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# Hierarchical material properties in the solution to Maxwell's equations: Theory and applications in applied geophysics

## Summary

A long–standing problem in computational electromagnetics is the search for efficient methods to minimize the resource burden in modeling scenarios where volumetrically insignificant scatterers, whether natural or anthropogenic, have a measureable and spatially extensive signature over length scales many times larger than the size of the scatterer itself. The problem is particularly acute when the scatterer is large in one (or more) physical dimension in comparison to the other(s). That is, the scatterer is either long and slender or wide and thin. Examples include conductive infrastructure (pipes, rails, cables), fractures, laminations, spatially extensive textural lineaments. Brute force three–dimensional modeling of these targets can be computationally explosive because they typically consume a disproportionately large number of elements in a given numerical discretization when compared to the elements required for the surrounding Earth model. In many cases, simulation of "real world" geoscenarios is simply intractable without significant and potentially compromising assumptions.

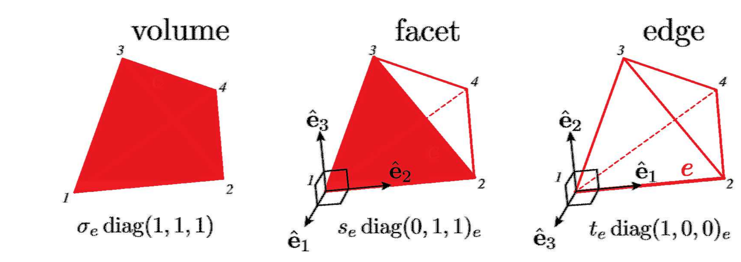
We review here a recently proposed (Weiss, 2017) mathematical architecture for meeting this computational challenge – the hierarchical material properties representation – and discuss its implementation for finite element analysis of Maxwell's equations. Briefly stated, the representation decomposes the electrical conductivity model into discrete elements which reside over the volumes, facets and edges of an unstructured discretization. Hence, long slender features are economically represented by sets of connected edges throughout the mesh whereas thin sheets are done by connected facets, thus providing a flexible structure for arbitrary model representation without an excessive number of very small volume elements to define the scatterer. This representation offers an intriguing approach toward quantification of the electromagnetic character of both singular scatterers and composites of many, and could provide a means for clarifying the mesoscale where various upscaling theories (homogenization, fractional calculus, etc.) meet their discrete many-body counterparts.

## Hierarchical Finite Element Method (*Hi*-FEM)

**Poster S2.2-P234**

### Hierarchical Decomposition of Material Properties

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{e=1}^{N_V}\sigma_e\boldsymbol{\psi}_e^V(\boldsymbol{x}) + \sum_{e=1}^{N_F} s_e\boldsymbol{\psi}_e^F(\boldsymbol{x}) + \sum_{e=1}^{N_E} t_e\boldsymbol{\psi}_e^E(\boldsymbol{x})$$




$$\boldsymbol{\psi}_e^V(\boldsymbol{x}) = \mathrm{diag}(1,1,1)\begin{cases}1 & \text{if } \boldsymbol{x}\in\text{volume } e\\ 0 & \text{otherwise}\end{cases}$$

$$\boldsymbol{\psi}_e^E(\boldsymbol{x}) = \mathrm{diag}(1,0,0)_e\begin{cases}1 & \text{if } \boldsymbol{x}\in\text{edge } e\\ 0 & \text{otherwise}\end{cases}$$

$$\boldsymbol{\psi}_e^F(\boldsymbol{x}) = \mathrm{diag}(0,1,1)_e\begin{cases}1 & \text{if } \boldsymbol{x}\in\text{facet } e\\ 0 & \text{otherwise}\end{cases}$$

This hierarchy of material distributions is made possible by using rank-2 tensor basis functions – an extension of the early work in 3D anisotropy by Weiss and Newman (Geophysics, 2002, 2003)

The tensor representation keeps the material properties local to the edges and facets in the Finite Element weak formulation / bilinear form.

Whereas $\sigma_e$ is the electrical conductivity we're accustomed to using, the quantities $s_e$ and $t_e$ are conductivity-thickness and -area products, respectively, for the anomalous conductivity associated with facets and edges.

