in [31]; see also [14]. It is also important to include both sides of the thermofield double to make sense of the $SL(2)$ constraints. Now, this Schwarzian action goes beyond ordinary long-distance hydrodynamics,

because it is including modes whose time variation rate is comparable to the temperature. Such modes are crucial for reproducing the out-of-time-order correlator in the chaos regime.

Two-dimensional black holes are a very useful testing ground for ideas for solving the information paradox. Any general idea should work in this simplest context. An important element seems to be a better understanding of the emergence of the charges $Q^\pm$. These are the symmetries that allow us to move into the interior! These charges are analogous to the edge modes of the electromagnetic field discussed in [21,22], or the “center” in [23], or horizon symmetries in [24] (see also [32]).¹⁹

There are several other questions remaining to be answered. In two dimensions similar holomorphic reparametrizations give rise to a Virasoro algebra with a central charge. Here we have mentioned neither the algebra nor the central charge. It would be nice to see whether and how it can be defined. Several papers have discussed a central charge for AdS_2 , including [7,8,33,34], but we have not understood how they are connected to the present discussion.

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Appendix A. Massive fields in AdS_2 and their coupling to gravity

In this appendix we study in some detail the effect that sources for massive fields have on the time dependence of the Hamiltonian and the $SL(2)$ charges. A subtlety is that the Schwarzian is not equal to the ADM Hamiltonian while such sources are turned on; it differs by a term involving T_{zz} . For the $SL(2)$ charges, one has to add a similar term to the naive matter charges to get exact conservation.

We will start by considering free fields. We imagine we add classical sources at the boundary by specifying the boundary conditions $\chi_r(u)$; see (4.3). As we explained in Sect. (4), we can go from the effective action (4.2) to (4.4). We can add this to the Schwarzian action (3.9) and then vary the resulting effective action for $t(u)$ to obtain a new classical equation,

$$C \frac{[\text{Sch}(t, u)]'}{t'} = -\frac{1}{t'} \left\{ \chi_r'(u) \mathcal{O}(u) + \partial_u [(\Delta - 1) \chi_r(u) \mathcal{O}(u)] \right\} \quad (\text{A.1})$$

$$\text{with } \mathcal{O}(u) \equiv 2D t'(u)^\Delta \int du' \frac{t'(u')^\Delta \chi_r(u')}{(t(u) - t(u'))^{2\Delta}}, \quad (\text{A.2})$$

where $\mathcal{O}$ can also be interpreted as the classical expectation value of the operator dual to the source $\chi_r(u)$. $\mathcal{O}$ and χ_r are related to the small- z behavior of the field by

$$\chi(z, t) = \left[\chi_r(u) - \frac{\epsilon^{2\Delta-1} \mathcal{O}(u)}{2\Delta-1} \right] \left(\frac{z}{t'(u)} \right)^{1-\Delta} + \cdots + \frac{\mathcal{O}(u)}{2\Delta-1} \left(\frac{z}{t'(u)} \right)^\Delta + \cdots \quad (\text{A.3})$$

The explicit ϵ term is to ensure $\chi(\epsilon t'(u), t(u)) = \epsilon^{1-\Delta} \chi_r(u)$ so that (4.3) is satisfied.

¹⁹ It was emphasized in [24] that the number of charges is infinite in more than two dimensions. In two dimensions we simply have a finite number of charges, the $SL(2)$ charges discussed here.

We now want to relate (A.1) to the energy conservation condition. In the bulk, given any vector ζ^μ , we can construct a current $(*j_\zeta)_\mu = \epsilon_\mu^\nu T_{\nu\delta} \zeta^\delta$, which is conserved when ζ is a Killing vector. In general, the ADM mass M is given by the first equality in (3.15) as a function of the dilaton. We expect that its first derivative should give us the flux of energy into the system,

$$\partial_u M = \frac{1}{8\pi G} \partial_u (t' \partial_z \phi - z' \partial_t \phi) = (*j)_\mu (t', z')^\mu = \sqrt{h} T_{um}, \quad (\text{A.4})$$

where u is the coordinate along the boundary and n is the normal direction. Here we have equated the flux with the energy corresponding locally to a direction tangent to the boundary curve. We also assumed $\partial_u \phi_b = 0$. For a scalar field we see that

$$\partial_u M = \sqrt{h} T_{um} = \sqrt{h} \partial_u \chi \partial_n \chi = \partial_u \chi_r [\epsilon^{1-2\Delta} (\Delta - 1) \chi_r - \mathcal{O}], \quad (\text{A.5})$$

where we used (A.3). Note that, as expected, energy is conserved as long as the sources are time independent. The first term diverges as $\epsilon \rightarrow 0$ and can be cancelled by a counter term. Here we assumed that $1 < \Delta < 3/2$ in order to avoid further divergent terms.

