SAND2018-8965PE

# Comparison of Heat Flux Predictions over the Blottner Cylinder and Sphere

Sandia National Laboratories
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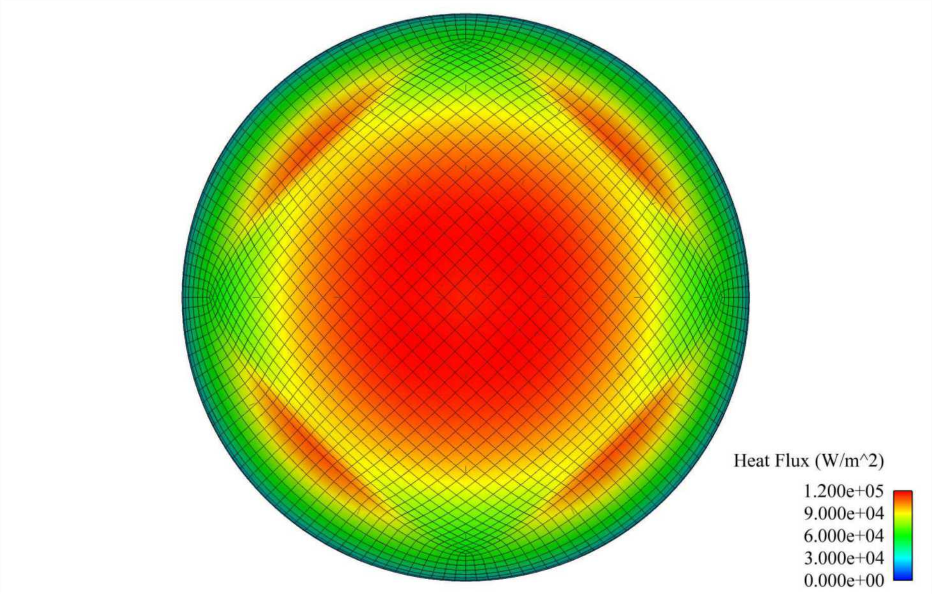
# Problem Background

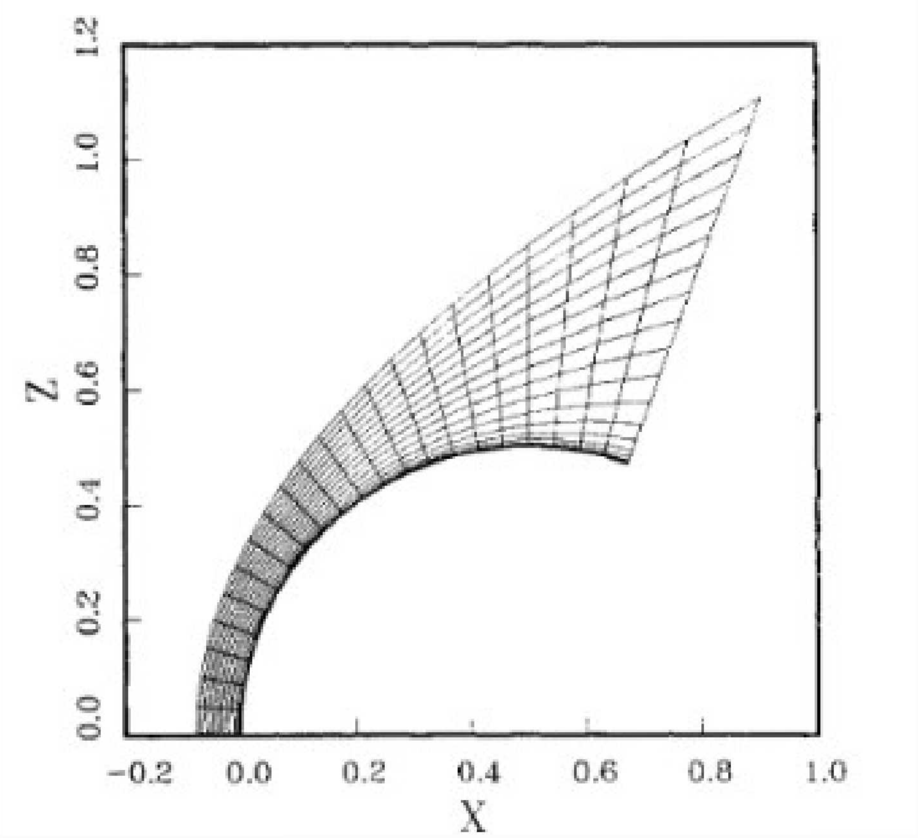
- Heat flux exhibits 'grid imprinting'
  - Topology of grid imposed on heat flux profile
- What is the origin of this grid imprinting
  - Shock?
  - Boundary layer?
- Use error estimation in finite element framework to investigate in 2D and 3D




SUPG Heat Flux from Kirk et. al.[1]

# Blottner Sphere Case

- Using Blottner's sphere[2] as a test case:
  - Mach 5
  - Reynolds number $1.89 \times 10^6$
  - Ratio of wall to stagnation temperature: 0.21
  - Radius 0.0635m
- Blottner obtained high quality shock fitting results on this case
- Run
  - cylinders in 2D
  - Spheres in 3D




Example Mesh from Blottner[2]

# Research Questions

- Can we get good quality high-order heat fluxes on this case?
- Can we use features of the numerical solution to understand the error sources?
- What is the origin of this ‘grid imprinting’?

# Mathematical Model
# Navier-Stokes Equations

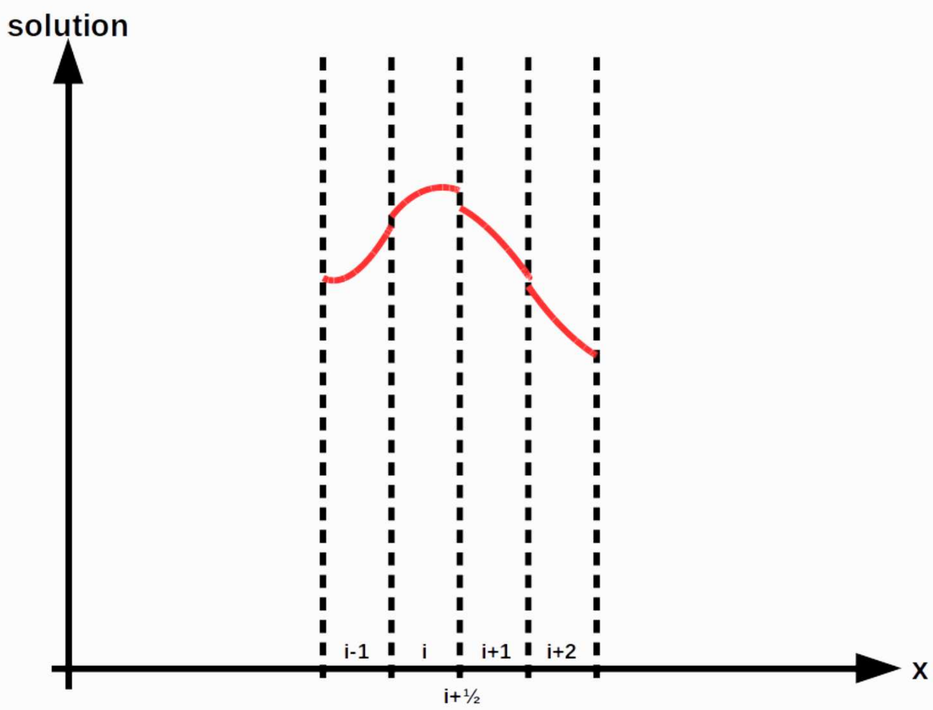
- Solve the steady Navier-Stokes equations in conservation form,

$$\nabla \cdot \begin{bmatrix} \rho \mathbf{V} \\ \rho \mathbf{V}\mathbf{V} + p\mathcal{I} \\ \rho \mathbf{V} H \end{bmatrix} - \nabla \cdot \begin{bmatrix} 0 \\ \tau \\ q + p\mathbf{V} + \tau \mathbf{V} \end{bmatrix} = 0$$

- Where: $q = k\nabla T$; $\tau = \mu \left[ \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T \right) - \frac{2}{3} (\nabla \cdot \mathbf{V}) \mathcal{I} \right]$
- Closed under:
  - calorificially perfect air
  - Sutherland's Law viscosity
  - constant Prandtl number (Pr = 0.72)

# Numerical Method
# Discontinuous Galerkin

- Approximate solution inside elements with polynomials
- Discontinuous solution across element faces
  - Roe approximate Riemann flux for advective flux
  - Bassi-Rebay-2 (BR2) for viscous flux
- DG solutions using MIT's Solution Adaptive Numerical Simulator (SANS)



# Numerical Method
# Artificial Viscosity

- Based on the work of Barter and Darmofal[3], and Yano[4]
- Solve an additional PDE for a smooth shock-sensor field

$$0 = \nabla \cdot (\eta \nabla s) + \left[ C_3 \tilde{S}_k (u) - s \right]$$

- Where $\eta_{ij} = C_2 \mathcal{H}_{ik} \mathcal{H}_{kj}$
- Robin boundary conditions

$$\eta_{ij} \frac{\partial s}{\partial x_j} n_i + \sqrt{C_2} n_i \mathcal{H}_{ij} n_j s = 0$$

- Source term is resolution indicator based on inter-element pressure jumps

# Numerical Method
# Artificial Viscosity

- Interpret sensor as inverse local Peclet number,

$$\epsilon_{ij} = \frac{2}{p}\lambda\left(u\right)\mathcal{H}_{ij}\max\left\{0, s\right\}$$

- Apply viscosity to *enthalpy* not internal energy,

$$F_{AV}\left(u, \nabla u\right) = \epsilon\nabla\begin{bmatrix}\rho \\ \rho\mathbf{V} \\ \rho H\end{bmatrix}$$

# Numerical Method
# Error Estimation[4]

- Interested in some output from the PDE solution: $J$
- Error in output:

$$\mathcal{E}_{true} = J\left(u\right) - J\left(u_{h,p}\right)$$

- Or, using Dual Weighted Residual (DWR):

$$\mathcal{E}_{true} = -R_{h,p}\left(u_{h,p}, \psi\right)$$

- The true adjoint is from solving:

$$\bar{R}'_{h,\hat{p}}\left[u, u_{h,p}\right]\left(w, \psi\right) = \bar{J}'_{h,\hat{p}}\left[u, u_{h,p}\right]\left(w\right) \quad \forall w \in \mathcal{W}$$

# Numerical Method
# Error Estimation[4]

- This requires the exact solution to the PDE
  - We don't have this
- Approximate the adjoint, by solving

$$R'_{h,\hat{p}}\left[u_{h,p}\right]\left(w_{h,\hat{p}},\psi_{h,\hat{p}}\right)=J'_{h,\hat{p}}\left[u_{h,p}\right]\left(w_{h,\hat{p}}\right)\quad\forall w_{h,\hat{p}}\in\mathcal{W}_{h,\hat{p}}$$

- Now, we approximate the error by,

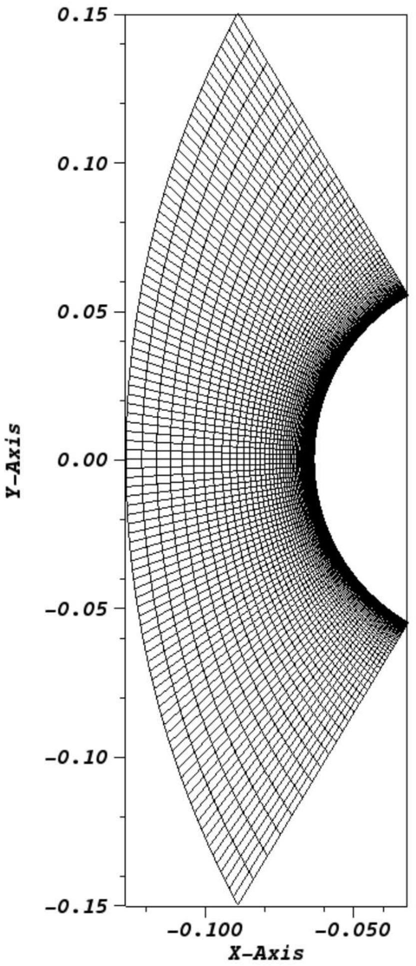
$$\mathcal{E}\approx -R_{h,p}\left(u_{h,p},\psi_{h,\hat{p}}\right).$$

- And the local error by,

$$\eta_k=\left|R_{h,p}\left(u_{h,p},\psi_{h,\hat{p}}\big|_k\right)\right|$$

# 2D Grids

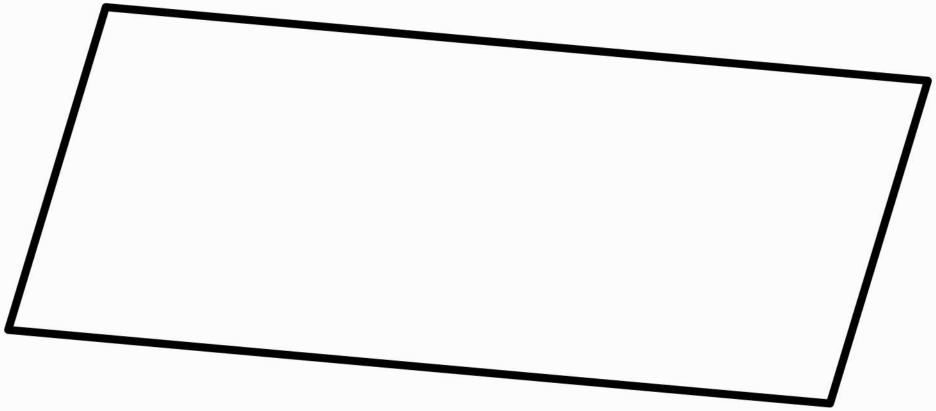
- 4 Fixed Q1 Grids
- Grid 1: 2500 elements
- Grid 2: 10,000 elements
- Grid 3: 40,000 elements
- Grid 4: 160,000 elements



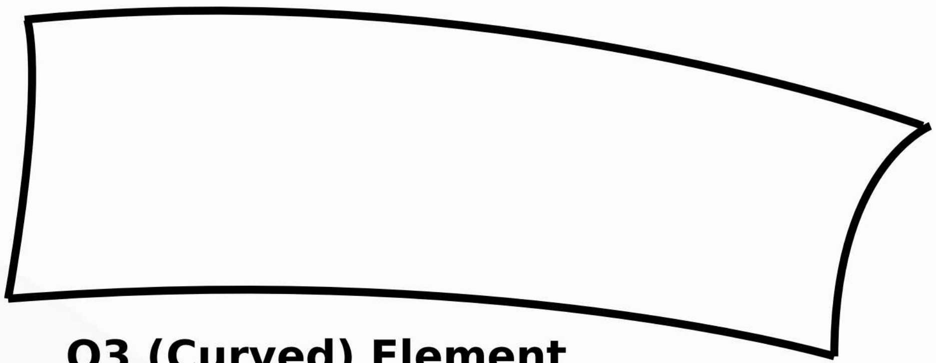

Grid 2

# Element Order

- Finite volume methods typically use straight sided elements
- DG Method can use curved elements
  - Element faces approximated using higher order polynomials
- Curved elements better approximate curved boundaries

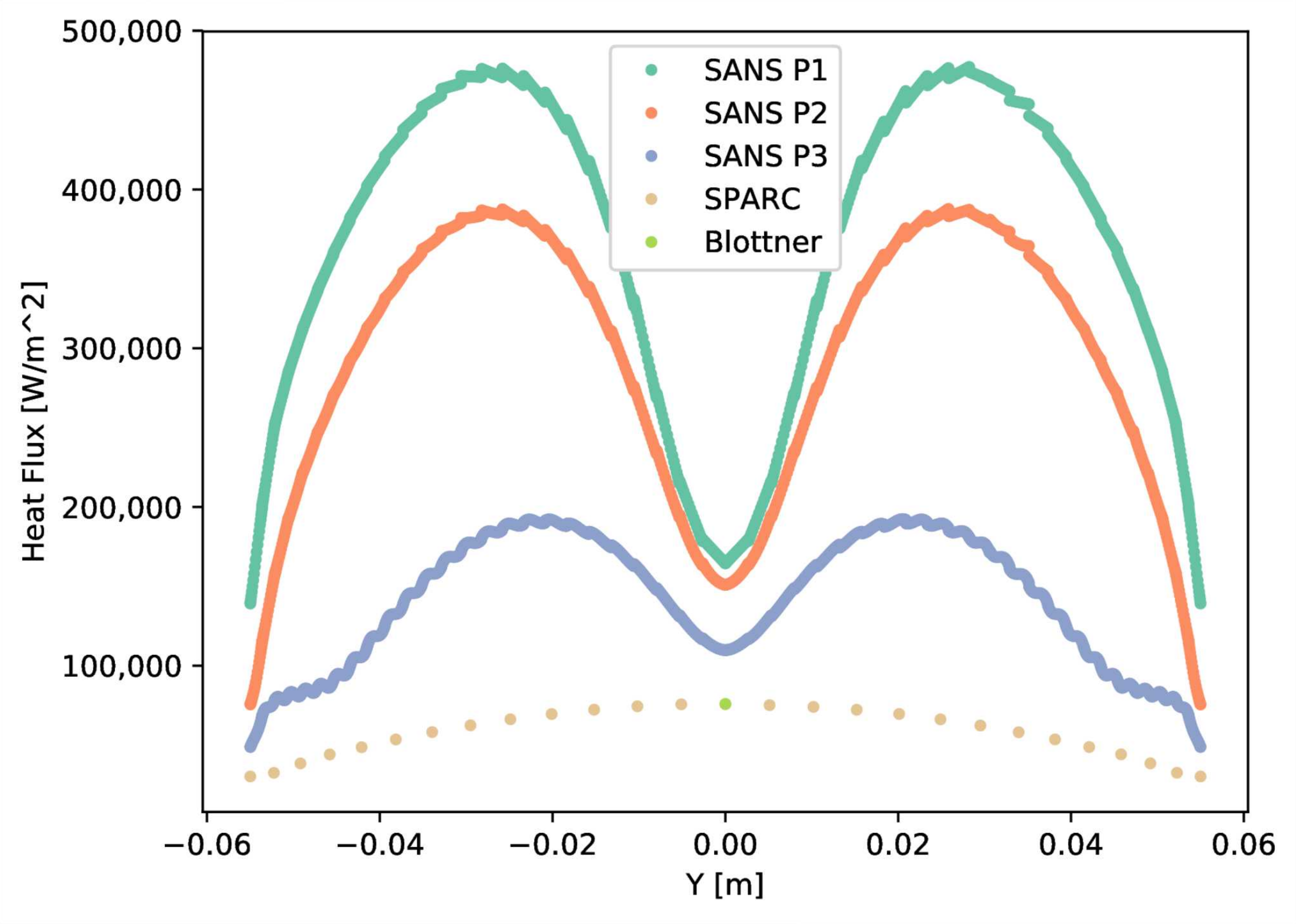


**Q1 (Straight Sided) Element**

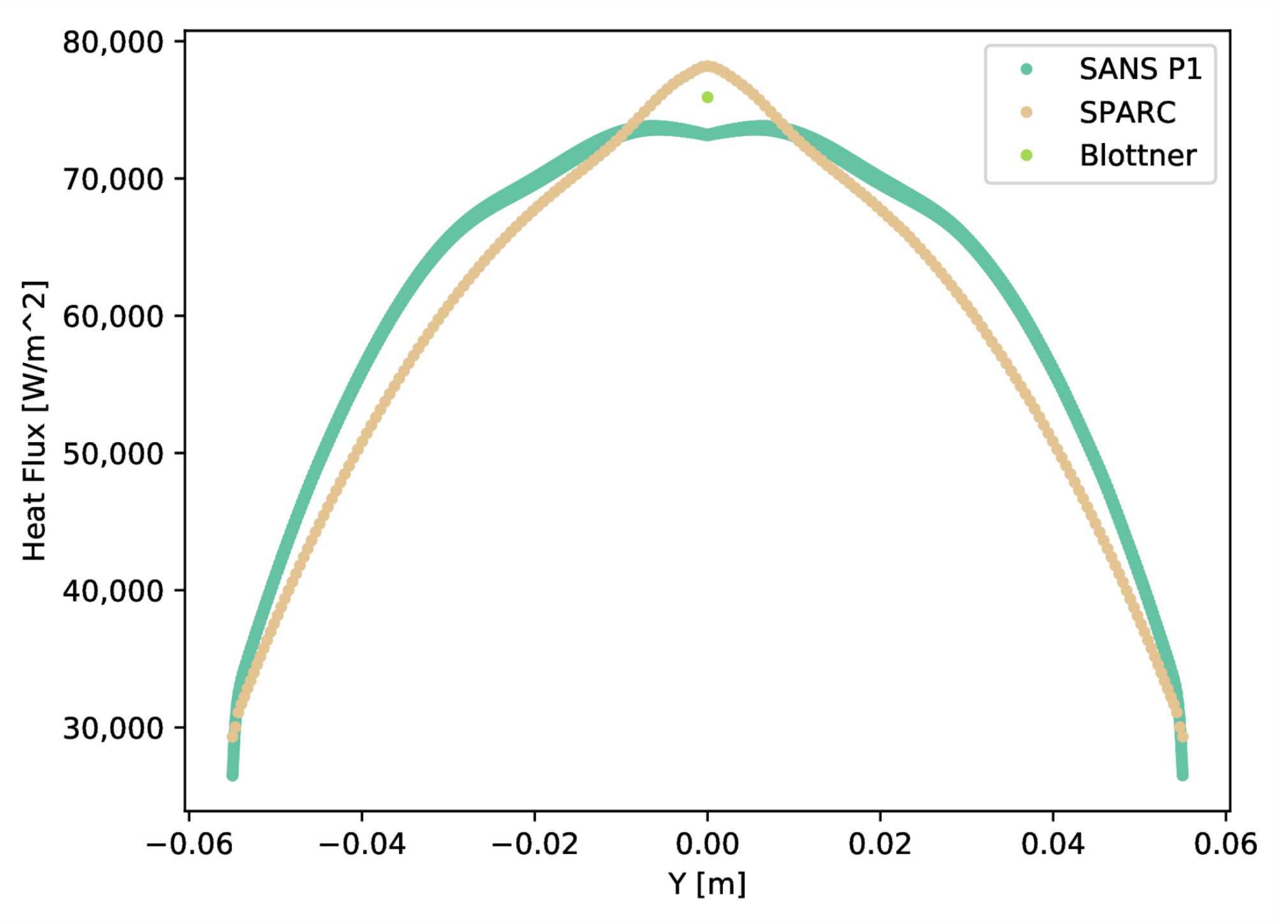


**Q3 (Curved) Element**

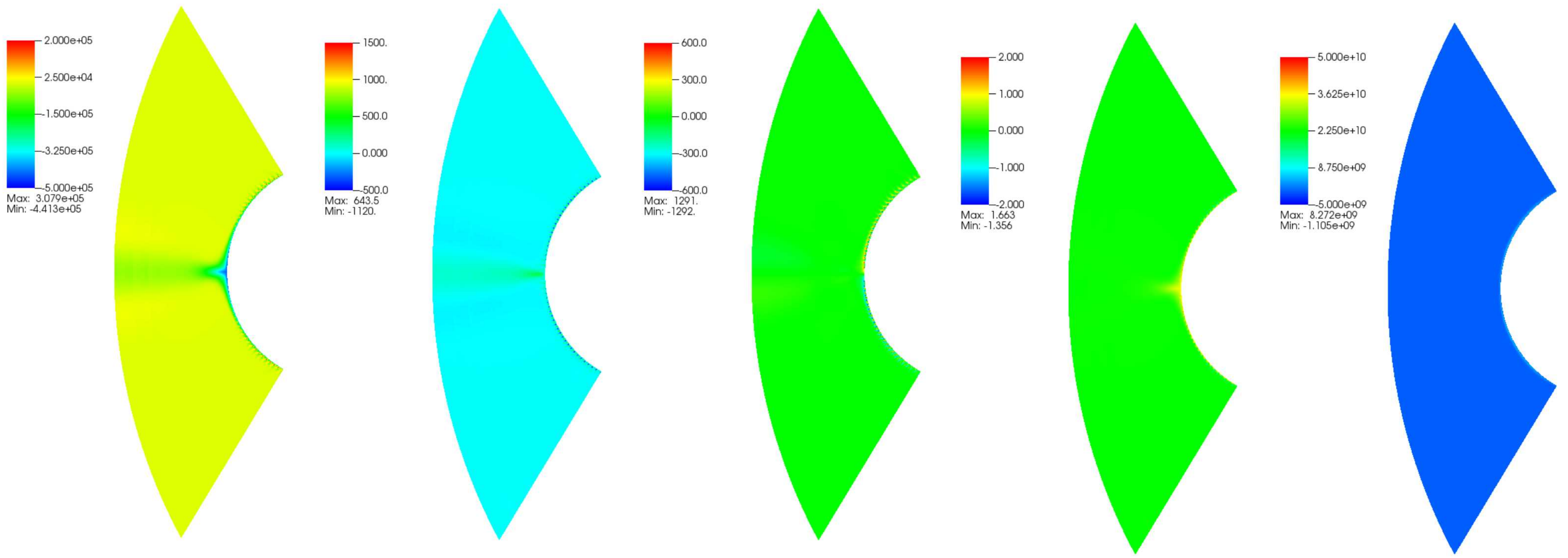
# Heat Flux
# 2,500 Q1 Quad Elements



# Heat Flux
# 160,000 Q1 Quad Elements

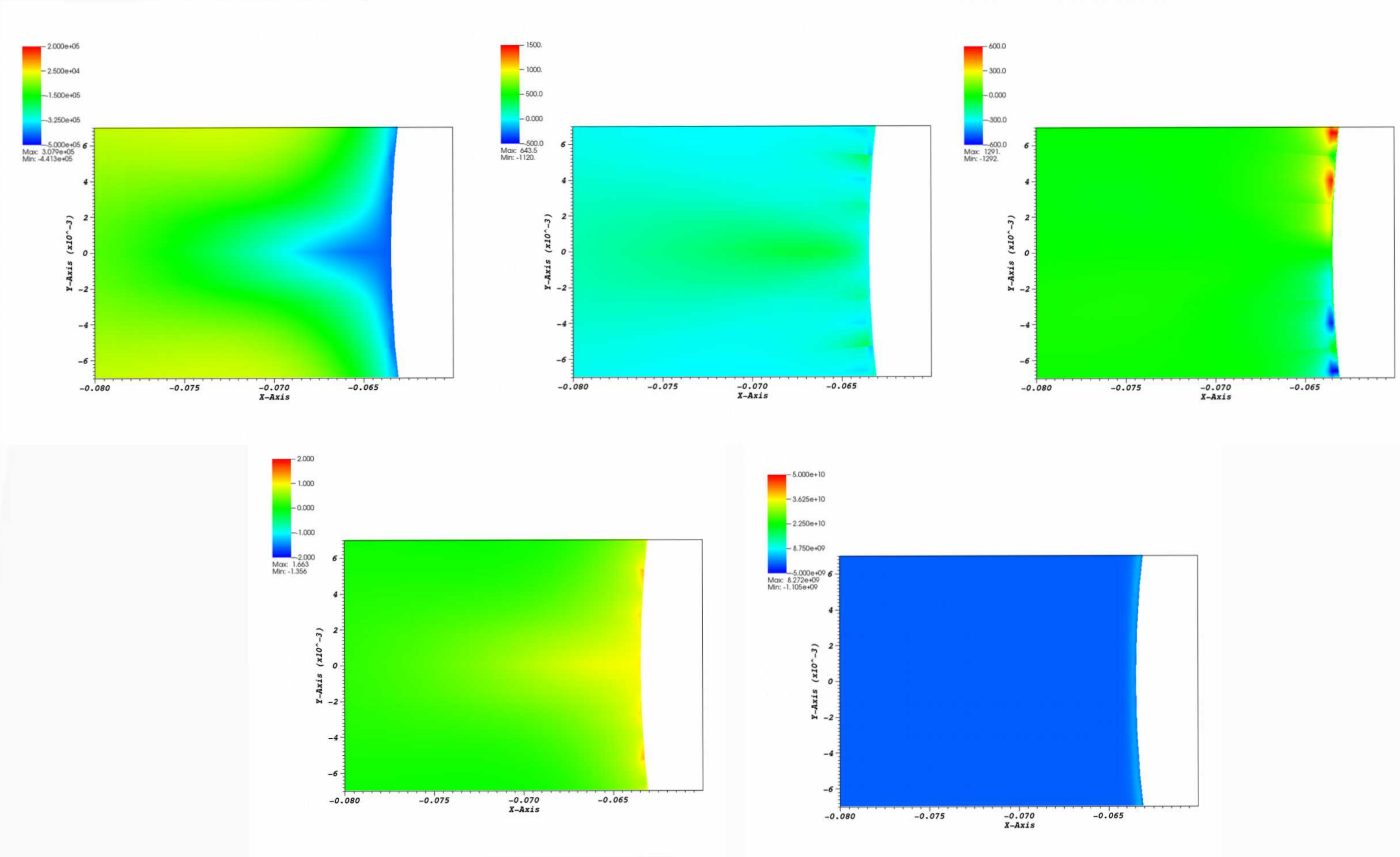


# P1 Adjoint
# 2,500 Q1 Quad Elements




Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

# P1 Adjoint
# 2,500 Q1 Quad Elements

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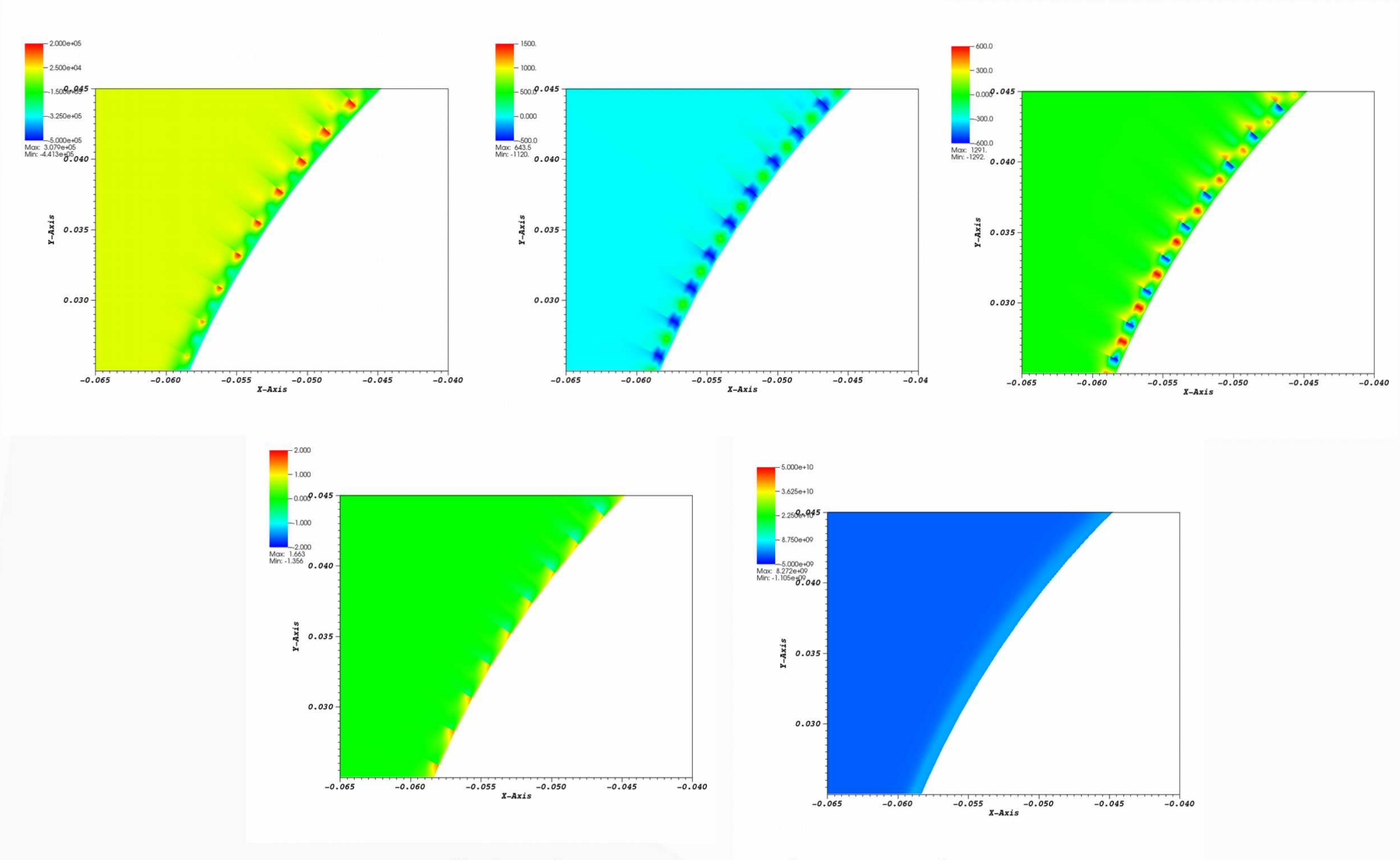


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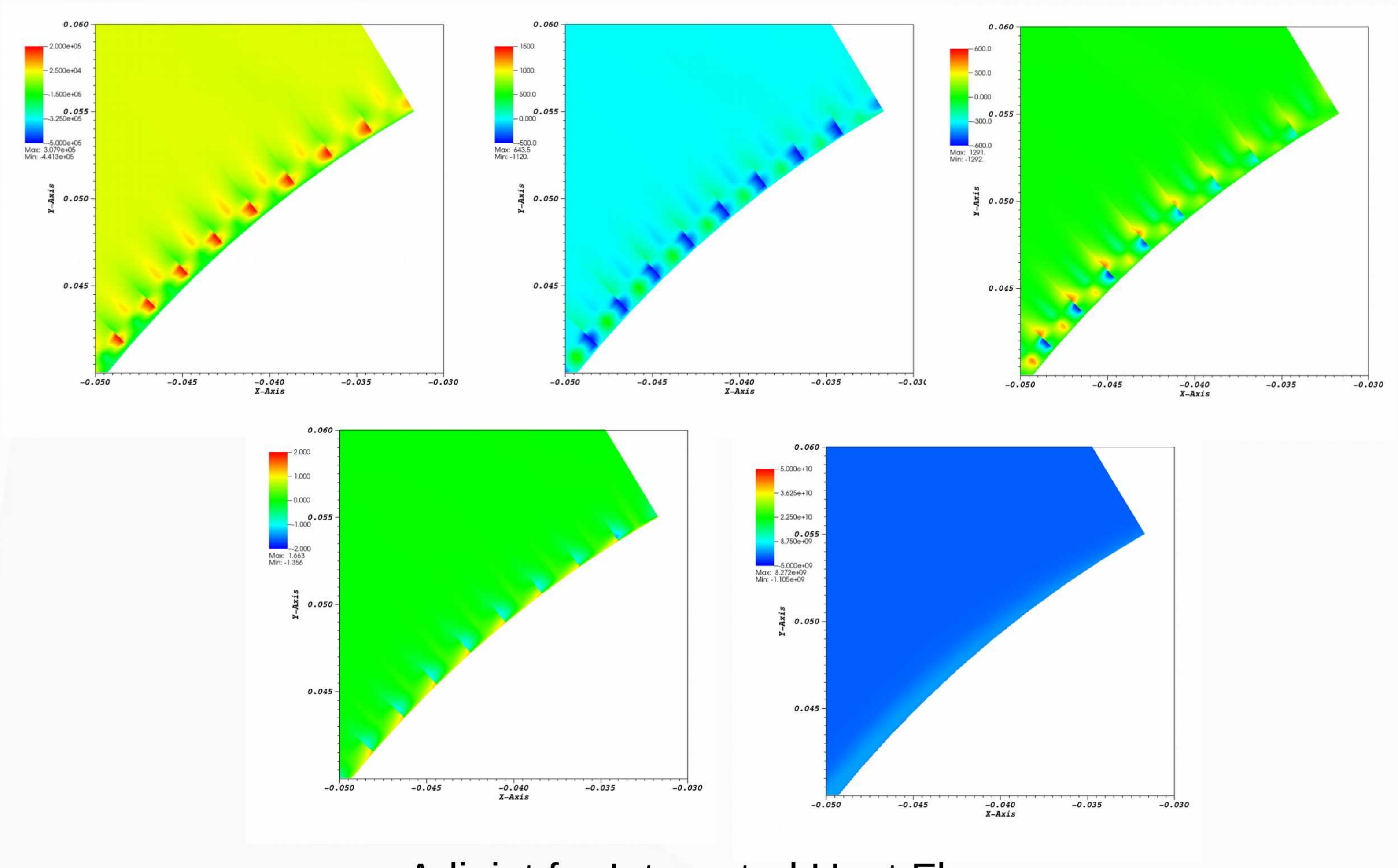


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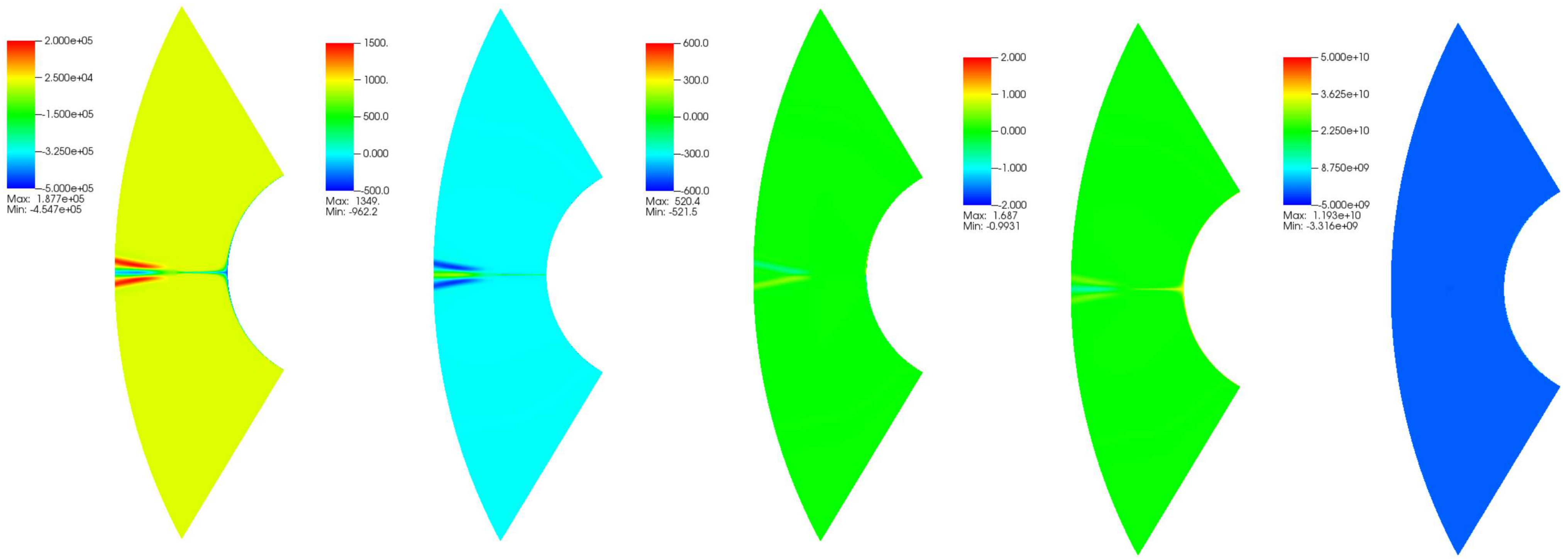


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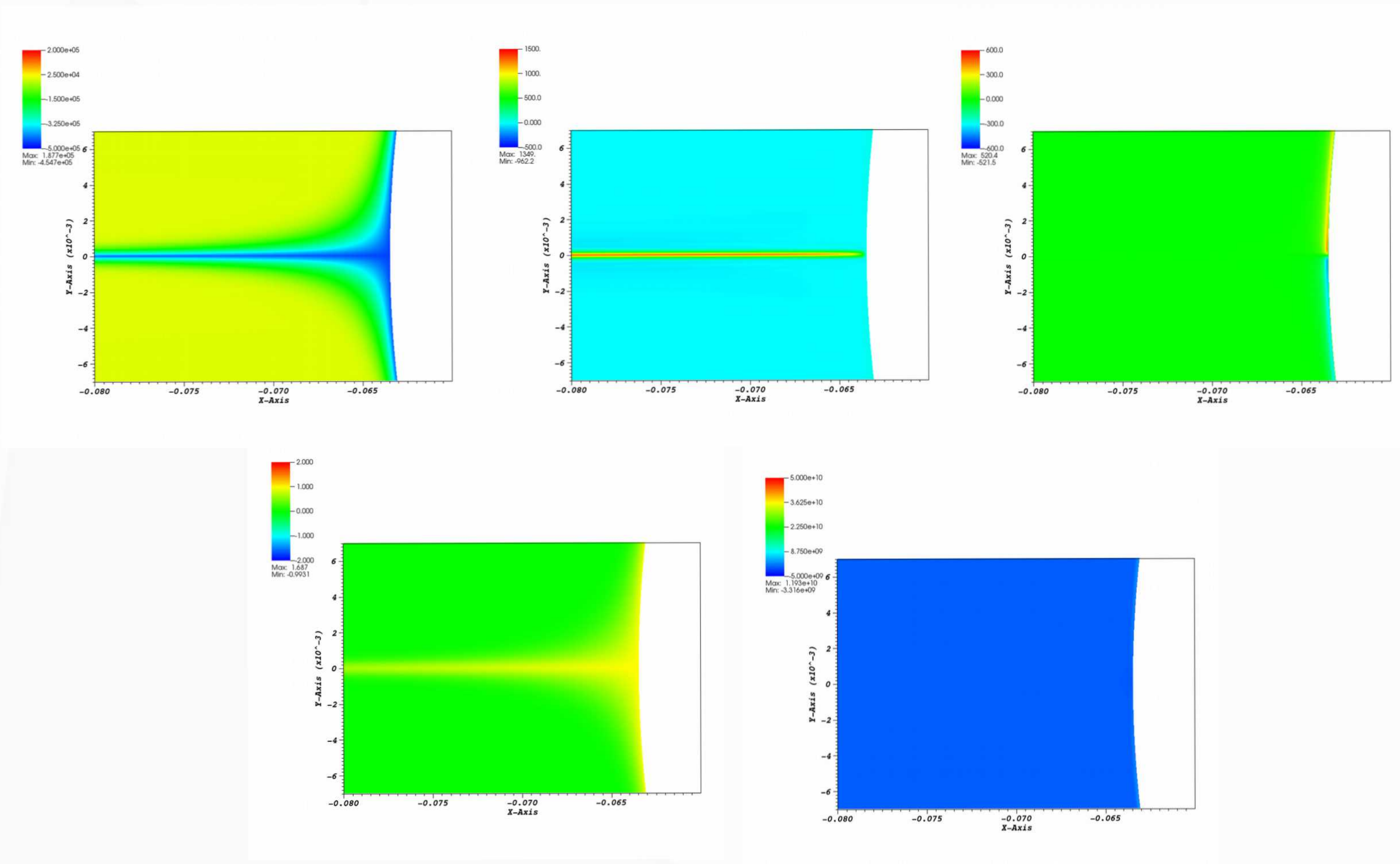