### Assembly and solution of the linear system

Poisson Eq for electro/magnetostatics

$$-\nabla\cdot(\boldsymbol{\sigma}\cdot\nabla u) = f \qquad \int_\Omega \nabla v\cdot(\boldsymbol{\sigma}\cdot\nabla u)\;dx^3 = \int_\Omega vf\;dx^3$$

$$\boldsymbol{\sigma} = \mathrm{diag}(0,\sigma,\sigma) \qquad \nabla v\cdot(\boldsymbol{\sigma}\cdot\nabla u) = \sigma\nabla_{23}v\cdot\nabla_{23}u$$

$$\boldsymbol{\sigma} = \mathrm{diag}(\sigma,0,0) \qquad \nabla v\cdot(\boldsymbol{\sigma}\cdot\nabla u) = \sigma\nabla_1 v\cdot\nabla_1 u$$

…thus ensuring that the facet and edge material properties are local and not distributed over the tetrahedral volume.

Variational formulation: $\int_\Omega \nabla v\cdot(\boldsymbol{\sigma}\cdot\nabla u)\;dx^3 = \int_\Omega vf\;dx^3$

3D inner products collapse to 2D and 1D inner products

$$\int_\Omega \nabla v\cdot\left[\sum_{e=1}^{N_V}\sigma_e\boldsymbol{\psi}_e^V(\boldsymbol{x})\right]\nabla u\;dx^3 = \sum_{e=1}^{N_V}\sigma_e\int_{V_e}\nabla v\cdot\nabla u\;dx^3 = \sum_{e=1}^{N_V}\sigma_e\,\mathbf{v}_e^{\mathrm{T}}\mathbf{K}_e^4\mathbf{u}_e$$

$$\int_\Omega \nabla v\cdot\left[\sum_{e=1}^{N_F}s_e\boldsymbol{\psi}_e^F(\boldsymbol{x})\right]\nabla u\;dx^3 = \sum_{e=1}^{N_F}s_e\int_{F_e}\nabla_{23}v\cdot\nabla_{23}u\;dx^2 = \sum_{e=1}^{N_F}s_e\,\mathbf{v}_e^{\mathrm{T}}\mathbf{K}_e^3\mathbf{u}_e$$

$$\int_\Omega \nabla v\cdot\left[\sum_{e=1}^{N_E}t_e\boldsymbol{\psi}_e^E(\boldsymbol{x})\right]\nabla u\;dx^3 = \sum_{e=1}^{N_E}t_e\int_{E_e}\nabla_1 v\cdot\nabla_1 u\;dx = \sum_{e=1}^{N_E}t_e\,\mathbf{v}_e^{\mathrm{T}}\mathbf{K}_e^2\mathbf{u}_e$$

Global stiffness matrix is a sum of 3D, 2D and 1D element stiffness matrices.

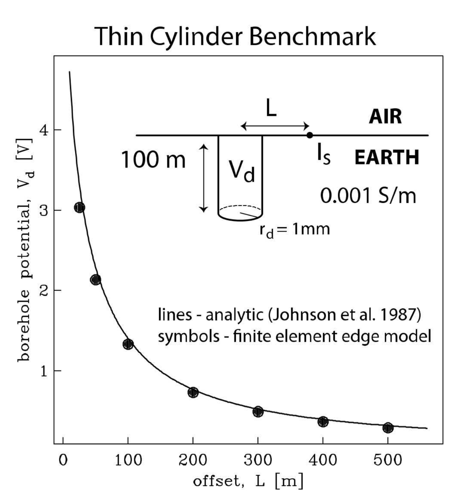
$$\mathbf{K}\mathbf{u} = \mathbf{b}$$

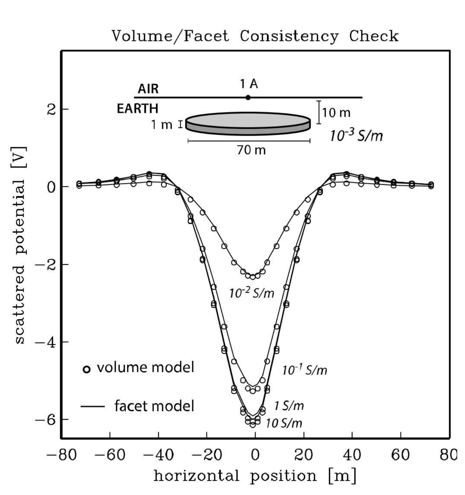
$$\mathbf{K} = \sum_{e=1}^{N_V}\sigma_e\mathbf{K}_e^4 + \sum_{e=1}^{N_F}s_e\mathbf{K}_e^3 + \sum_{e=1}^{N_E}t_e\mathbf{K}_e^2$$

