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Approximating Two-Stage Chance-Constrained Programs with Classical Probability Bounds

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Chance Constraint Setting

Consider a linear Joint Chance Constraint:

$$P(x_t \leq y_t^\omega + w_t^\omega, \forall t \in T) \geq 1 - \varepsilon$$

Background:

- Two-stage stochastic program with recourse
- Possibly integer restrictions
- i.i.d. samples of uncertainty w_t^ω
- First stage decision, x_t, second-stage decision, y_t^ω

Challenges

- CC models are computationally intractable
- A known NP-hard problem
- Existing algorithms not scalable to practical sized problems
- Feasible region is non-convex

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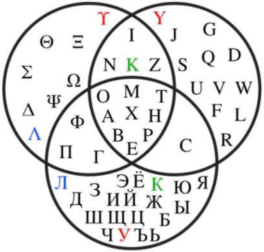
Approximations with classical probability bounds

Satisfying a JCC is an intersection of events. Failing a JCC is a union of events.



We can rewrite the JCC as follows:

$$\mathbb{P}\left(\bigcup_{t \in T} F_t\right) \leq \varepsilon.$$

where F_t denotes the set of scenarios that we "fail"; i.e., $F_t = \{\omega : x_t > y_t^\omega + w_t^\omega\}$..

Approximations with Classical Probability Bounds

$$\mathbb{P}(\bigcup_{t \in T} F_t) \leq \varepsilon.$$

Consider an optimization model with a JCC with a maximization objective.

- Lower Bound (LB): Approximate the LHS using a quantity **larger** than $\mathbb{P}(\bigcup_{t \in T} F_t)$. Feasible region is **restricted**.
- Upper Bound (UB): Approximate the LHS using a quantity **smaller** than $\mathbb{P}(\bigcup_{t \in T} F_t)$. Feasible region is **enlarged**.

Approximations with Classical Probability Bounds

$\mathbb{P}(\bigcup_{t\in T} F_t) = S_1 - S_2 + \cdots (-1)^{|T|-1} S_T$
where, $S_k = \mathbb{P}(\sum_{1\leq i_1<\cdots<i_k\leq |T|} F_{i_1} \cap \cdots \cap F_{i_k})$.

Bonferroni bounds:

$$\mathbb{P}(\bigcup_{t\in T} F_t) \leq \quad S_1 \leftarrow {\color{red}LB} \tag{1a}$$

$$\mathbb{P}(\bigcup_{t\in T} F_t) \geq S_1 - S_2 \leftarrow {\color{red}UB}. \tag{1b}$$

Approximations with Classical Probability Bounds

$\mathbb{P}(\bigcup_{t\in T} F_t) = S_1 - S_2 + \cdots (-1)^{|T|-1} S_T$
where, $S_k = \mathbb{P}(\sum_{1\leq i_1<\cdots<i_k\leq |T|} F_{i_1} \cap \cdots \cap F_{i_k})$.

Tighter bounds from Sathe et al. [1980]:

$$\mathbb{P}(\bigcup_{t\in T} F_t) \leq \quad S_1 - \frac{2}{T} S_2 \leftarrow LB \tag{2a}$$

$$\mathbb{P}(\bigcup_{t\in T} F_t) \geq \quad \frac{S_1 + 2S_2}{T^2} \leftarrow UB. \tag{2b}$$

Approximations with Classical Probability Bounds

$\mathbb{P}(\bigcup_{t\in T} F_t) = S_1 - S_2 + \cdots (-1)^{|T|-1} S_T$
where, $S_k = \mathbb{P}(\sum_{1\leq i_1 < \cdots < i_k \leq |T|} F_{i_1} \cap \cdots \cap F_{i_k})$.