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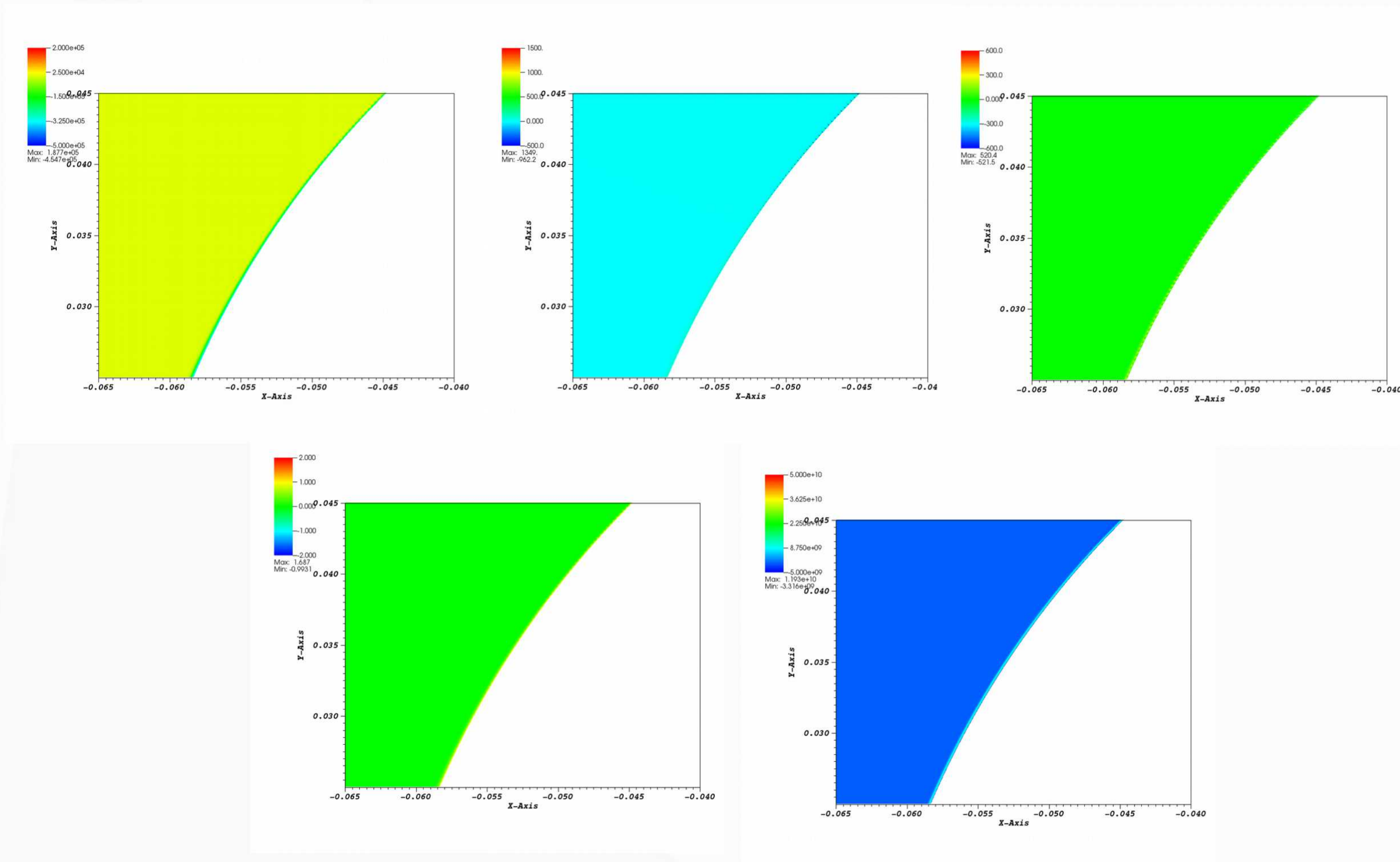


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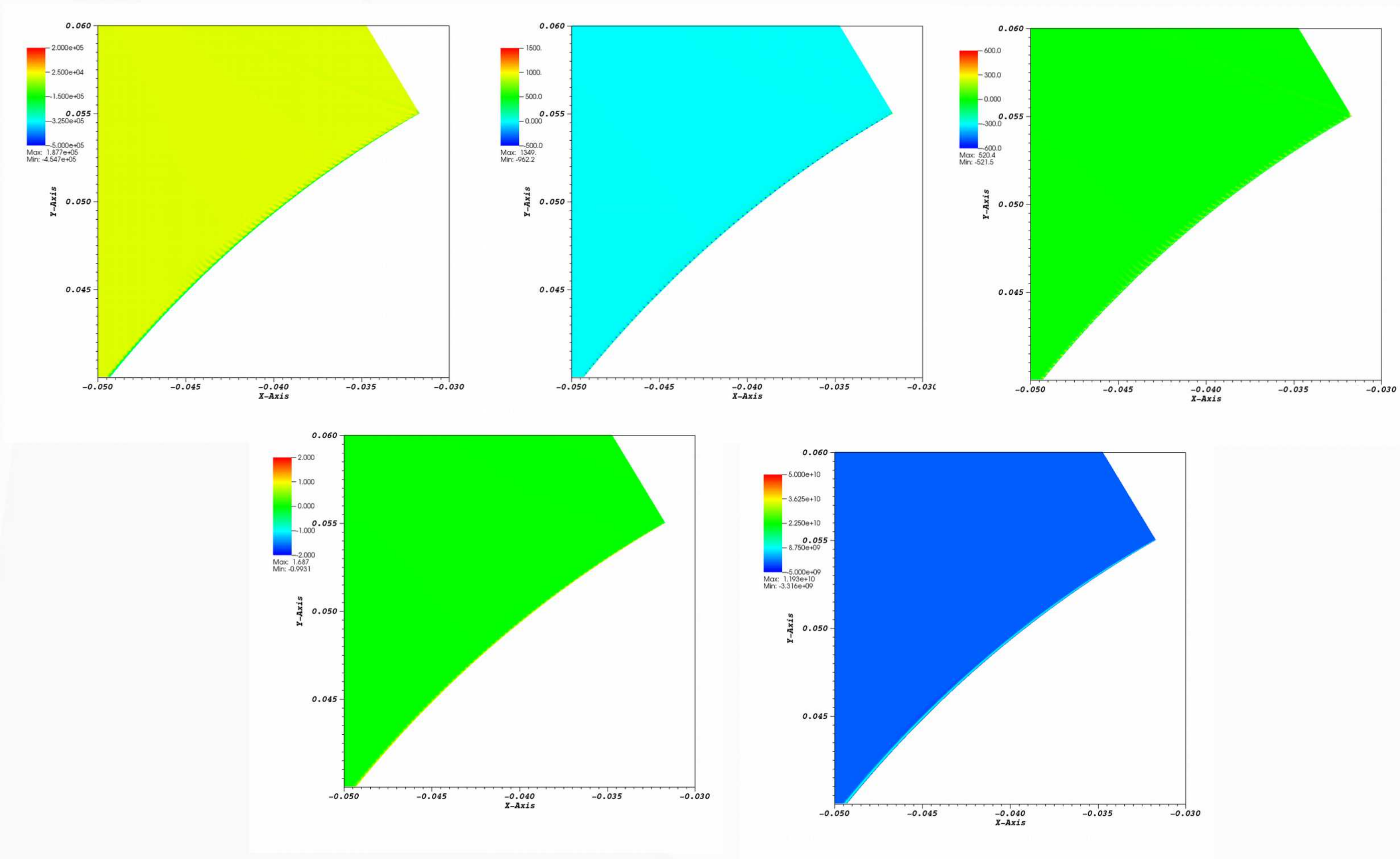


Adjoint for Integrated Heat Flux
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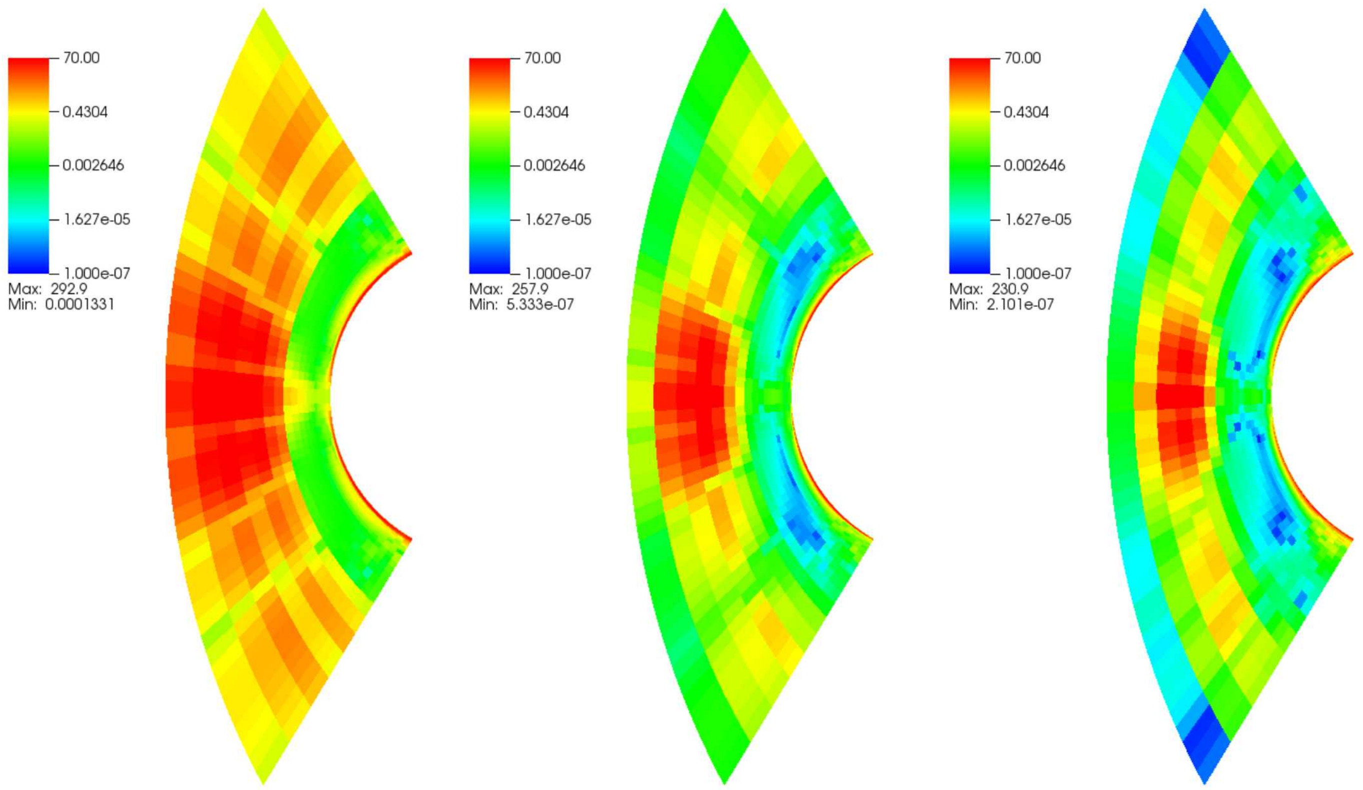
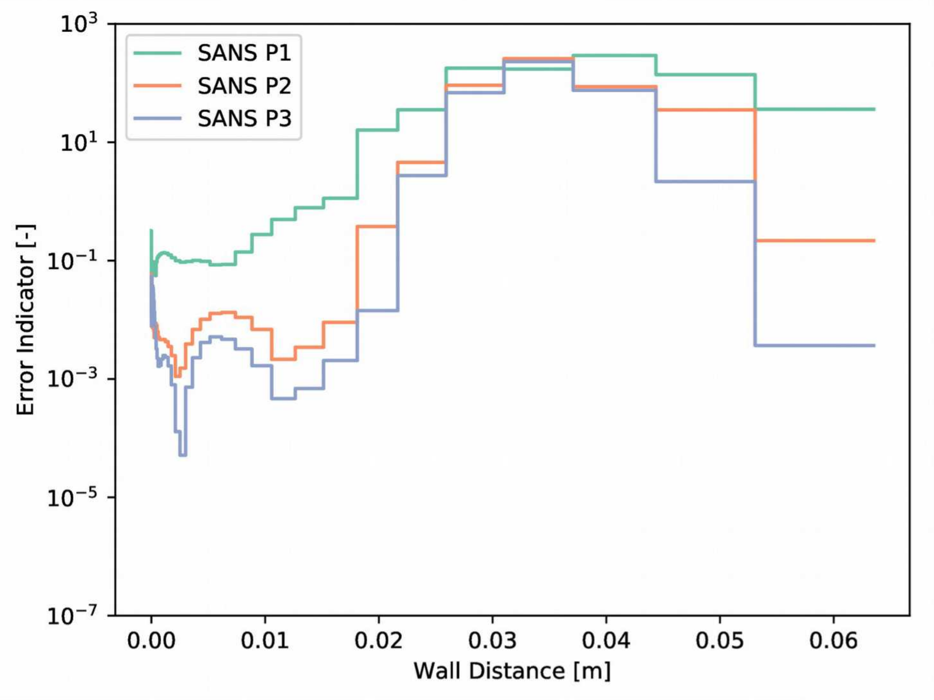
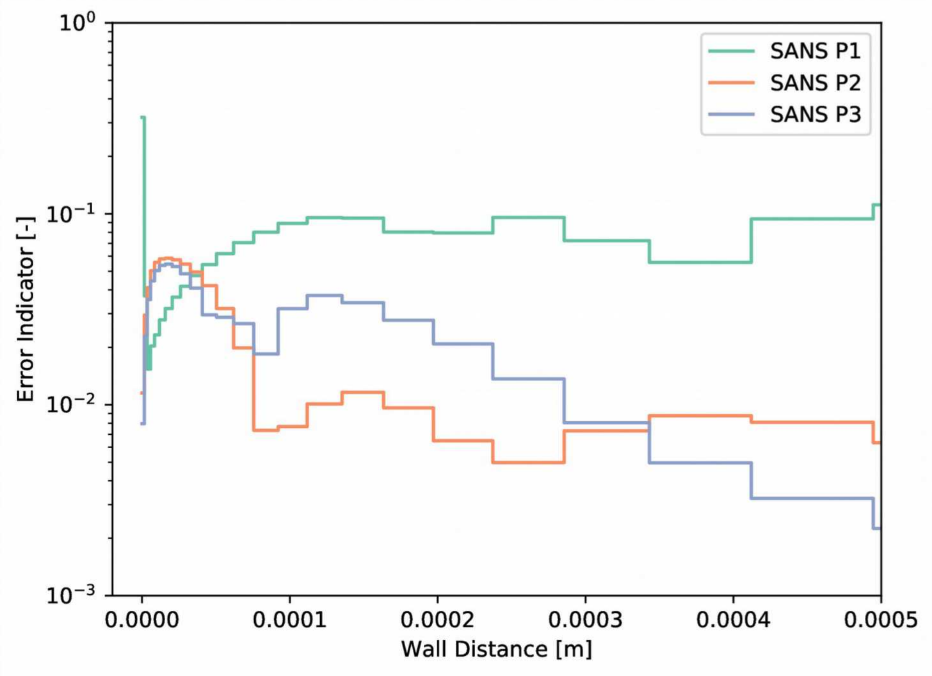
# Error Indicator
# 2,500 Q1 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1, P2, P3

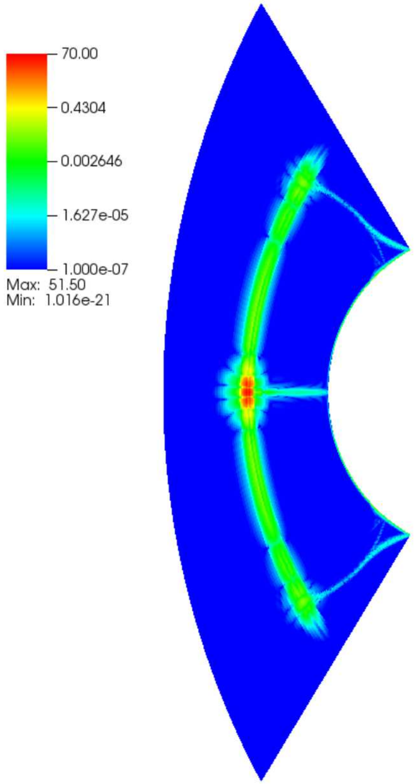
# Error Indicator on Stagnation Line 2,500 Q1 Quad Elements



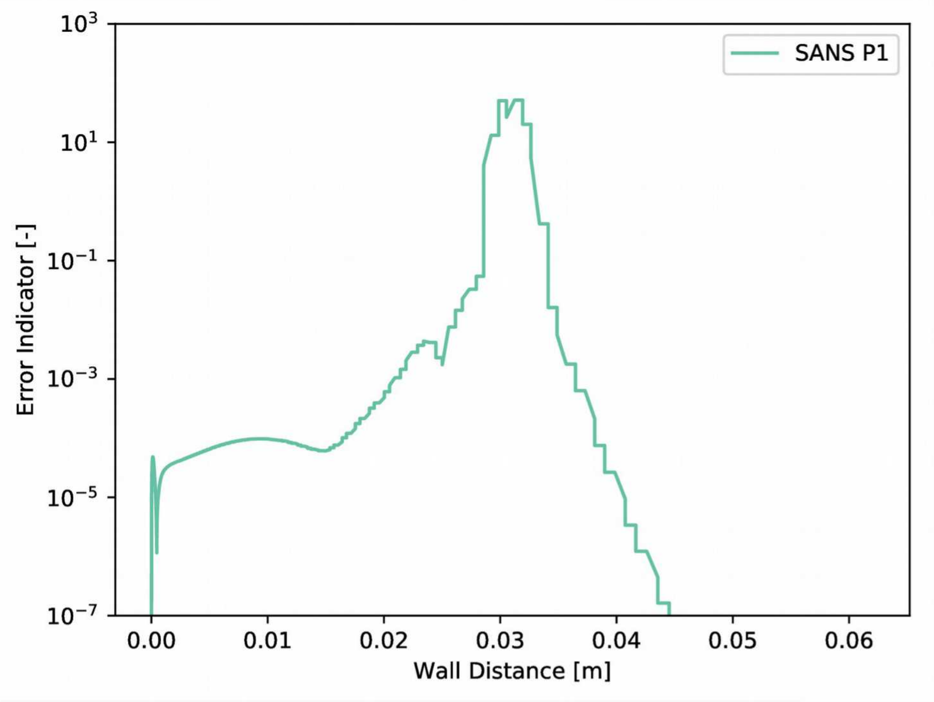

Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

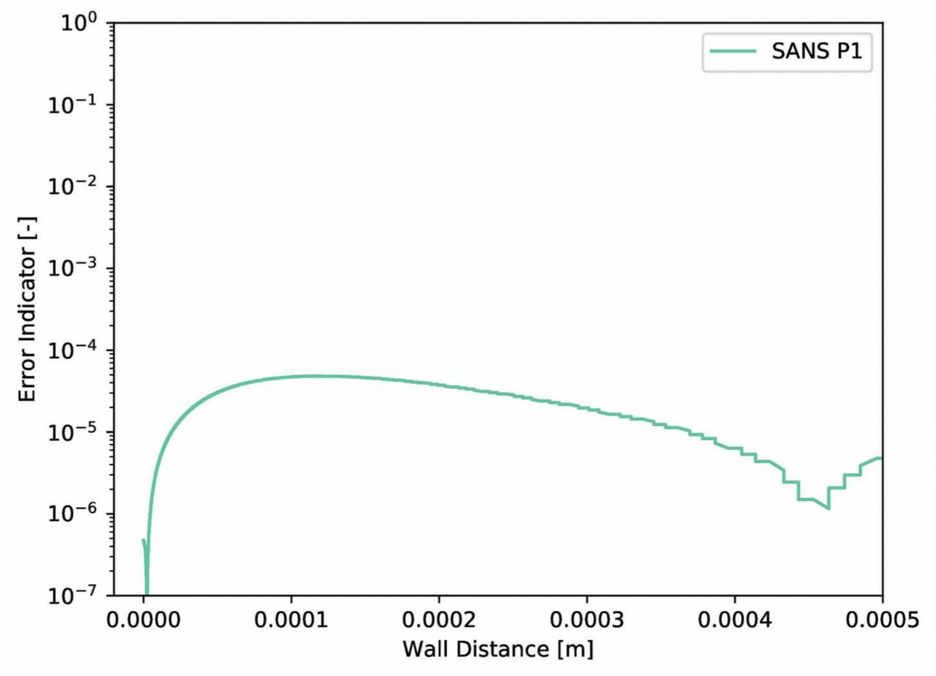
# Error Indicator
# 160,000 Q1 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1

# Error Indicator on Stagnation Line 160,000 Q1 Quad Elements

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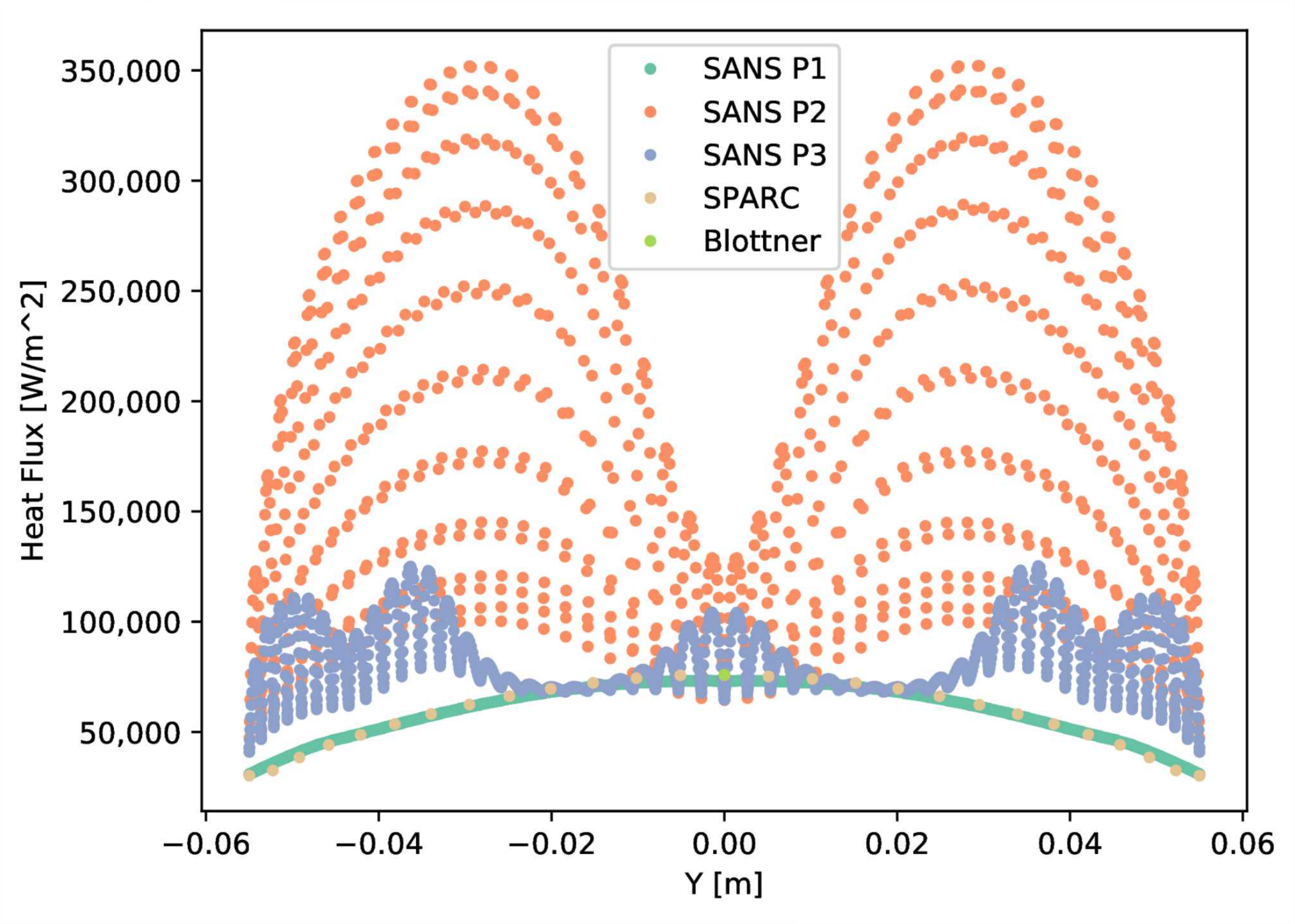
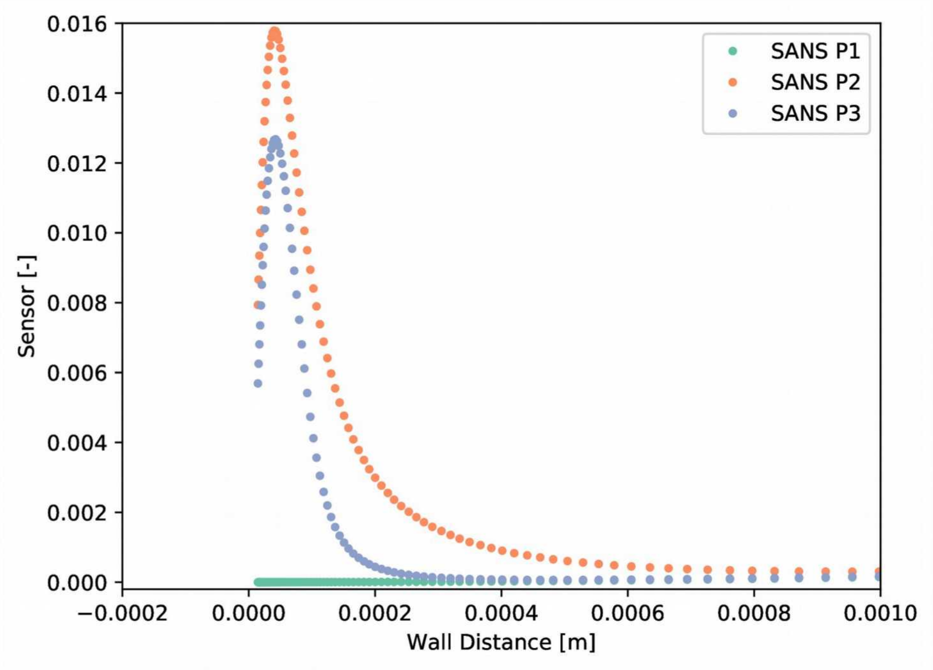
# Heat Flux
# 2,500 Q3 Quad Elements



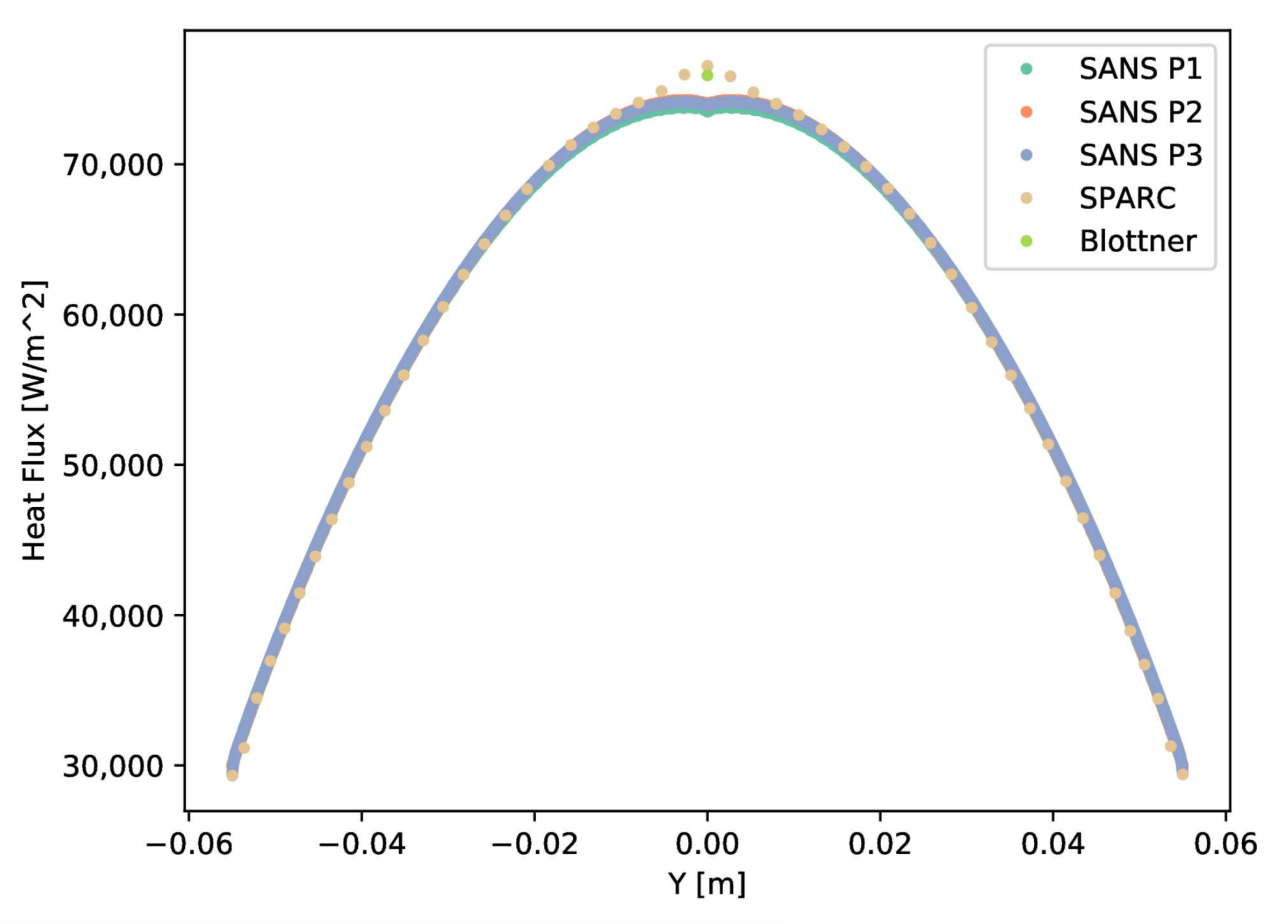
# Stagnation Streamline
# 2,500 Q3 Quad Elements

- Boundary layer occupies a coarse region of the grid
- Sensor PDE flags the boundary layer region as having insufficient resolution
- Artificial viscosity in boundary layer results in poor heat flux prediction