Solve iteratively with Jacobi scaled conjugate gradients and on-the-fly matrix assembly (Weiss, 2001)

## Benchmarking

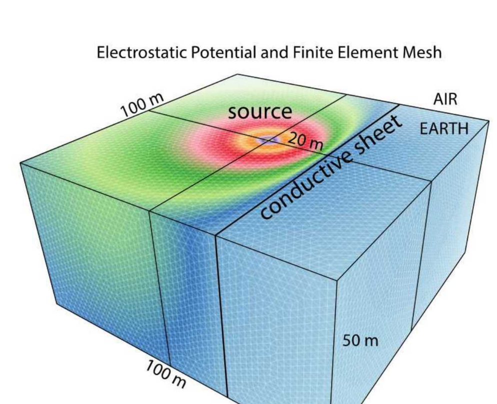
### Analytic Comparison and Facet/Volume Compatibility

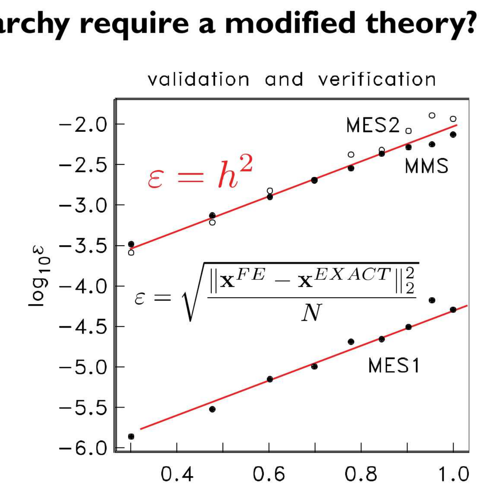



Benchmarking and internal consistency checks show that for thin conductors, the facet/edge representation achieves acceptable accuracy over a range of geometries and material properties. *Weiss. (2017)*

### Convergence Analysis: Does the hierarchy require a modified theory?



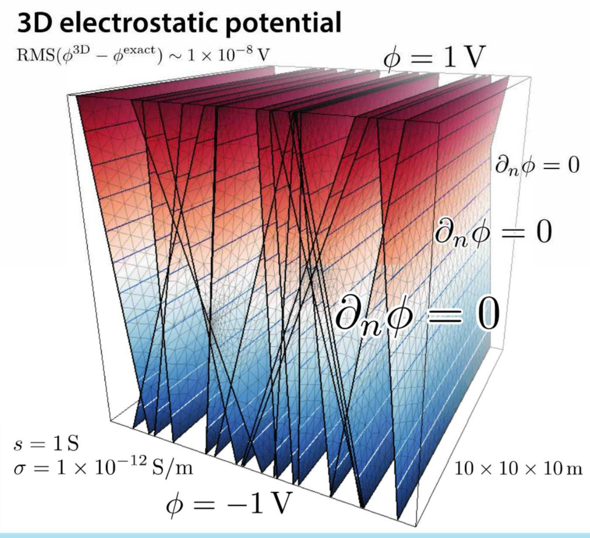




*Weiss and van Bloemen Waanders (2018, in review)*

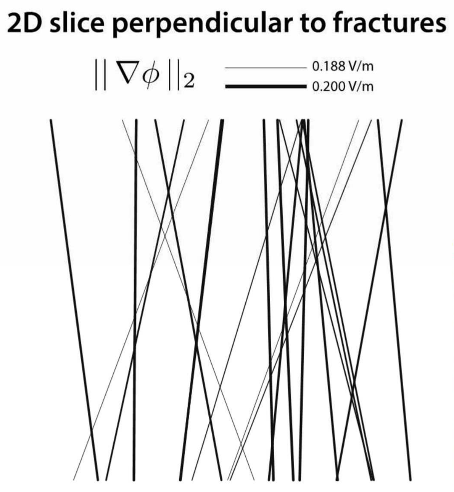
No. Error analysis for methods of exact solution (MES1: monopole in a wholespace; MES2: vertical conductive sheet) and manufactured solution (MMS: $u(\mathbf{r}) = \exp\left[-\left(r/a\right)^2\right]$, $a = 20\,\mathrm{m}$, $\sigma = 1\,\mathrm{S}/m$ ) shows the expected reduction in overall RMS error with mesh refinement *h*.