Comparing this with (A.1) we conclude that

$$M = C \text{Sch}(t, u) + \epsilon^{1-2\Delta} (\Delta - 1) \frac{\chi_r(u)^2}{2} + (\Delta - 1) \chi_r(u) \mathcal{O}(u). \quad (\text{A.6})$$

In fact, we can compute the relation between the mass M and the Schwarzian directly by using the definition of the ADM mass

$$(8\pi G)M \equiv -\partial_n \phi + \phi \sqrt{h} = t' \partial_z \phi - z' \partial_t \phi + \frac{\bar{\phi}_r}{\epsilon^2} = \quad (\text{A.7})$$

$$= t' \left(\frac{\phi}{z} + \partial_z \phi \right) - \frac{z'}{z} \partial_t (z\phi) - \left[\frac{t'}{z} - \frac{1}{\epsilon} \right] \frac{\bar{\phi}_r}{\epsilon}, \quad (\text{A.8})$$

where $\sqrt{h} = 1/\epsilon$ is the boundary metric and $\phi = \bar{\phi}_r/\epsilon$ at the boundary (3.5). We have added and subtracted various terms. In the first term we use the T_{zz} equation

$$\partial_t^2 \phi - \frac{1}{z^2} \phi - \frac{1}{z} \partial_z \phi = 8\pi G T_{zz}. \quad (\text{A.9})$$

In the third term in (A.8) we expand the constant proper length condition as $t'/z = \frac{1}{\epsilon} - \frac{1}{2} \epsilon \frac{t''^2}{t'^2}$. We also assume that

$$\phi = \frac{\phi_-}{z} + \phi_{sl}, \quad (\text{A.10})$$

where ϕ_{sl} is less singular than $1/z$, so that $z\phi$ has a finite limit. Then we see that $\phi_- = t' \bar{\phi}_r$ to leading order. We can also convert ∂_t into $\frac{1}{t'} \partial_u$ for the terms that are finite. All these terms together then give

$$M = \frac{1}{8\pi G} \left[t' (\partial_t^2 (z\phi) - 8\pi G z T_{zz}) - \frac{t''^2}{t'^2} \bar{\phi}_r + \frac{1}{2} \frac{t''^2}{t'^2} \bar{\phi}_r \right] = C \text{Sch}(t, u) - t' z T_{zz}. \quad (\text{A.11})$$

This derivation of the relation between the Schwarzian and the mass is valid also for a self-interacting matter theory (that is not directly coupled to the dilaton in the Lagrangian). Evaluating $z T_{zz}$ for a free field, we obtain the extra terms in (A.6).

We can similarly consider the expressions for the SL(2) charges. We expect that the total SL(2) charges should be preserved *even with time-dependent boundary conditions*. We first define a naive matter SL(2) charge, $q_M^{(k)}$, as the integral of $*j_\zeta$ over a spatial slice with $\zeta^\mu = (k(t), \epsilon k'(t))$ near the boundary, with $k(t) = 1, t, t^2$, for each of the SL(2) generators. For a free field, the fluxes are then given as

$$\partial_u q_M^{(k)} = -\sqrt{h} T_{n\mu} \zeta^\mu = \frac{k}{t'} \chi_r'(u) \mathcal{O}(u) - (\Delta - 1) \left(\frac{k}{t'} \right)' \chi_r \mathcal{O} + \partial_u \left((1 - \Delta) \epsilon^{1-2\Delta} \frac{\chi_r^2 k}{2t'} \right). \quad (\text{A.12})$$

We now define a new matter charge that includes some extra terms of the form

$$\begin{aligned} Q_M^{(k)} &= q_M^{(k)} - k z T_{zz}, \\ Q_M^{(k)} &= \int *j_\zeta + (\Delta - 1) \epsilon^{1-2\Delta} \frac{\chi_r^2 k}{2t'} + (\Delta - 1) \frac{k}{t'} \chi_r \mathcal{O}. \end{aligned} \quad (\text{A.13})$$

The extra terms are boundary terms that we can add in the definition of the charge. Then we see that the total SL(2) charges defined as

$$Q_T^{(k)} = Q^{(k)} + Q_M^{(k)}, \quad \partial_u Q_T^{(k)} = 0 \quad (\text{A.14})$$

are conserved, once we use the equations of motion (A.1). Here, $Q^{(k)}$ are the charges constructed purely out of the $t(u)$ variable as in (5.5), $(Q^-, Q^0, Q^+) = (Q^{(1)}, Q^{(t)}, Q^{(t^2)})$. They obey

$$\partial_u Q^{(k)} = C \frac{k [\text{Sch}(t, u)]'}{t'}, \quad k = 1, t, t^2. \quad (\text{A.15})$$

Appendix B. Gravitational shock wave scattering

Here we consider the scattering of two pulses in two-dimensional gravity. This is simplest to discuss in a frame where one pulse is highly boosted, created by $V(-\hat{u})$ where $\hat{u}$ large, and the other is unboosted, created by $V(0)$. As in higher dimensions, the scattering can then be described by studying the propagation of the probe V particle on the background created by W [28].

