PDE constrained optimization for digital image correlation

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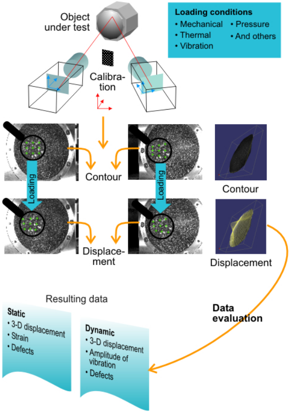
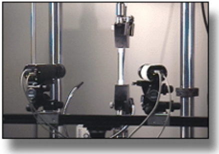
Digital Image Correlation (DIC)

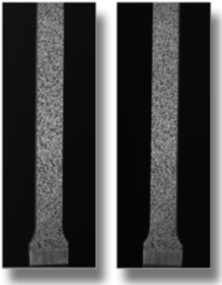
- DIC is a full-field image analysis method based on grey value digital images
- Determine the deformation, strain of an object subjected to a load
- Courtesy Dantec Dynamics

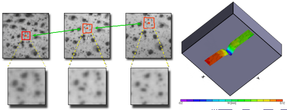


Basic DIC process

- Two cameras photograph a "speckled" dogbone subjected to a load
- A speckling (or contrast) occurs by spraying a mist of black paint over a white dogbone
- DIC tracks the speckles by comparing the sequence of photos, or images
- Extract deformation, strain
- Courtesy Correlated Solutions







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Problem of interest

- Given a sequence of images, we are interested in the deformation and strain
- The rate of deformation—velocity—is given by the optical flow constraint
$$\frac{\partial}{\partial t}\phi + \mathbf{b} \cdot \nabla \phi = 0$$
where ϕ represents the image intensity
- Velocity $\mathbf{b}$ is assumed to be spatially varying but not temporally
- **Goal**: Estimate velocity $\mathbf{b}$ given image intensity data $\hat{\phi}$

Deformation : $\tau \mathbf{b}$

Strain : $\tau^2 (\nabla \mathbf{b})^T \nabla \mathbf{b} - \mathbf{I}$

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Modeling assumption

- Optical flow constraint $\frac{\partial}{\partial t}\phi + \mathbf{b} \cdot \nabla\phi = 0$ assumes that the trajectory

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{b}\big(\mathbf{x}(t)\big) & 0 < t \leq \tau \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

  is well-behaved
- Measured image intensity $\hat{\phi}$ is subject to error
- We model this error by assuming that the trajectory is given by Itō's stochastic differential equation

$$\begin{cases} d\mathbf{X}_t = \mathbf{b}\,dt + \sqrt{2\sigma}\,d\mathbf{W}_t & 0 < t \leq \tau \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

  with the "diffusion enhanced" optical flow constraint $\frac{\partial}{\partial t}\phi + \mathbf{b} \cdot \nabla\phi = \sigma\Delta\phi$

Model fit

- Image intensity evolves according to the diffusion equation

$$\begin{cases} \dfrac{\partial}{\partial t}\phi + \mathbf{b} \cdot \nabla \phi = \sigma \, \Delta \phi & 0 < t \leq \tau \text{ over } \Omega_t \,, \\ \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \,, & \end{cases}$$

- The difference, or "model fit"

$$\phi(\mathbf{x}, t) - \hat{\phi}(\mathbf{x}, t) = \mathbb{E}^{\mathbf{x}} \left[\phi_0(\mathbf{X}_t) \right] - \phi_0(\mathbf{X}_t)$$

is the intensity fluctuation about the mean of a trajectory

One dimension

- In one dimension, the term

$$\phi(x,t) = \mathbb{E}^x\left[\phi_0(X_t)\right] = \frac{1}{\sqrt{4\pi\sigma\, t}}\int_{\mathbb{R}} e^{-(x-y-bt)^2/(4\sigma\, t)}\phi_0(y)\,dy$$

  is the expectation of the random variable $\phi_0(X_t)$ conditioned on the point x
- By Itō's Lemma, the data $\hat{\phi}(\mathbf{x},t) = \phi_0(\mathbf{X}_t)$ (a random variable!) is also given by a stochastic differential equation
- Can use inference methods but we use a PDE constrained optimization approach

PDE constrained optimization

$$(\phi_*, \mathbf{b}_*) = \underset{(\phi,\mathbf{b})\in(\mathcal{P}\times\mathcal{B})}{\operatorname{argmin}} \left(\|\phi - \hat{\phi}\|^2_{L^2(\Omega)\times(0,\tau)} + \frac{\beta}{2}\tau\|\mathbf{b}\|^2_{L^2(\Omega)} \right)$$

$$\text{subject to} \begin{cases} \dfrac{\partial}{\partial t}\phi + \mathbf{b}\cdot\nabla\phi = \sigma\,\Delta\phi & 0 < t \le \tau \\ \phi(\mathbf{x},0) = \phi_0(\mathbf{x}) & \mathbf{x}\in\Omega \\ \nabla\cdot\mathbf{b} = 0 & \text{over } \Omega \end{cases}$$