### Reduction to Analytics for Problems Possessing Symmetry

3D electrostatic potential




2D slice perpendicular to fractures

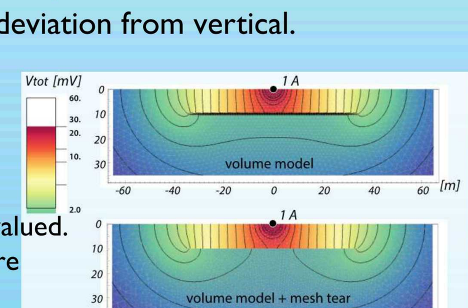



*Beskardes and Weiss (2018)*

2D fracture systems driven by boundary conditions in a perfectly resistive medium can admit analytic solutions. RMS errors for HiFem 3D solutions are on the order of $10^{-8}$V for the problem shown above, where the fracture current is a function of the cosine of its angular deviation from vertical.

### What about thin resistive features?

Thin resistive features result in strong potential Differences between adjacent surfaces and in the infinitely thin limit means that the potential is double-valued. We accommodate this through tears in the mesh where physically coincident surfaces are twice discretized to represent potentials associated with each side of the surface.

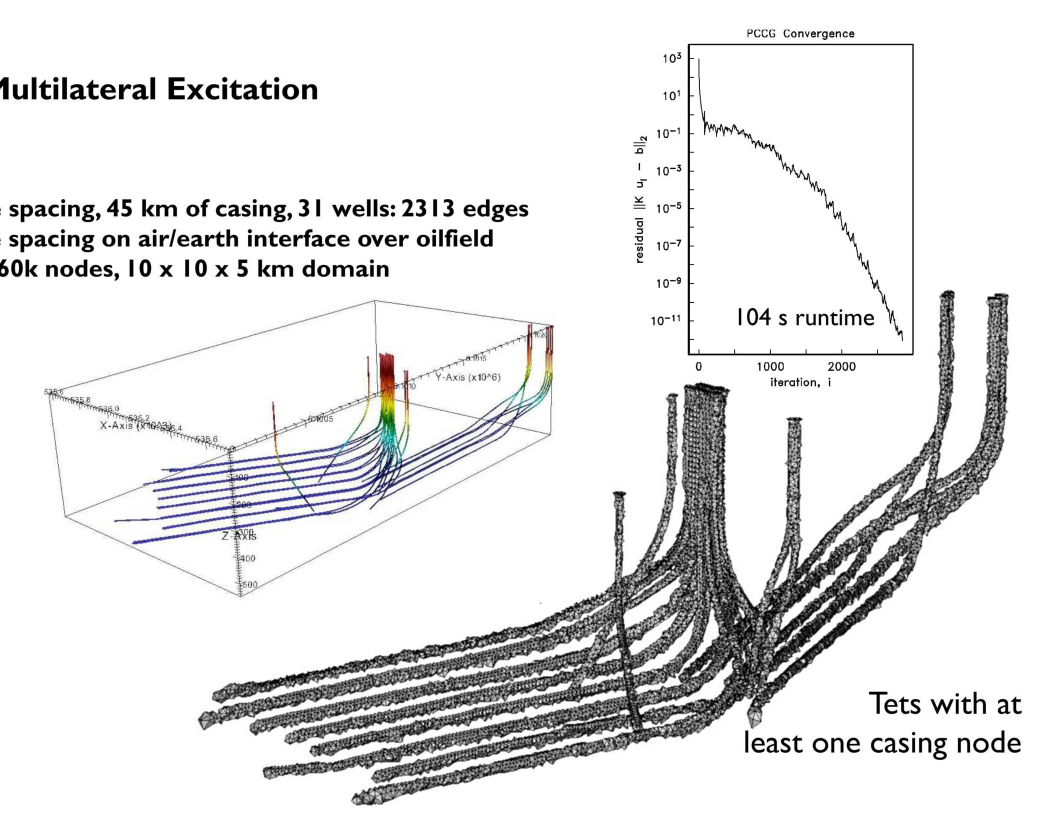



## Examples in Applied Geophysics

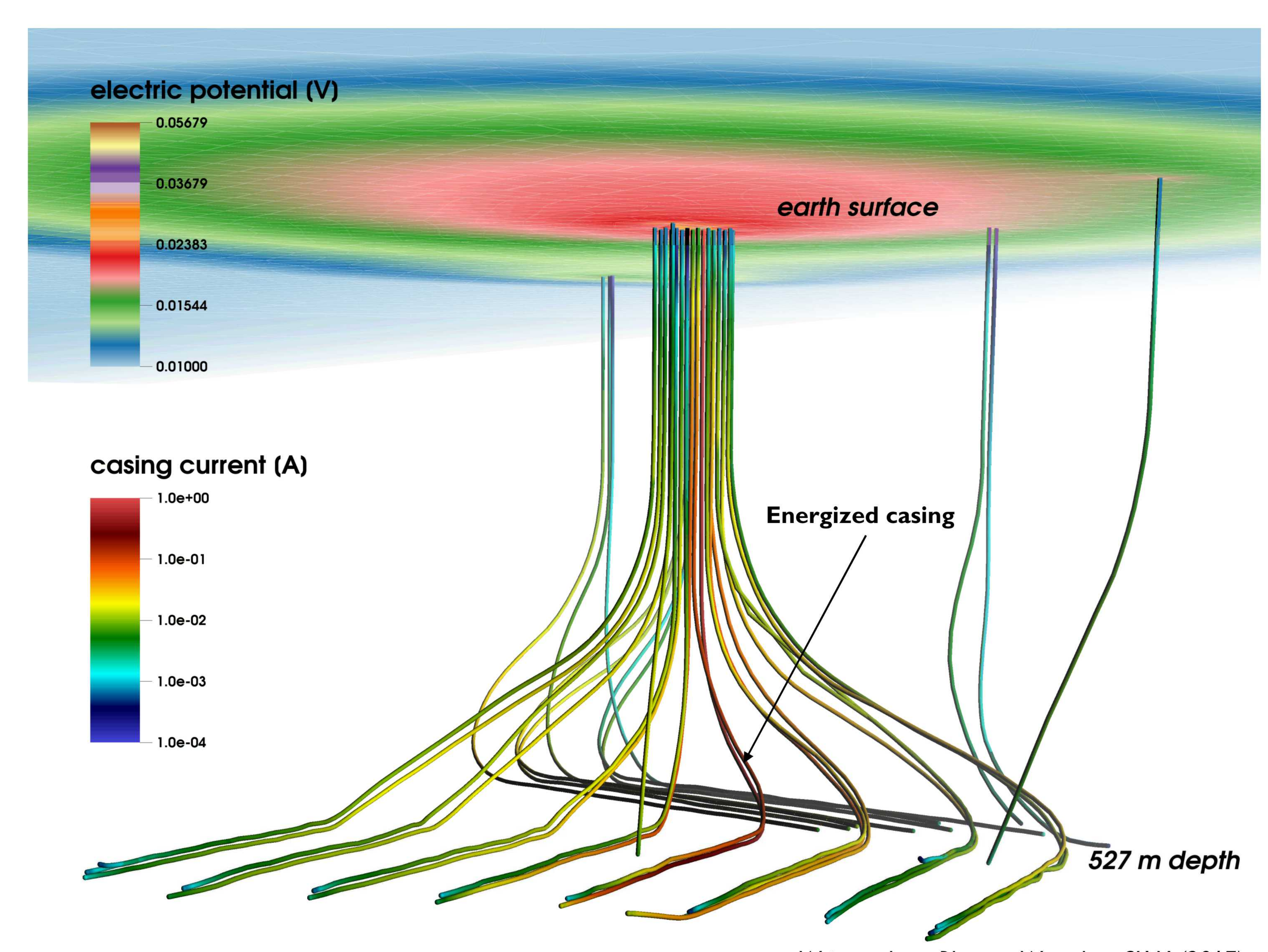
*Please see Poster S2.2-P234 for more applications in fracture characterization!!!*