In the theory (3.1), the metric is always exactly AdS_2 , so there must be a set of coordinates in which this background is trivial, and the particles simply pass through each other, without detecting any local gravitational effect. If this is the case, how can there be any scattering at all? The answer is that these coordinates are related to the physical boundary coordinate $\hat{u}$ in a nontrivial way [3,29]. This is simplest to explain with a drawing, which we attempt in Fig. 5.

We can also relate this discussion to the standard shock wave picture. The backreaction of V can be described by a metric

$$ds^2 = -\frac{4dx^+ dx^-}{(1 + x^+ x^-)^2} + 4X^- \delta(x^+) (dx^+)^2, \quad (\text{B.1})$$

where X^- is proportional to the large p_+ momentum of V . In these coordinates, the dilaton is a function of $x^+ x^-$ only, and the physical time coordinate $\hat{u}$ at the right boundary ($x^+ x^- = -1, x^+ > 0$) is given by $e^{\hat{u}} = x^+$. However, it is not manifest that they describe a piece of AdS_2 . It is simple to check that we can rewrite (B.1) as

$$ds^2 = -\frac{4d\tilde{x}^+ d\tilde{x}^-}{(1 + \tilde{x}^+ \tilde{x}^-)^2}, \quad \tilde{x}^+ = \begin{cases} x^+ & x^+ < 0 \\ \frac{x^+}{1+X^-x^+} & x^+ > 0 \end{cases} \quad \tilde{x}^- = x^- - X^- \theta(x^+). \quad (\text{B.2})$$

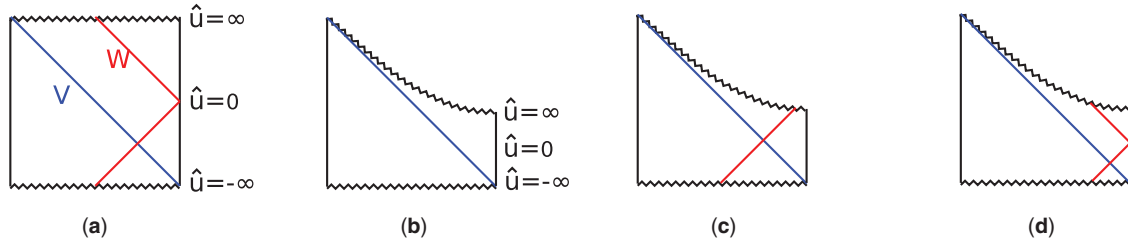


Fig. 5. In (a) we show the trajectories of the V , W quanta without backreaction. In (b) we show the backreaction of the V particle. This is still a piece of AdS_2 , but it is a smaller piece. In (c) we add back W in the arrangement appropriate for the operator ordering $V(-\hat{u})W(0)$. The trajectories are (almost) the same as (a) relative to the fixed AdS_2 coordinates of the diagram, but they change relative to the physical $\hat{u}$ coordinate. In (d) we show the other ordering $W(0)V(-\hat{u})$. Now the red line touches the boundary at time $\hat{u} = 0$. Although it is difficult to see in this frame, the V line has moved down slightly, so that it no longer reaches the boundary.

These coordinates make it clear that we have a piece of AdS_2 . In fact, these are the coordinates of the fixed AdS_2 space on which the drawings in figure 5 are represented. In these coordinates, the dilaton profile does not simply depend on $\tilde{x}^+\tilde{x}^-$. Their relationship to the time $\hat{u}$ at the right boundary depends on X^- . At the boundary we have

$$e^{\hat{u}} = x^+ = \frac{\tilde{x}^+}{1 - X^- \tilde{x}^+}. \quad (\text{B.3})$$

In summary, the particles simply pass through each other in the bulk, but nevertheless there is an effect on the boundary time. In this way, the gravity dual manages to encode chaos in non-interacting particles.