- Ito & Kunish (1997) analyzed the steady-state version of the above optimization problem[1] (but no mention of DIC)
- The solenoidal constraint needed because otherwise the velocity is unique up to a rotational vector
- Spaces $\mathcal{P}, \mathcal{B}$ depend upon the boundary conditions for the intensity ϕ and solenoidal constraint

[1] Thanks to Drew Kouri for help digesting the paper

Optimization Problem

- Ito & Kunish (1997) showed the (steady-state) problem is well-posed given homogeneous Dirichlet boundary conditions for the intensity ϕ
- Important that the model fit $\|\phi - \hat{\phi}\|^2_{L^2(\Omega)\times(0,\tau)}$ is not large
- Ito & Kunish analyzed gradient and SQP methods
- Summer intern Carlos Garavito demonstrated that the functional is sensitive and robust via a slew of stochastic simulations—distance traveled (velocity $\times$ time) must be sufficient to induce a minimum

Initialization

Important issue not discussed by Ito & Kunish is that of initialization—how do you pick an initial velocity that satisfies the diffusion equation?

- Constrained optimization problem

$$\mathbf{b}_0 = \operatorname*{arg\,min}_{\mathbf{b}\in H_{\text{div}}(\Omega)} \|R(\mathbf{b})\|^2_{L^2(\Omega)} \quad \text{subject to } \nabla\cdot\mathbf{b} = 0 \quad \text{in} \quad \Omega$$

  where

$$R(\mathbf{b}) = \frac{\partial\psi}{\partial t} + \mathbf{b}\cdot\nabla\psi - \sigma\Delta\psi$$

- Technical detail: Need to go from discrete data $\hat{\phi}(\mathbf{x}_i, 0)$ to the initial condition ψ_0

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Optimality system

Given $\psi \in H_0^1(\Omega)$, find $(\mathbf{b}_0, u) \in H_{\text{div}}(\Omega) \times L^2(\Omega)$

$$\int_\Omega (\nabla\psi \cdot \mathbf{w})(\nabla\psi \cdot \mathbf{b}_0)dx - \int_\Omega u\nabla \cdot \mathbf{w}dx = \int_\Omega \left(\sigma\Delta\psi - \frac{\partial\psi}{\partial t}\right) \nabla\psi \cdot \mathbf{w}dx \quad \mathbf{w} \in H_{\text{div}}(\Omega)$$

$$\int_\Omega v(\nabla \cdot \mathbf{b}_0)dx = 0 \quad v \in L^2(\Omega)$$

- Velocity, or drift, is contained in the quantity $\sigma\Delta\psi - \frac{\partial\psi}{\partial t} = \mathbf{b} \cdot \nabla\psi - R(\mathbf{b}) \approx \mathbf{b} \cdot \nabla\psi$ when $R(\mathbf{b}) \approx 0$ i.e., model fit is small
- Our assumption is that a trajectory is “modeled” by an Itō SDE so that the residual R is small; otherwise we need to add other Markovian noise, e.g., a jump process

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Ill-posed problem

- LBB satisfied but coercivity is not because

$$a(\mathbf{w},\mathbf{w}) := \int_\Omega (\nabla\phi\cdot\mathbf{w})^2 dx \ngeq \alpha\|\mathbf{w}\|_{H_{\text{div}}(\Omega)}$$

  for some $\alpha > 0$
- One possible regularization is via the mixed formulation of the Poisson problem

$$a_\mu(\mathbf{w},\mathbf{w}) := a(\mathbf{w},\mathbf{w}) + \mu^{-2}\Big\{(\mathbf{w},\mathbf{w})_{L^2(\Omega)} + (\nabla\cdot\mathbf{w},\nabla\cdot\mathbf{w})_{L^2(\Omega)}\Big\}$$

- Use of Raviart-Thomas elements leads to a stable discrete problem

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Standard DIC approach

- Taylor series approximation for the intensity about the initial condition ϕ_0

$$\phi(\mathbf{x}, t) = \phi_0(\mathbf{x}) - \nabla\phi_0(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x})t + r\big(\mathbf{b}(\mathbf{x})\big)$$

- A least squares approximation leads to the rank one DIC optimality system

$$\big(\nabla\phi_0 \otimes \nabla\phi_0\big)\mathbf{b}_\star = -t^{-1}\nabla\phi_0(\phi - \phi_0) \quad 0 < t < \tau$$

- Regularize by considering a collection of points about **x**
- Standard DIC approach does NOT recognize that the problem is ill-posed, does not enforce a constraint on the rotational component, solves a discrete under-determined least squares problem

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