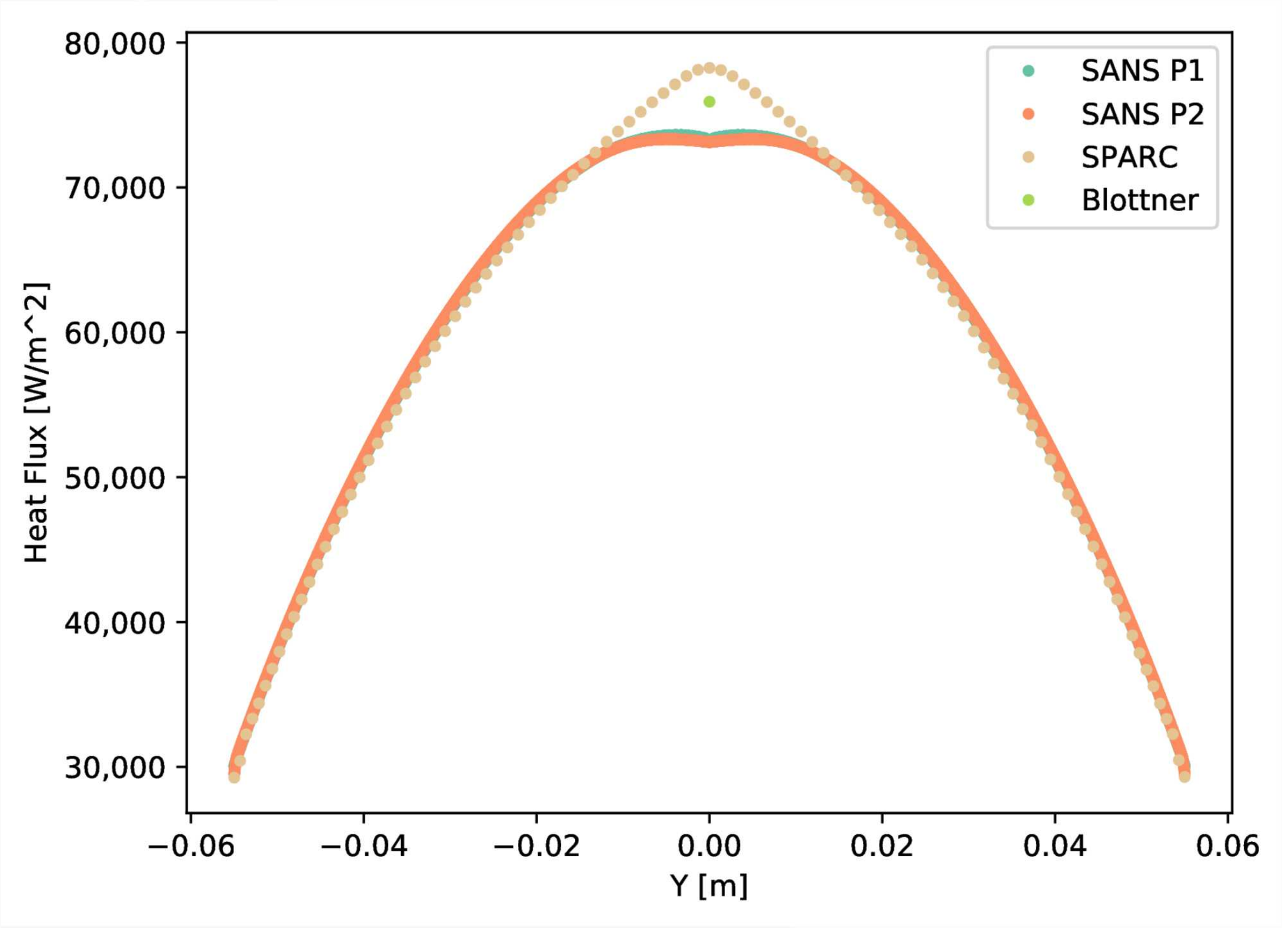



Stagnation stream line focus on boundary layer

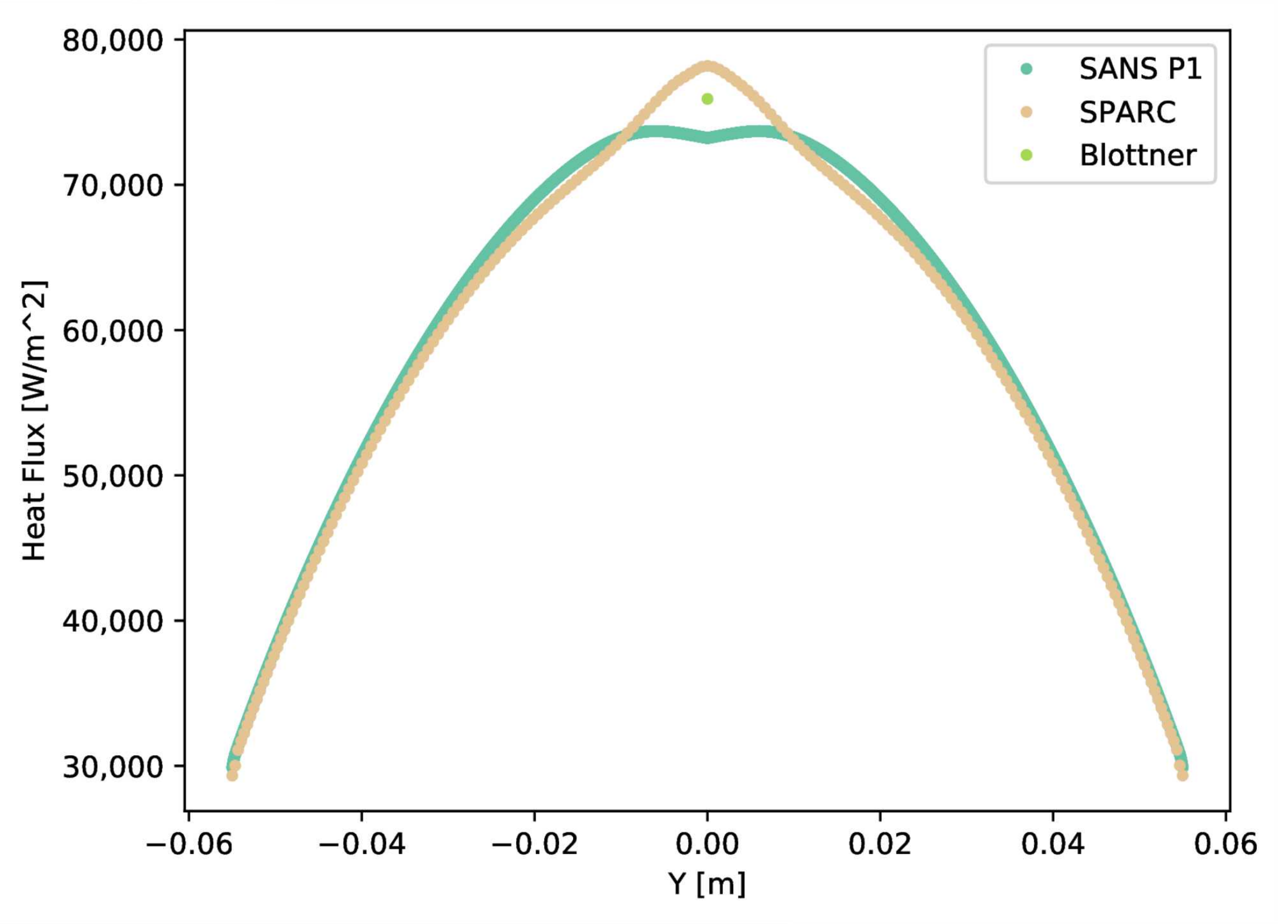
# Heat Flux
# 10,000 Q3 Quad Elements



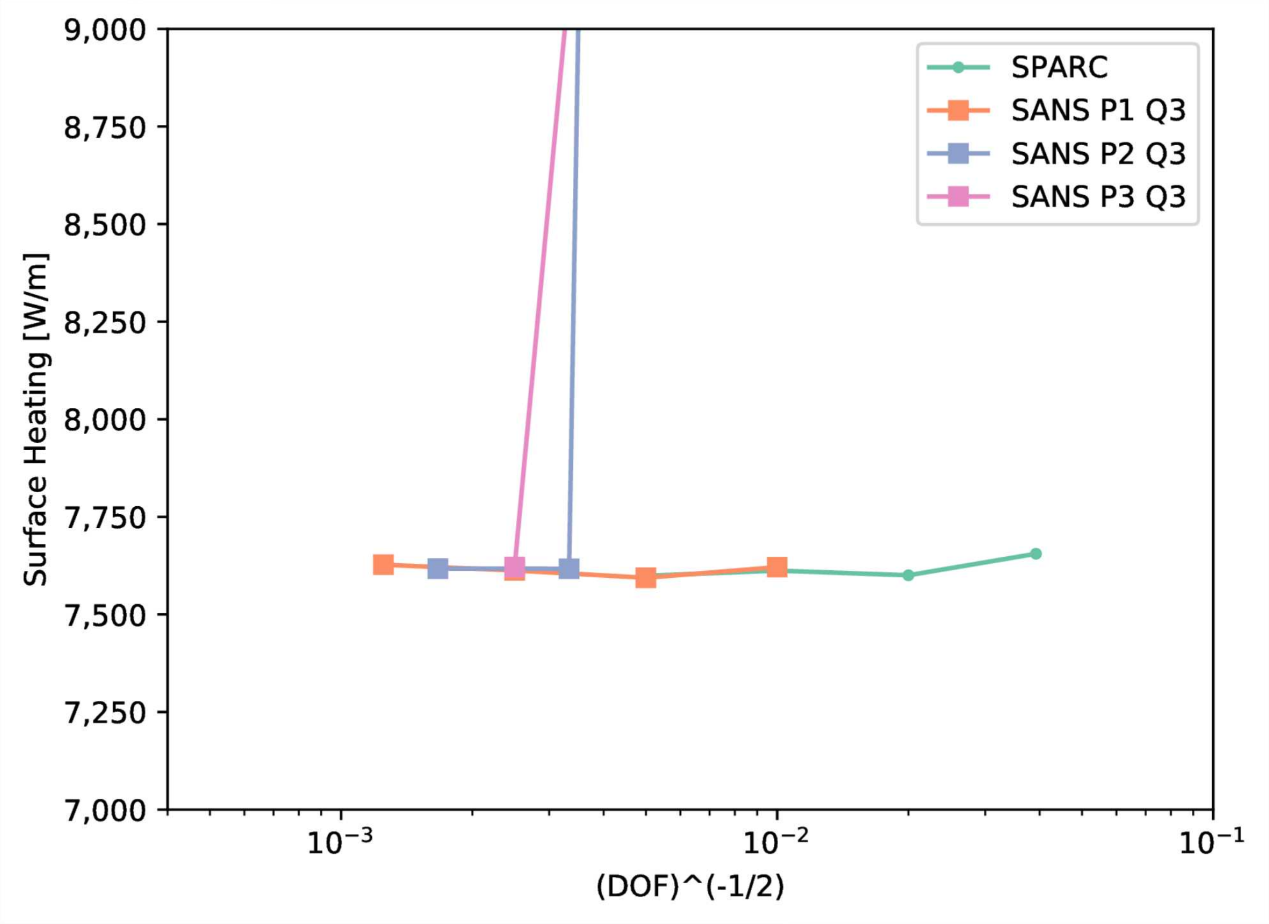
# Heat Flux
# 40,000 Q3 Quad Elements



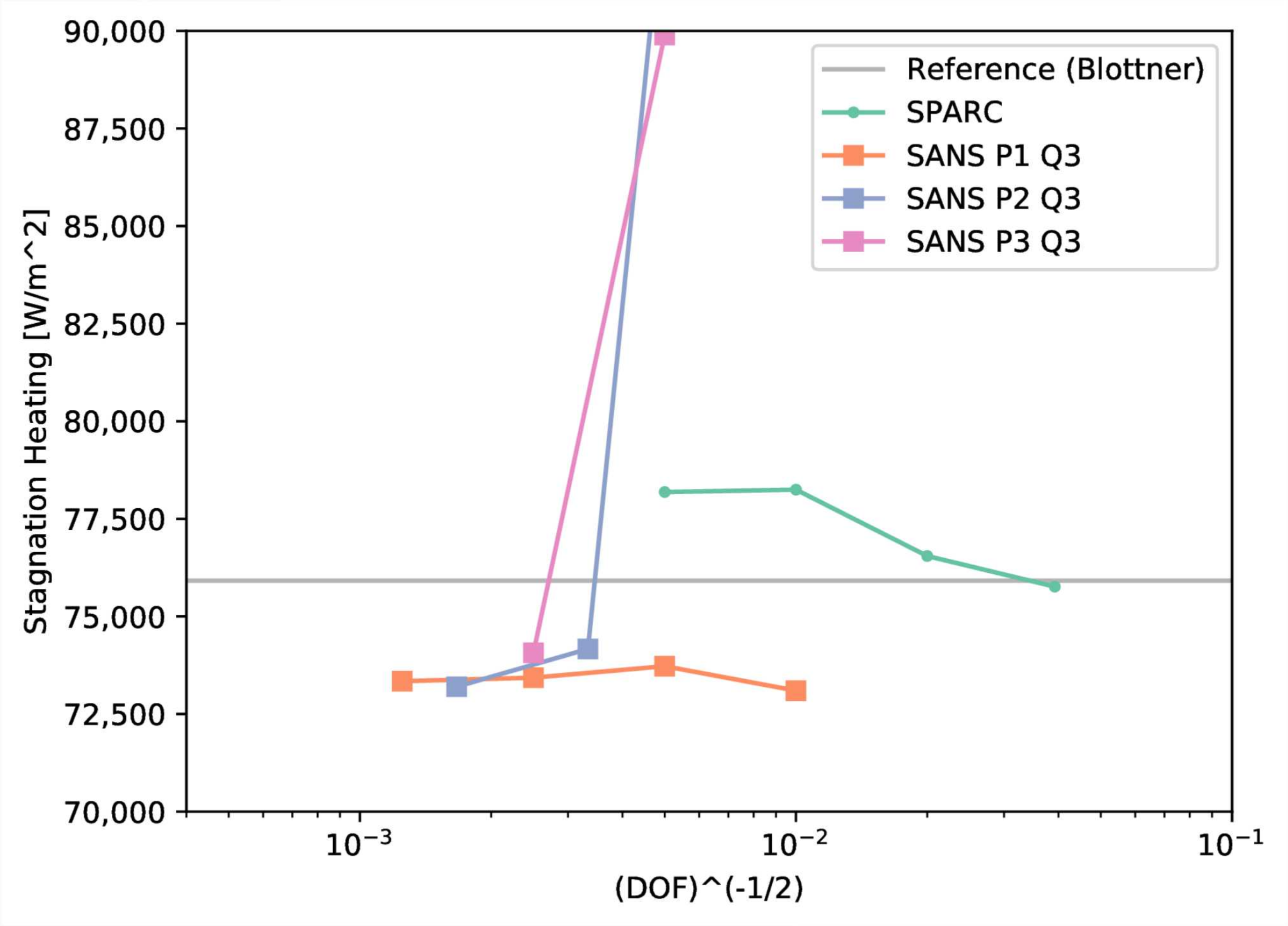
# Heat Flux
# 160,000 Q3 Quad Elements



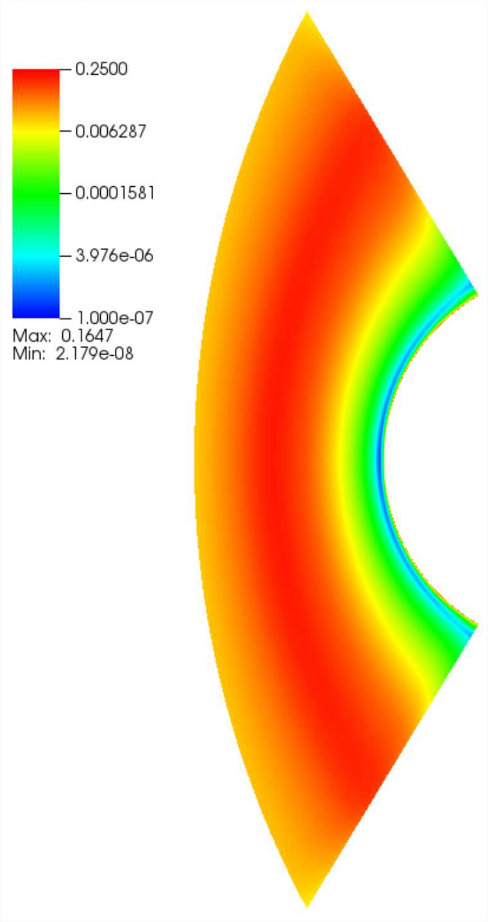
# Integrated Heat Flux



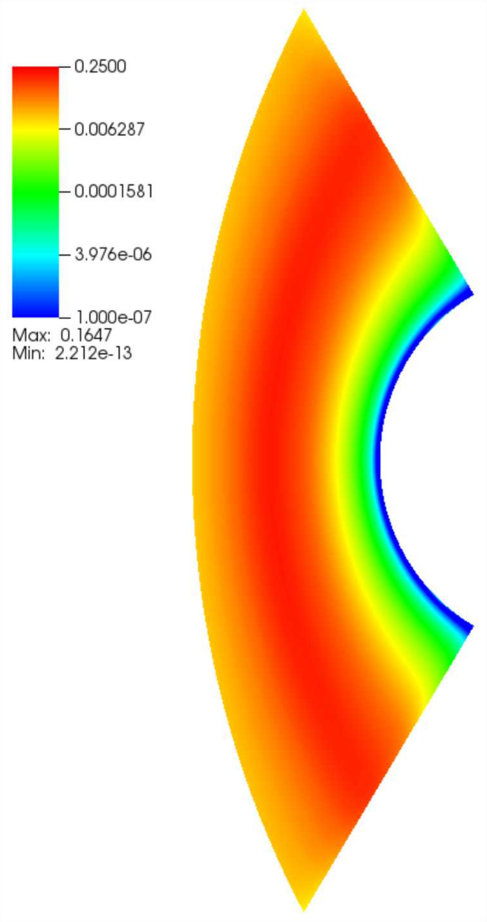
# Stagnation Heat Flux



# Why Is Q3 So Much Better?



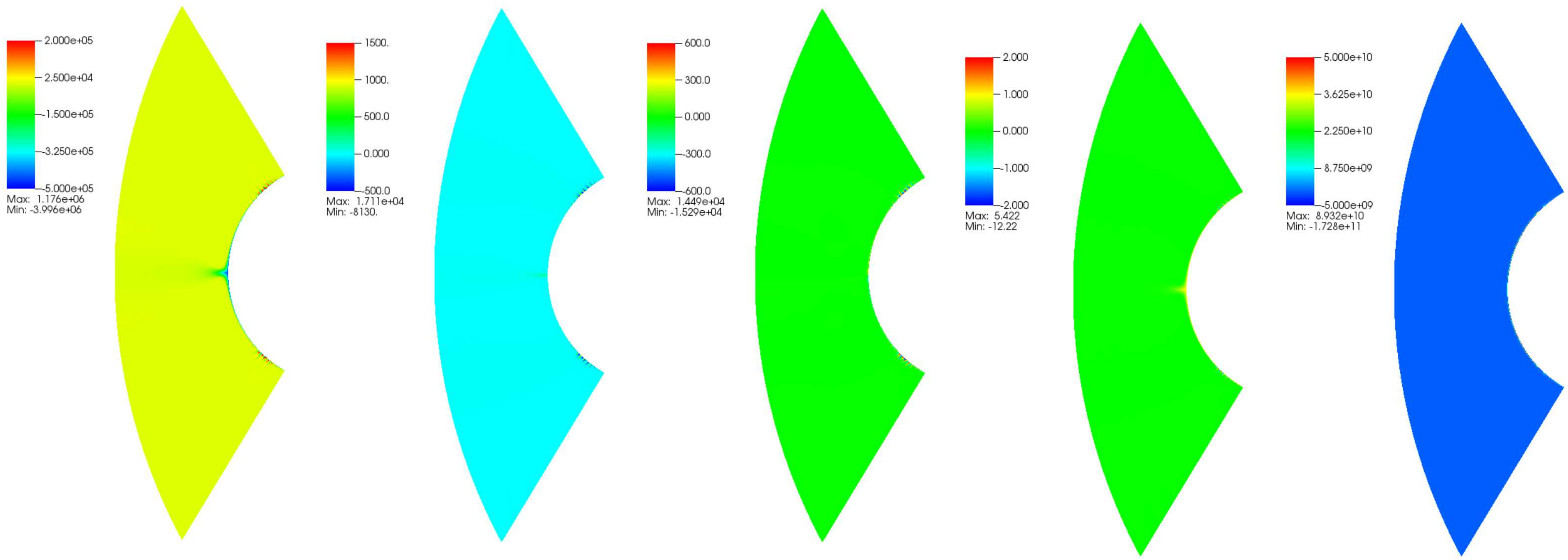

Q1, 10,000 Elements




Q3 10,000 Elements

- Sensor equation flags Q1 elements in boundary layer
  - Poor approximation to curved geometry
- Artificial viscosity generated in boundary layer for Q1 case

# P1 Adjoint
# 2,500 Q3 Quad Elements




Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

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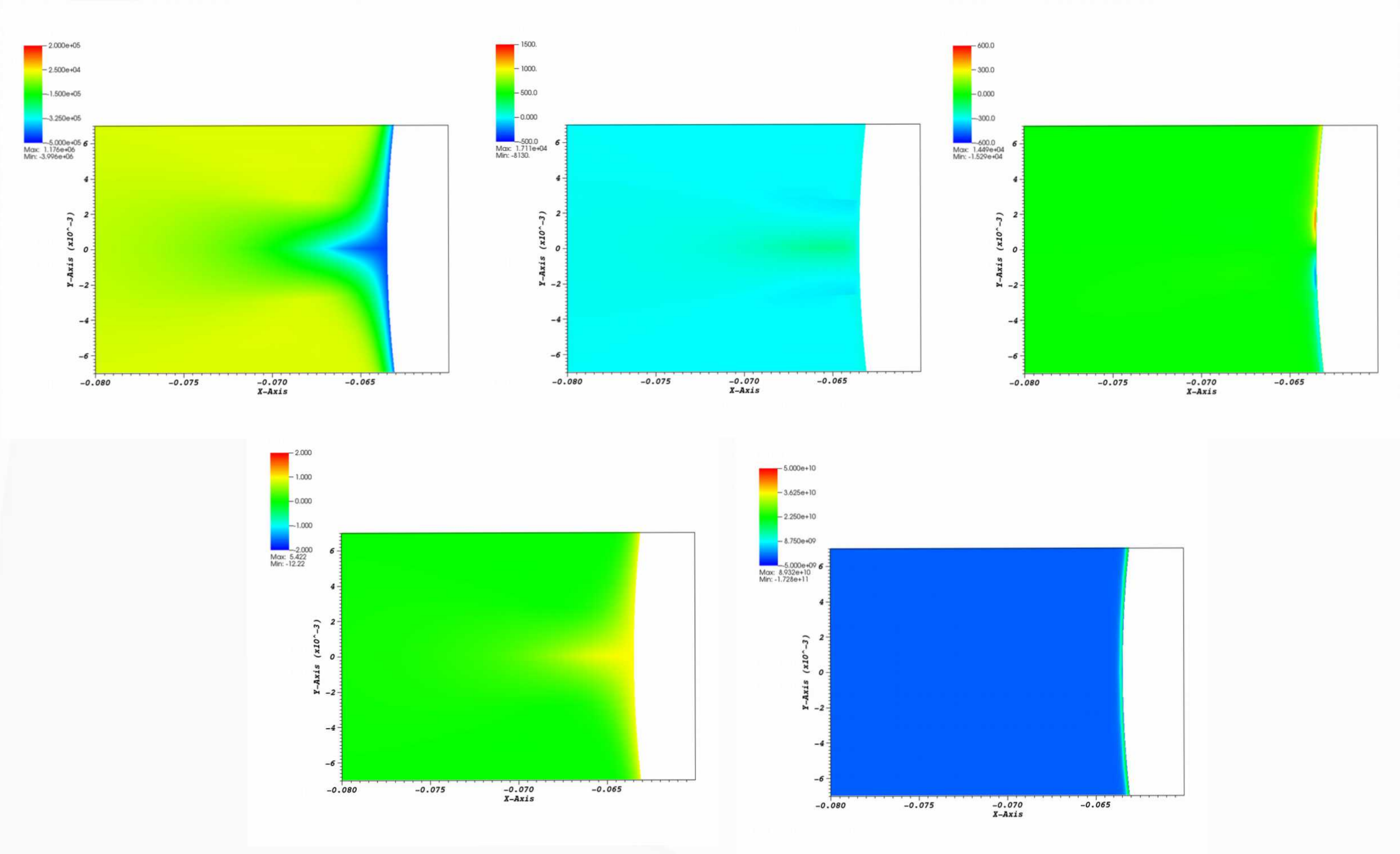

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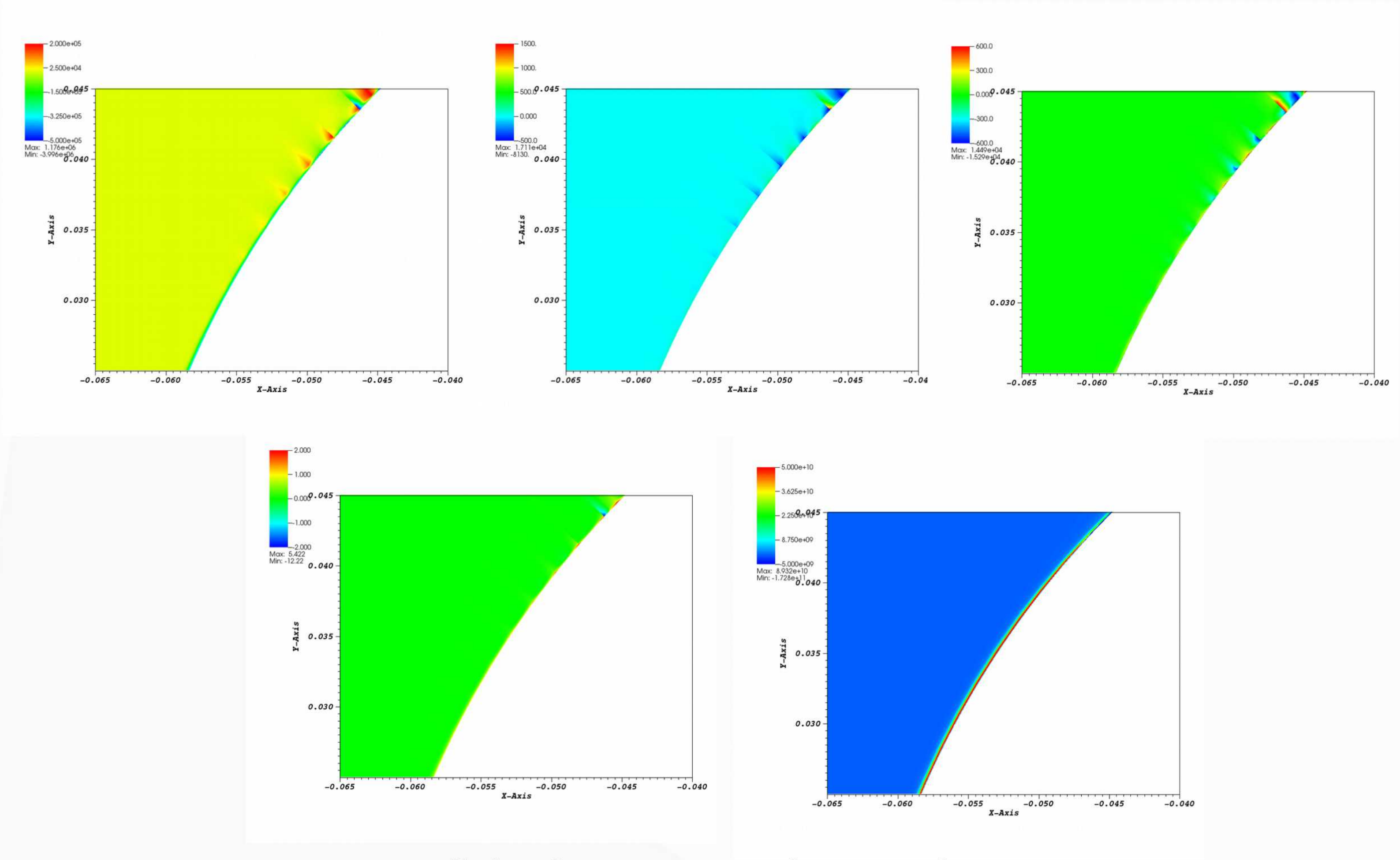


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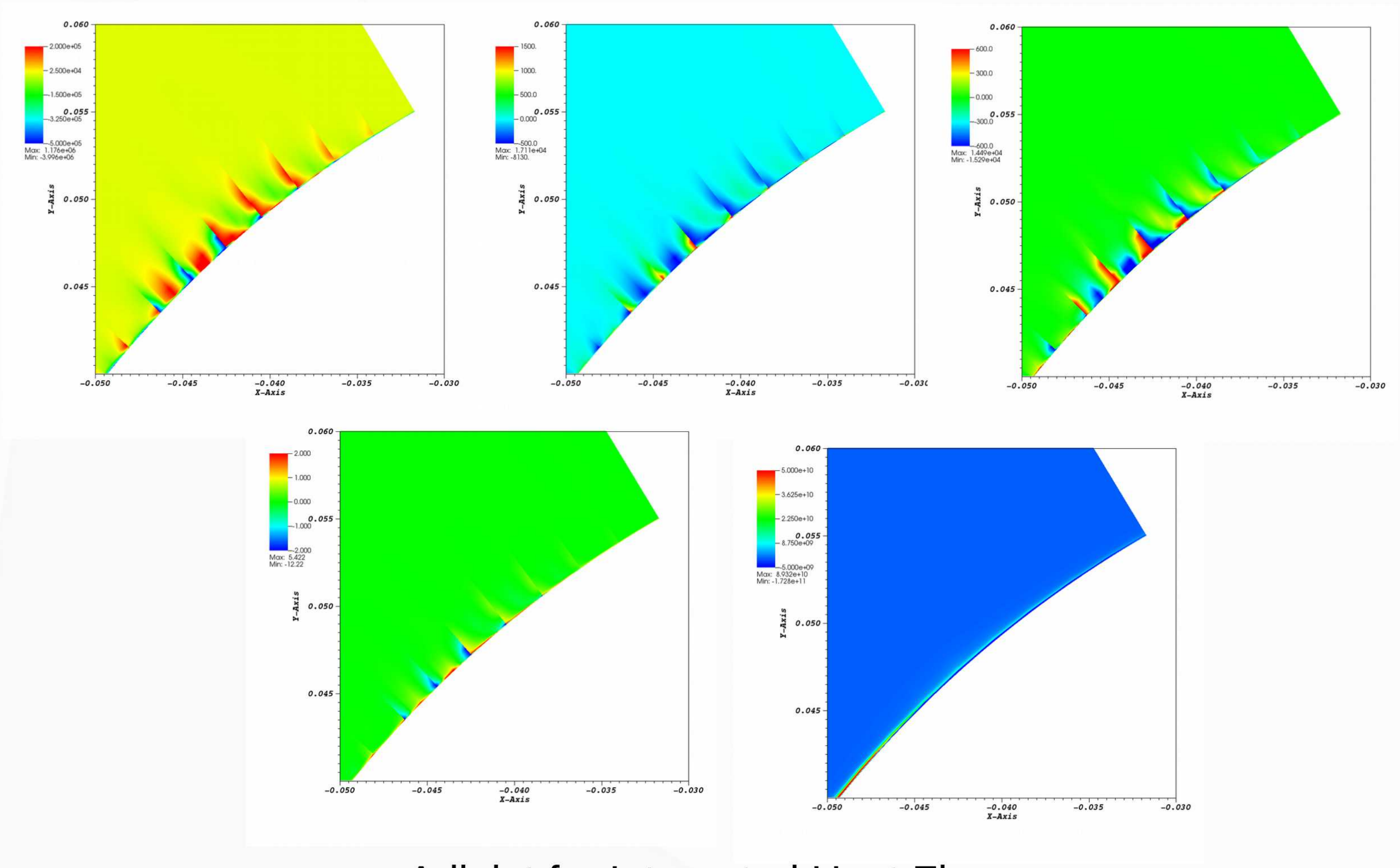


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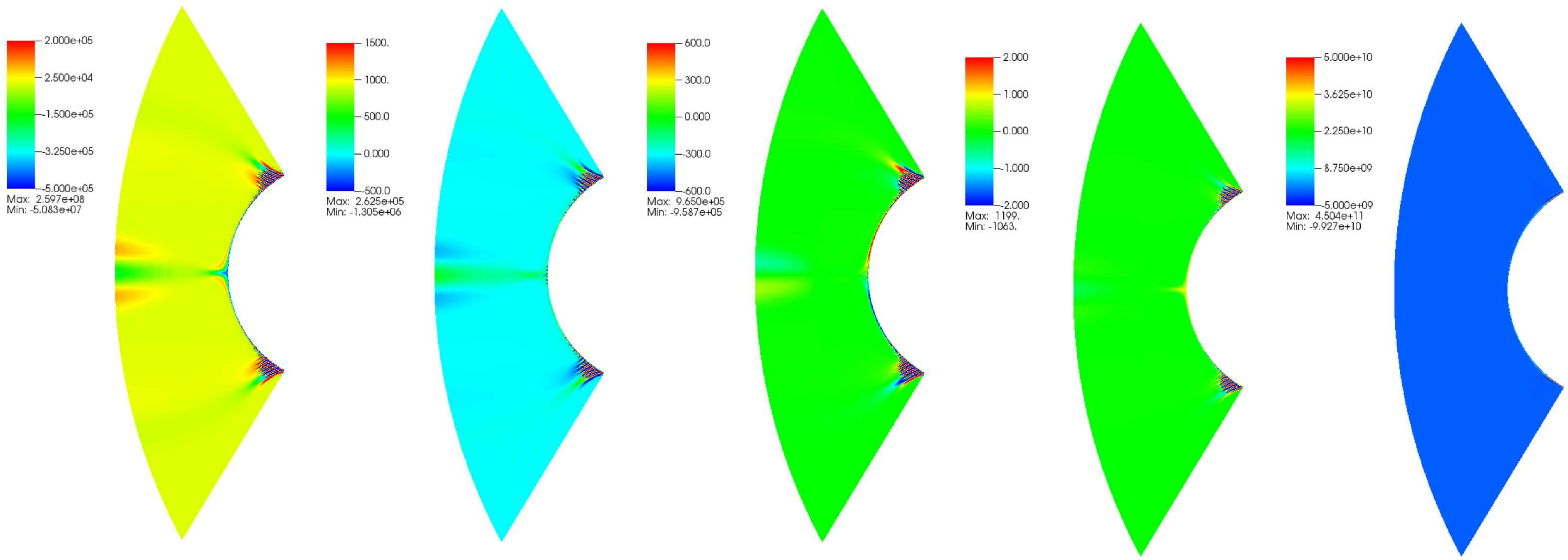
# P3 Adjoint
# 2,500 Q3 Quad Elements

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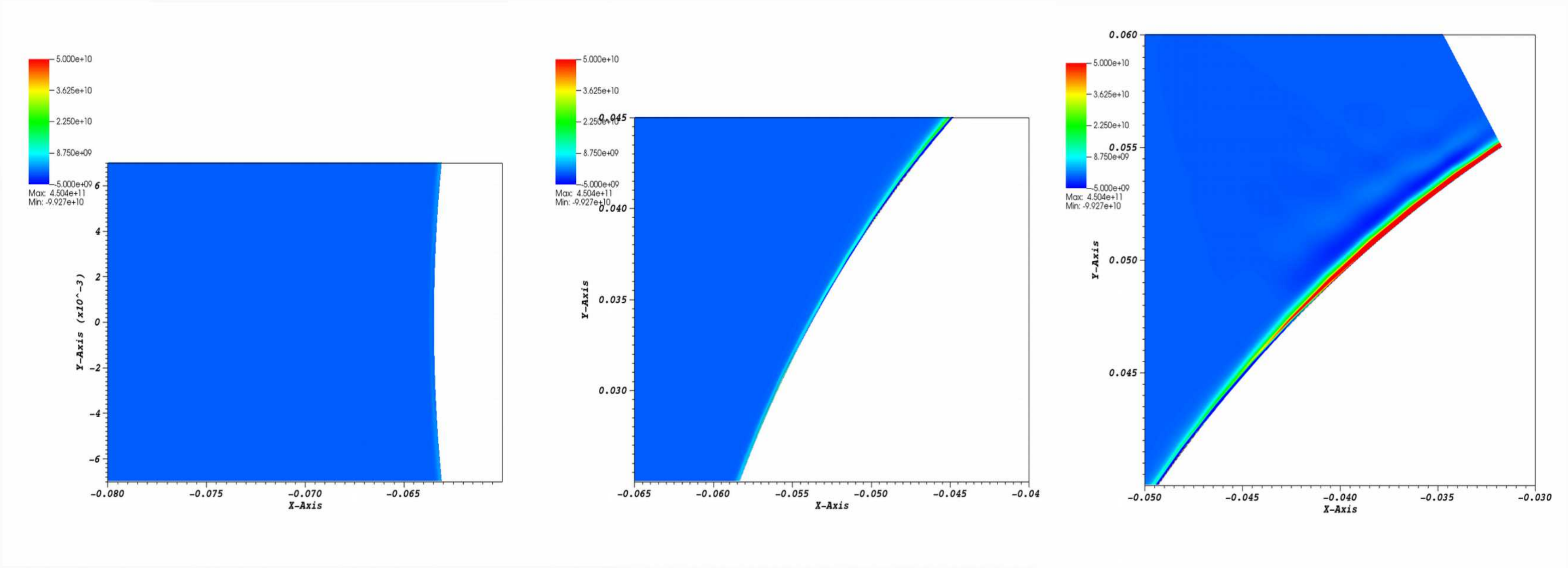
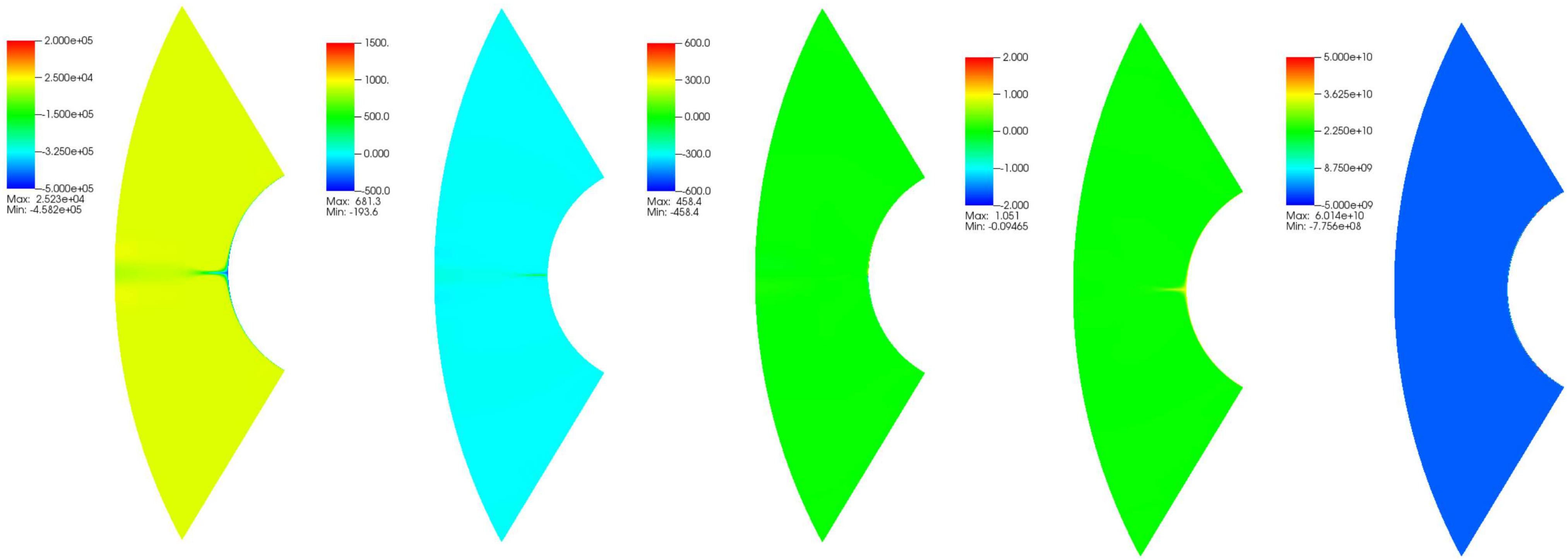
# P3 Adjoint
# 2,500 Q3 Quad Elements




Adjoint of Sensor Equation for Integrated Heat Flux, taken 2 locations around cylinder

# P1 Adjoint
# 10,000 Q3 Quad Elements

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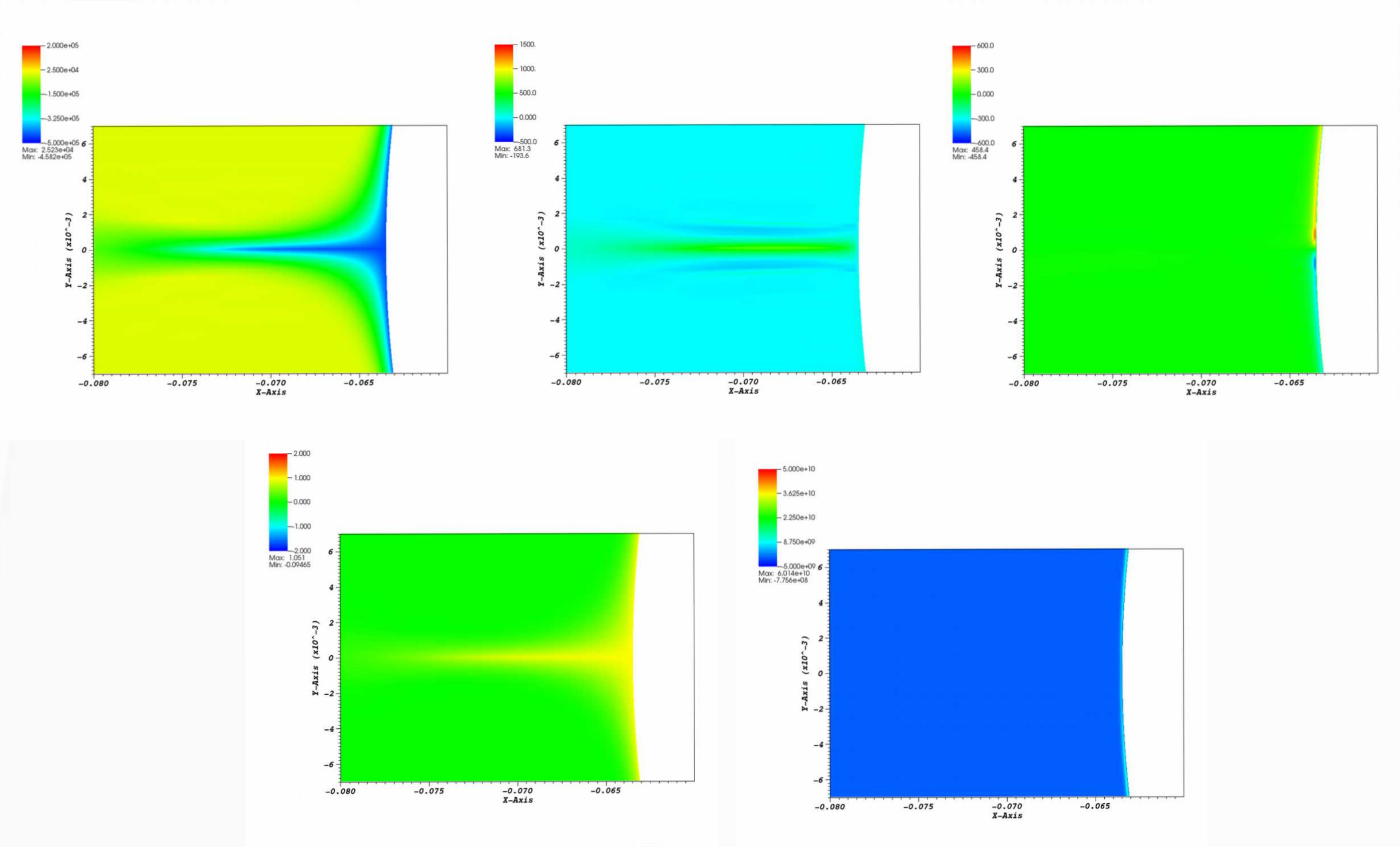
# P1 Adjoint
# 10,000 Q3 Quad Elements



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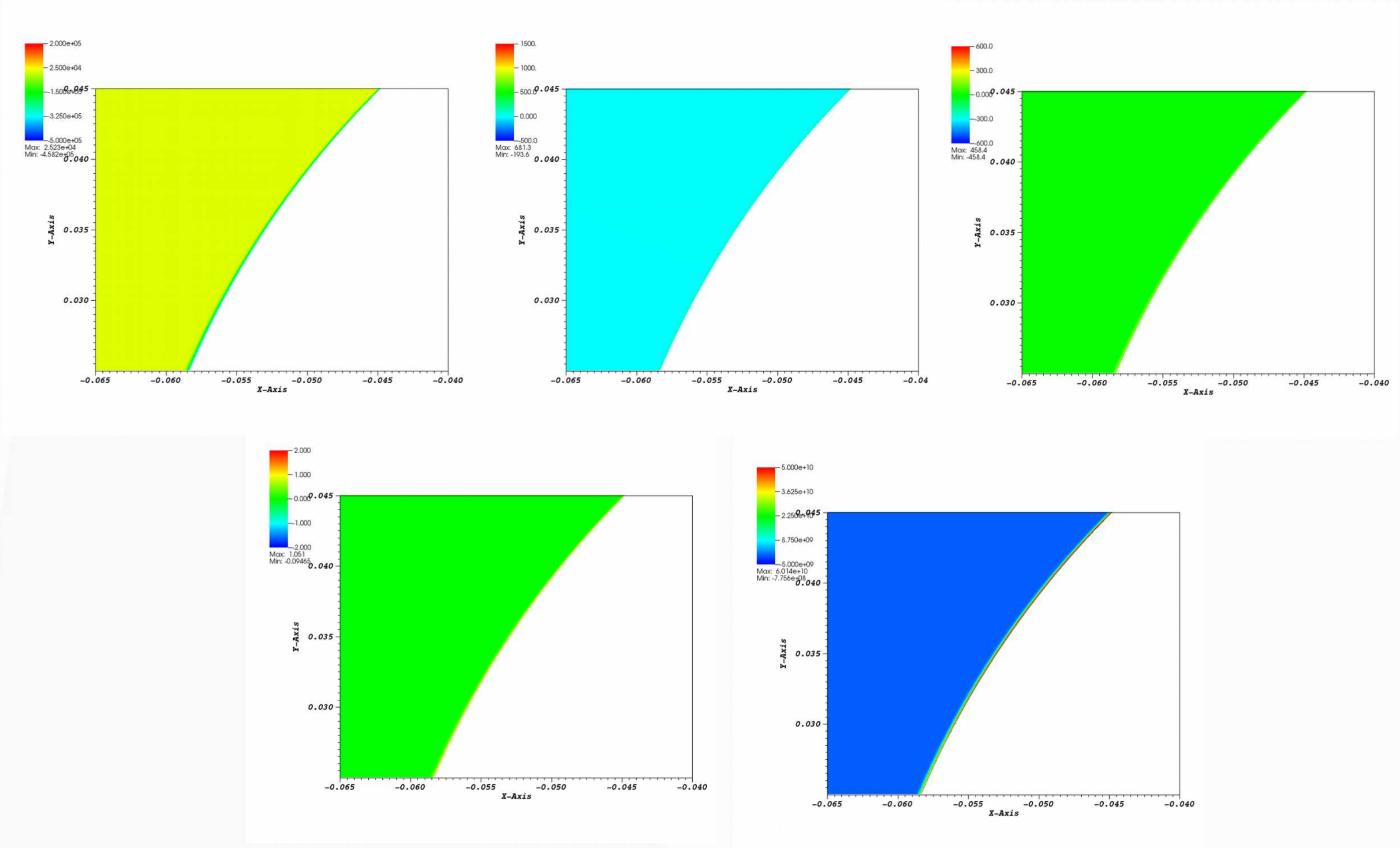


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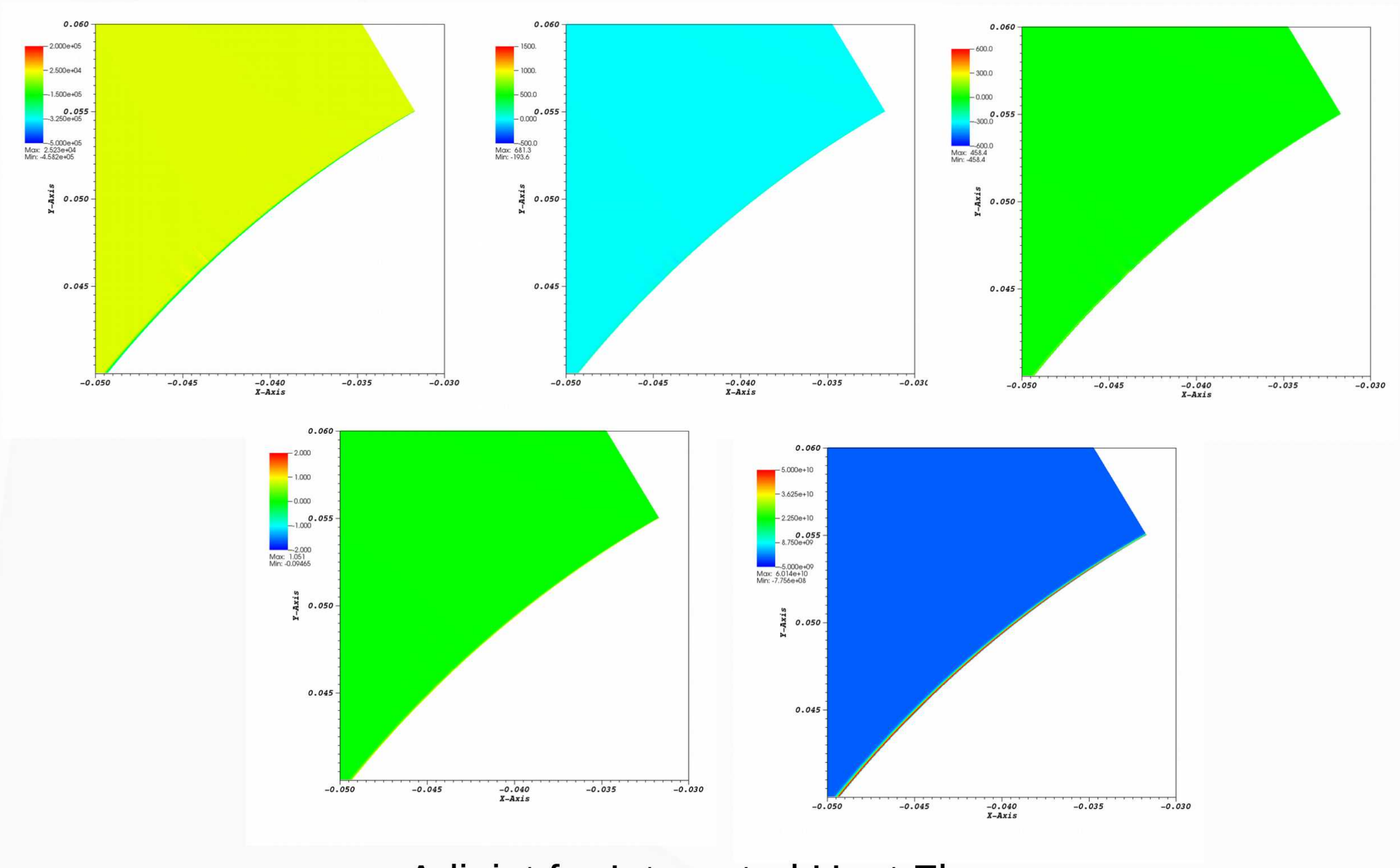


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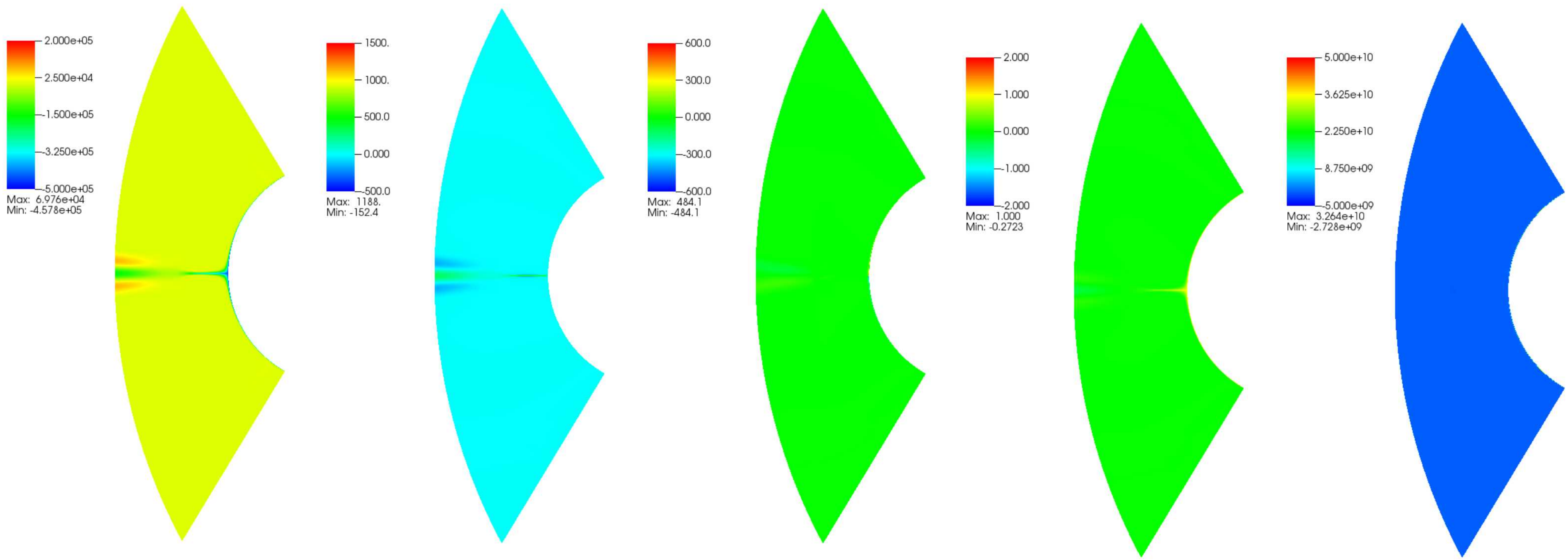
# P1 Adjoint
# 40,000 Q3 Quad Elements

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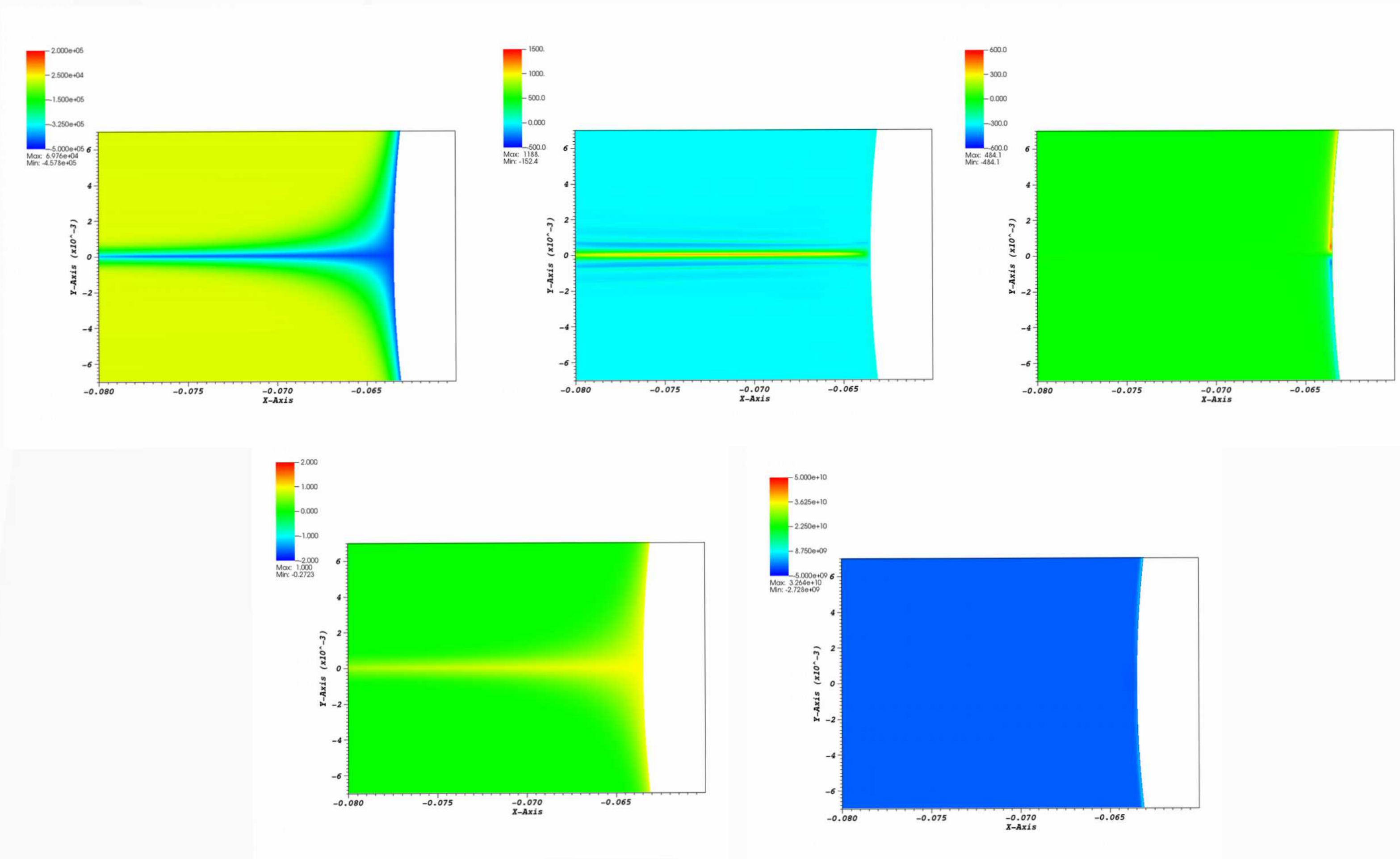