And more from Dawson and Sankoff [1967]:

$$\mathbb{P}(\bigcup_{t\in T} F_t) \geq \frac{{S_1}^2}{S_1 + 2S_2}. \leftarrow UB \tag{3a}$$

can be linearized for JCC $\leq \varepsilon$:

$$2\varepsilon S_2 \geq \alpha_n S_1 + \beta_n, n = 0, 1, \ldots |N| - 1, \leftarrow UB \tag{3b}$$

Optimizing over JCCs

$u_t^\omega = 1$: failure at t in scenario ω
$v_{tt'}^\omega = 1$: failure at t and t' in scenario ω

$$x_t - y_t^\omega - w_t^\omega \leq M_t^\omega u_t^\omega, \forall t \in T, \omega \in \Omega$$

$$\text{McCormick envelope} \begin{cases} v_{t,t'}^\omega \leq u_t^\omega, (t,t') \in T, t < t', \omega \in \Omega \\ v_{t,t'}^\omega \leq u_{t'}^\omega, \forall (t,t') \in T, t < t', \omega \in \Omega \\ v_{t,t'}^\omega \geq u_t^\omega + u_{t'}^\omega - 1, \forall (t,t') \in T, t < t', \omega \in \Omega \end{cases}$$

$$u_t^\omega = \{0,1\}, \forall t \in T, v_{t,t'}^\omega = \{0,1\}, \forall (t,t') \in, \omega \in \Omega$$

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A case study

$$
\begin{aligned}
\max_{x,y} \quad & \sum_{t \in T} (R_t x_t - \mathbb{E}[B_t y_t^\omega]) && \text{(4a)}\\
\text{s.t.} \quad & \mathbb{P}(y_t^\omega + w_t^\omega \geq x_t, \forall t \in T) \geq 1 - \varepsilon && \text{(4b)}\\
& 0 \leq y_t^\omega \leq \Delta, \forall t \in T, \omega \in \Omega && \text{(4c)}\\
& x_t \geq 0, \forall t \in T. && \text{(4d)}
\end{aligned}
$$

Computational results

We compare two sampling procedures: (a) ARMA(2,2) process, and (b) normal random variables. Both samples have the same hourly means and variances.

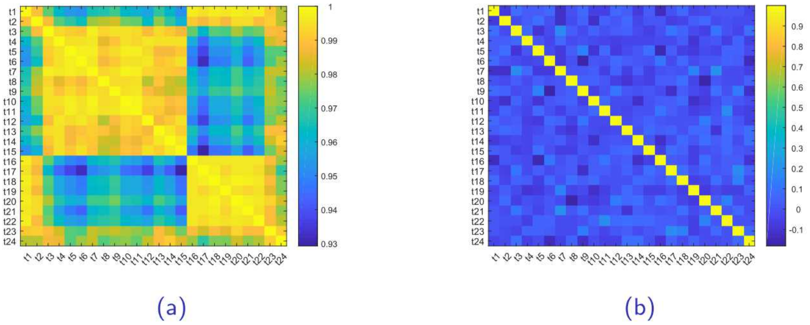

Figure: Correlation structure of w_t

Computational results: ARMA (large correlation)

ε	Bounding constraint	Optimal objective value Lower bound	Upper bound	MIP gap	Time (seconds)	Gap from optimal
0.01	(1a)	**8,351.3**	8,351.3	0%	2	3.3%
	(1b)	21,282.8	**21,282.8**	0%	12	59.4%
	(2a)	**8,351.3**	8,365 .8	0.1%	2100	3.3%
	(2b)	8,339.6	**10,682.1**	21.9%	2100	19.2%
	(3a)	8,339.7	**8,726.7**	4.5%	2100	1.1%
	(3b)	8,688.9	**8,702.1**	0.2%	2100	0.8%
0.03	(1a)	**8,374.6**	8,374.6	0%	2	8.5%
	(1b)	22,353.2	**22,353.2**	0%	14	59.0%
	(2a)	**8,339.6**	8,755.4	4.7%	2100	8.9%
	(2b)	8,339.6	**13,321.2**	37.4%	2100	31.3%
	(3a)	9,137.3	**9,311.4**	1.9%	2100	1.7 %
	(3b)	9,074.4	**9,252.2**	1.9%	2100	1.1%

Table: Tightest lower and upper bounds for $\varepsilon = 0.01$ are 8,351.3 and 8,702.1; true optimal value is 8,634.1
Tightest lower and upper bounds for $\varepsilon = 0.03$ are 8,374.6 and 9,252.2; true optimal value is 9,154.9

Computational results: Gaussian (weak correlation)