Appendix C. Corrections to the matter two-point functions

It is natural to ask about the form of the leading correction to the matter two-point functions due to further couplings to the dilaton field such as

$$\int d^2x \sqrt{g} [(\nabla \chi)^2 + m^2 \chi^2 + \alpha \phi \chi^2]. \quad (\text{C.1})$$

We can now consider the background value for $\phi = \phi_h \cosh \rho$ to find the leading correction to the thermal two-point function. This can be found by noticing that the integral we need to do for the dilaton field has the same form as the one expected for the insertion of a bulk-to-boundary propagator for a $\Delta = -1$ boundary operator. Thus the correction to the correlator has the form $\int du \langle O_\Delta(u_1) O_\Delta(u_2) V_{-1}(u) \rangle$. Since the three-point function is fixed by conformal symmetry, we find that

$$\langle O(1)O(2) \rangle = \left(\frac{\pi}{\beta \sin \frac{\pi \tau_{12}}{\beta}} \right)^{2\Delta} \left[1 + c_0 \alpha \frac{\phi_r}{\beta} \left(2 + \pi \frac{(1 - 2\tau_{12}/\beta)}{\tan \frac{\pi \tau_{12}}{\beta}} \right) \right], \quad (\text{C.2})$$

where c_0 is a numerical constant. We see that this correction depends on a new parameter α that depends on the details of the theory. This has the same form as the corrections found in [16] for the Sachdev–Ye–Kitaev model.

Appendix D. A case where the Schwarzian is not dominating

Throughout this paper we have considered nearly AdS_2 situations where the Schwarzian is the leading irrelevant deformation. We now ask the question of whether this always happens or whether there are also situations where other corrections dominate.

For simplicity we will focus only on systems that have a large- N expansion, or a weakly coupled gravity description. By assumption we have an IR fixed point, and therefore we assume that there are no relevant operators turned on. We can consider the effects of turning on irrelevant single trace operators which correspond to changing the boundary values of massive bosonic fields in AdS_2 .

We will see that if we turn on an operator with

$$1 < \Delta < 3/2 \quad (D.1)$$

then its effects dominate over the ones due to the Schwarzian action and the IR dynamics is different from that described in this article.

As before, we still have the zero modes in the IR parametrized by the field $t(u)$. But due to the presence of the irrelevant operator with dimension Δ we get an effective action given by (4.4)

$$-I_{eff} = \lambda^2 \int du du' \left[\frac{t'(u)t'(u')}{(t(u) - t(u'))^2} \right]^\Delta, \quad (D.2)$$

where λ is the coefficient of the operator in the action $\int du \lambda O(u)$. As a simple check that we obtain some effect that dominates over the Schwarzian, we consider the finite-temperature configuration with $t = \tan \frac{\pi u}{\beta}$, where we obtain a free energy of the form

$$\begin{aligned} \log Z &= \lambda^2 \beta \int_\epsilon^{\beta-\epsilon} du \left[\frac{\pi}{\beta \sin \frac{\pi u}{\beta}} \right]^{2\Delta} \\ &= \beta \lambda^2 \frac{1}{\Delta - \frac{1}{2}} \frac{1}{\epsilon^{2\Delta-1}} + \lambda^2 \beta^{2-2\Delta} \frac{\pi^{2\Delta-\frac{1}{2}} \Gamma(\frac{1}{2} - \Delta)}{\Gamma(1 - \Delta)}. \end{aligned} \quad (D.3)$$

The first term is a UV divergence, but is proportional to β so that it is a correction to the ground state energy. The second term is finite. We see that if $\Delta < 3/2$, this second term dominates, for large β , over the Schwarzian answer (3.13), which goes as β^{-1} .

Thus in the range (D.1) this operator gives the leading IR correction. The effective action for the reparametrizations (D.2) is non-local. This can be checked more explicitly by setting $t = u + \varepsilon(u)$ and expanding in Fourier space. We end up with an action of the form $I_{eff} \propto \int dp |p|^{1+2\Delta} \varepsilon(p) \varepsilon(-p)$, which indeed has a non-local form for the range (D.1).

The Sachdev–Ye–Kitaev model [11,35] has no operators in the range (D.1) [11], and therefore the Schwarzian dominates in the IR.

Appendix E. Lack of reparametrization symmetry in “conformal” quantum mechanics

There are simple quantum mechanical theories that display an $SL(2)$ conformal symmetry. An example is a Lagrangian of the form

$$S = \int dt \left[\left(\frac{dX}{dt} \right)^2 - \frac{\ell^2}{X^2} \right]. \quad (E.1)$$

Under a transformation of the form

$$t \rightarrow t(\tilde{t}), \quad X(t(\tilde{t})) = (t')^{\frac{1}{2}} \tilde{X}(\tilde{t}), \quad (\text{E.2})$$

(E.1) changes to

$$S \rightarrow \int d\tilde{t} \left[\left(\frac{d\tilde{X}}{d\tilde{t}} \right)^2 - \frac{\ell^2}{\tilde{X}^2} - \frac{1}{2} \text{Sch}(t, \tilde{t}) \tilde{X}^2 \right]. \quad (\text{E.3})$$

We see that if $t(\tilde{t})$ is an $\text{SL}(2)$ transformation, then the action is invariant. However, if it is a more general reparametrization then the action is not invariant.

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