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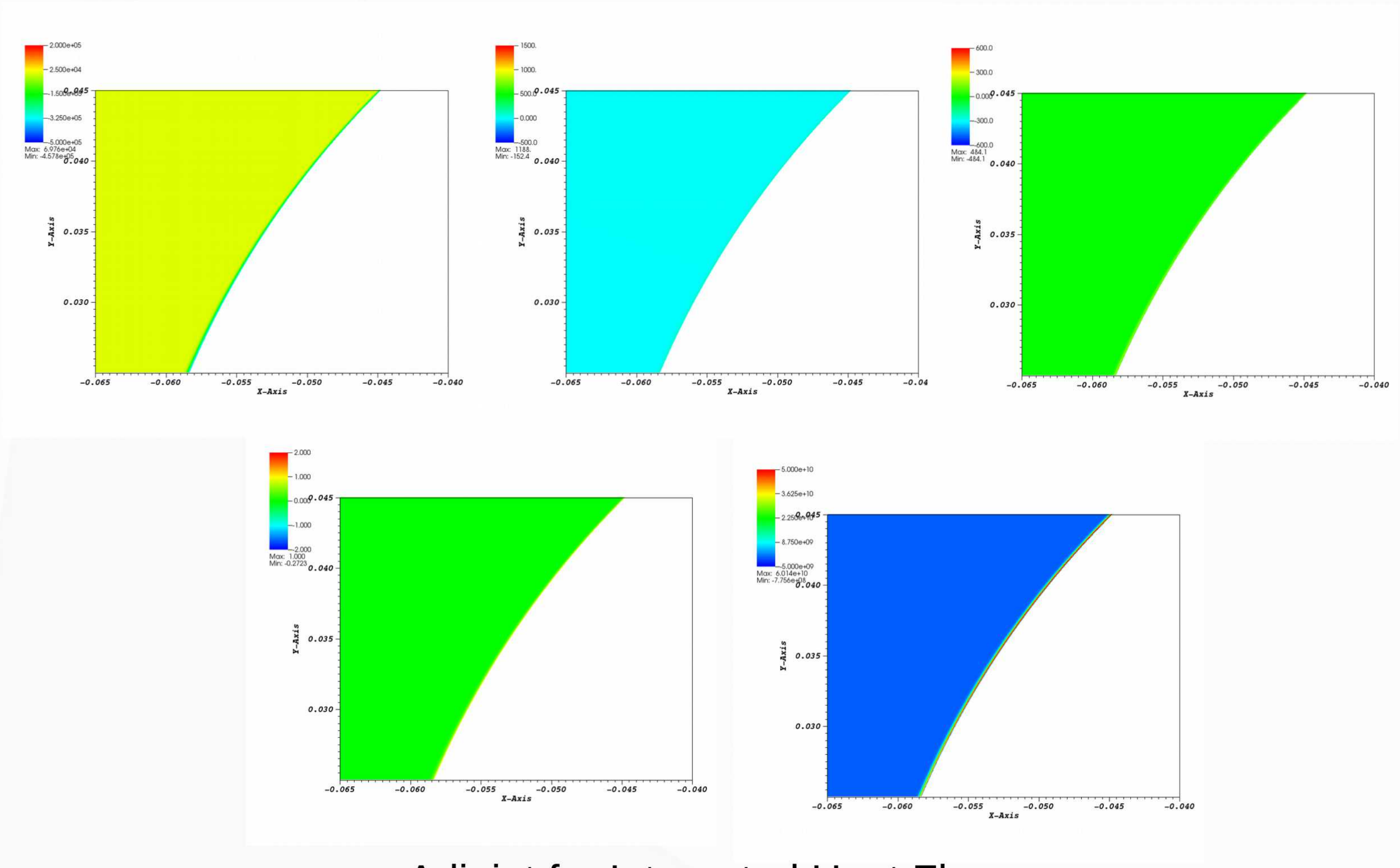


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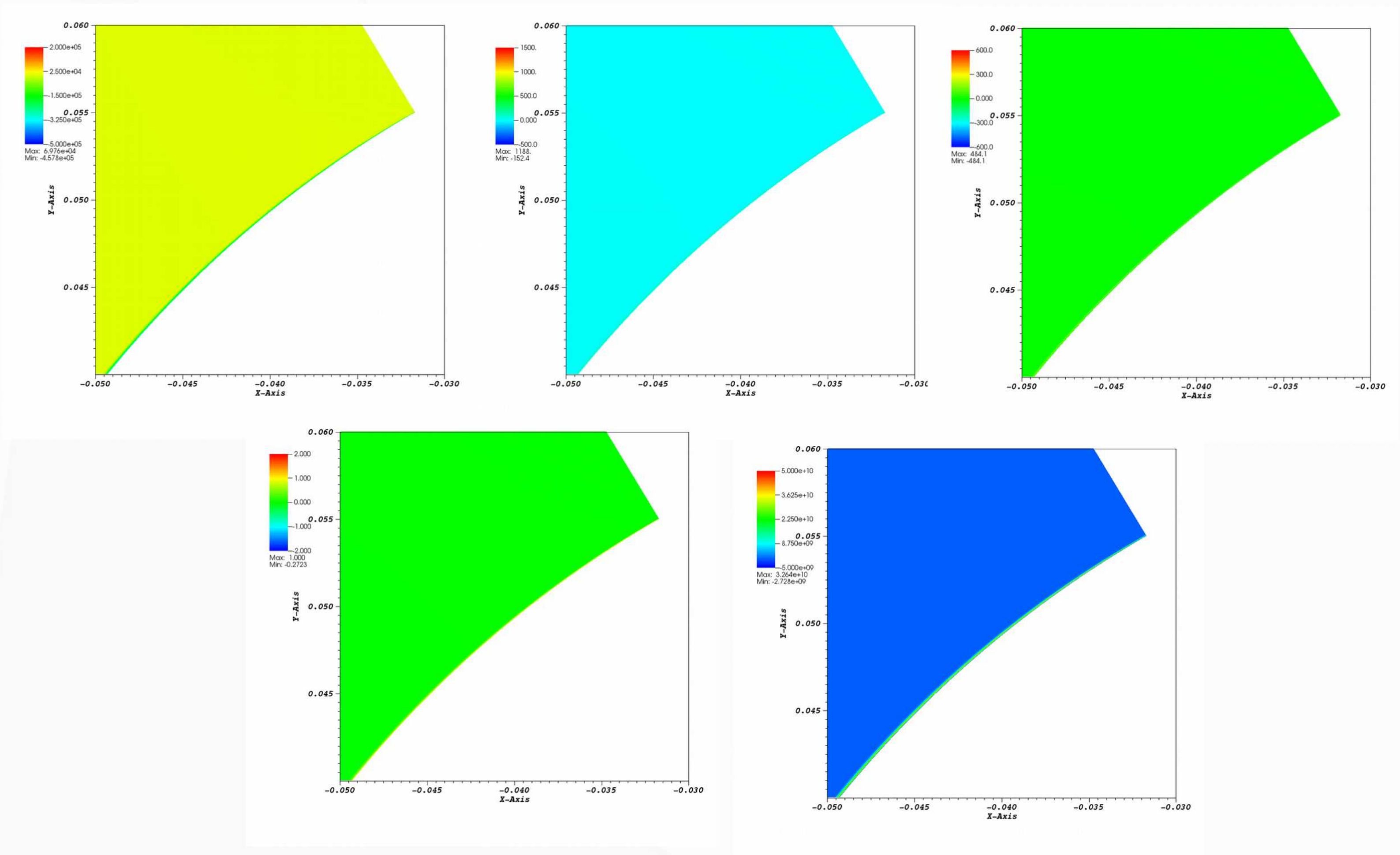

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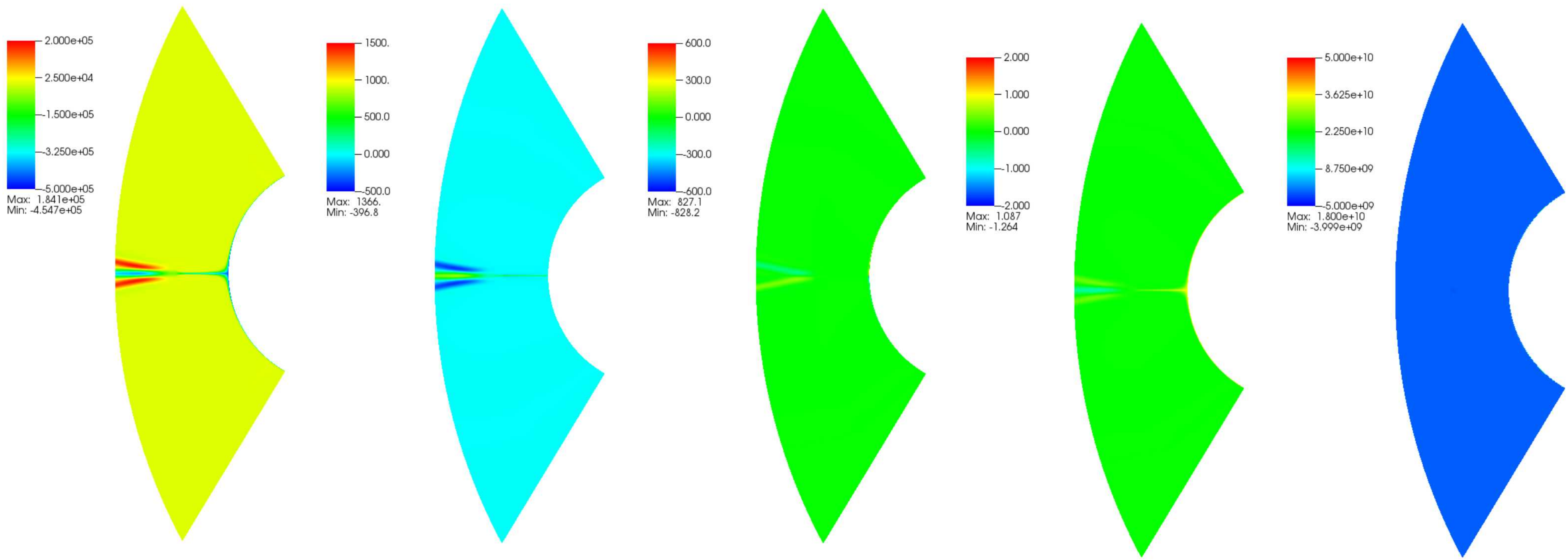
# 160,000 Q3 Quad Elements

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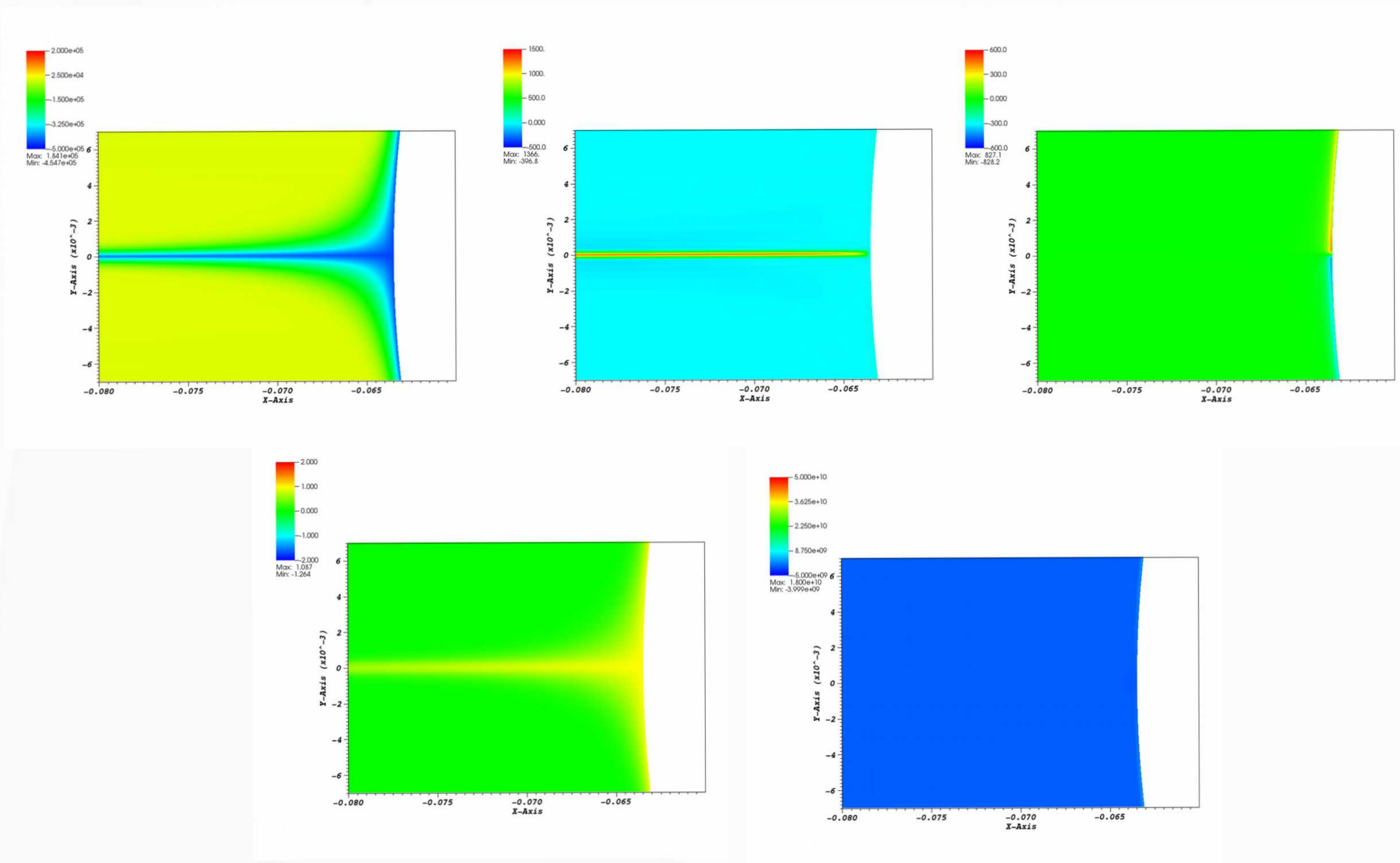
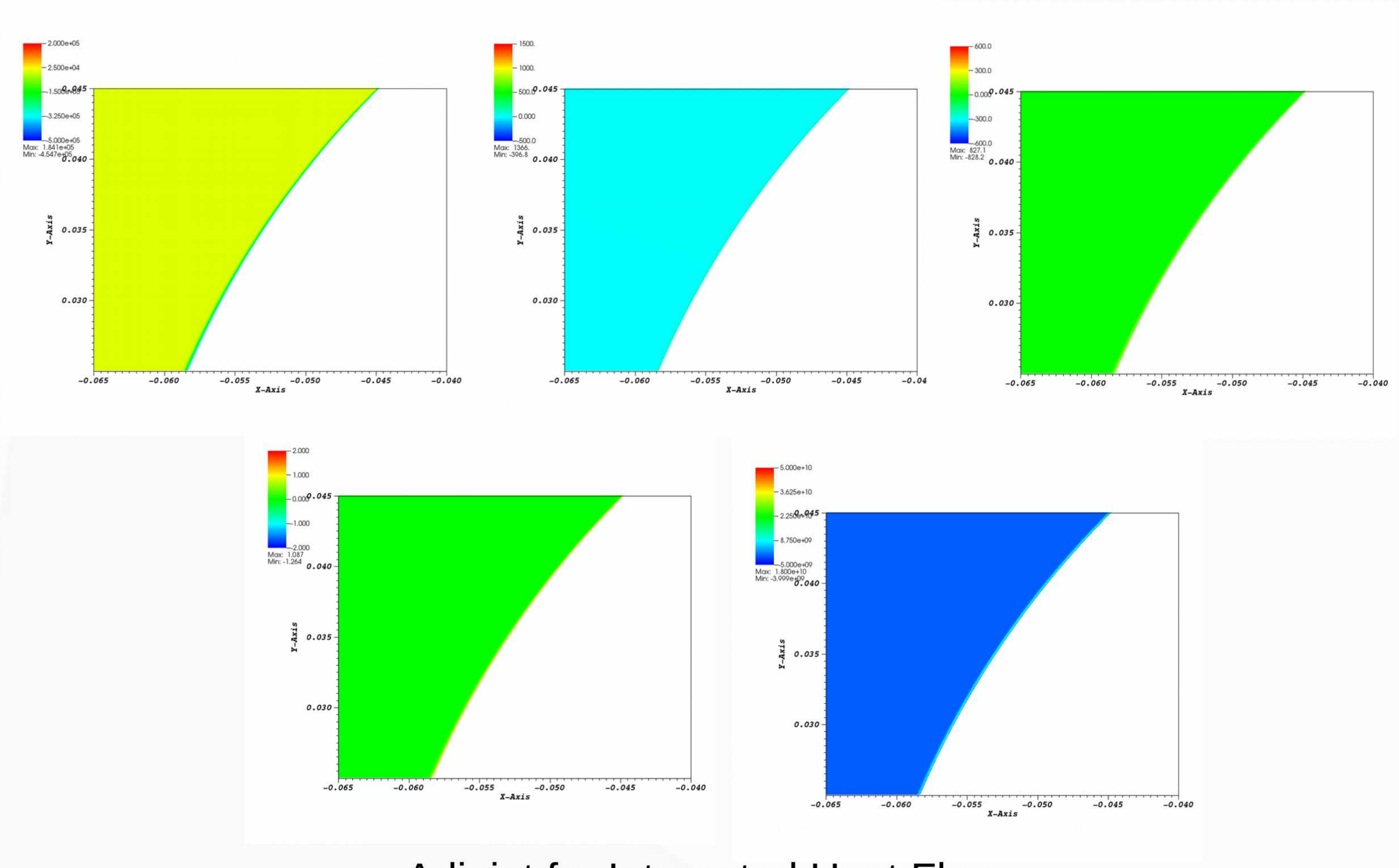
# P1 Adjoint
# 160,000 Q3 Quad Elements



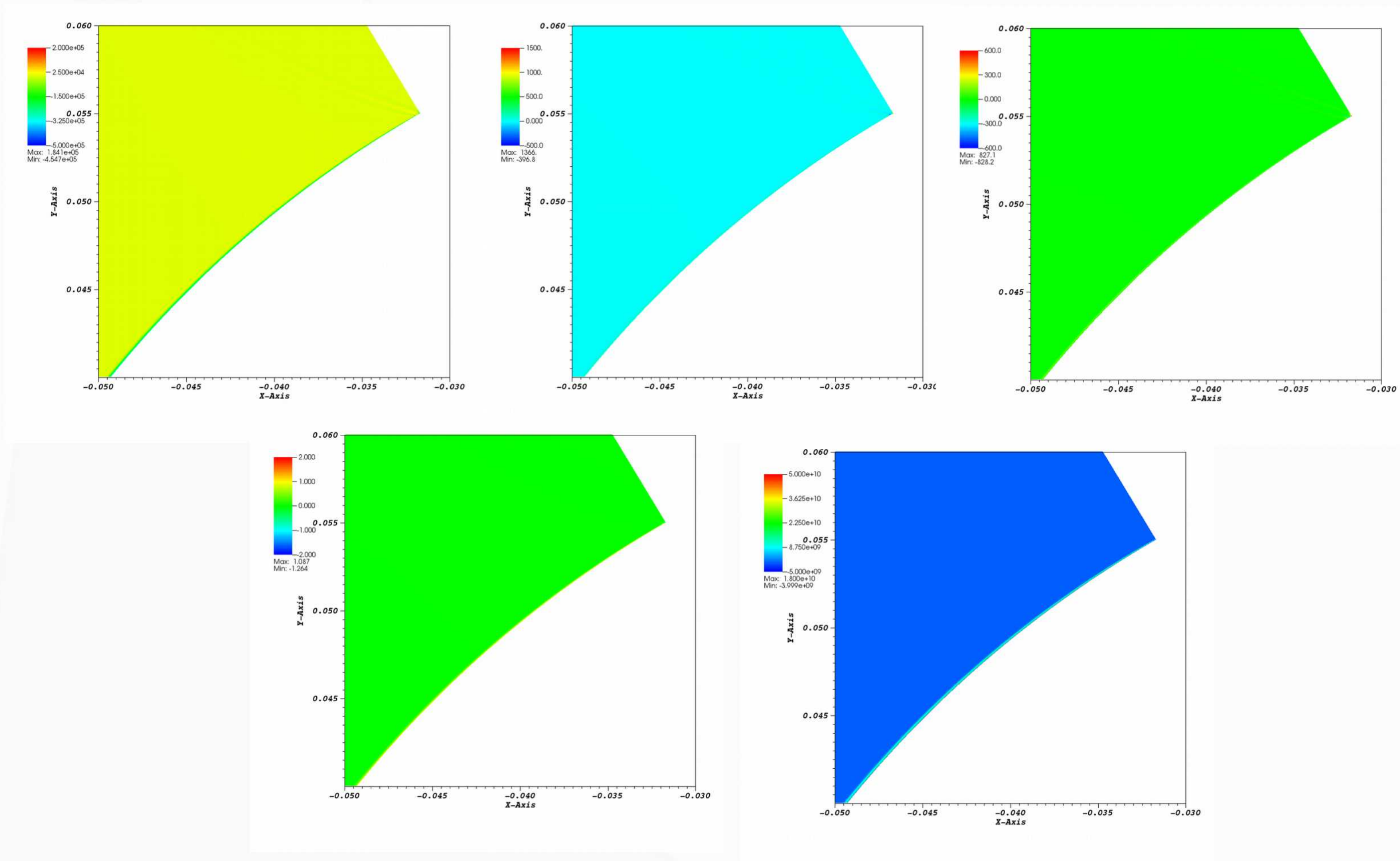

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 160,000 Q3 Quad Elements



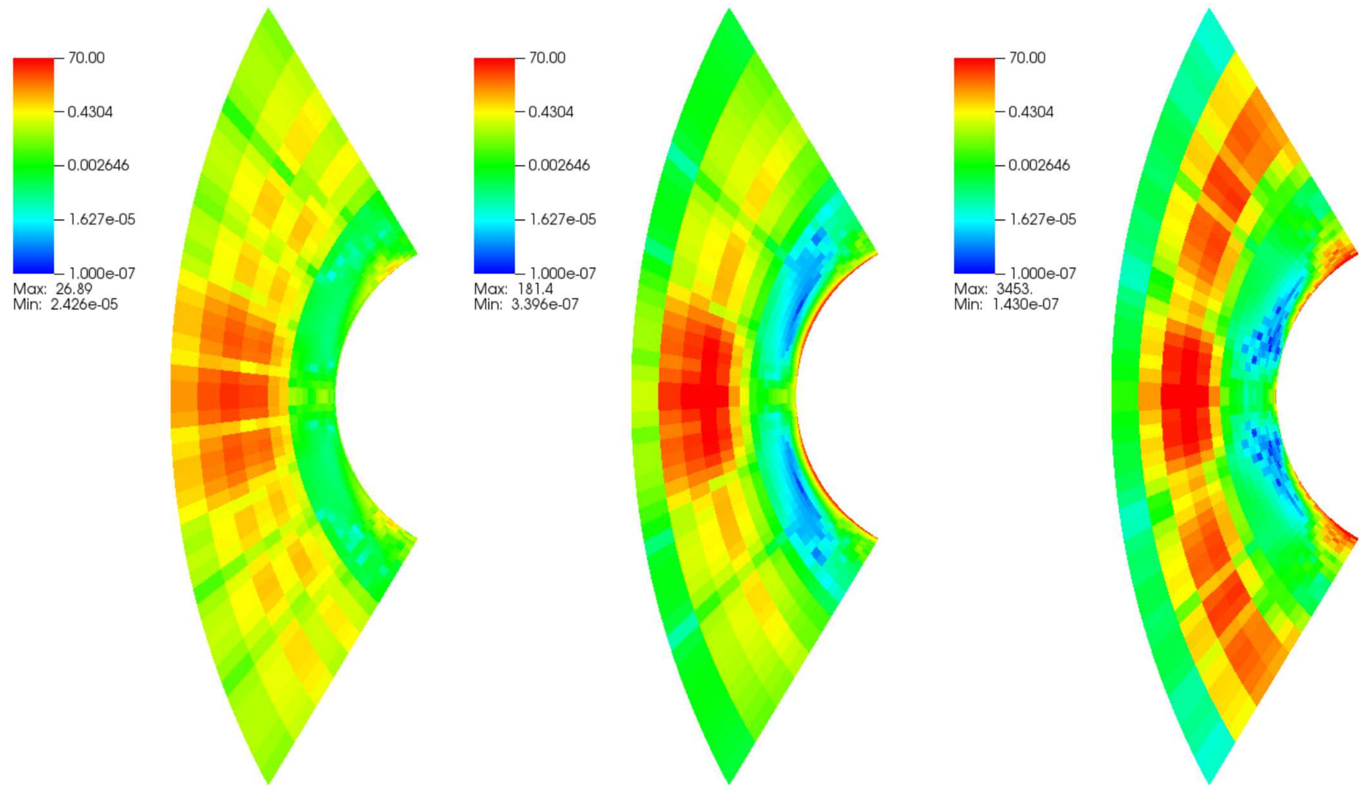
Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 160,000 Q3 Quad Elements



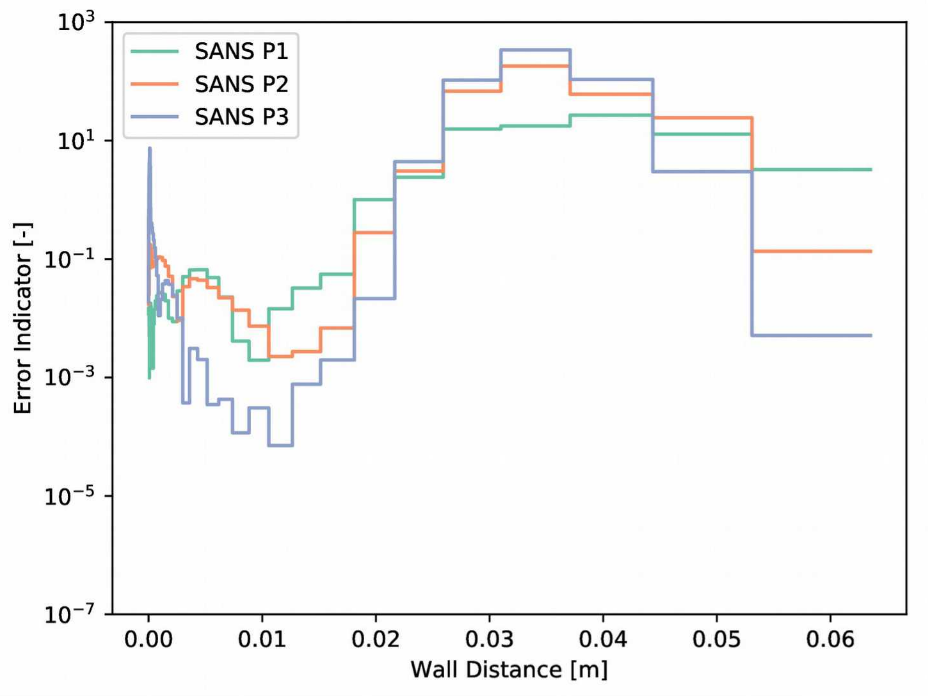

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

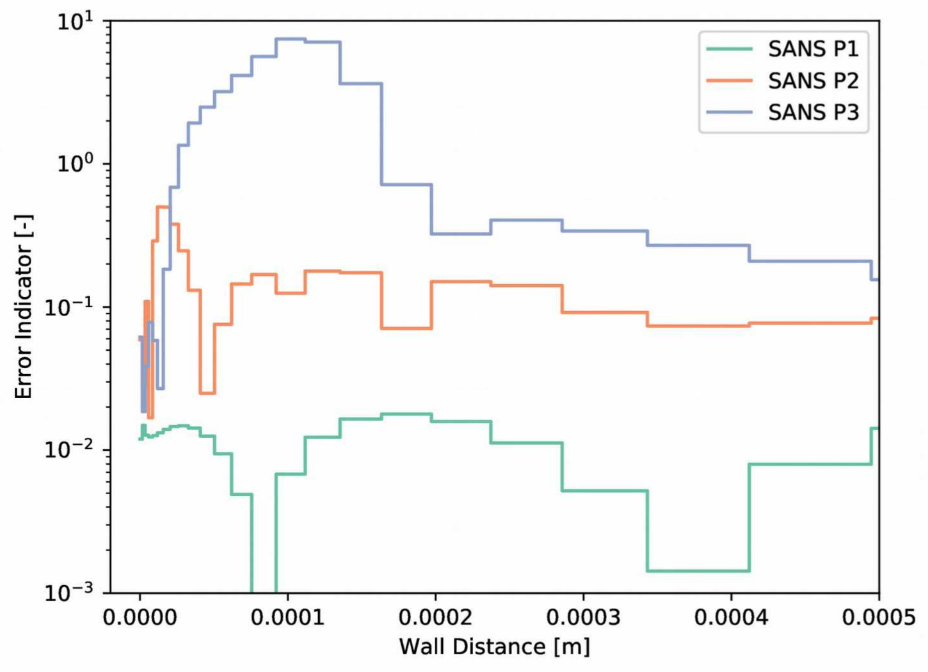
# Error Indicator
# 2,500 Q3 Quad Elements




Error Indicator for Integrated Heat Flux

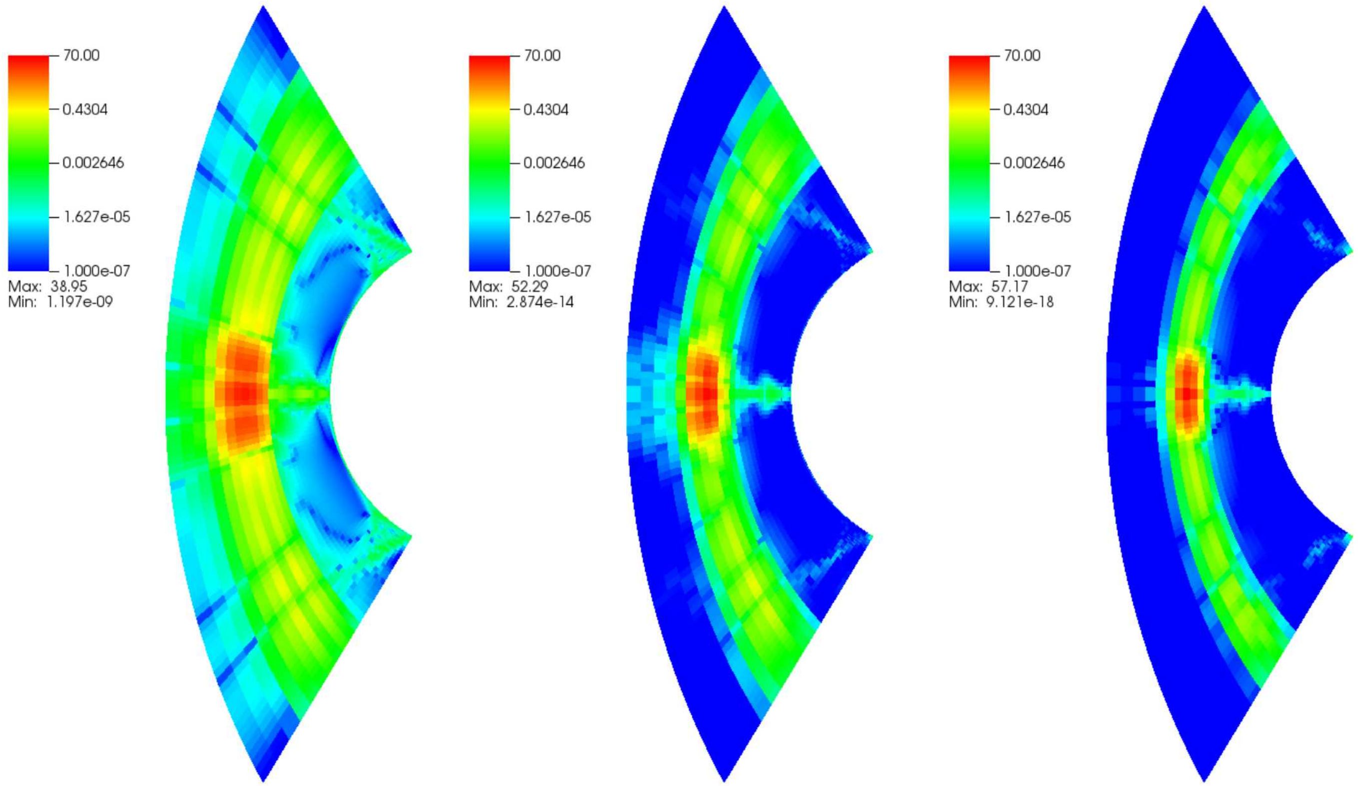
# Error Indicator on Stagnation Line 2,500 Q1 Quad Elements



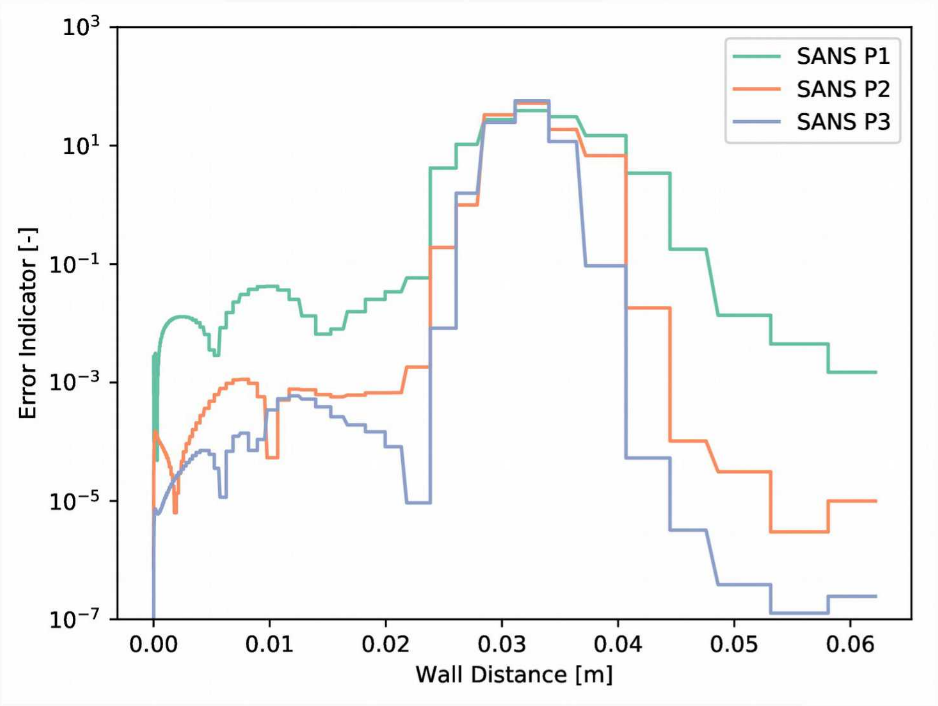

Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

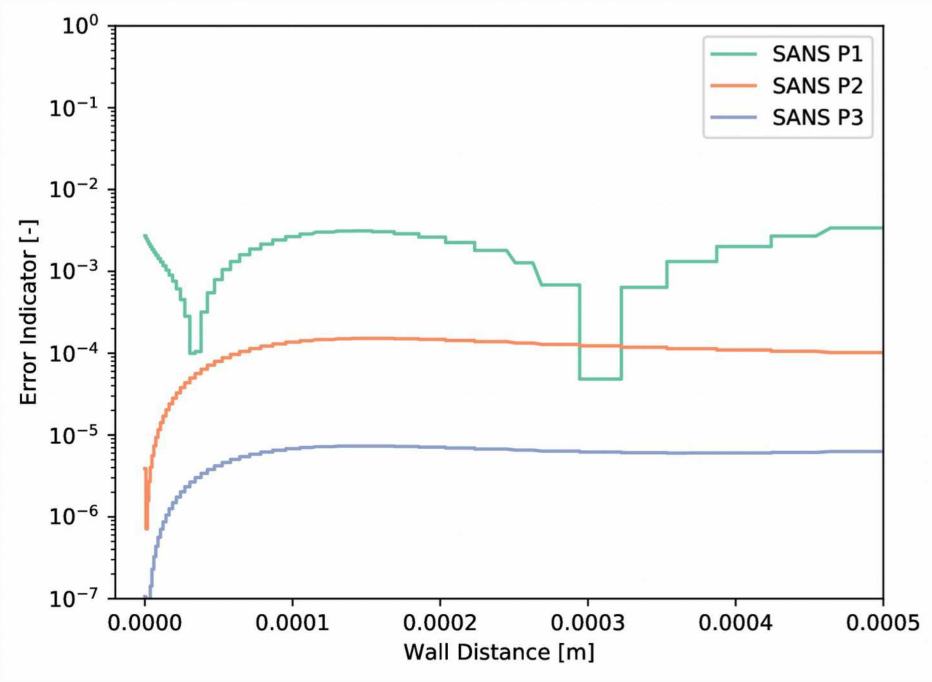
# Error Indicator
# 10,000 Q3 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1, P2, P3

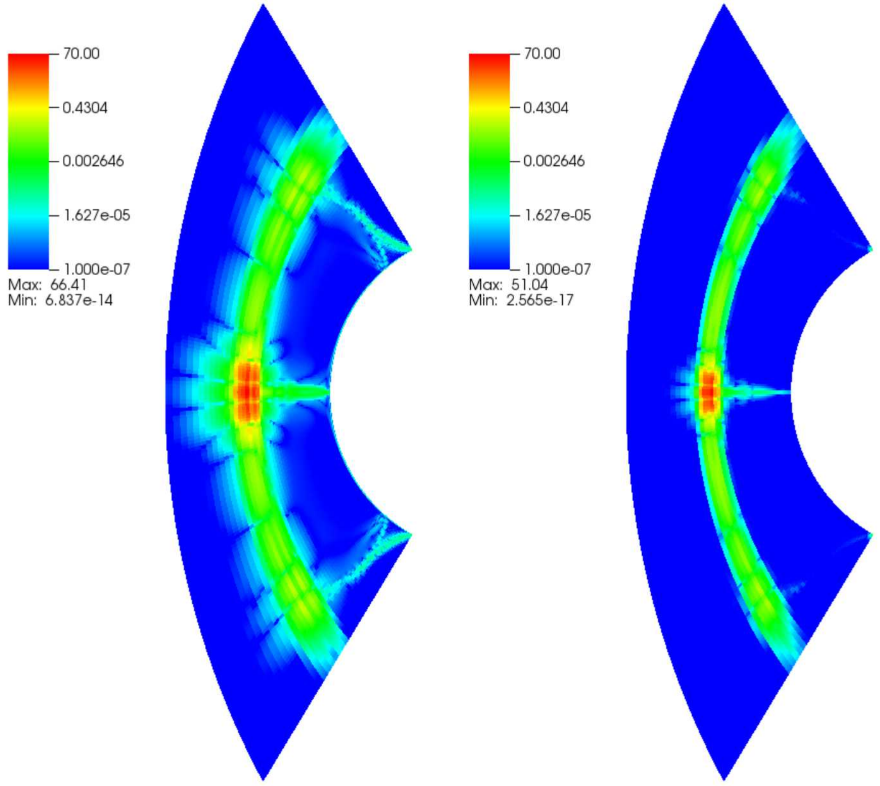
# Error Indicator on Stagnation Line
# 10,000 Q1 Quad Elements



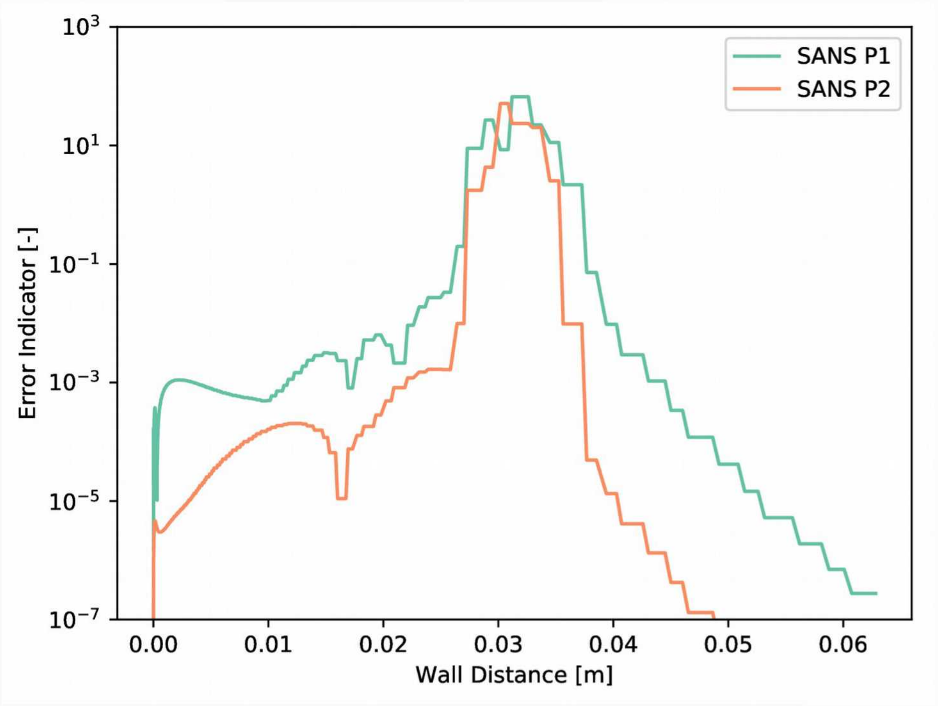

Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

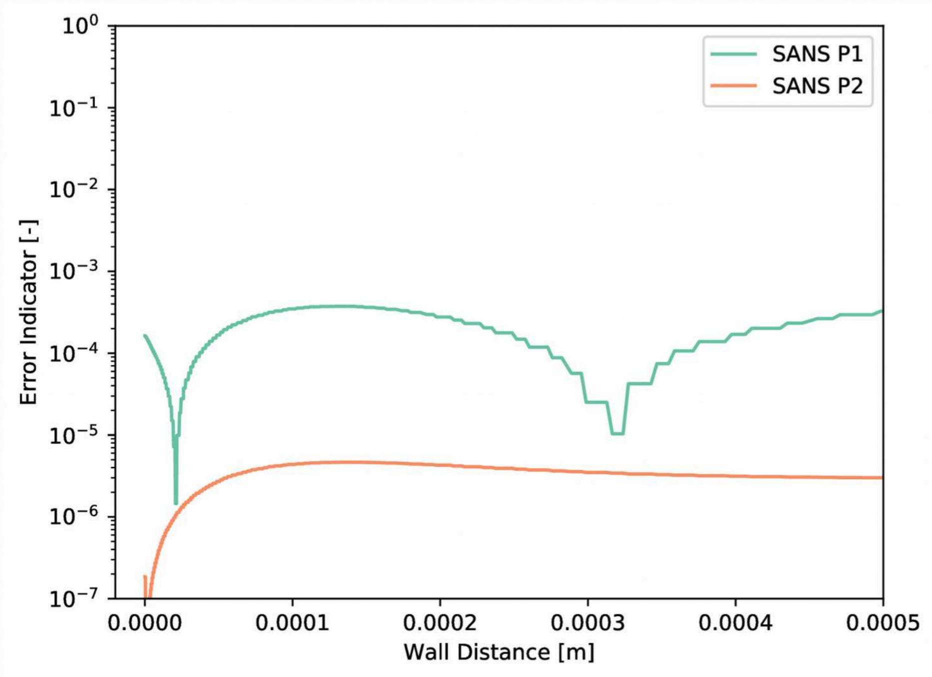
# Error Indicator
# 40,000 Q3 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1, P2