ε	Bounding constraint	Optimal objective value Lower bound	Upper bound	MIP gap	Time (seconds)	Gap from optimal
0.01	(1a)	**9,100.8**	9,100.8	0%	1	2.7%
	(1b)	21,606.6	**21,606.6**	0%	18	56.7%
	(2a)	**9,102.0**8	9,113.3	0.1%	2100	2.7%
	(2b)	9092.3	**11,365.5**	20%	2100	17.7%
	(3a)	9,434.3	**9,486.3**	0.5%	2100	1.4%
	(3b)	9,421.5	**9,452.3**	0.3%	2100	1.1%
0.03	(1a)	**9,124.3**	9,124.3	0%	2	7.7%
	(1b)	22,762.1	**22,762.1**	0%	21	56.6%
	(2a)	**9,124.8**	9,198.4	0.8%	2100	7.7%
	(2b)	9,092.3	**13,907.6**	34.9%	2100	28.9%
	(3a)	9,092.3	**10,062.6**	9.6%	2100	1.8%
	(3b)	9,092.3	**10,004.8**	9.1%	2100	1.2%

Table: Tightest lower and upper bounds for $\varepsilon = 0.01$ are 9,100.8 and 9,449.9; true optimal value is 9,353.2
Tightest lower and upper bounds for $\varepsilon = 0.03$ are 9,124.3 and 10,004.8; true optimal value is 9,884.0

Computational results

- Bonferroni lower bound and Dawson & Sankoff upper bound consistently perform better than others
- Weaker correlation in uncertainty leads to easier-to-solve models
- MIQCP formulation of Dawson & Sankoff bound is challenging

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Possible reasons for long computation time of naive solve

- No extended variable formulation above
- Big M
- Less reliable regime, more combinations to choose from

Computational results: ARMA (large correlation) with 500 scenarios

ε	Bounding constraint	Optimal objective value Lower bound	Upper bound	MIP gap	Time (seconds)	Gap from optimal
0.01	(1a)	**8,453.4**	8,453.4	0%	1	2.9%
	(1b)	21,582.9	**21,582.9**	0%	129	59.7%
	(2a)	**8,701.0**	8,701.0	0%	1717	0%
	(2b)	10,462.7	**11,318.4**	7.5%	2100	23.1%
	(3a)	8,348.9	**40,116.9**	79.2%	2100	78.3%
	(3b)	8,348.9	**8,772.9**	4.8%	2100	0.8%
0.03	(1a)	**8,542.5**	8,542.5	0%	3	7.3%
	(1b)	22,570.6	**22,570.6**	0%	175	59.2%
	(2a)	**8,348.9**	9,396.1	11.1%	2100	9.4%
	(2b)	8,348.9	**15,127.8**	44.8%	2100	39.1%
	(3a)	8,348.9	**41,151.4**	79.8%	2100	77.6 %
	(3b)	8,348.9	**9,352.9**	10.7%	2100	1.5%

Table: Tightest lower and upper bounds for $\varepsilon = 0.01$ are 8,701.0 and 8,772.9; true optimal value is 8,701.0
Tightest lower and upper bounds for $\varepsilon = 0.03$ are 8,542.5 and 9,352.9; true optimal value is 9,211.3

Computational results: Gaussian (weak correlation) with 500 scenarios

ε	Bounding constraint	Optimal objective value Lower bound	Upper bound	MIP gap	Time (seconds)	Gap from optimal
0.01	(1a)	**9,005.1**	9,005.1	0%	1	3.7%
	(1b)	21,503.7	**21,503.7**	0%	75	56.5%
	(2a)	**8866.9**	8,889.3	1.3%	2100	5.1%
	(2b)	8,866.9	**11,071.9**	19.9%	2100	15.6%
	(3a)	8,866.9	**40,126.1**	77.9%	2100	76.7%
	(3b)	9,343.6	**9,390.3**	0.5%	2100	0.5%
0.03	(1a)	**9,148.2**	9,148.2	0%	3	7.4%
	(1b)	22,565.4	**22,565.4**	0%	46	56.2%
	(2a)	**8,866.9**	9,315.3	4.8%	2100	10.2%
	(2b)	8,866.9	**13,711.9**	35.3%	2100	27.9%
	(3a)	8,866.9	**41,187.8**	78.5%	2100	76.0 %
	(3b)	8,866.9	**9,990.9**	11.2%	2100	1.2%

Table: Tightest lower and upper bounds for $\varepsilon = 0.01$ are 9,005.1 and 9,390.3; true optimal value is 9,346.4
Tightest lower and upper bounds for $\varepsilon = 0.03$ are 9,148.2 and 9,990.9; true optimal value is 9,874.1

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