### SAGD Multilateral Excitation

**20 m node spacing, 45 km of casing, 31 wells: 2313 edges**
**50 m node spacing on air/earth interface over oilfield**
**332k tets, 60k nodes, 10 x 10 x 5 km domain**

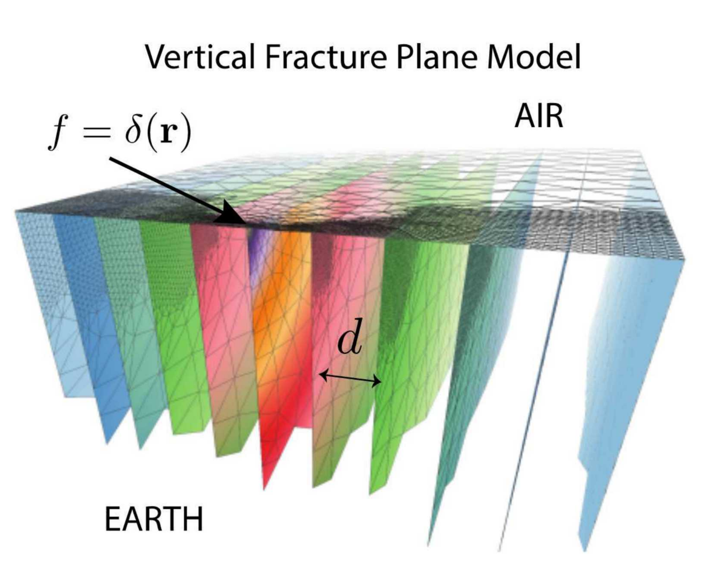


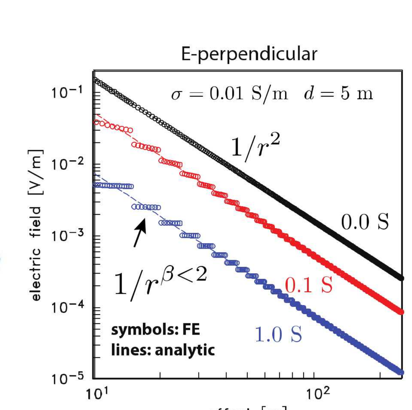

Discretization of complex reservoir completion scenarios is relatively simple do to from "as built" directional logs (LAS files) provided by the driller/service company because the casing need only be represented by a connected set of infinitely thin segments, rather than a complex and computationally costly set of nested cylinders. We generate such meshes using Cubit (cubit.sandia.gov) using an advancing front method with LAS data points as internal volumetric constraints.




*Weiss and van Bloemen Waanders, SIAM (2017)*

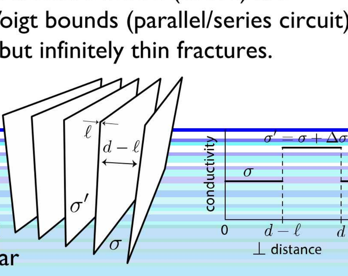
### Anomalous, non-classical power law response from fractures?

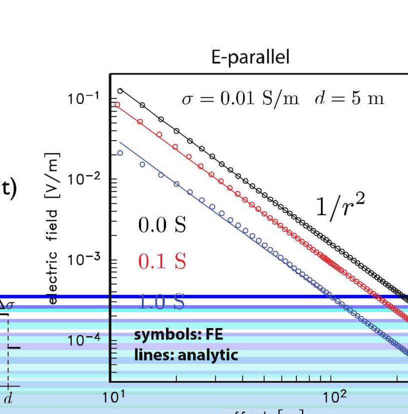



Analytic solutions for the discrete fracture model (above) are computed using the anisotropic Voigt bounds (parallel/series circuit) in the limit of finite conductance but infinitely thin fractures.