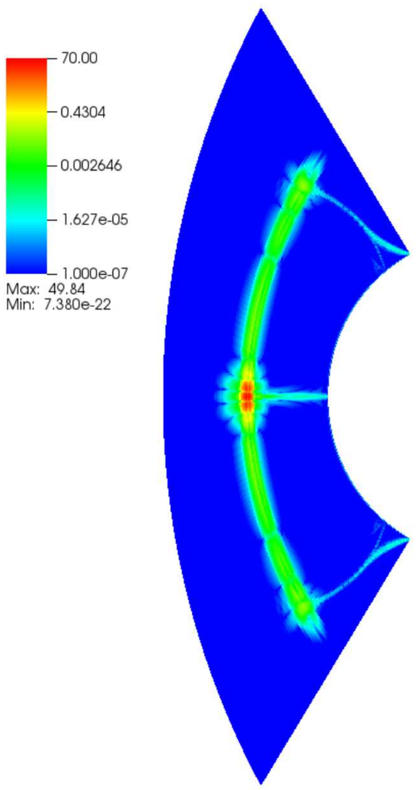
# Error Indicator on Stagnation Line 40,000 Q1 Quad Elements



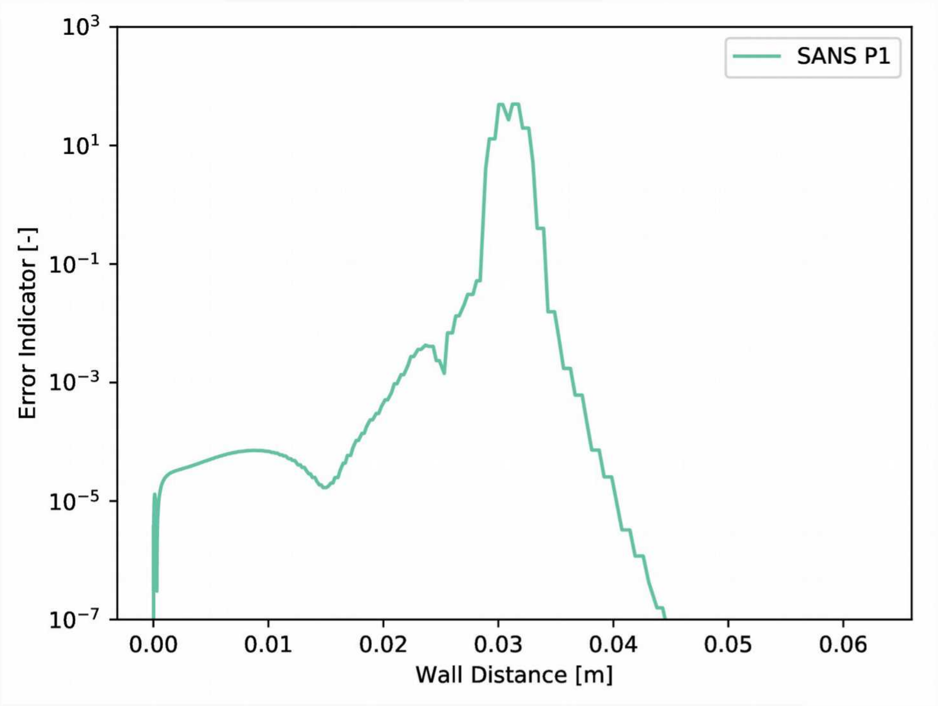

Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

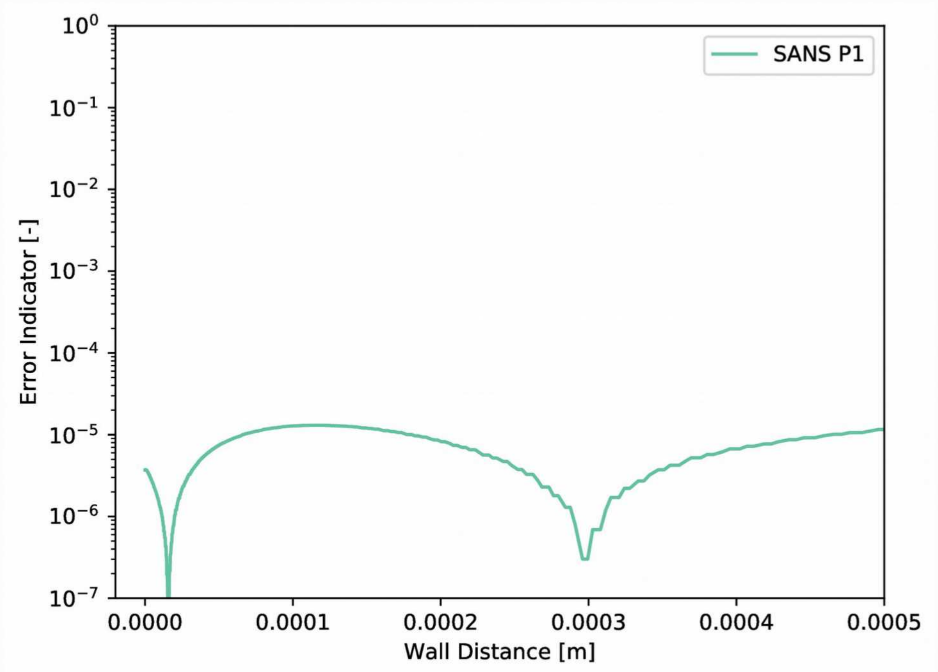
# Error Indicator
# 160,000 Q3 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1

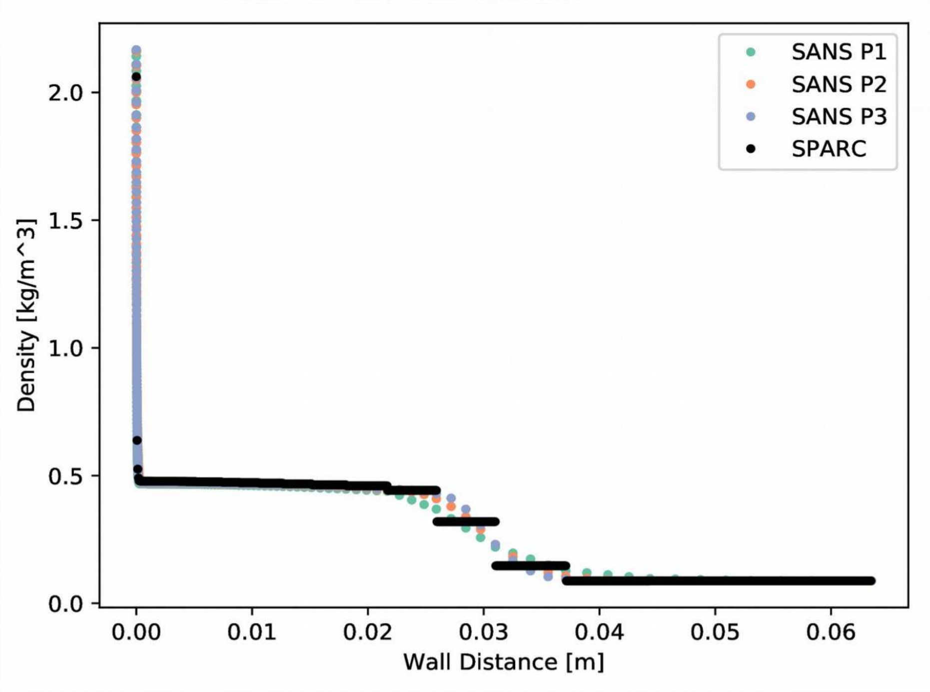
# Error Indicator on Stagnation Line 160,000 Q1 Quad Elements

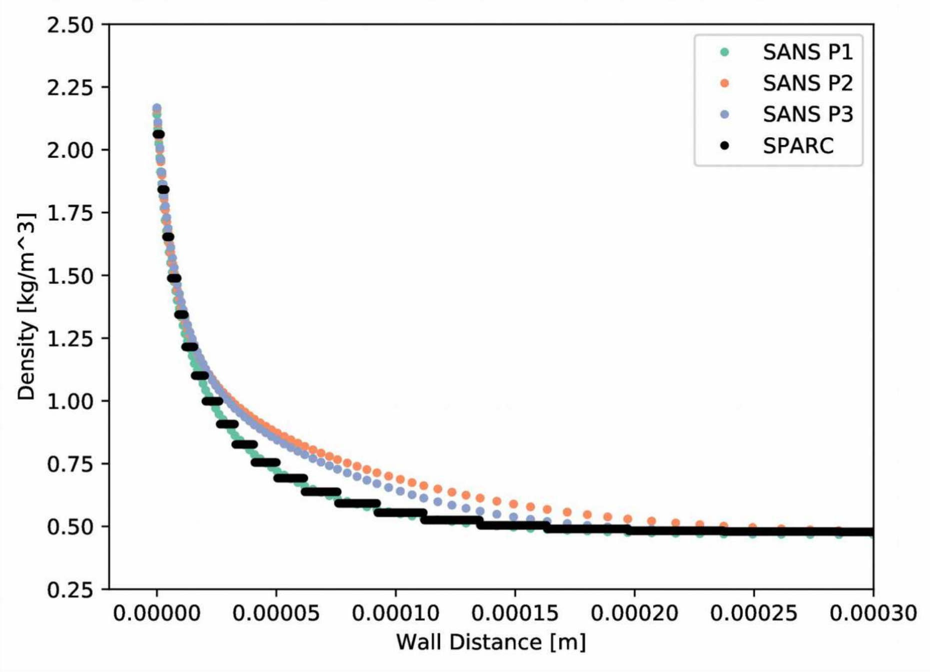



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

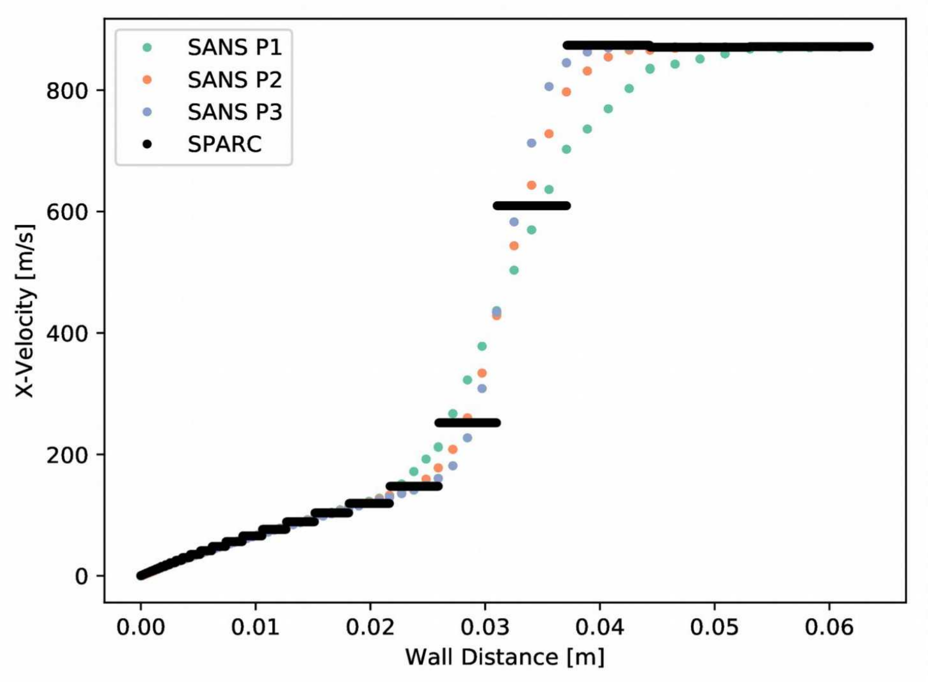
# Stagnation Streamline
# 2,500 Q3 Quad Elements

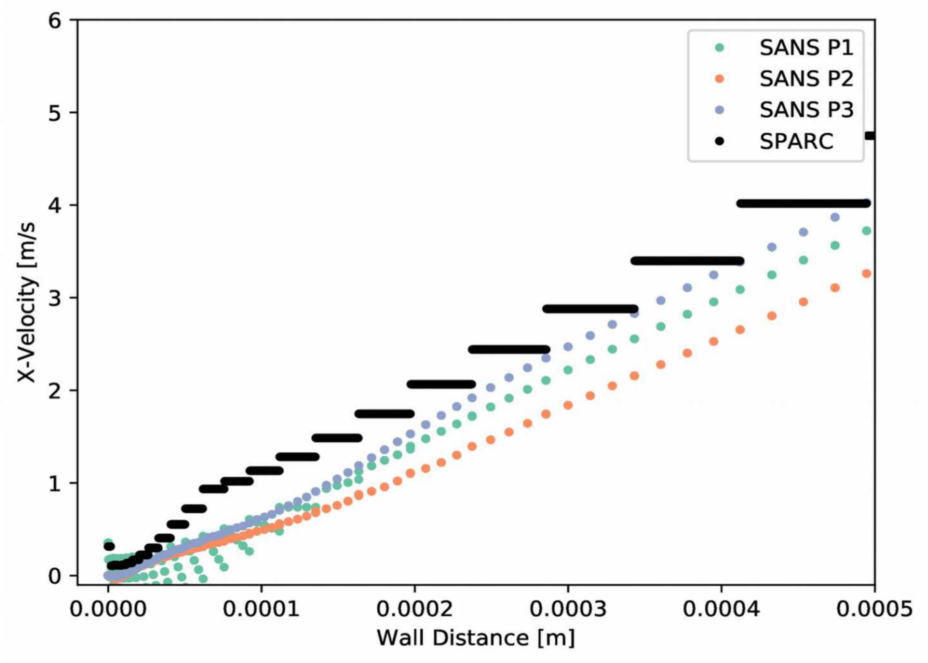



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

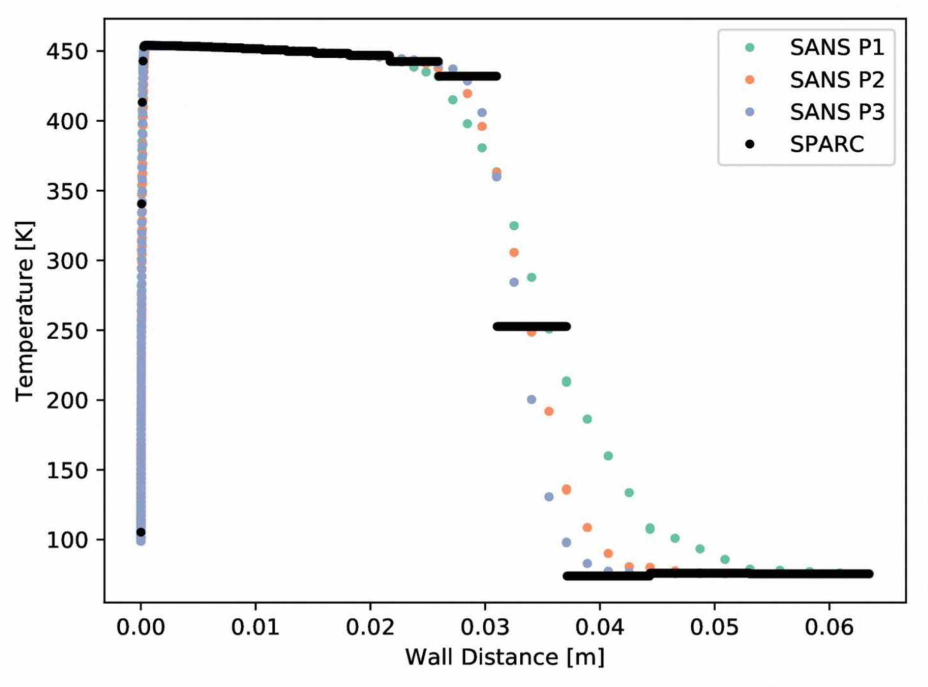
# Stagnation Streamline
# 2,500 Q3 Quad Elements

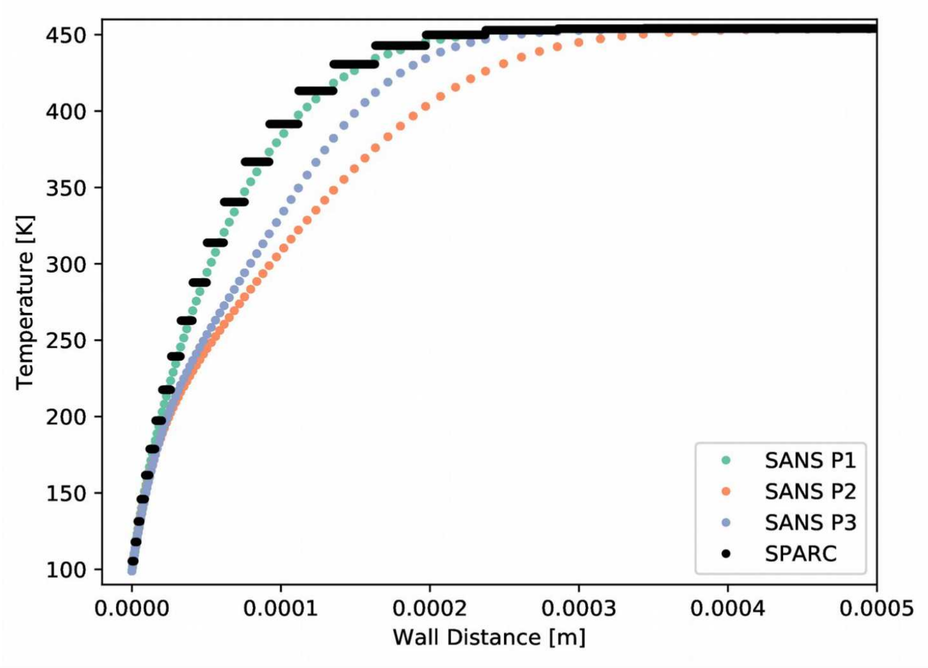



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

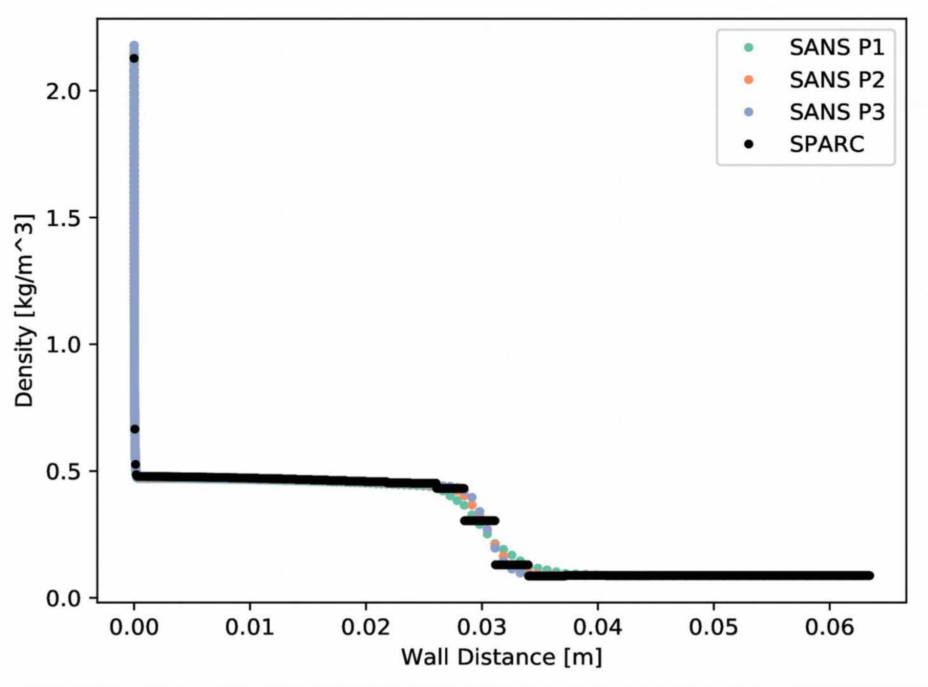
# Stagnation Streamline
# 2,500 Q3 Quad Elements

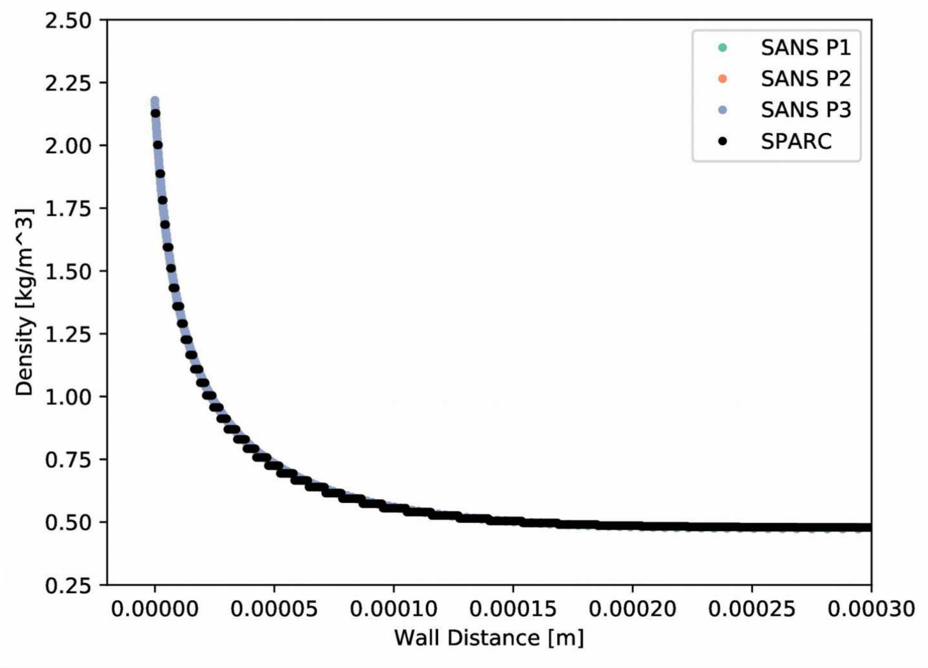



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

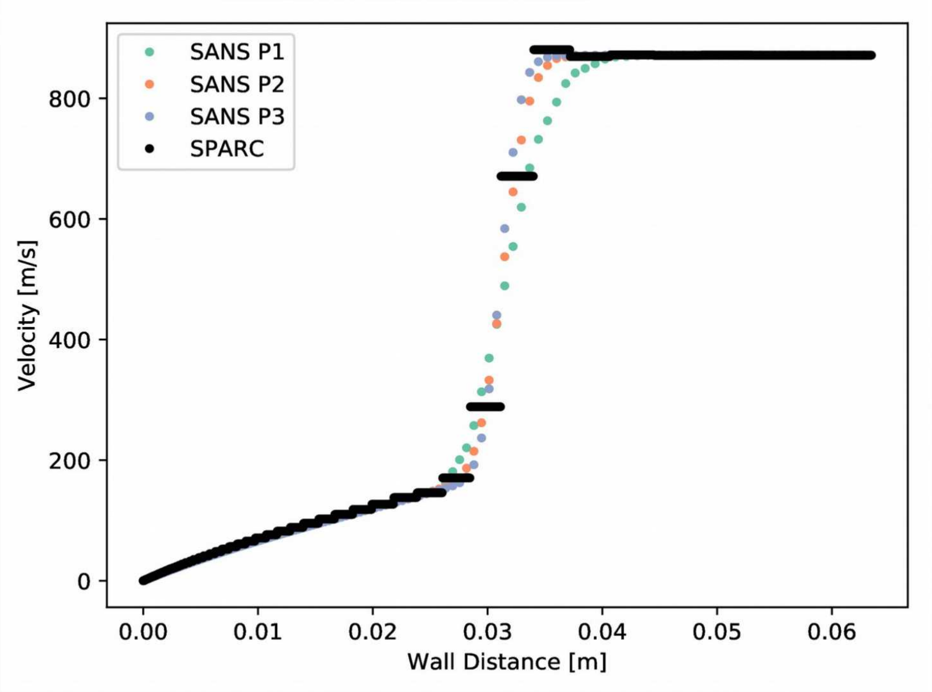
# Stagnation Streamline
# 10,000 Q3 Quad Elements

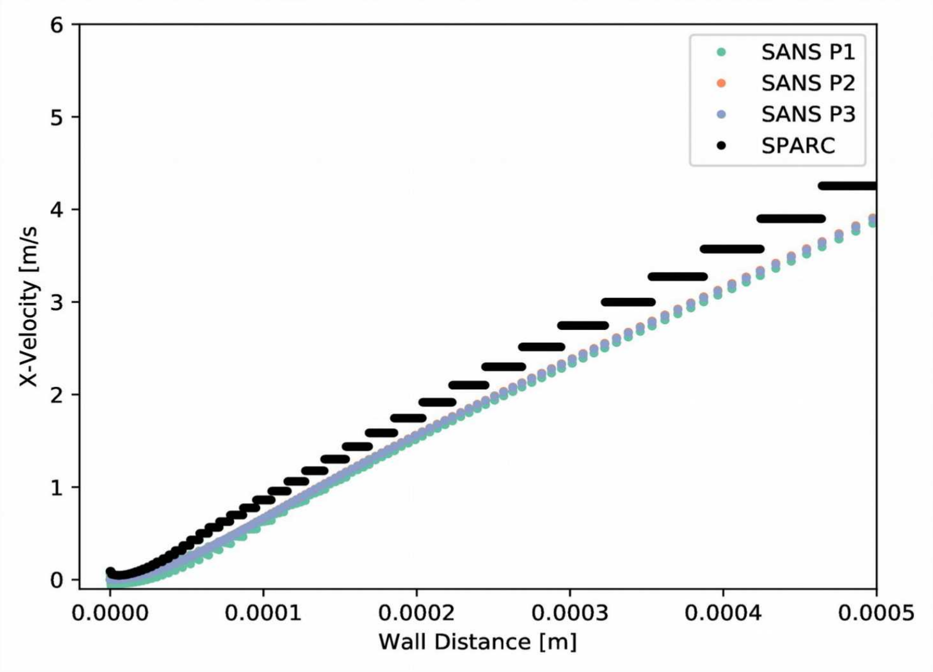



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

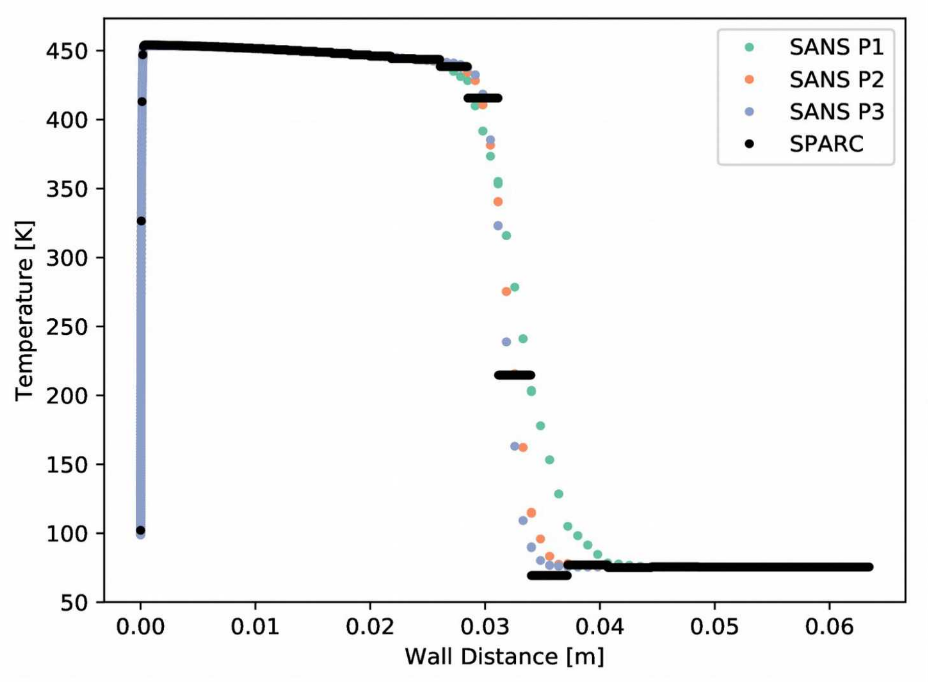
# Stagnation Streamline
# 10,000 Q3 Quad Elements

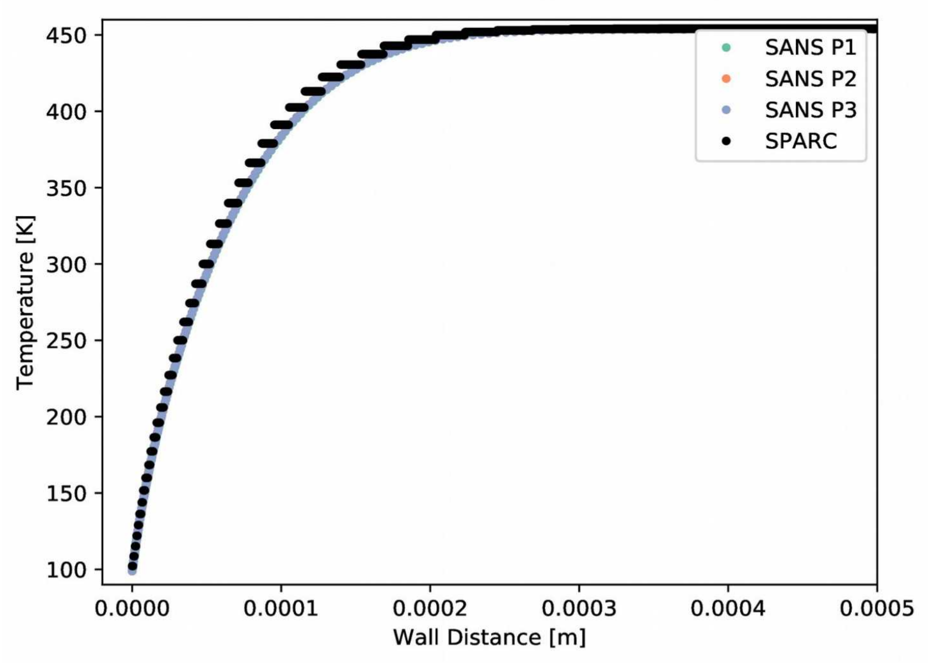



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

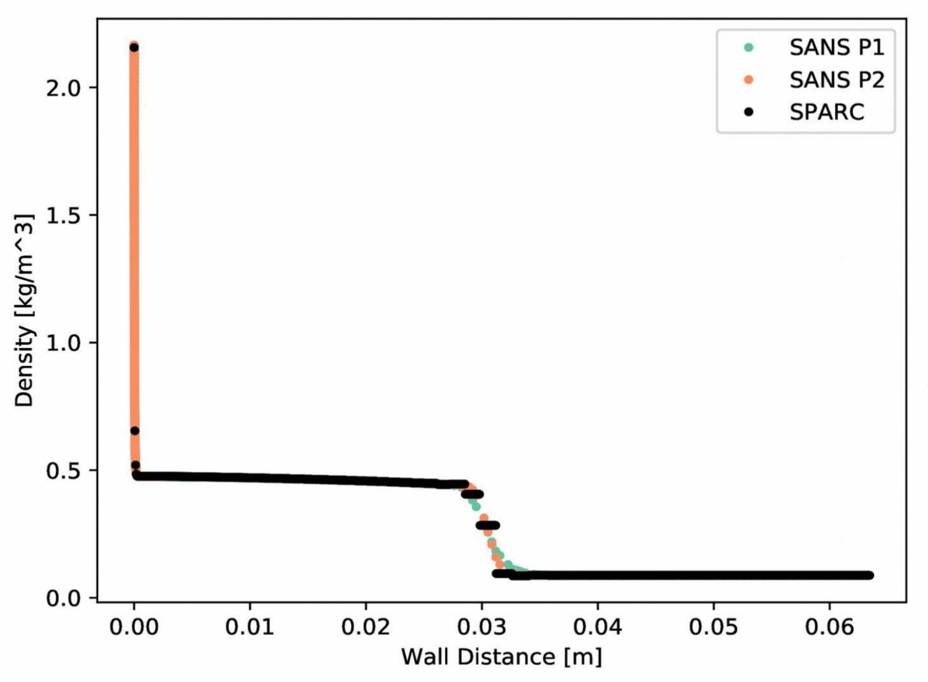
# Stagnation Streamline
# 10,000 Q3 Quad Elements

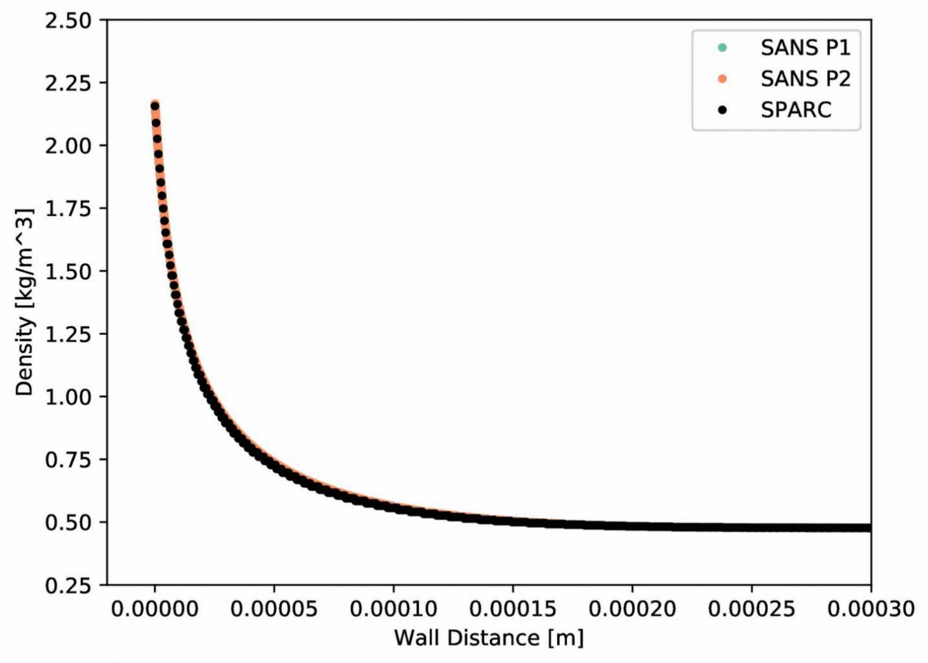



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

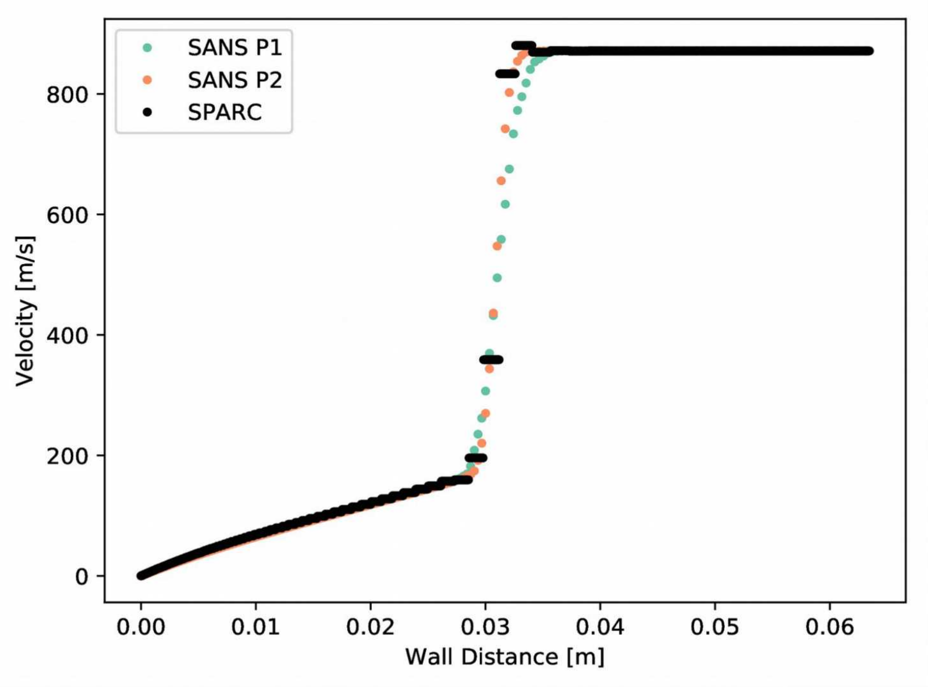
# Stagnation Streamline
# 40,000 Q3 Quad Elements

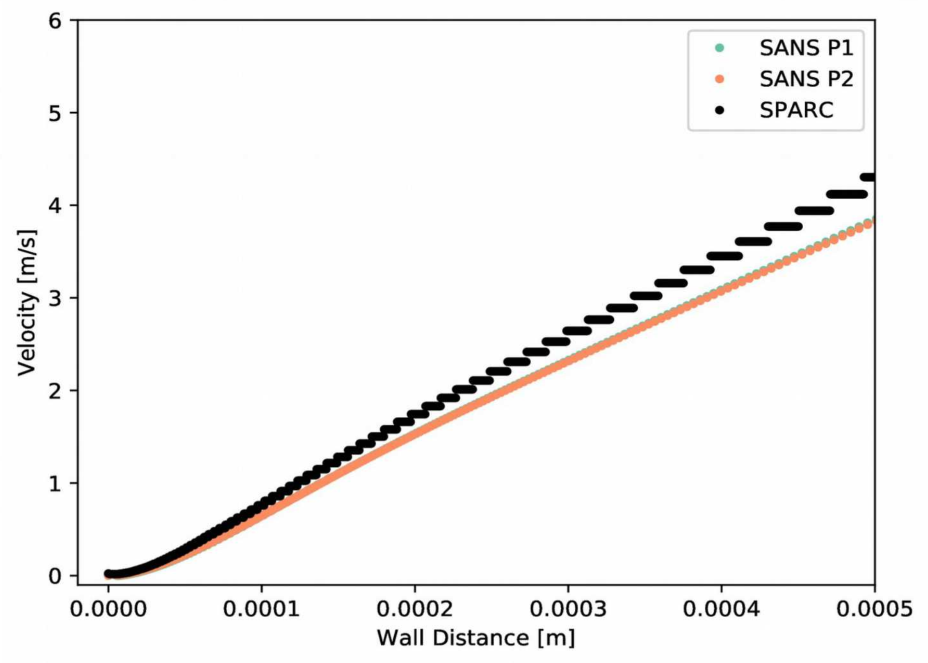



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

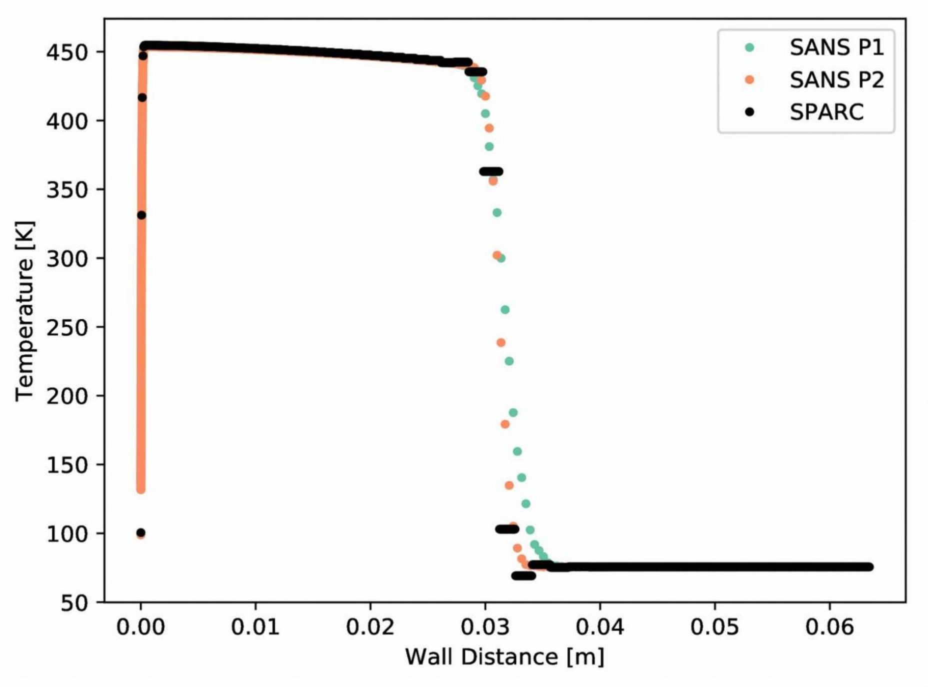
# Stagnation Streamline
# 40,000 Q3 Quad Elements

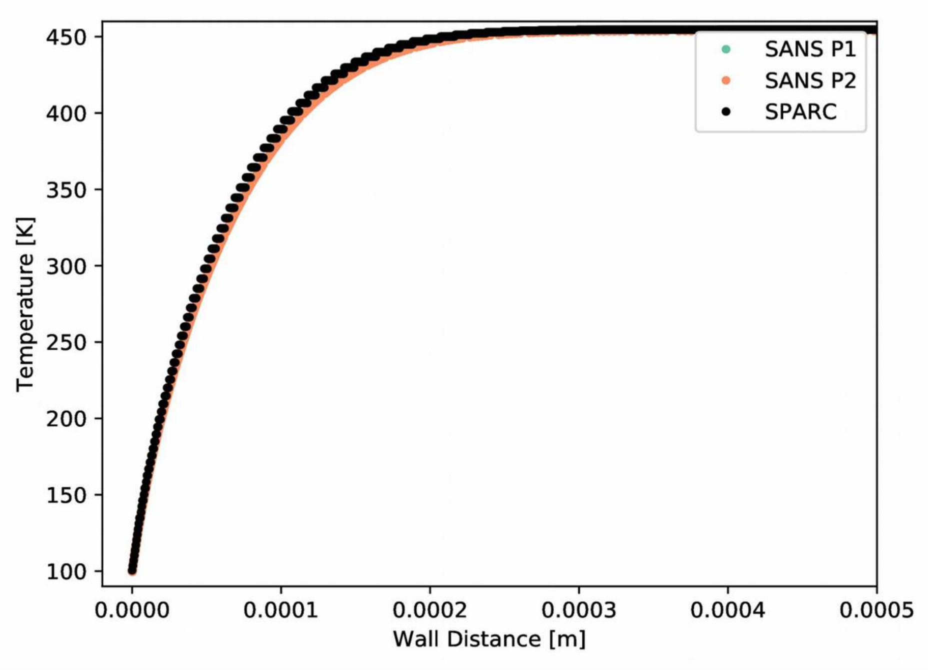



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

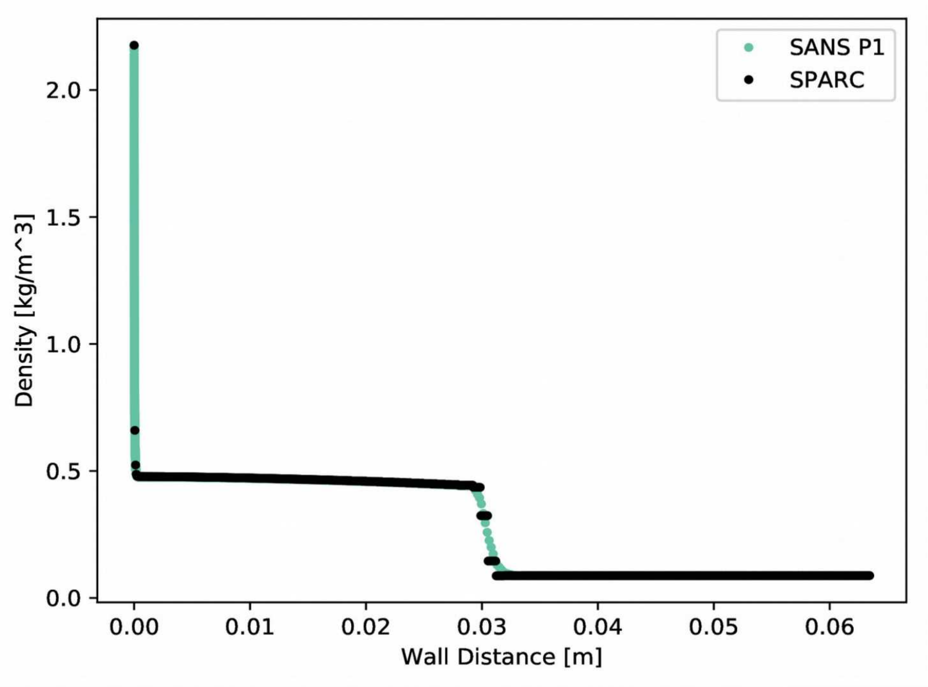
# Stagnation Streamline
# 40,000 Q3 Quad Elements

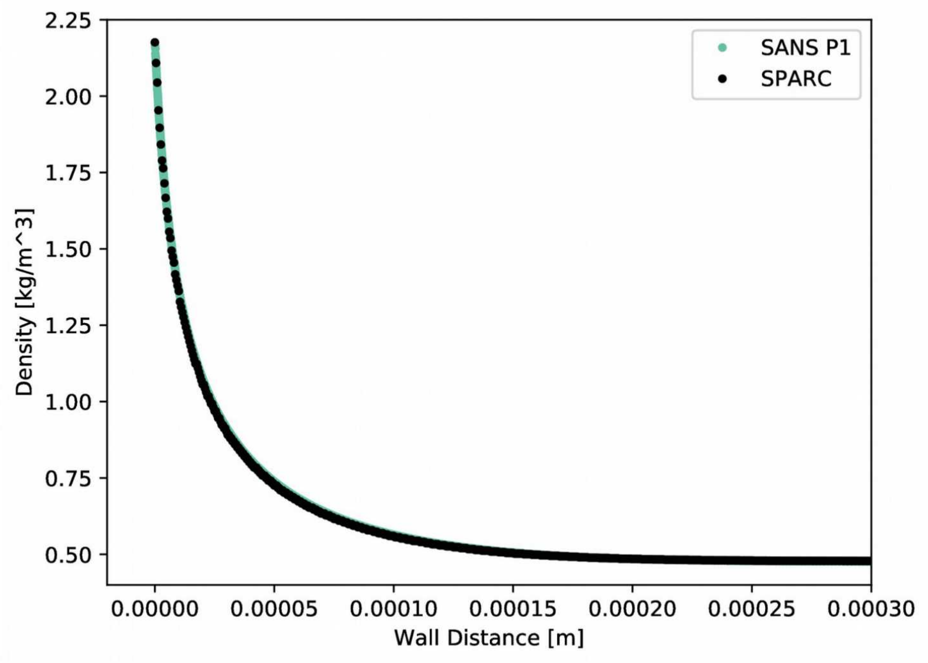



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

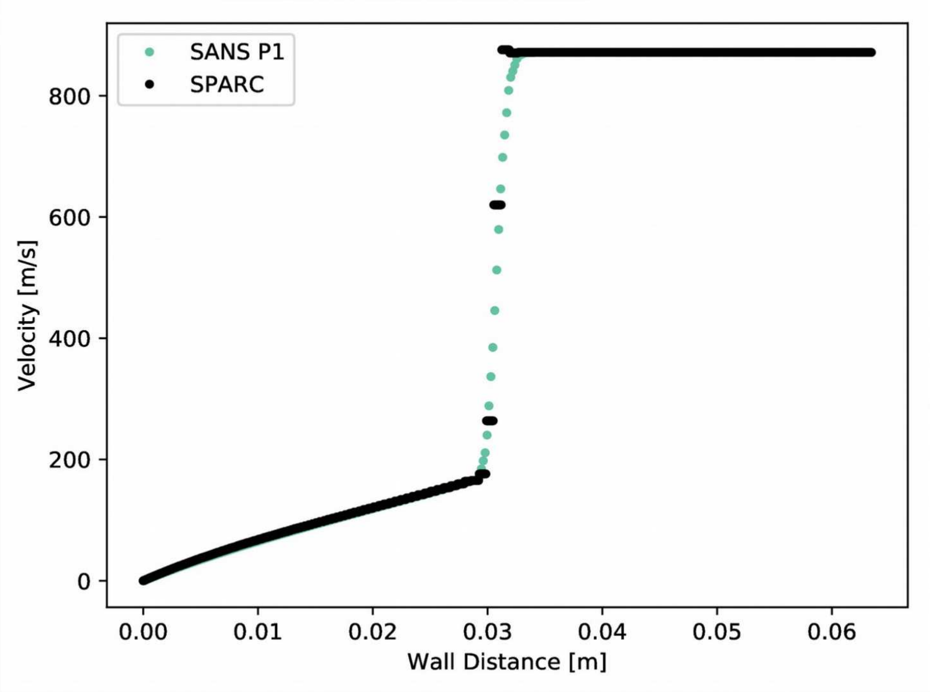
# Stagnation Streamline
# 160,000 Q3 Quad Elements

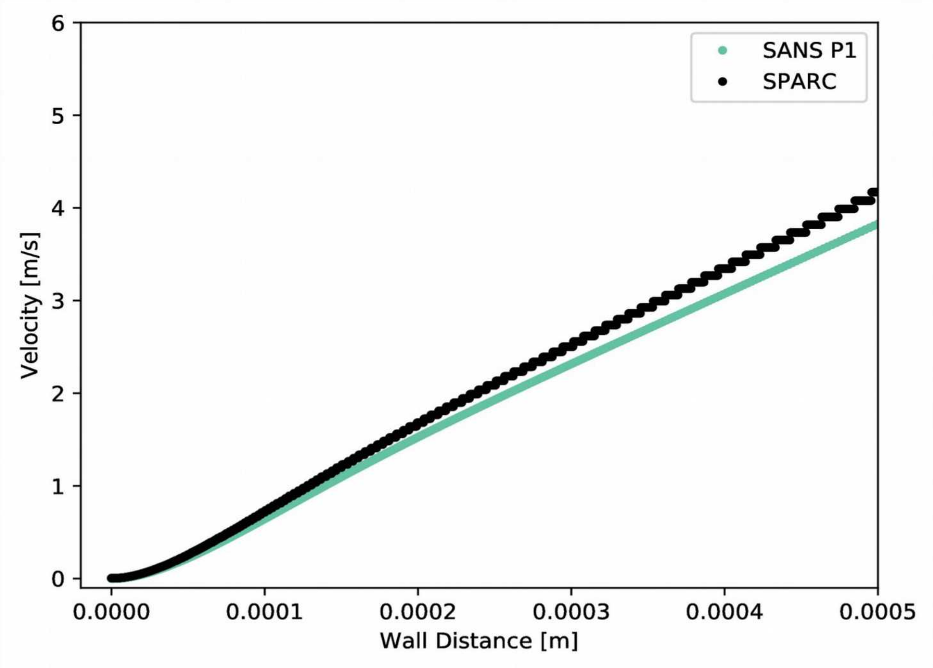



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

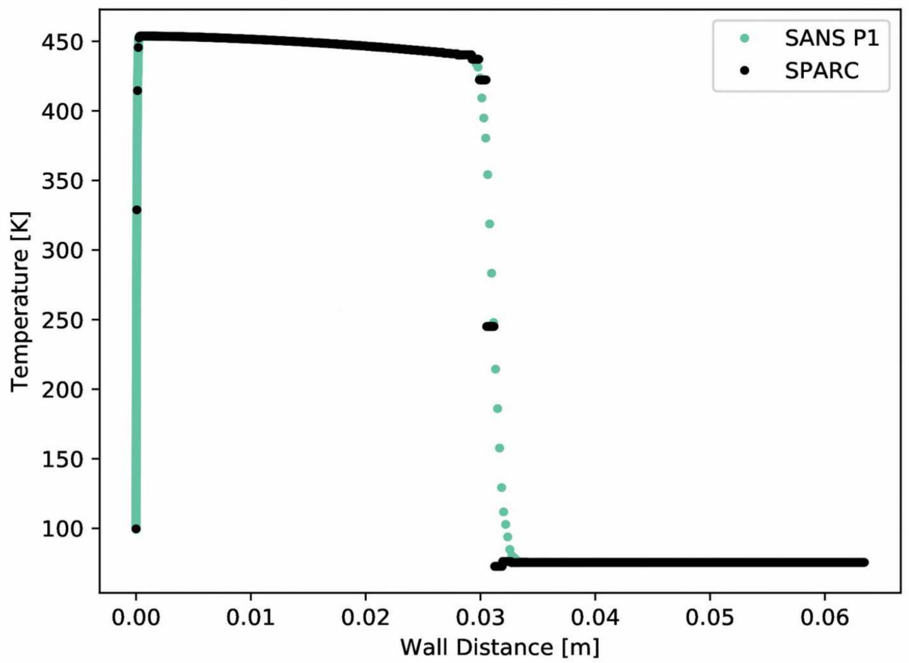
# Stagnation Streamline
# 160,000 Q3 Quad Elements

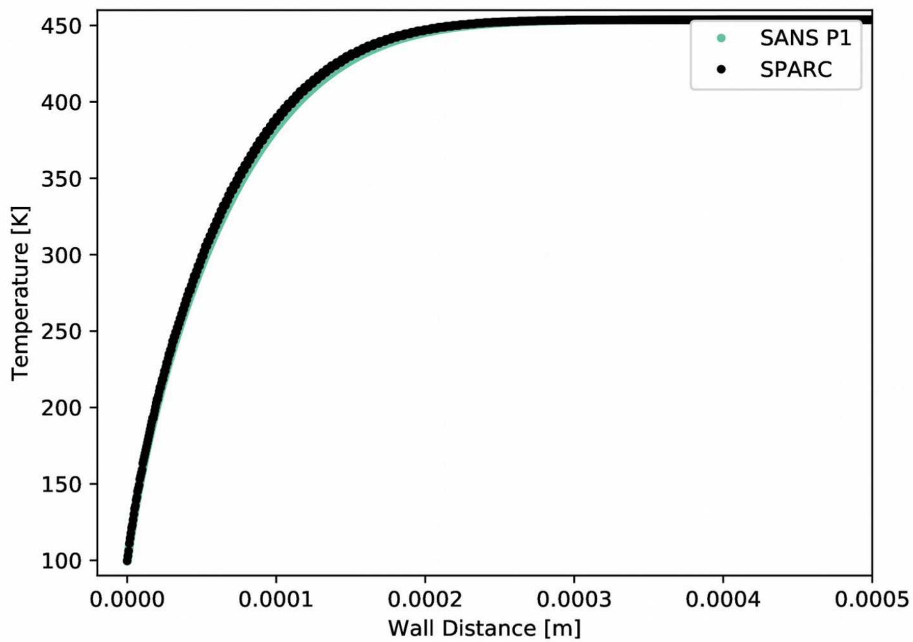



Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

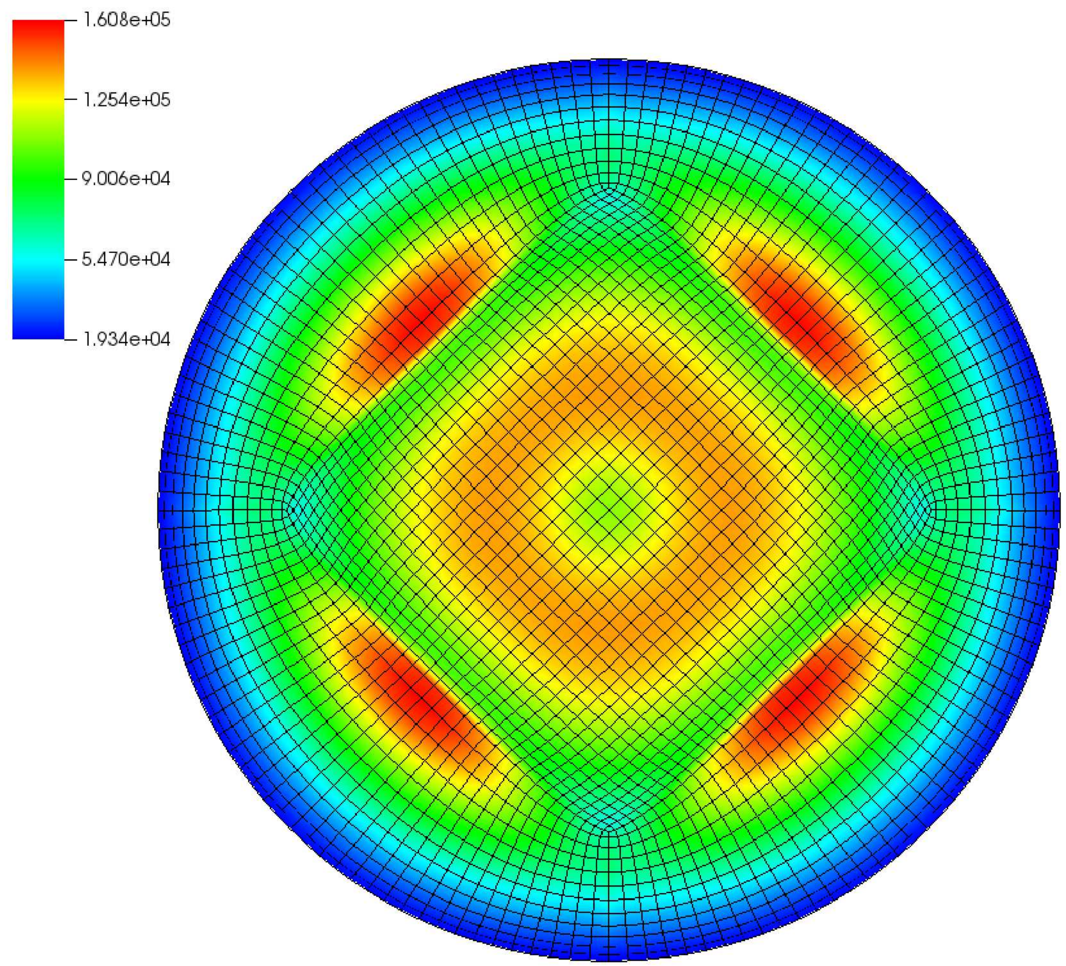
# Stagnation Streamline
# 160,000 Q3 Quad Elements



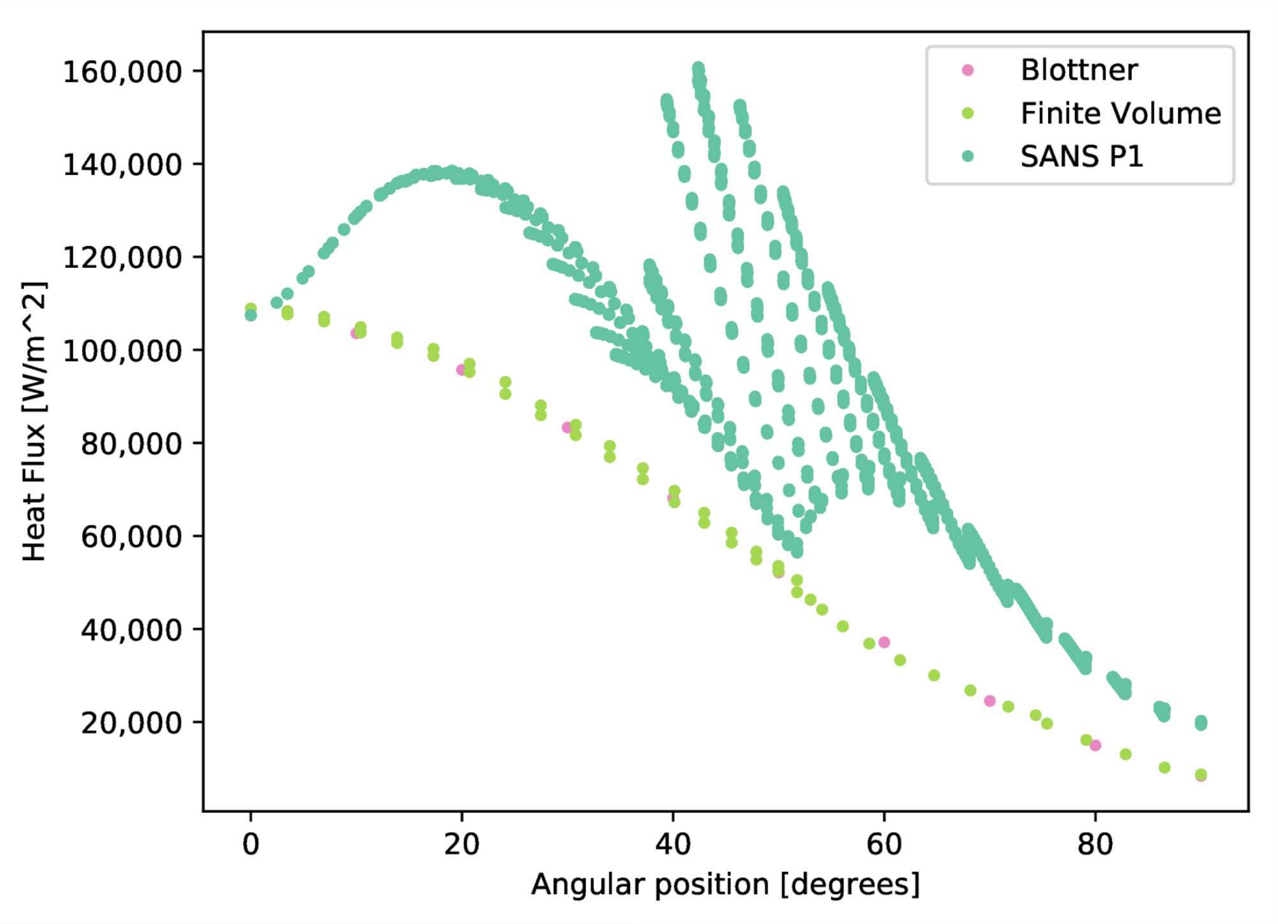




Left: Stagnation streamline to outflow boundary
Right: Stagnation stream line focus on boundary layer

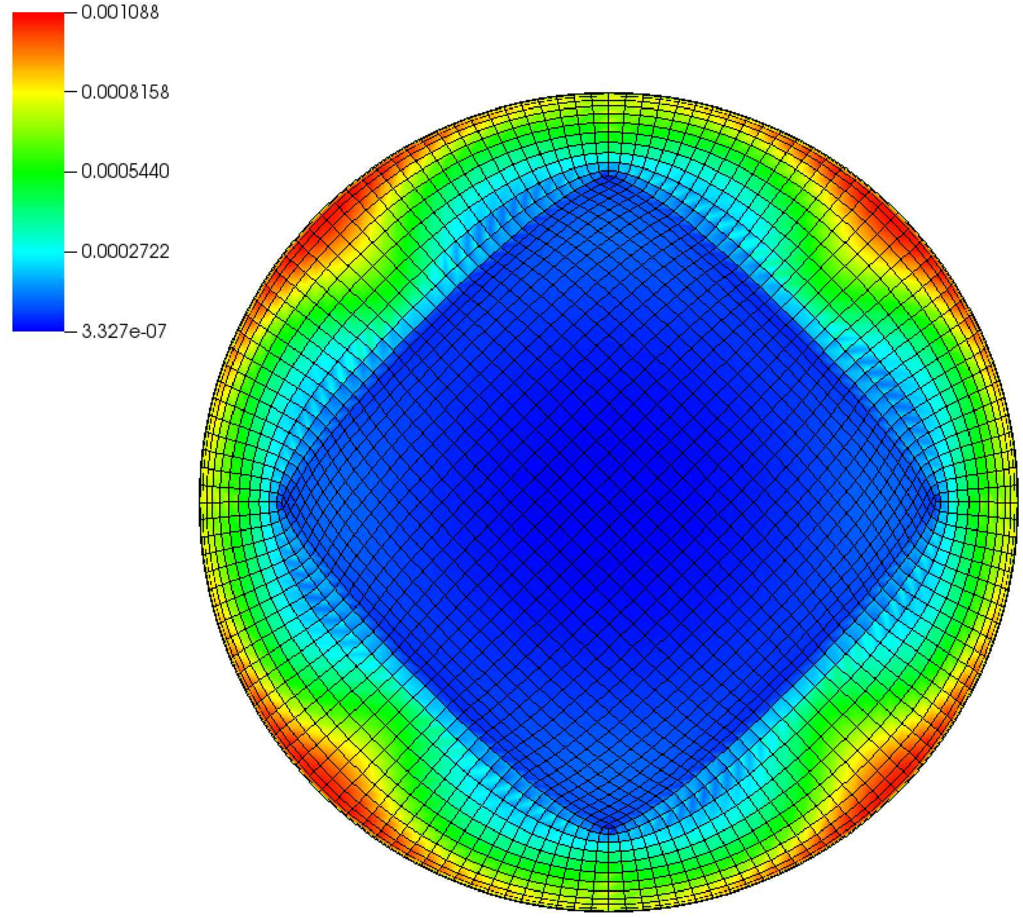
# Sphere P1 Surface Heat Flux 120,960 Q1 Hex Elements



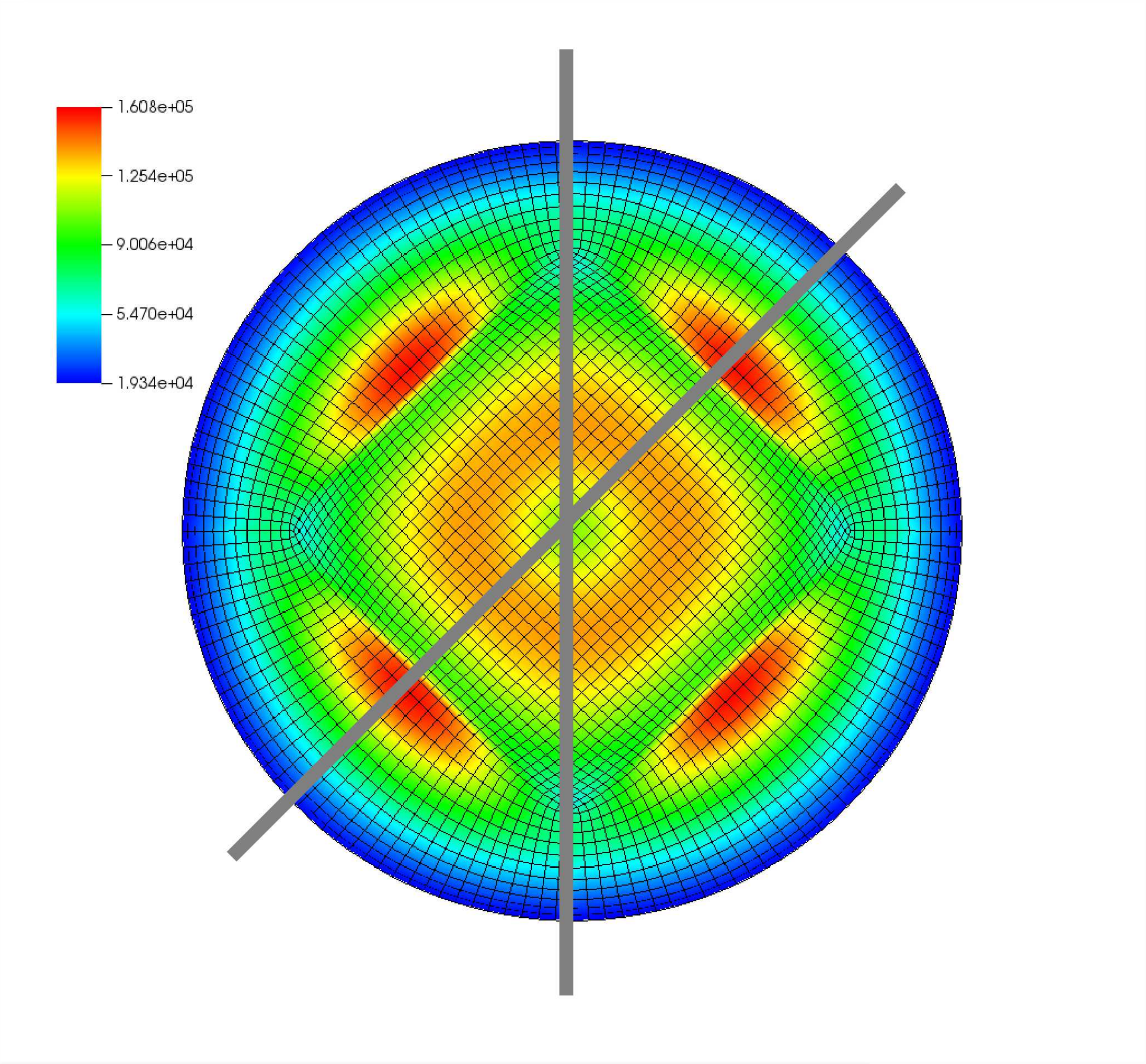
# Sphere P1 Surface Heat Flux
# 120,960 Q1 Hex Elements



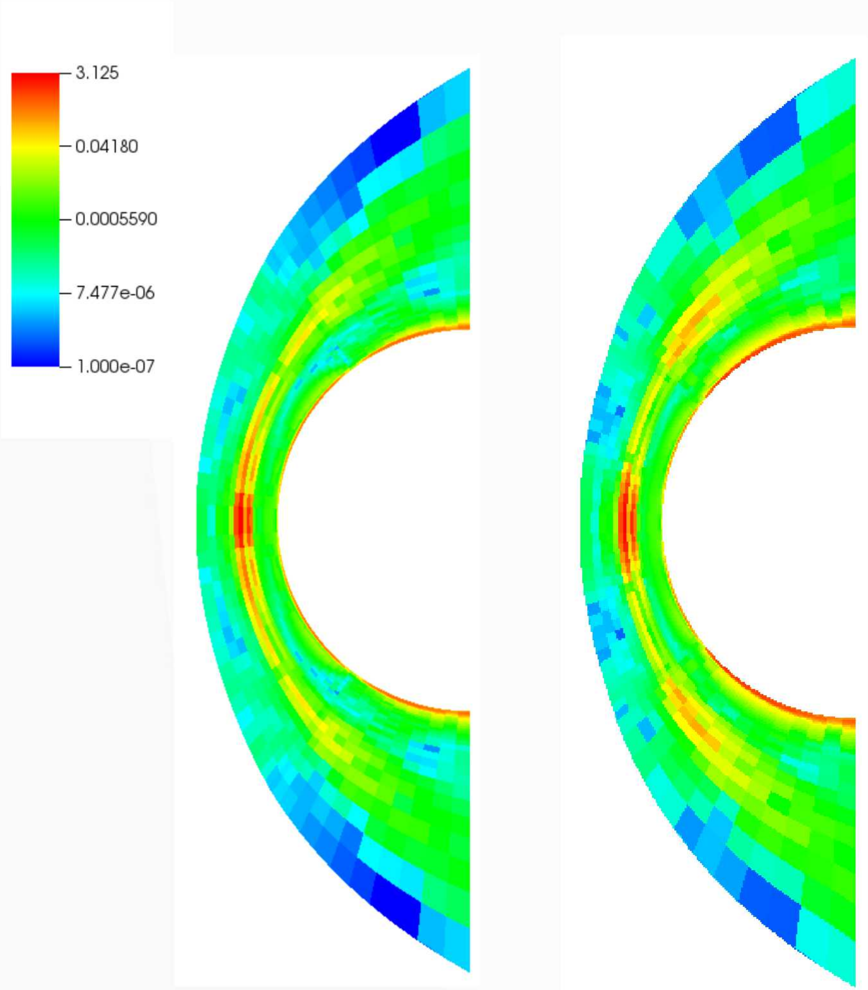
# Sphere P1 Surface Art. Viscosity 120,960 Q1 Hex Elements



# Sphere P1 Surface Heat Flux (20,960 Q1 Hex Elements

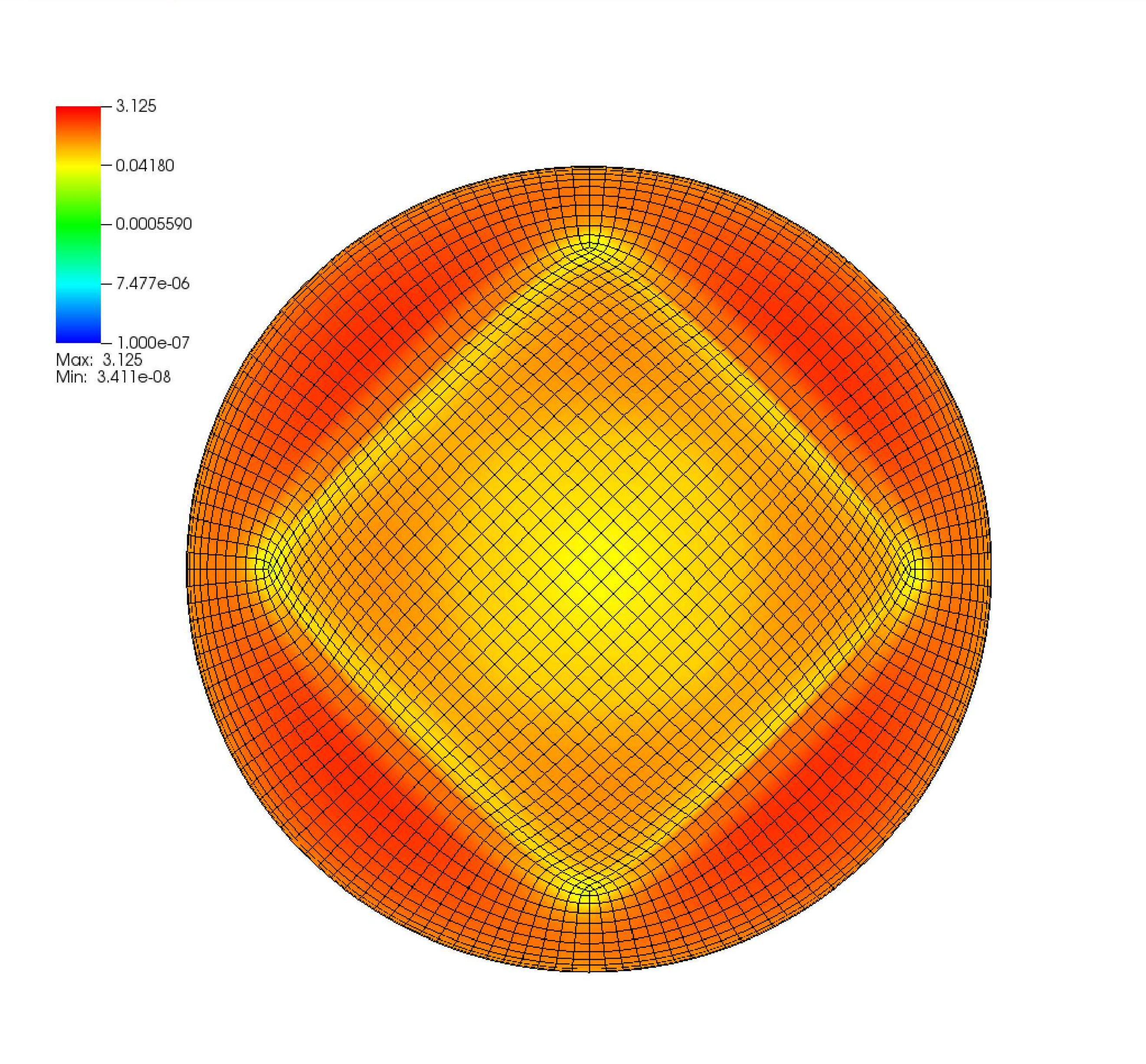


# Sphere P1 Error Indicator
# 120,960 Q1 Hex Elements

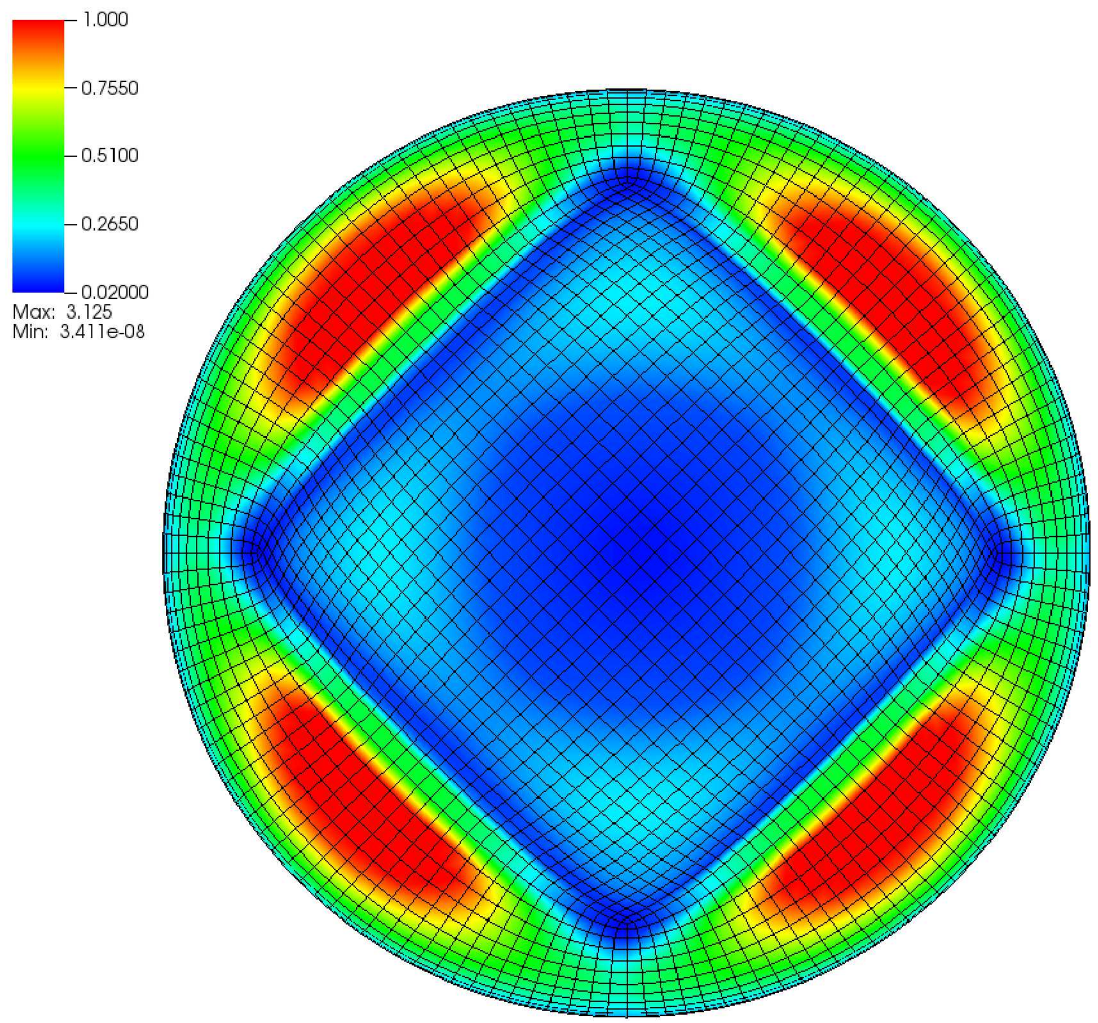



Error Indicator, Left to Right: Vertical and 45degree lines

# Sphere P1 Surface Error Indicator
# 120,960 Q1 Hex Elements



# Sphere P1 Surface Error Indicator 120,960 Q1 Hex Elements



# Research Questions

- Can we get good quality high-order heat fluxes on this case?
  - Grid must be curved
- Can we use features of the numerical solution to understand the error sources?
  - If boundary layer is too coarse:
    - poor heat flux results
    - lots of oscillation in the adjoints
    - much higher error indicators
  - If boundary layer sufficiently refined:
    - flux is converging
    - well behaved adjoints
    - error is primarily driven by the shock
- What is the origin of this ‘grid imprinting’?
  - Appears to be driven by the sudden change in grid size on surface and in boundary layer

# New Research Questions

- Does using curved grids in 3D alleviate ‘grid imprinting’?
- How much would output based mesh adaptation help?

# How does this inform future numerical methods?

- We need curved grids for higher-order DG
- On finer grids, error due to ability to capture shock is most significant
- Rapid increase in surface grid size lead to poor heat flux prediction in those regions

# Questions

# References

[1] B. S. Kirk, S. W. Bova, and R. B. Bond, *The influence of stabilization parameters in the SUPG finite element method for hypersonic flows*, in 48th AIAA Aerospace Sciences Meeting Including the New Horizons Forum and Aerospace Exposition, American Institute of Aeronautics and Astronautics, 2010.

[2] F. G. Blotter, *Accurate Navier-Stokes results for the hypersonic flow over a spherical nosetip*, Journal of Spacecraft and Rockets, 27 (1990), pp. 113–122.

[3] G. E. Barter and D. L. Darmofal, *Shock capturing with PDE-based artificial viscosity for DGFEM: Part I. formulation*, J. Comput. Phys., 229 (2010), pp. 1810–1827.

[4] M. Yano, *An optimization framework for adaptive higher-order discretizations of partial differential equations on anisotropic simplex meshes*, PhD thesis, Massachusetts Institute of Technology. Dept. of Aeronautics and Astronautics, 2012.