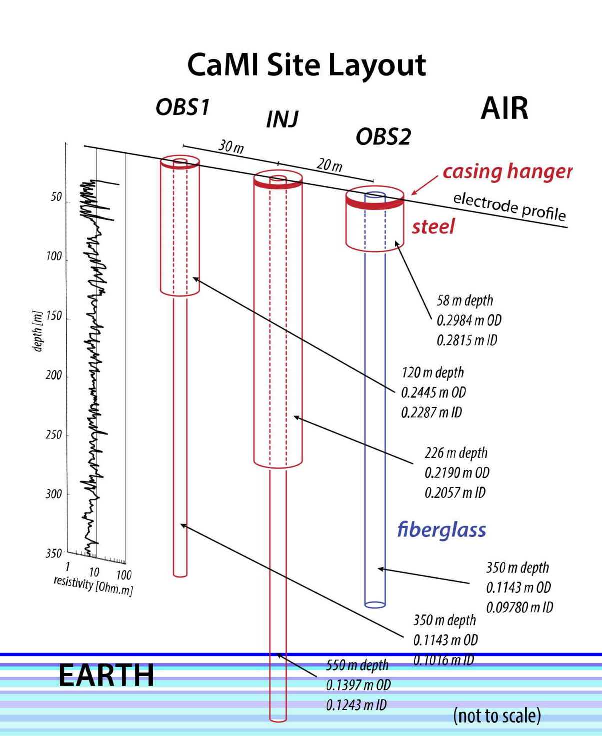
3D HiFEM results in the fracture parallel direction follow analytics (right, bottom), however, an anomalous power law decay is observed between fractures along surface profiles in the perpendicular direction (right, top)

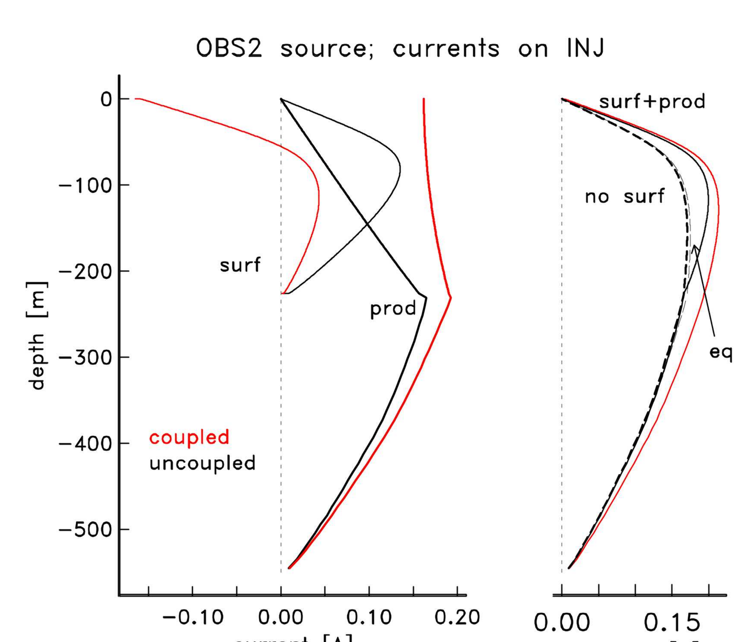





*Weiss and Beskardes. SEG Ann. Mtg. (2018, accepted)*

### Completion Design Effects on Energized Casing Response




*Weiss et al. SEG Ann. Mtg. (2018, accepted)*

Direct excitation of the OBS2 well results in wildly different patterns of current flow in the INJ well, depending on whether the surface and production casings are electrically coupled through the casing hanger (middle). The effective current system (right) bears little resemblance to that in a simplified, single casing by either ignoring the surface casing (right, "no surf"), or including it through an equivalent conductance (right, "eq").

## Conclusion

- The hierarchical material properties concept approximates small scale features by equivalent conductances on the facets and edges of an arbitrary discretization.
- This concept has been applied to finite element analysis of Maxwell's Equations in the electrostatic limit.
- The hierarchical finite element method (HiFEM) is shown to possess the expected $h^2$ error convergence for linear nodal elements, consistency with brute force discretization of fine model features, and agreement with various analytic solutions for slender, rod-like conductors and thin, sheet-like conductors.
- The HiFEM architecture has been applied to problems in applied geophysics for simulating well-casing response, fracture characterization and casing-design analysis at a minimal computational cost where, otherwise, it would be computationally explosive if tractable at all.
- Careful inspection of fracture response shows evidence of "anomalous" non-classical field decay, reminiscent of prior field studies and a possible segue toward reconciliation with alternative mathematical frameworks for macroscopic transport phenomena based on fractional calculus theory.

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**Essential References**

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