# Backup Slides

# Discontinuous Galerkin

- Find $u_{h,p} \in \mathcal{W}_{h,p}$ such that

$$R_{h,p}\left(u_{h,p}, w_{h,p}\right) = 0 \quad \forall w_{h,p} \in \mathcal{W}_{h,p}$$

- Where $\mathcal{W}_{h,p} = \left\{ w \in L^2\left(\Omega\right) \,:\, w|_k \in P^p, \forall k \in \mathcal{T}_h \right\}$
- Consider the residual as three parts

$$R_{h,p}\left(u_{h,p}, w_{h,p}\right) = R_{h,p}^{adv}\left(u_{h,p}, w_{h,p}\right)$$
$$+R_{h,p}^{visc}\left(u_{h,p}, w_{h,p}\right) \; + R_{h,p}^{source}\left(u_{h,p}, w_{h,p}\right)$$

# Discontinuous Galerkin
# Advective Term

- DG discretization of the advective term,

$$R^{adv}_{h,p}(u,w) = \sum_{k \in \mathcal{T}_h} \int_{\Omega_k} -(\nabla w)^T \cdot \mathbf{F}(u)\, d\Omega$$
$$+ \sum_{f \in \Gamma_I} \int_f \left(w^+ - w^-\right)^T \cdot \mathbf{H}\left(u^+, u^-, \mathbf{n}\right)\, d\Gamma$$
$$+ \sum_{f \in \Gamma_B} \int_f w^T \cdot \mathbf{H}^B\left(u^+, u^B, \mathbf{n}\right)\, d\Gamma$$

# Discontinuous Galerkin Viscous Term

- More complicated than advective term
  - Need to include effect of inter-element jumps on solution gradients
  - Use BR2 method [ref]
- Define:

$$F_{visc}\left(u,\nabla u\right)\equiv\mathcal{A}\left(u\right)\nabla u$$

- Then:

$$\begin{aligned}R_{h,p}^{visc}\left(u,w\right)=\sum_{k\in\mathcal{T}_h}\int_{\Omega_k}\left(\nabla w\right)^T\cdot\mathcal{A}\left(u\right)\nabla u\,d\Omega\\
-\sum_{f\in\Gamma_I}\int_f\left(\mathcal{A}^T\nabla w\right)^T\cdot u-w^T\cdot\left(\mathcal{A}\nabla u\right)-w\cdot\left(\mathcal{A}\eta_f r_f\left(u\right)\right)\,d\Gamma\\
-\sum_{f\in\Gamma_B}\int_f\left(\mathcal{A}_B^T\nabla w^+\right)^T\cdot\left(u^+-u_B\right)-\left(w^+n^+\right)^T\cdot\mathcal{A}_B\nabla u^B\,d\Gamma\\
-\sum_{f\in\Gamma_B}\int_f\left(w^+n^+\right)^T\cdot\mathcal{A}_B\eta_f r_f\left(\left(u^+-u_B\right)n^+\right)\,d\Gamma,\end{aligned}$$

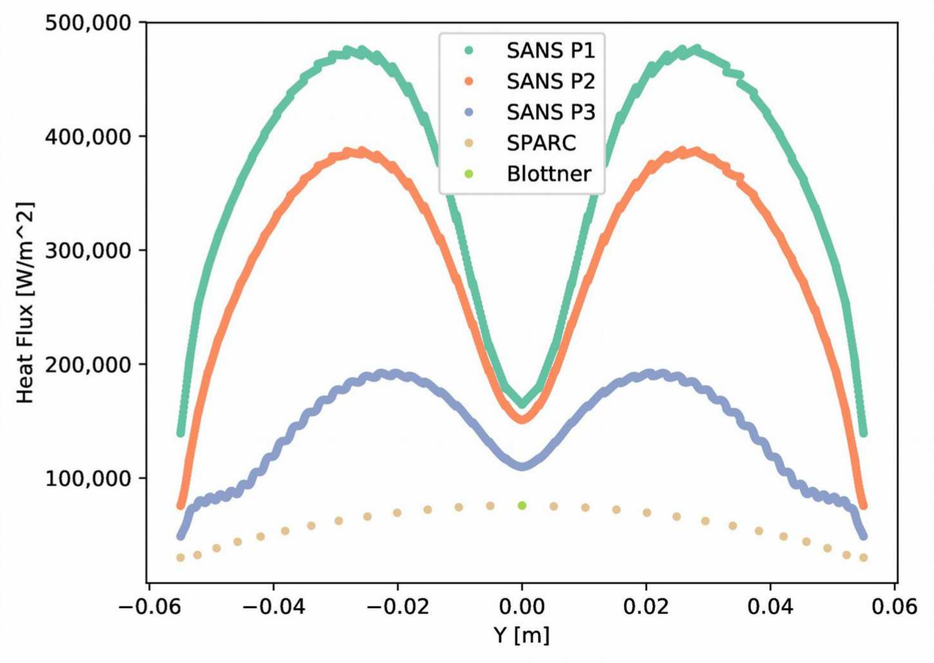
# Discontinuous Galerkin
# Source Term

- The source term is given by

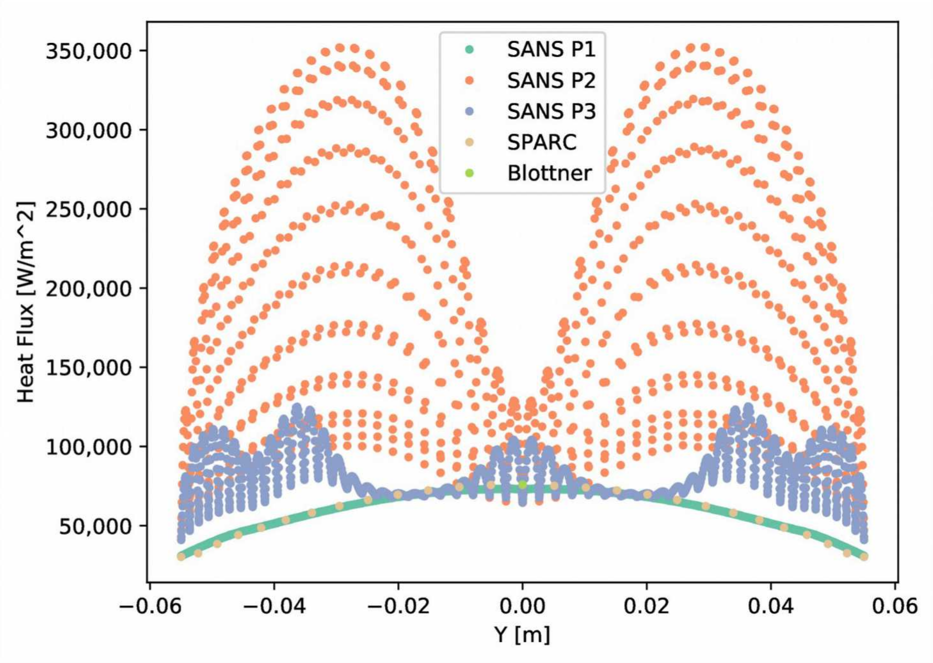
$$R_{h,p}^{source}\left(u,w\right)=\sum_{k\in\mathcal{T}_h}\int_{\Omega_k}w^T S\left(u\right)\,d\Omega$$

- If source term is a function of solution gradient, a more complex term is needed for adjoint consistency

# Heat Flux
# 2,500 Quad Elements

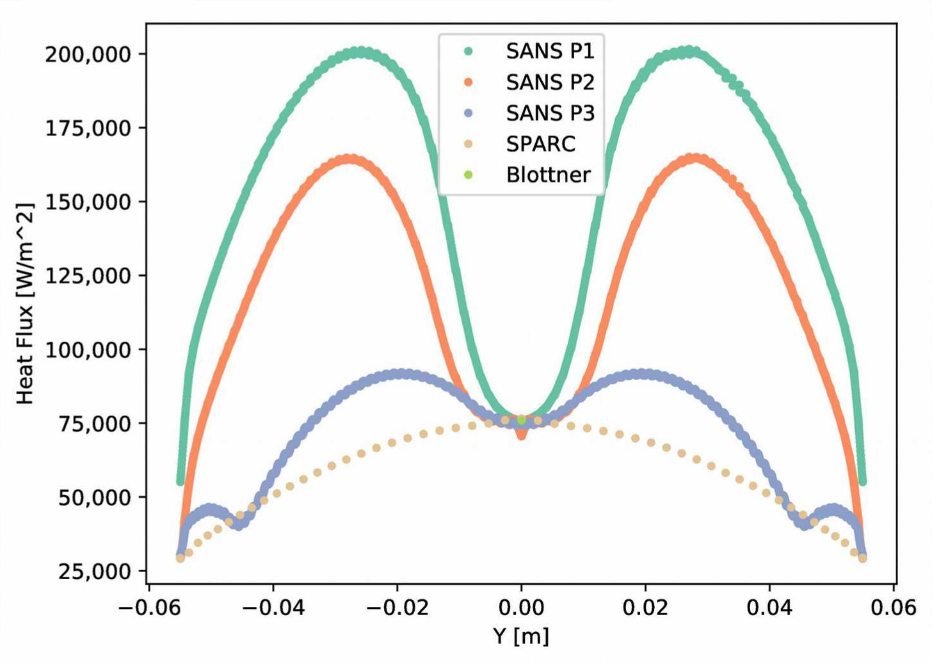



Q1 Grid

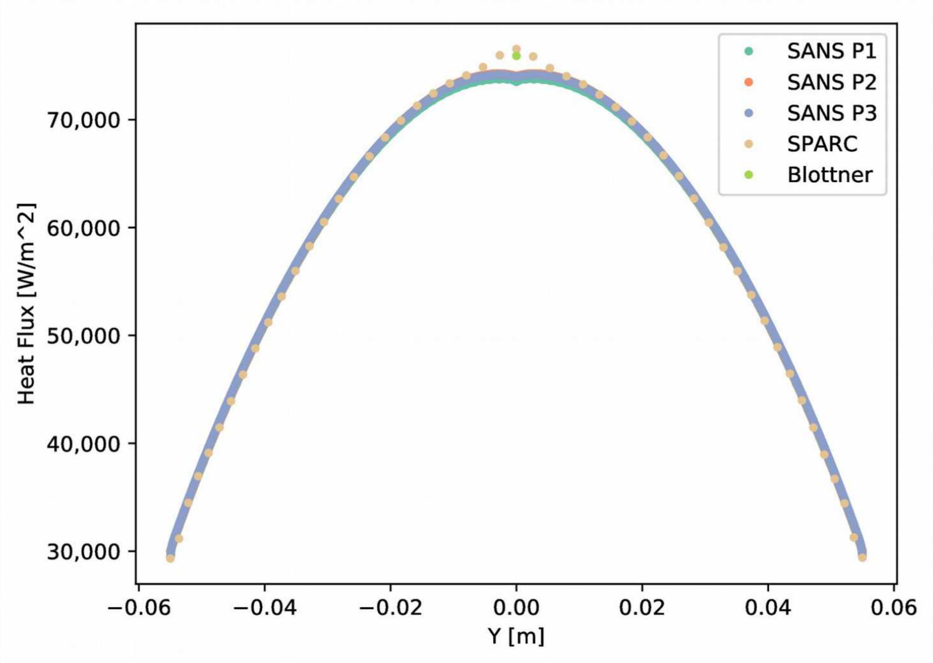



Q3 Grid

# Heat Flux
# 10,000 Quad Elements

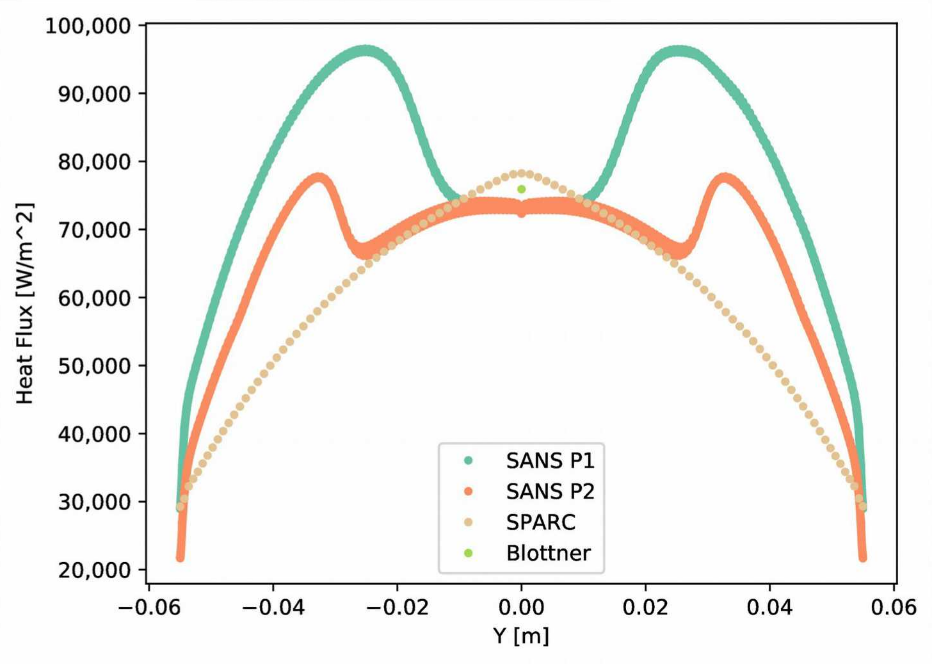



Q1 Grid

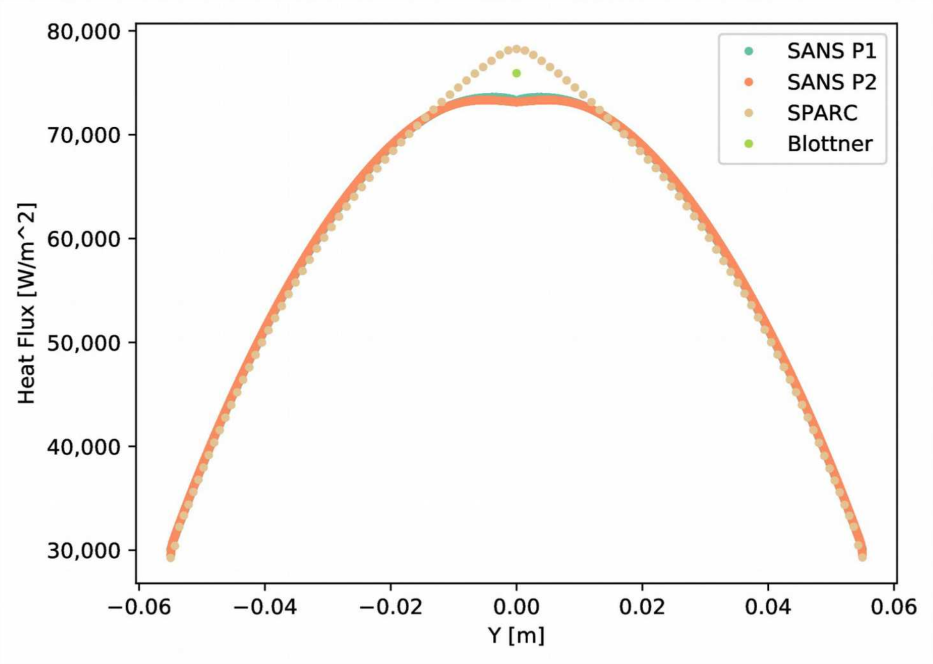



Q3 Grid

# Heat Flux
# 40,000 Quad Elements

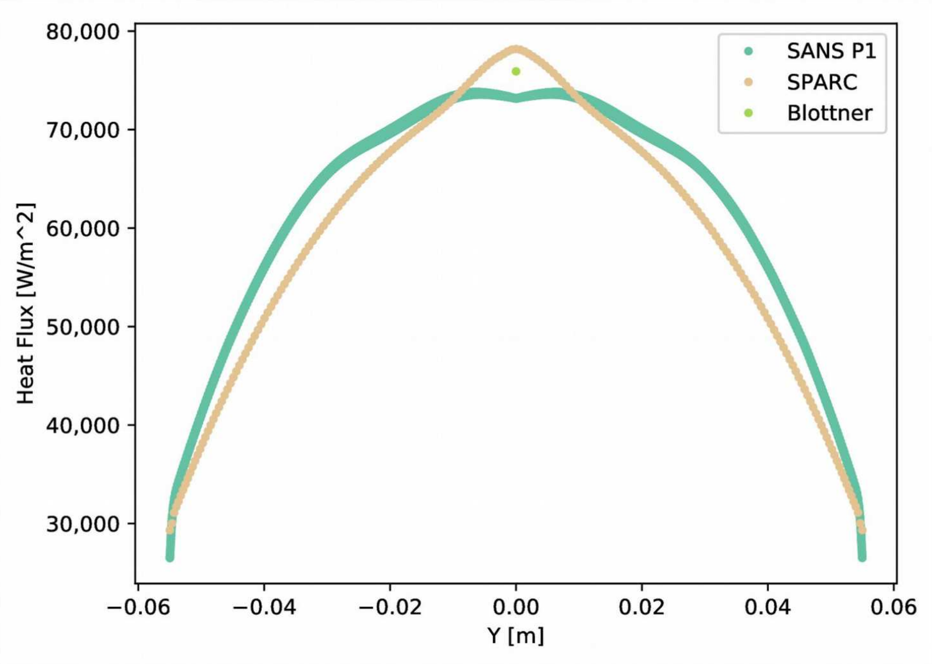



Q1 Grid

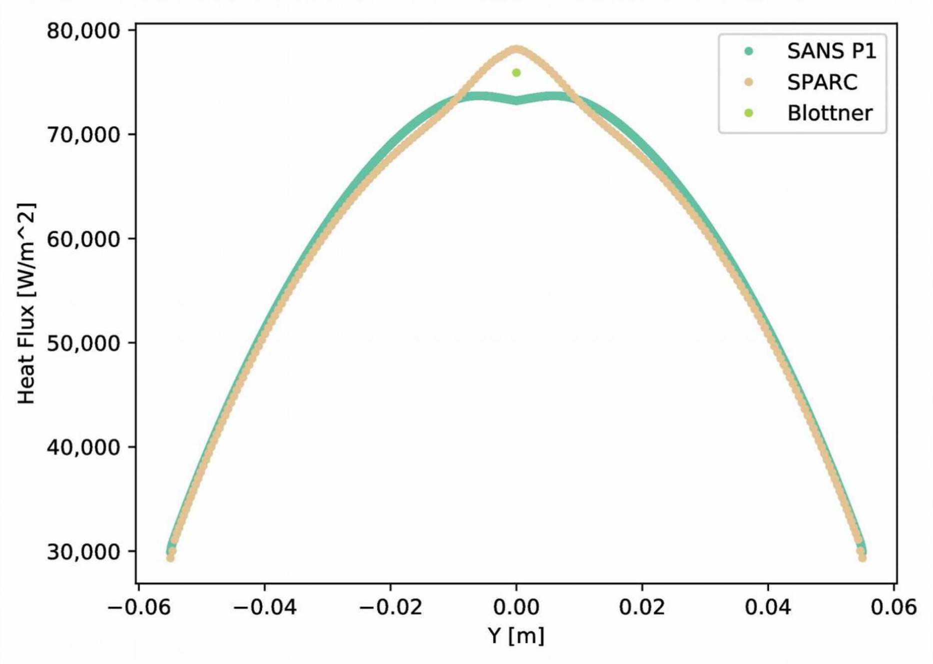



Q3 Grid

# Heat Flux
# 160,000 Quad Elements

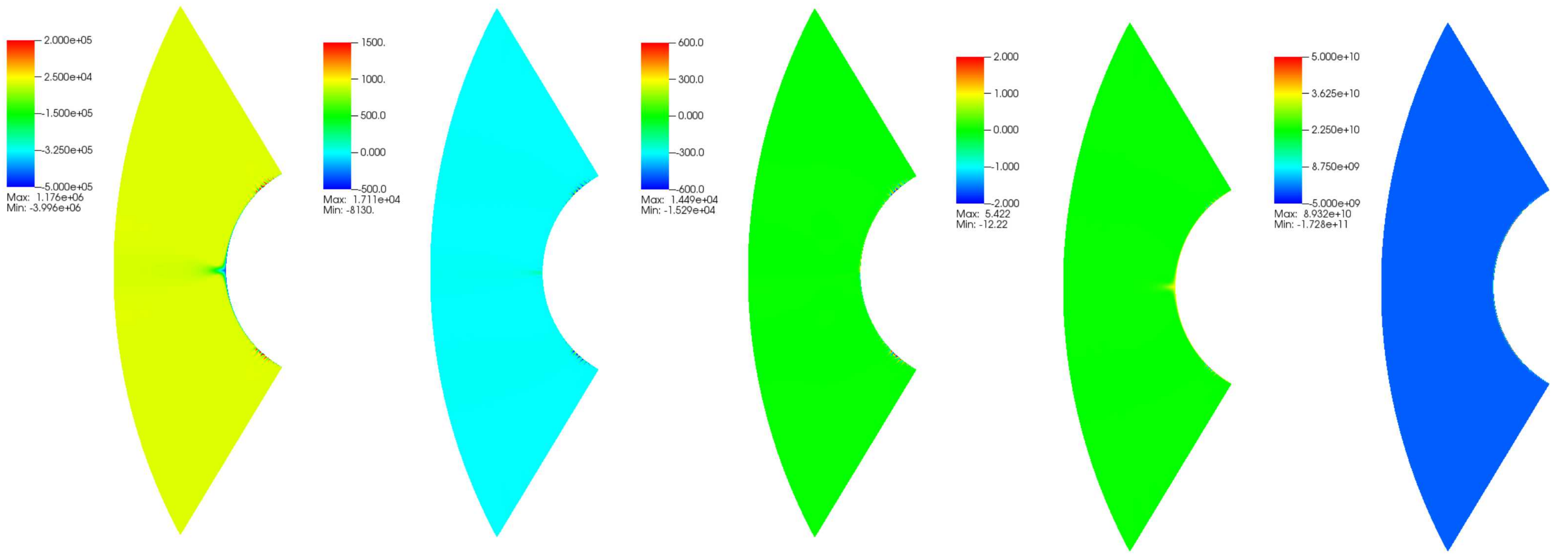



Q1 Grid



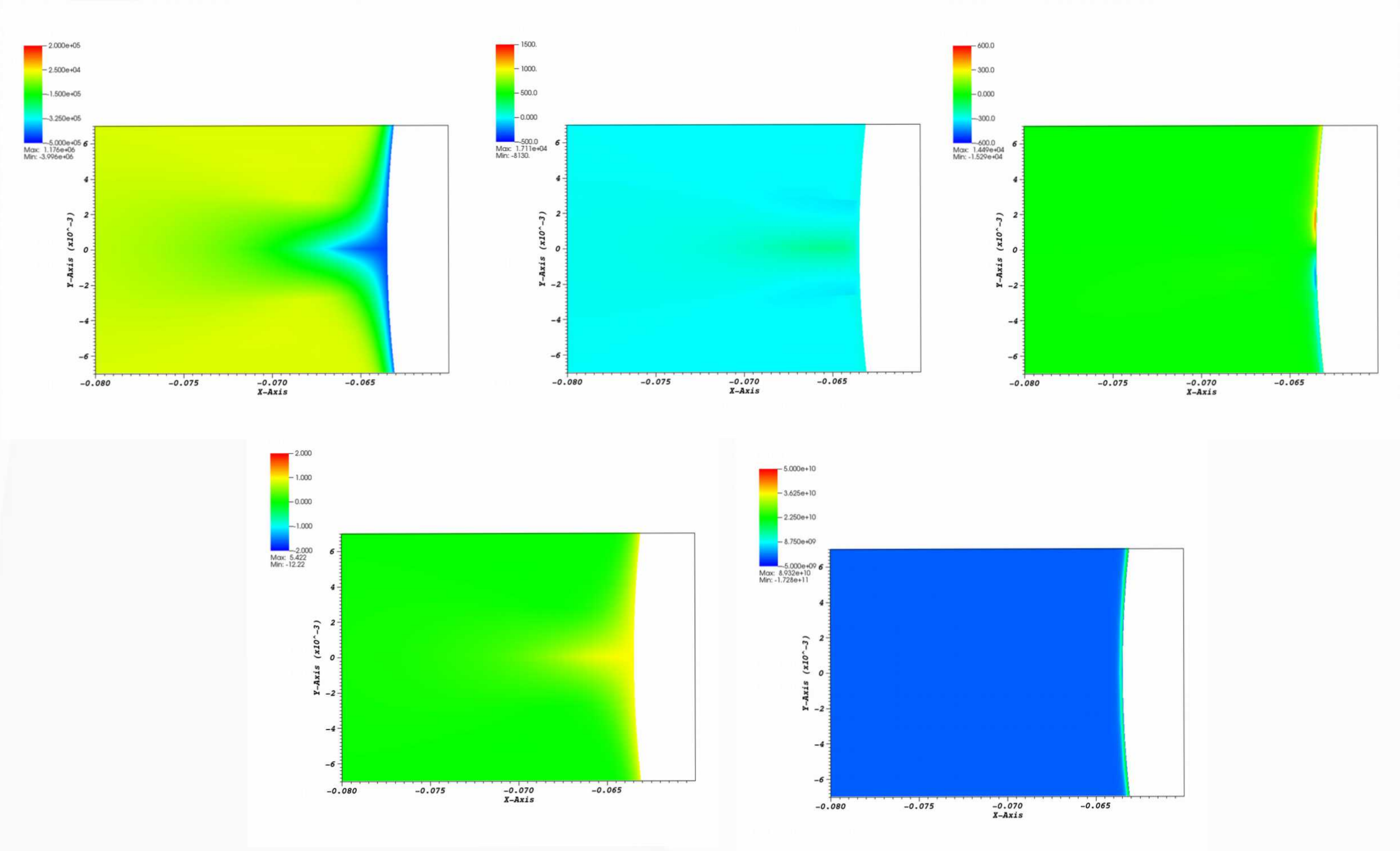

Q3 Grid

# P1 Adjoint
# 2,500 Q3 Quad Elements



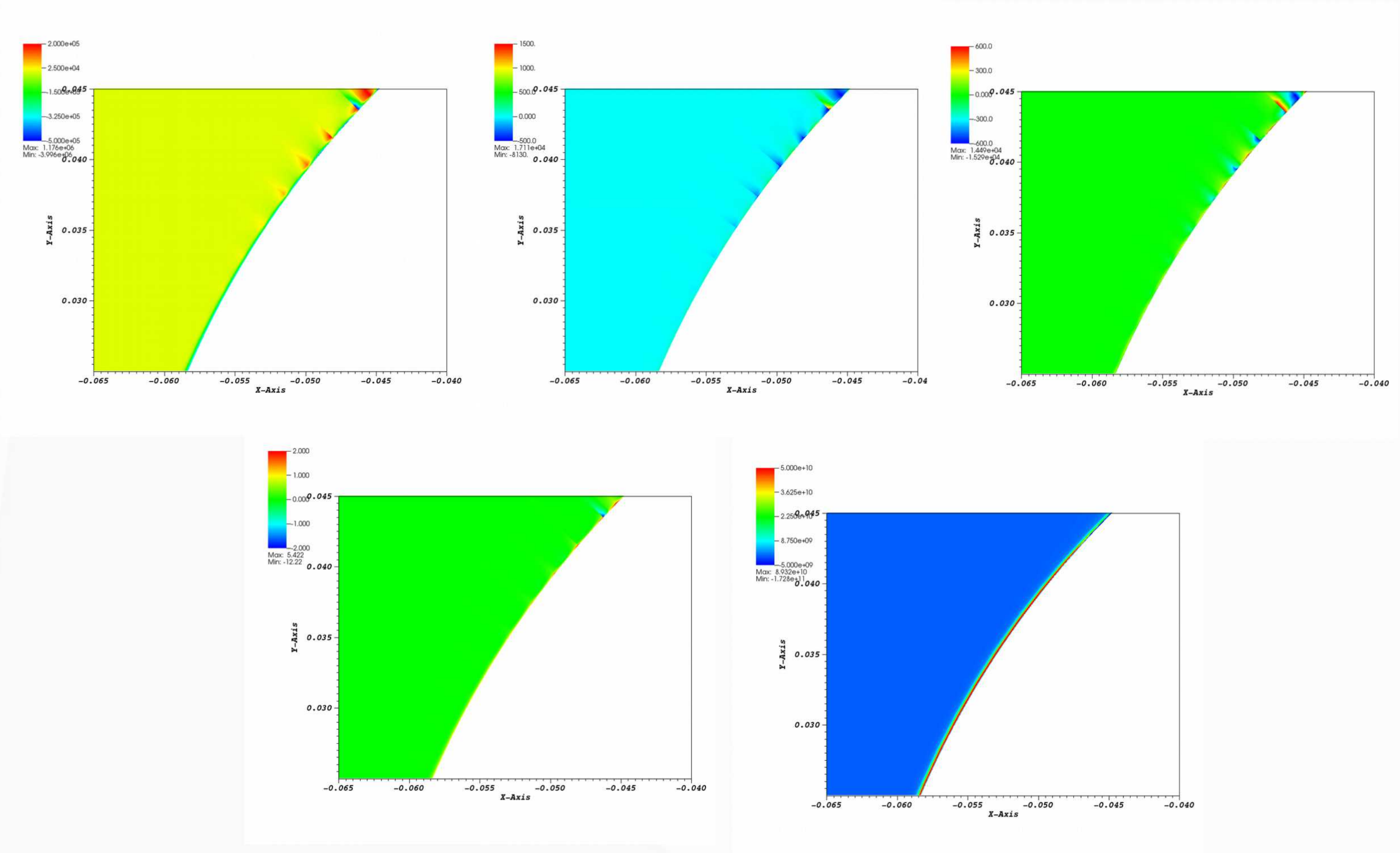

Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

# P1 Adjoint
# 2,500 Q3 Quad Elements



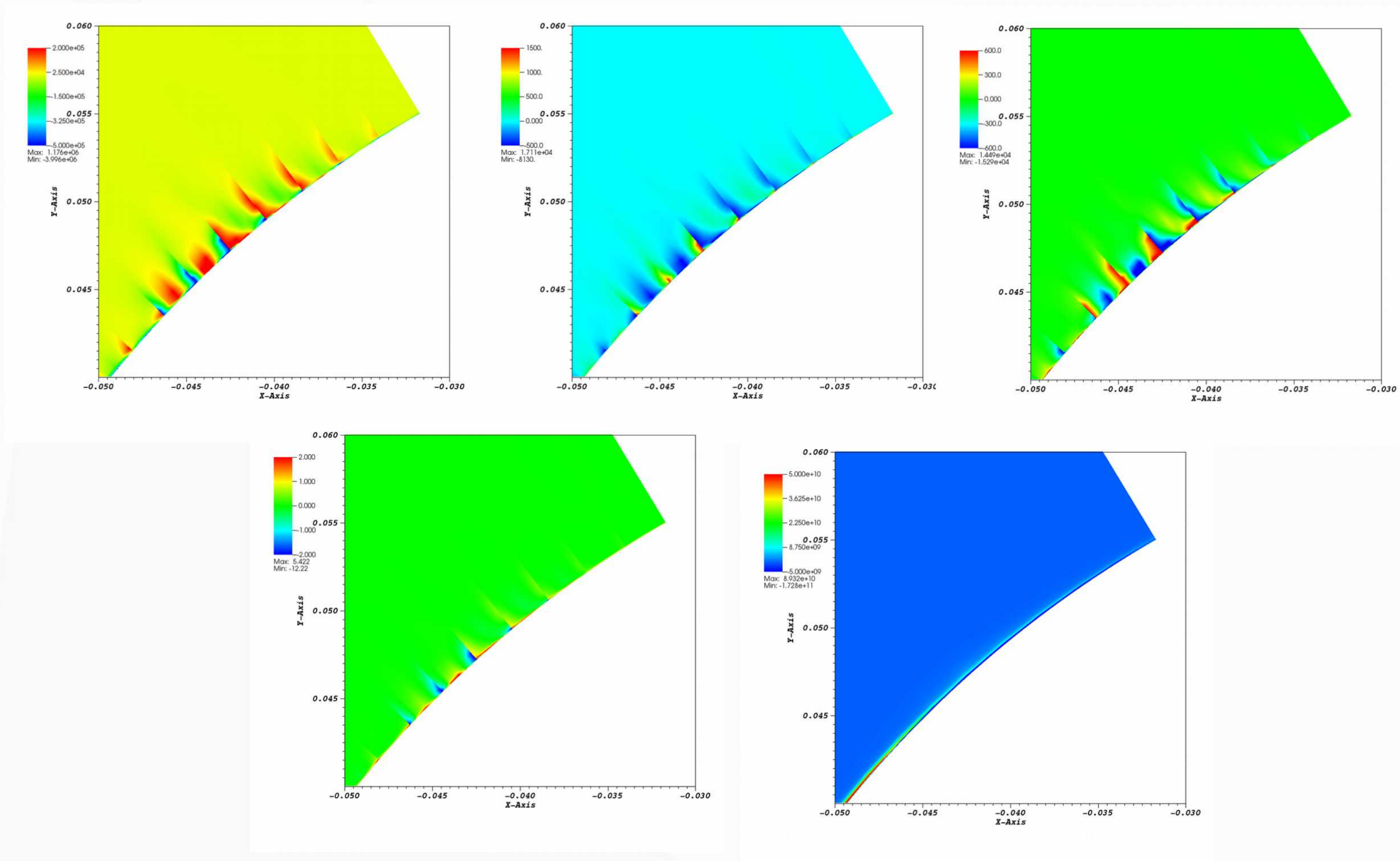

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 2,500 Q3 Quad Elements



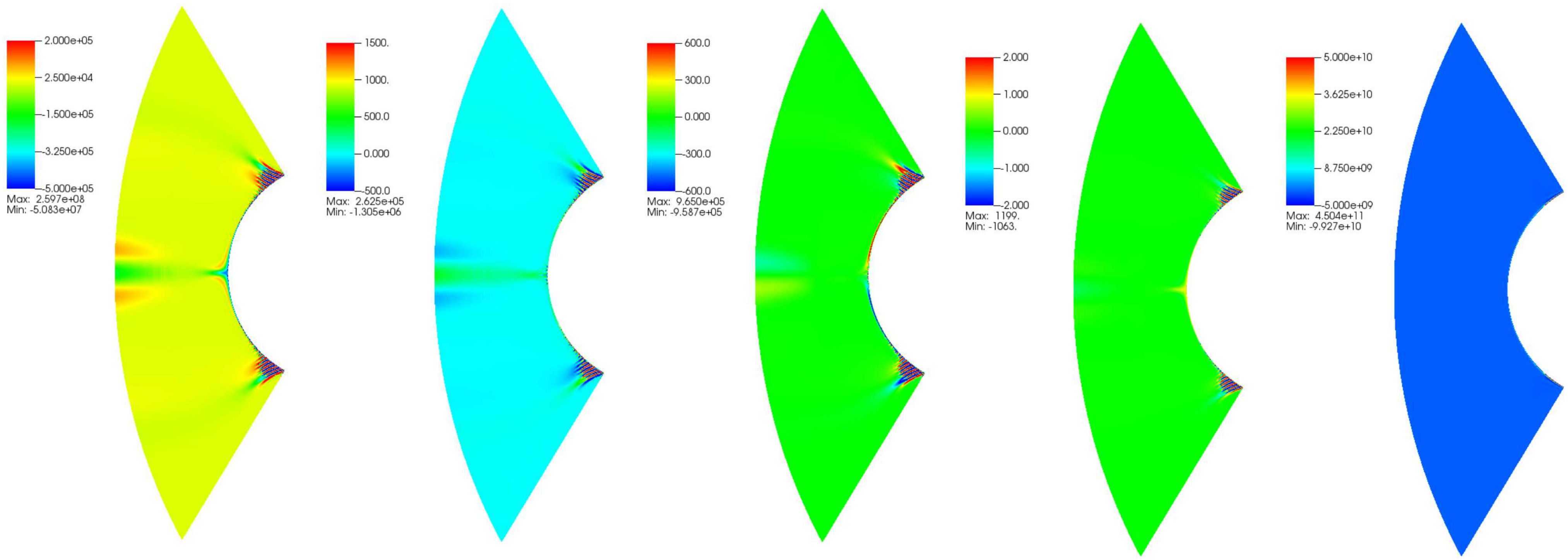

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 2,500 Q3 Quad Elements



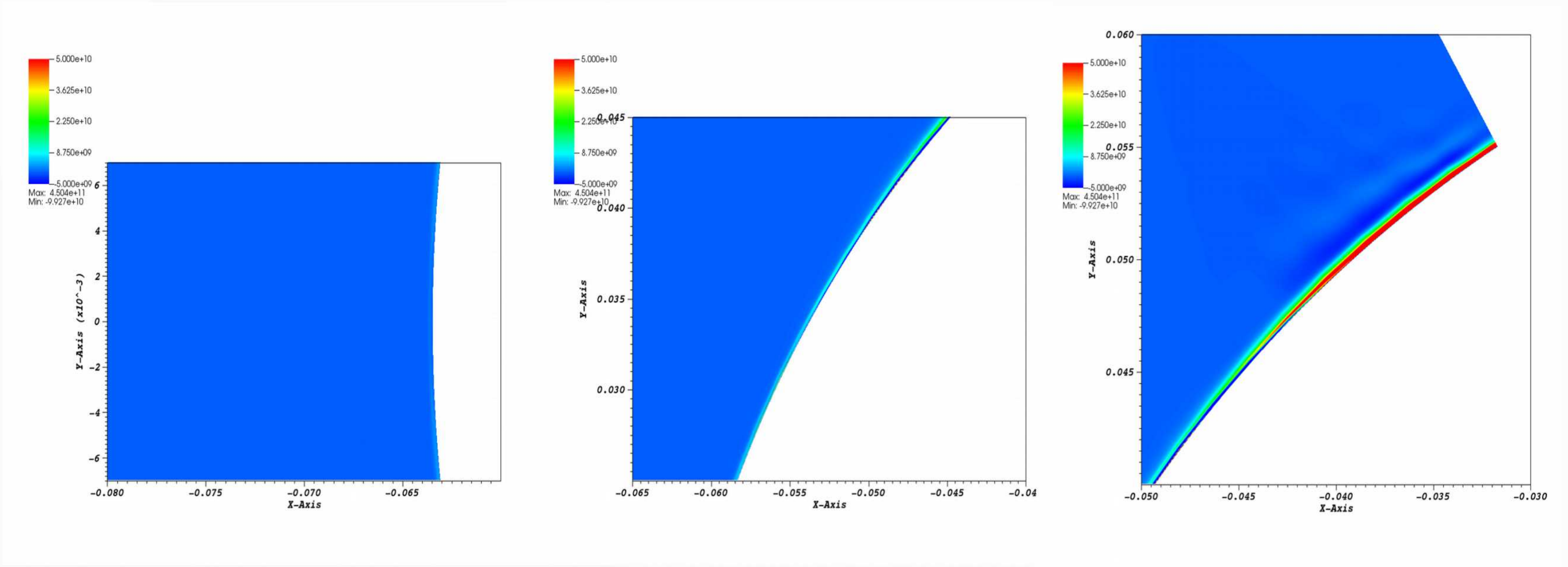

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P3 Adjoint
# 2,500 Q3 Quad Elements



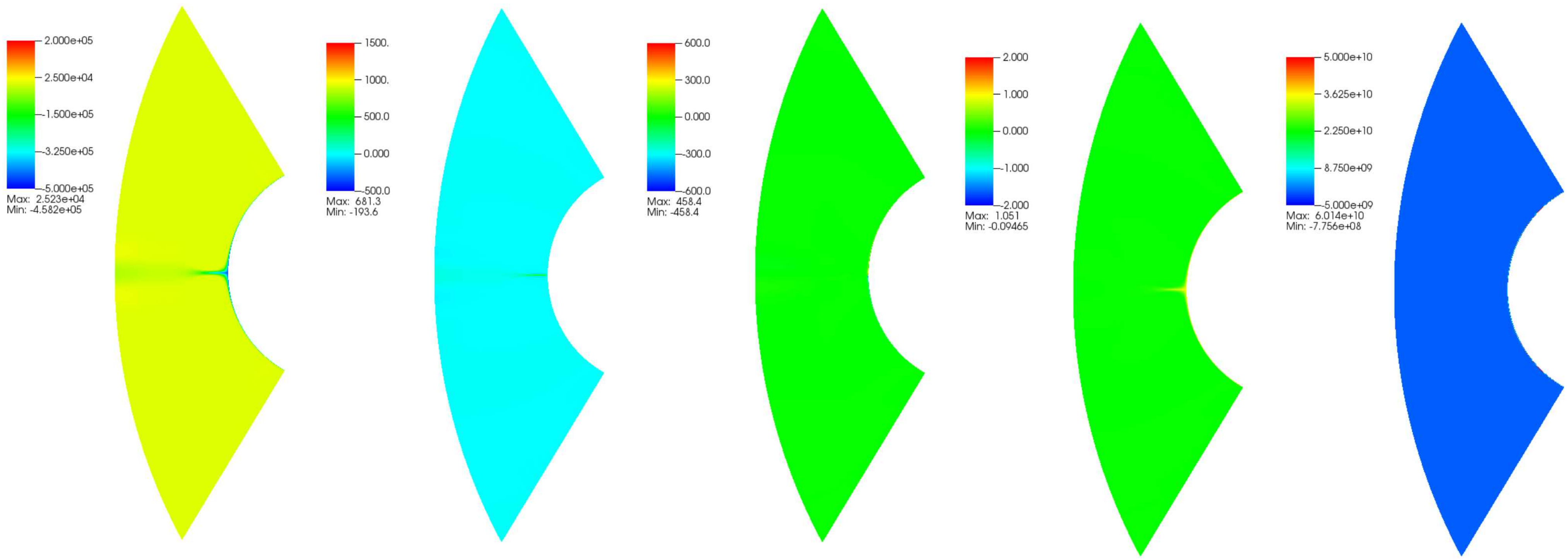

Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

# P3 Adjoint
# 2,500 Q3 Quad Elements



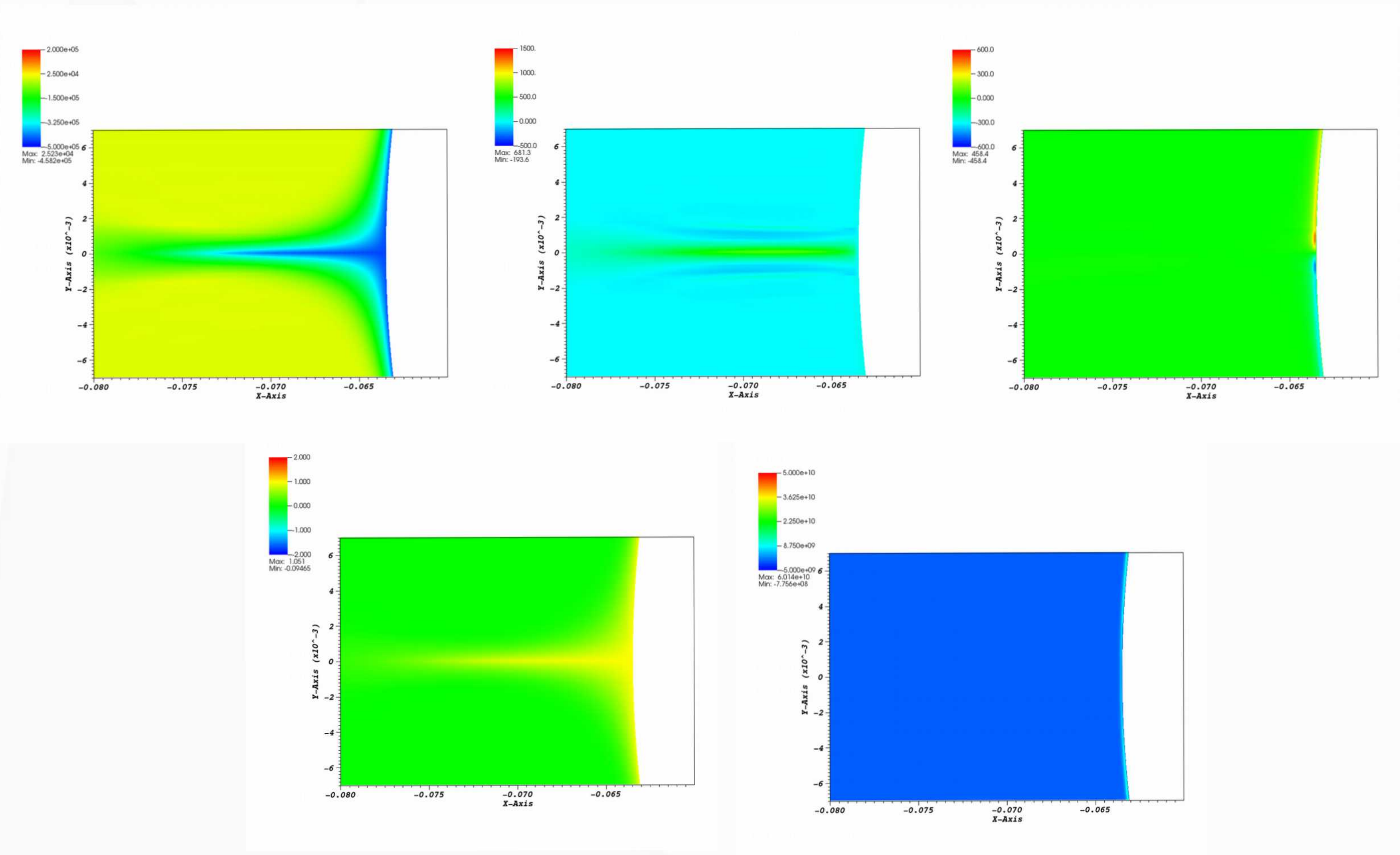
Adjoint of Sensor Equation for Integrated Heat Flux, taken 2 locations around cylinder

# P1 Adjoint
# 10,000 Q3 Quad Elements



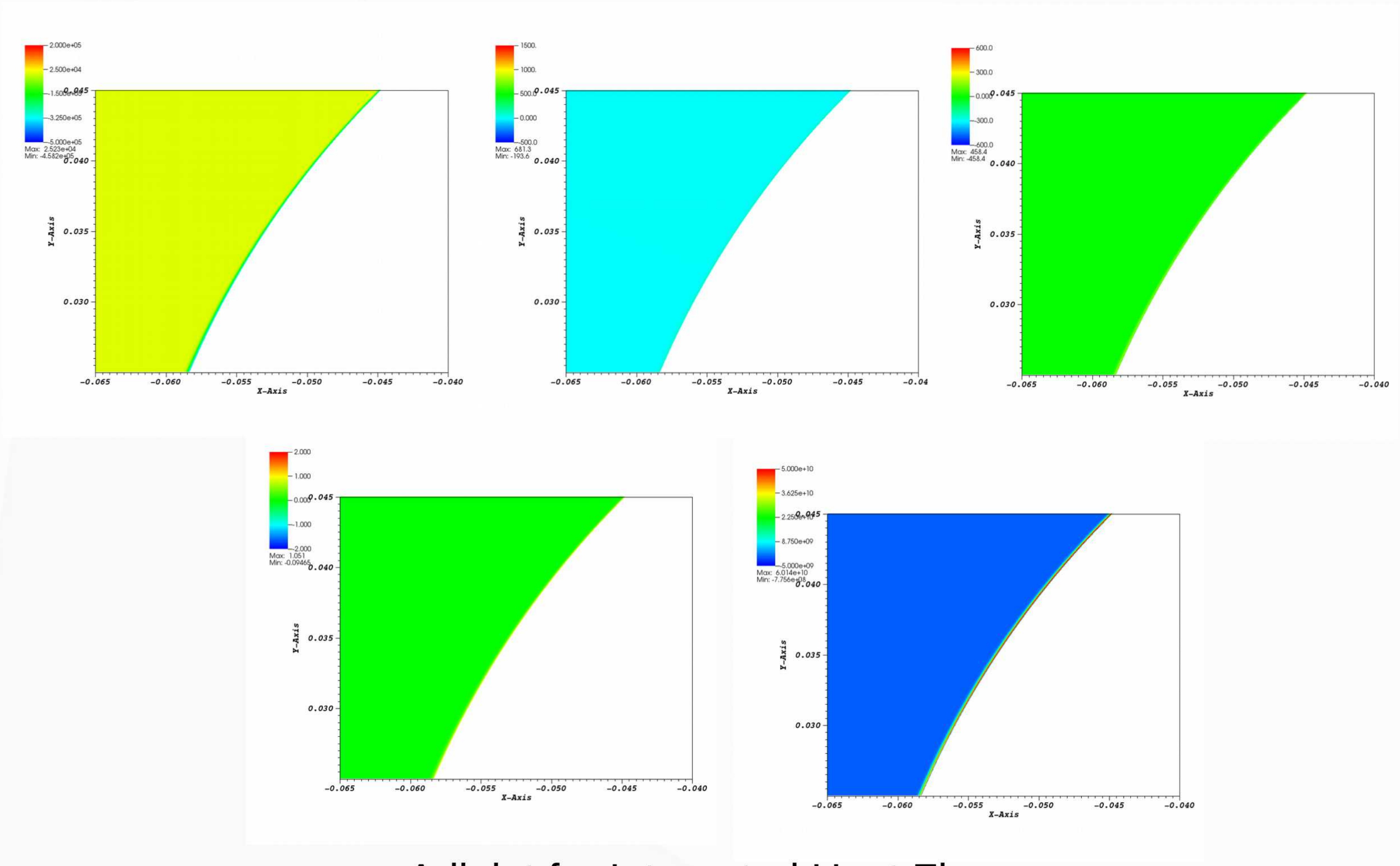

Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

# P1 Adjoint
# 10,000 Q3 Quad Elements



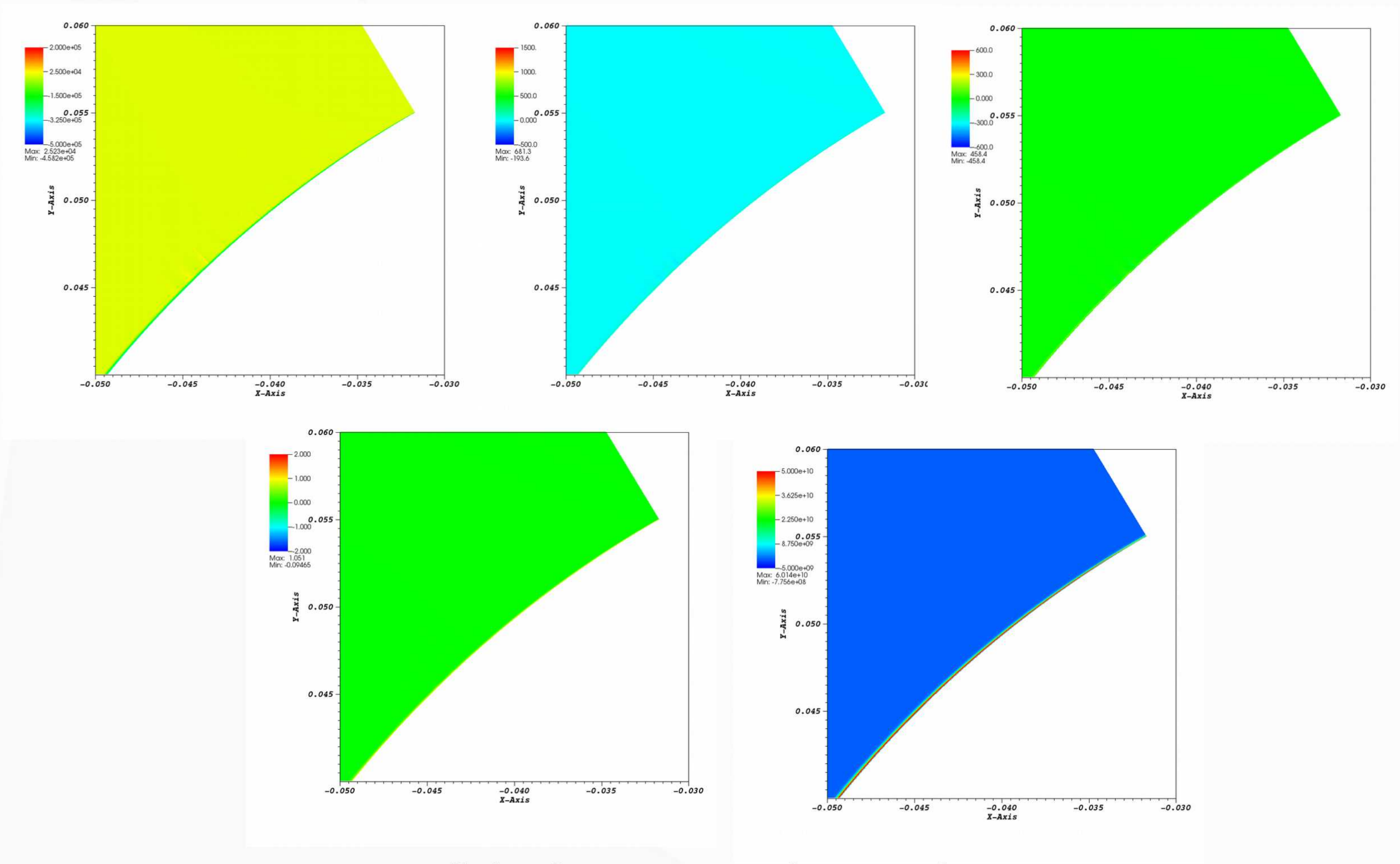
Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 10,000 Q3 Quad Elements



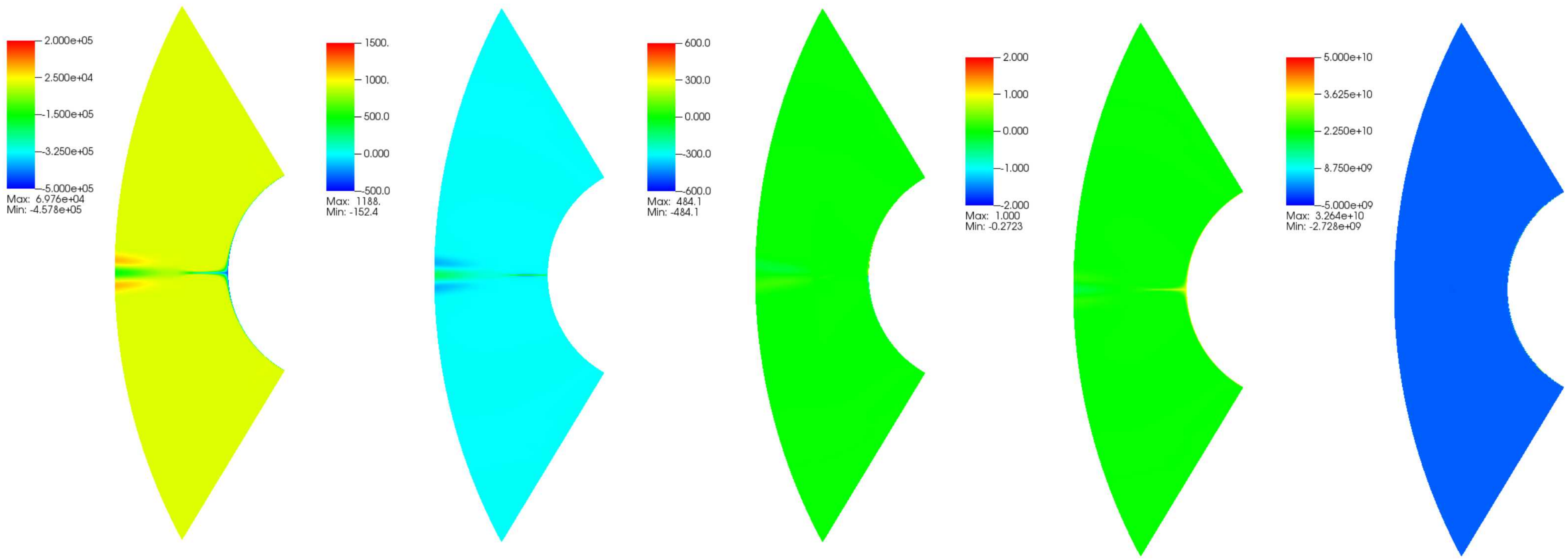

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 10,000 Q3 Quad Elements



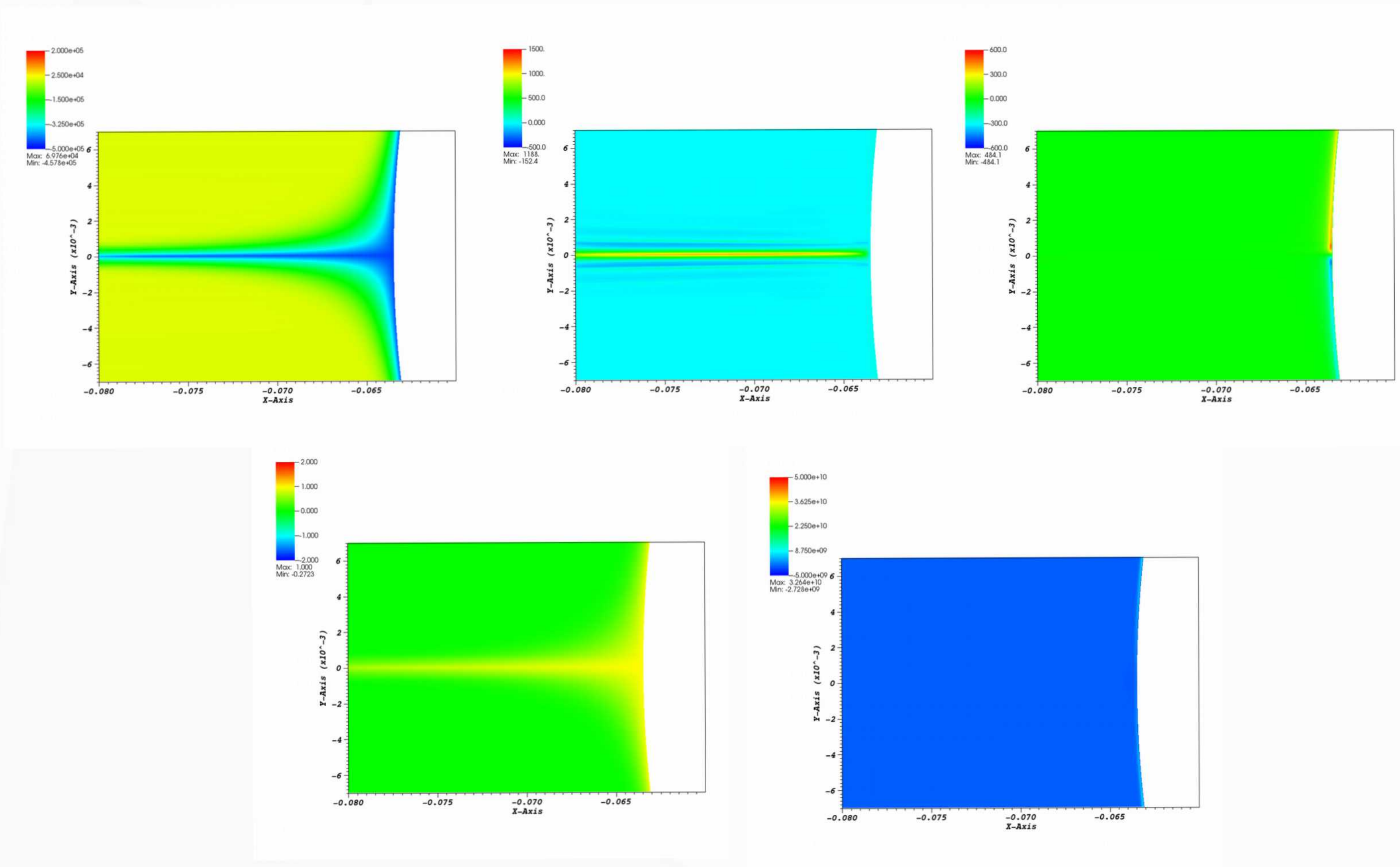
Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 40,000 Q3 Quad Elements



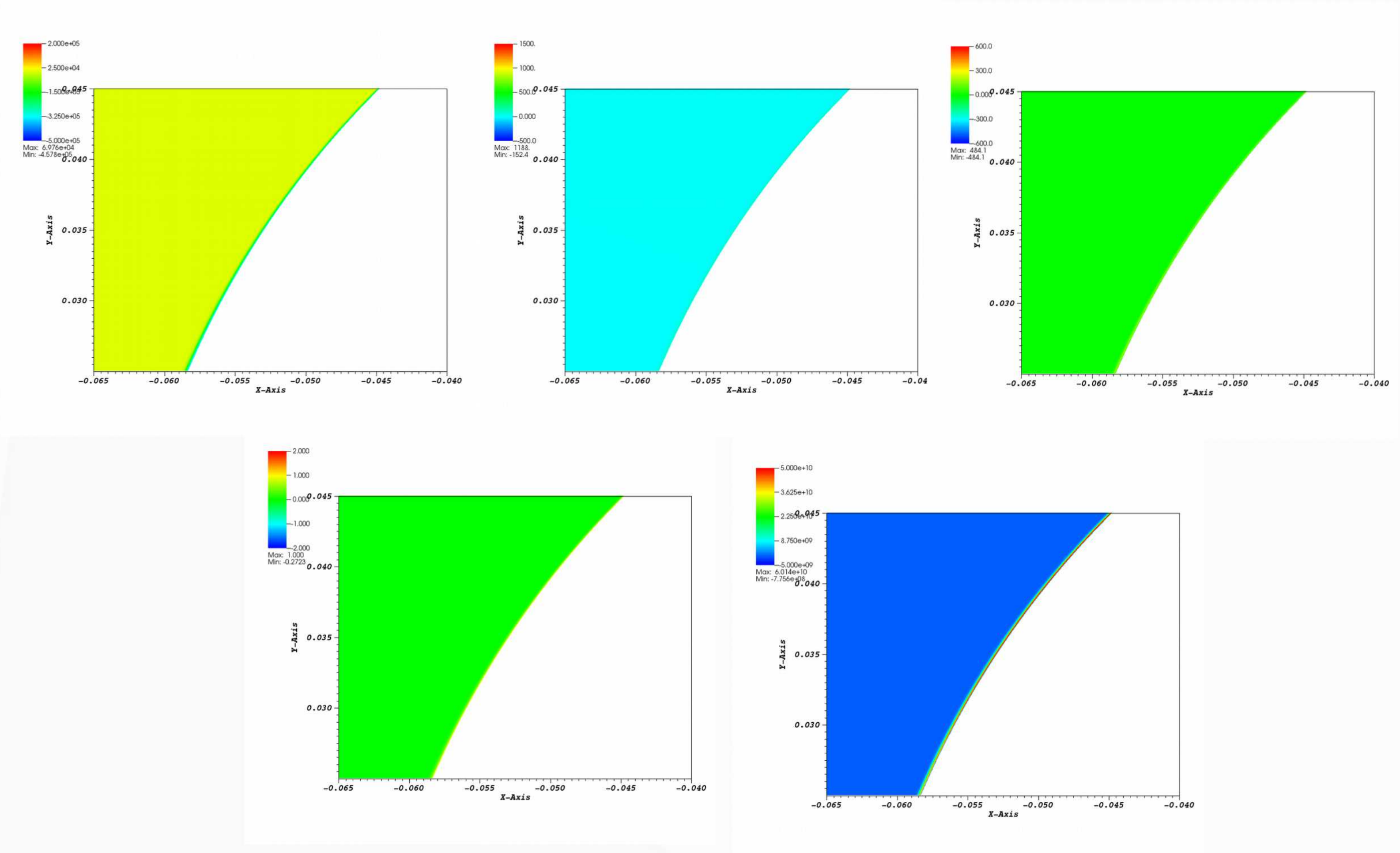

Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

# P1 Adjoint
# 40,000 Q3 Quad Elements



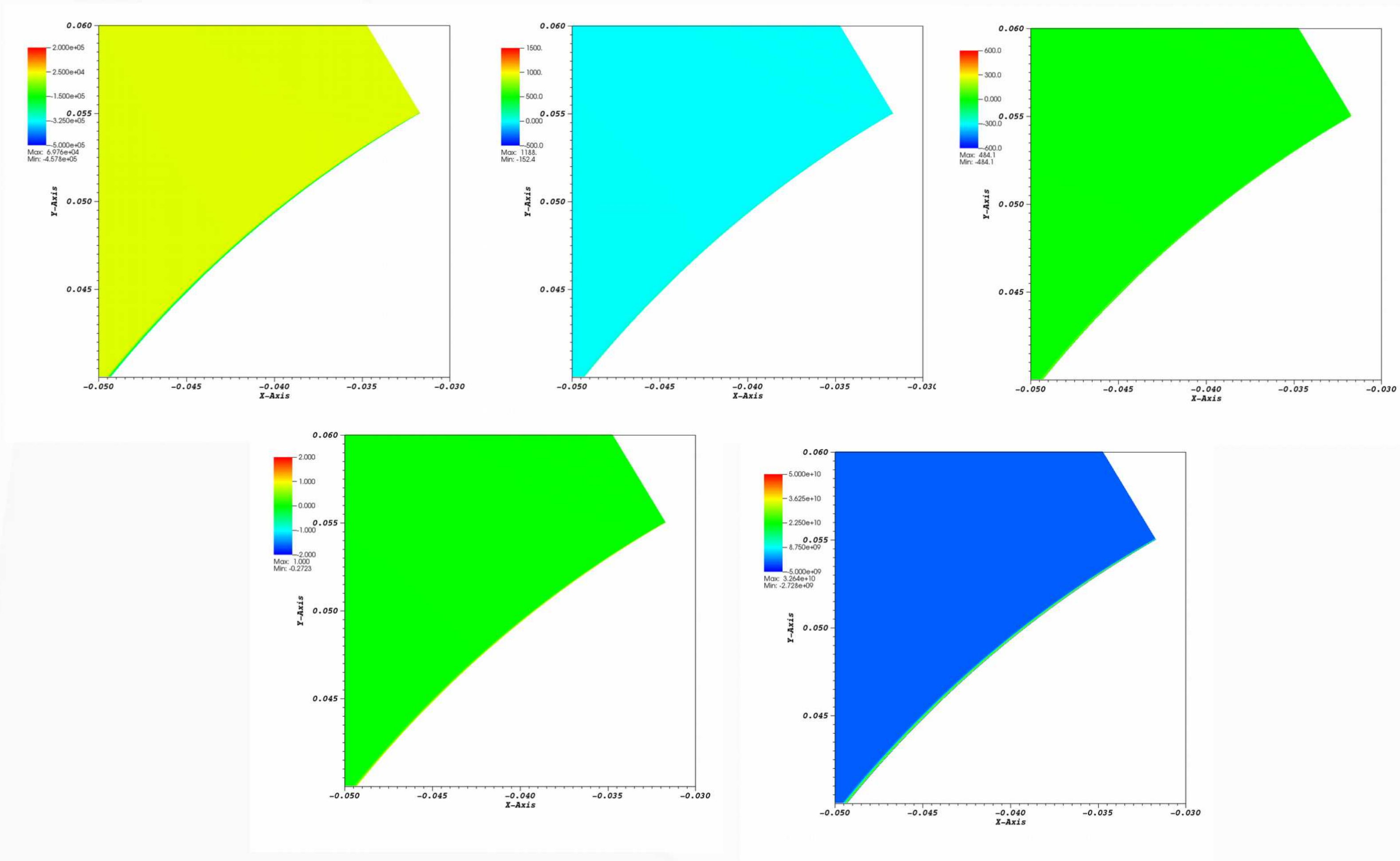
Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 40,000 Q3 Quad Elements



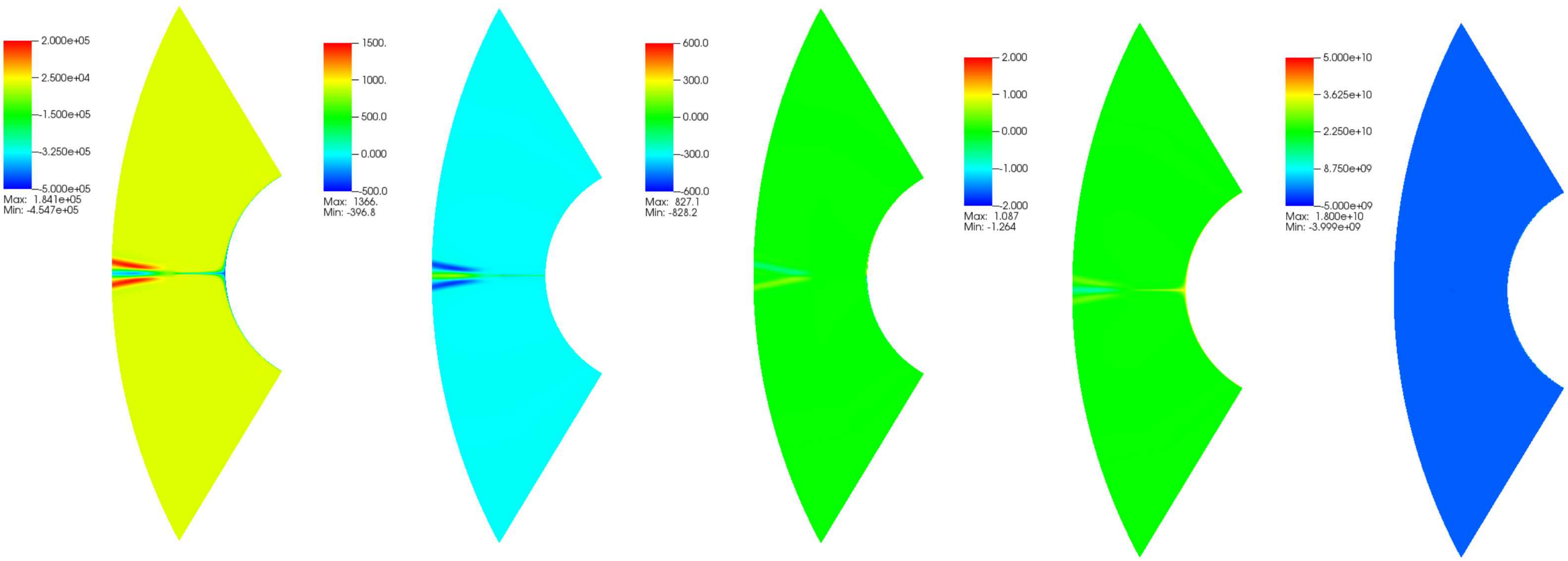

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 40,000 Q3 Quad Elements



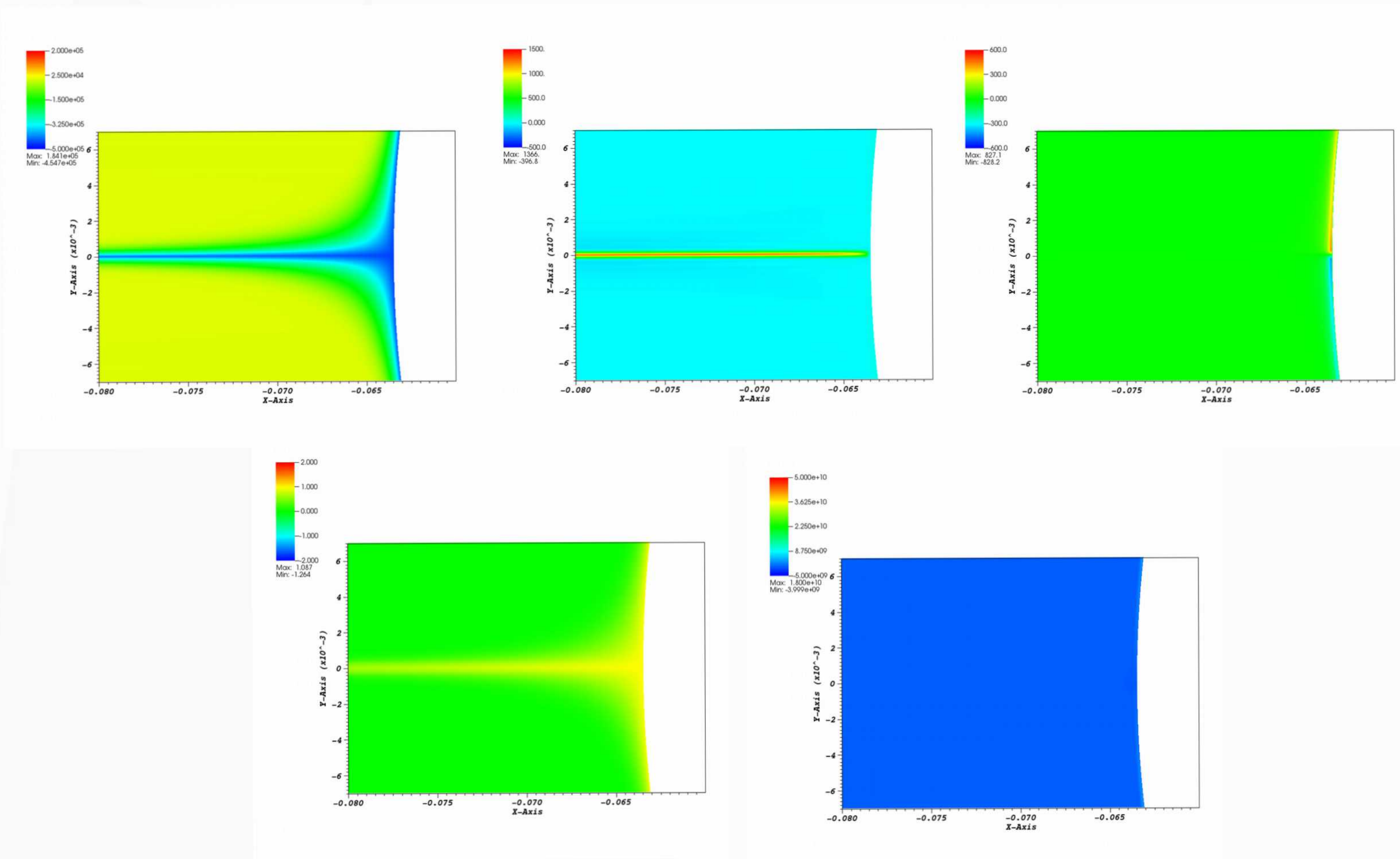

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 160,000 Q3 Quad Elements



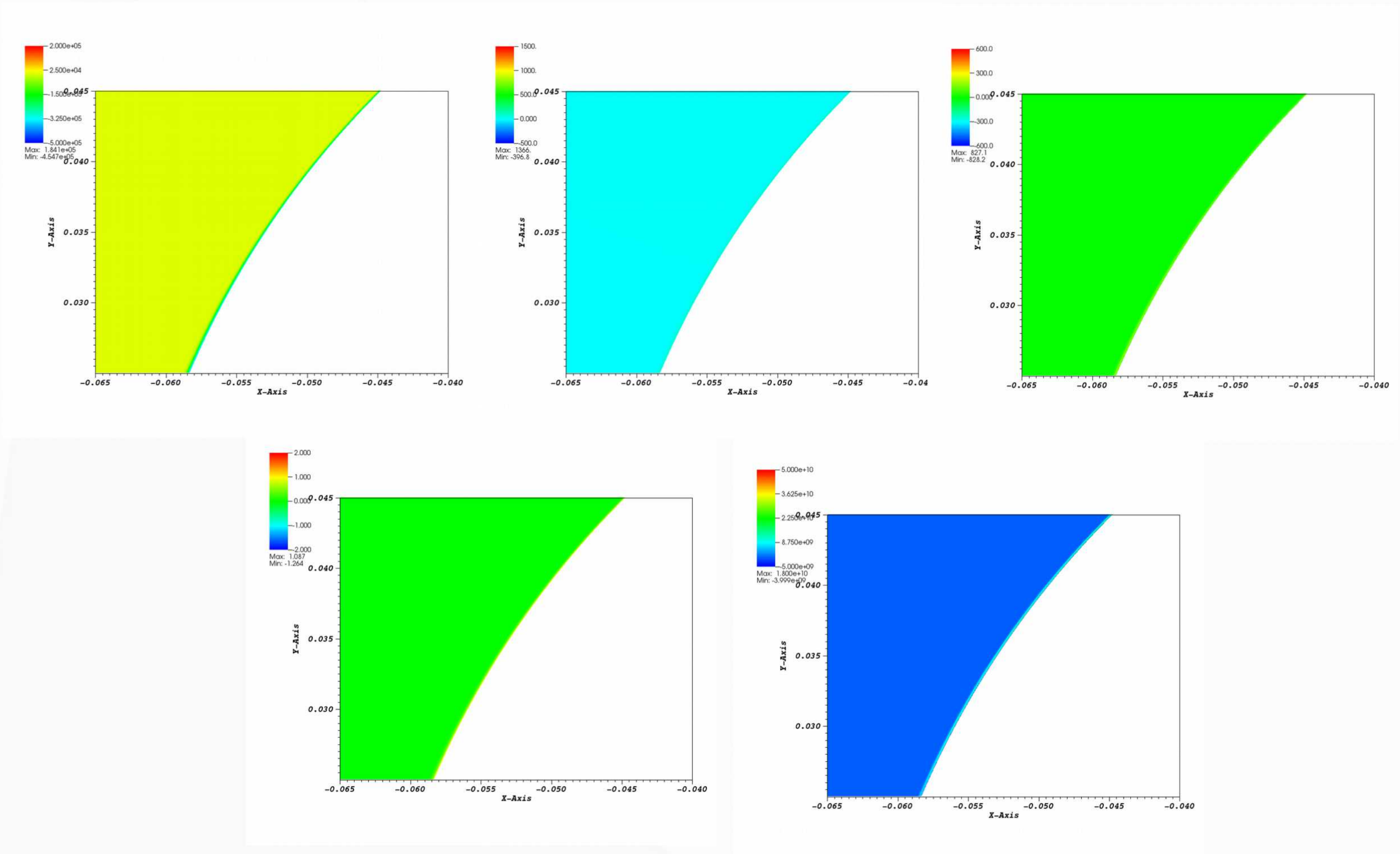

Adjoint for Integrated Heat Flux, Left to Right: Continuity, X-Momentum, Y-Momentum, Total Internal Energy, Sensor Equations

# P1 Adjoint
# 160,000 Q3 Quad Elements



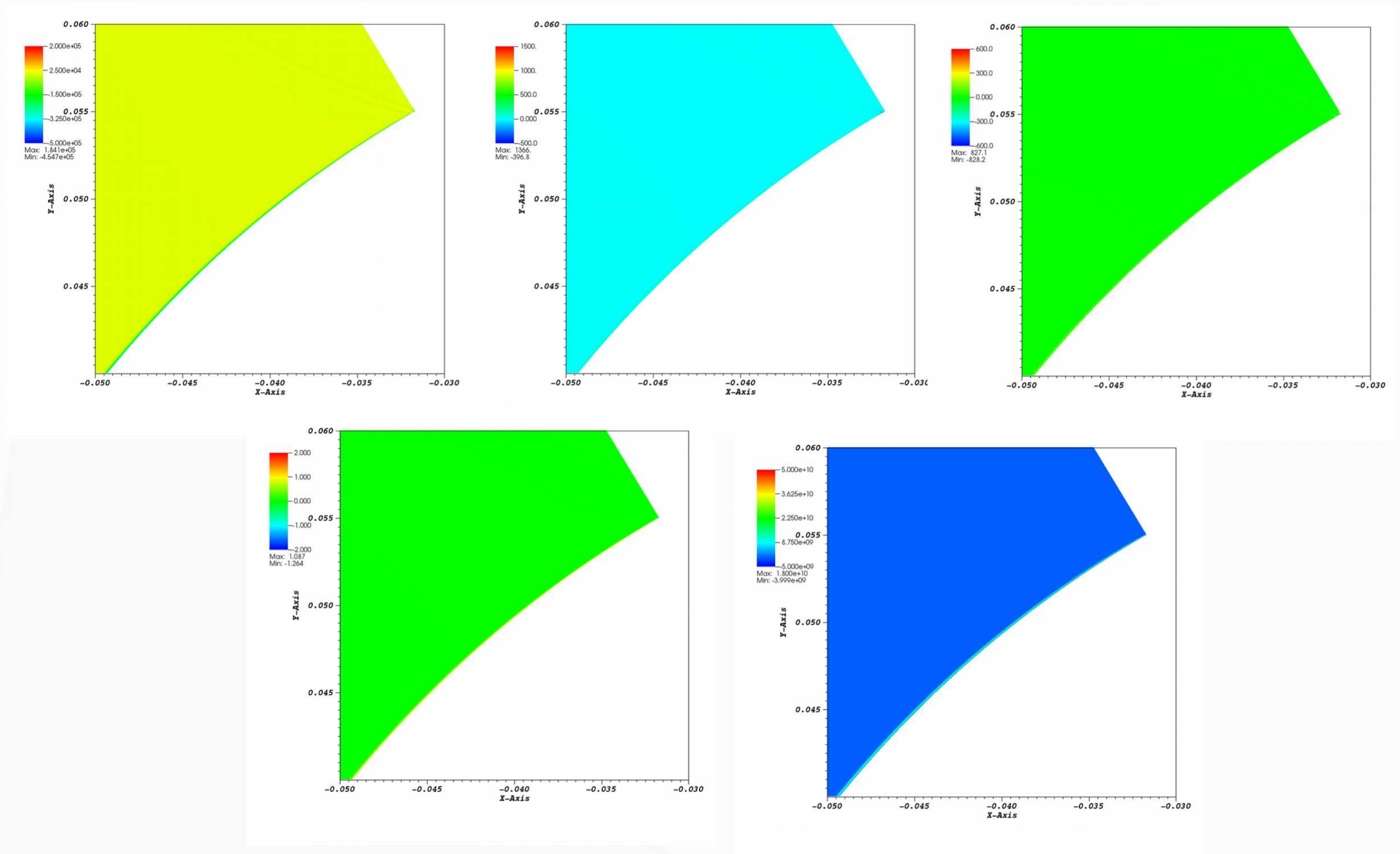

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 160,000 Q3 Quad Elements



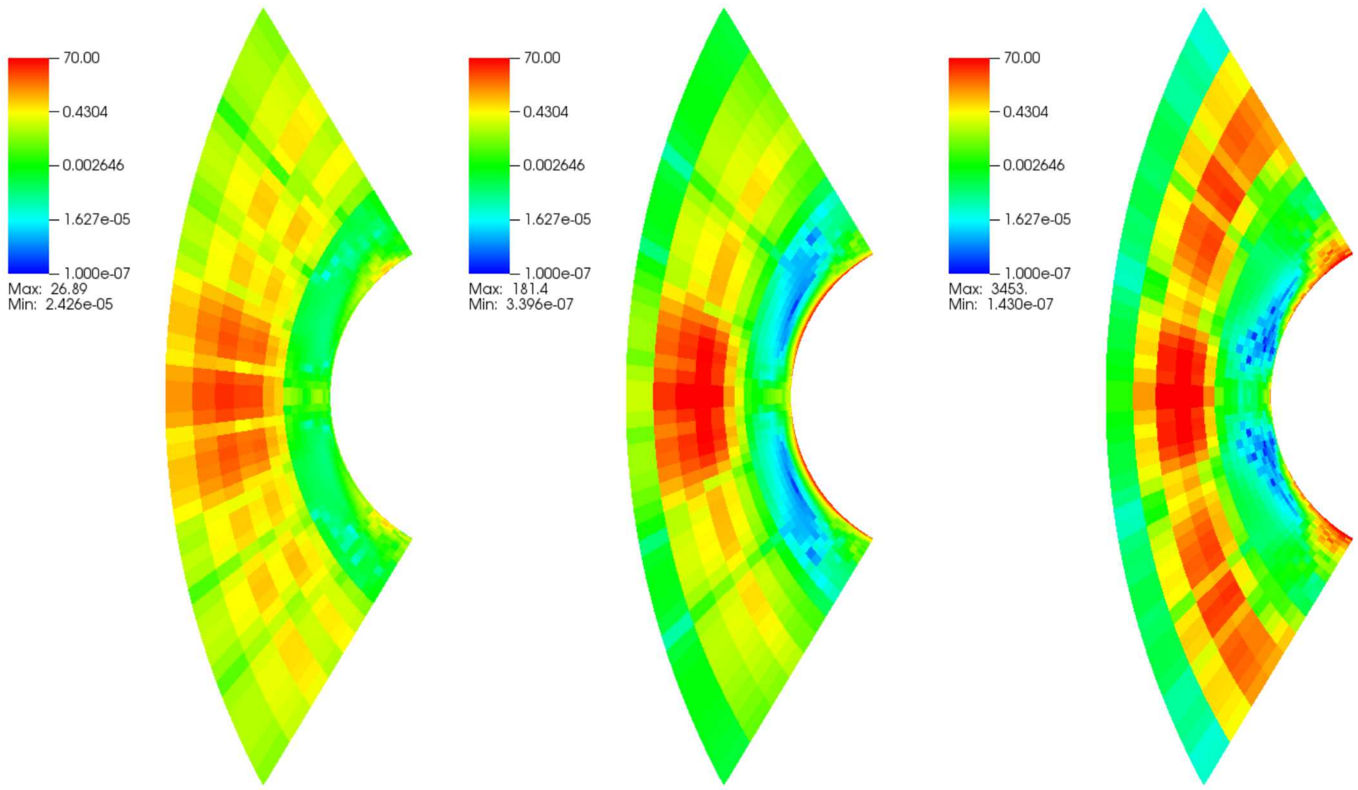

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# P1 Adjoint
# 160,000 Q3 Quad Elements



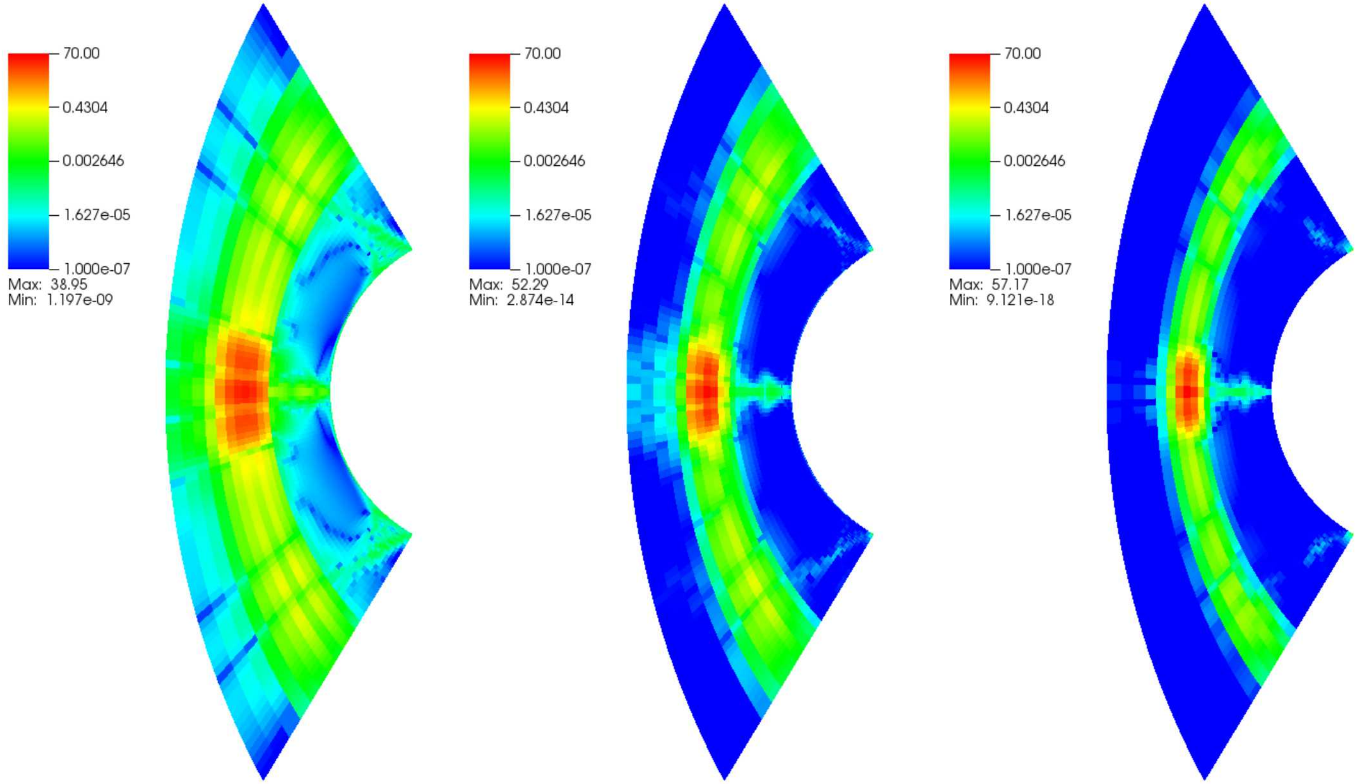

Adjoint for Integrated Heat Flux
Top Row Left to Right: Continuity, X-Momentum, Y-Momentum
Bottom Row Left to Right: Total Internal Energy, Sensor Equations

# Error Indicator
# 2,500 Q3 Quad Elements



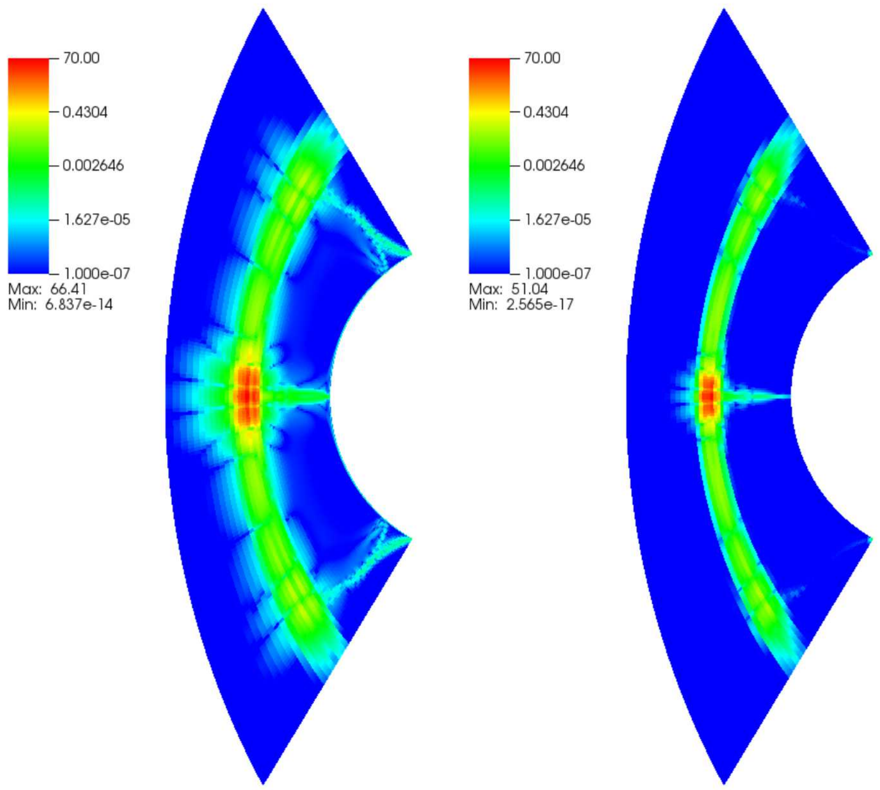

Error Indicator for Integrated Heat Flux

# Error Indicator
# 10,000 Q3 Quad Elements



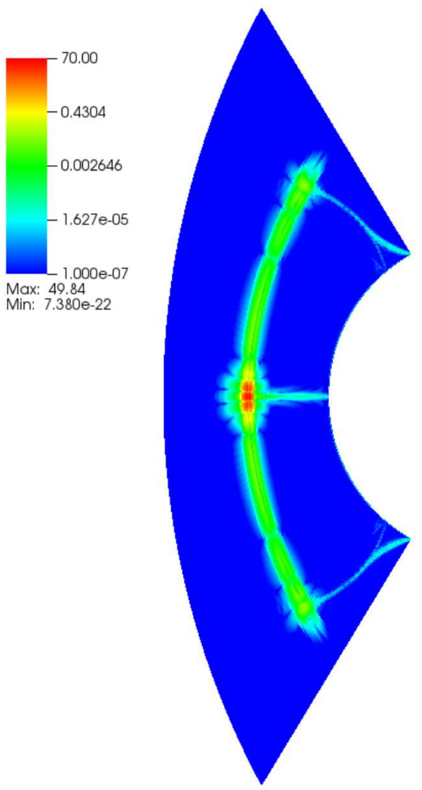

Error Indicator for Integrated Heat Flux, Left to right: P1, P2, P3

# Error Indicator
# 40,000 Q3 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1, P2

# Error Indicator
# 160,000 Q3 Quad Elements




Error Indicator for Integrated Heat Flux, Left to right: P1