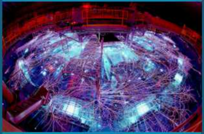
SAND2018-12620C

Dynamics of zonal flows and drift-wave turbulence in the presence of nonlinear wave–wave scattering



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June 11th, 2018
SIAM Conference of Nonlinear Waves and Coherent Structures

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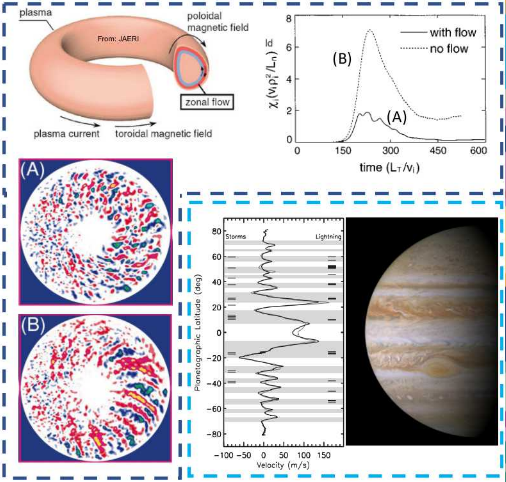
Interaction of small-scale turbulence with large-scale mean fields (structures) is essential to understand.

- In magnetic fusion experiments, large-scale zonal flows (ZFs) are driven by small-scale drift–wave (DW) turbulence.
- ZFs are important because they can shear turbulent eddies and suppress turbulence.[1]
  - Turbulent transport is reduced.
  - Confinement is improved.
- In the case of planetary atmospheres, large-scale structures (zonal jets) can spontaneously emerge from small-scale turbulence (Rossby waves).[2]

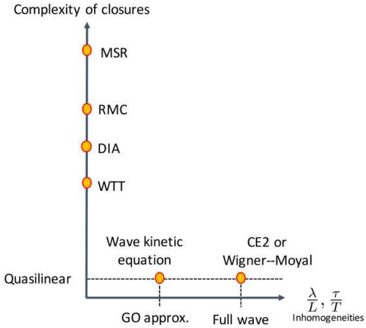


[1] Z. Lin, Science **281**, 1835 (1998); P. W. Terry, Rev. Mod. Phys. **72**, 109 (2000).
[2] A. R. Vasavada and A. P. Showman, Rep. Prog. Phys. **68**, 1935 (2005).

Much work has been done on homogeneous turbulence. Inhomogeneous turbulence has been studied more until recently.

- Most work on turbulence considers the medium to be homogeneous and stationary, for example:
  - WTT, DIA, RMC, and MSR.[3]

- Theories of inhomogeneous turbulence generally use simpler statistical closures.
  - The quasilinear (QL) approximation has been widely used.[4]
  - This approximation neglects wave–wave scattering.



- Quasilinear theories can be classified according to the degree of inhomogeneities.[5]
  - The **wave kinetic equation (WKE)** assumes a scale separation between the medium and the turbulence.
  - The **CE2** and the **Wigner–Moyal** (WM) formalisms do not assume any scale separation.

[3] J. A. Krommes, Phys. Rep. **360**, 1 (2002).

[4] K. Srinivasan and W. R. Young, J. Atmos. Sci. **69**, 1633 (2012); P. H. Diamond, et al., Plasma Phys. Control. Fusion **47**, R35 (2005); B. F. Farrell and P. J. Ioannou, J. Atmos. Sci. **64**, 3652 (2007).

[5] J. B. Parker, J. Plasma Phys. **82**, 595820602 (2016); also, don't miss J. B. Parker's talk during session MS36 next Wednesday.

New insights on DW–ZF interactions have been obtained by using Weyl phase-space representation.[6,7]

- **Key idea:** describe fluctuations as abstract vectors of a Hilbert space.

$$i\partial_t |\tilde{w}\rangle = \hat{H}\, |\tilde{w}\rangle + i\, |\tilde{\xi}\rangle$$

$$\hat{H} \doteq -\beta \hat{p}_x \hat{p}_{\mathrm{D}}^{-2} + \hat{U}\hat{p}_x + \hat{U}'' \hat{p}_x \hat{p}_{\mathrm{D}}^{-2} - i\mu_{\mathrm{dw}}$$

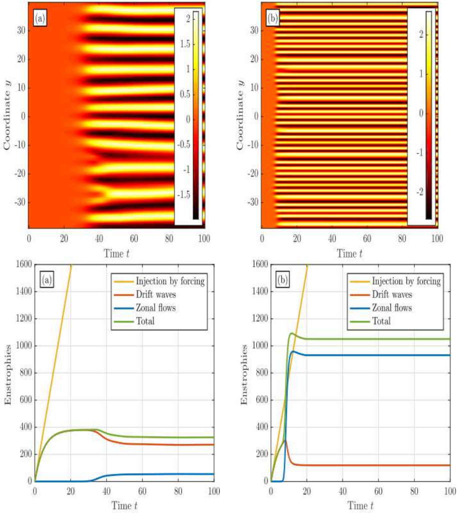
- Upon using the Weyl calculus, one obtains a WM equation for DW–ZF interactions:[8]

$$\partial_t W = \{\{\mathcal{H}, W\}\} + [[\Gamma, W]] + F - 2\mu_{\mathrm{dw}} W$$

$$\partial_t U + \mu_{\mathrm{zf}} U = \frac{\partial}{\partial y} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{p_x}{p_{\mathrm{D}}^2} \star W \star \frac{p_y}{p_{\mathrm{D}}^2}$$

- An improved WKE is obtained in the GO limit. In contrast to the traditional WKE, it conserves total enstrophy and energy, and it describes different dynamics.[7,8]

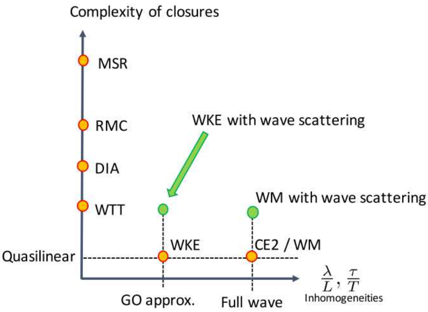
- The ray phase-space approach has led to new insights on DW–ZF interactions.[8]



[6] D. E. Ruiz, J. B. Parker, E. L. Shi, and I. Y. Dodin, Phys. Plasmas **23**, 122304 (2016).
[7] H. Zhu, Y. Zhou, D. E. Ruiz, and I. Y. Dodin, Phys. Rev. E **97**, 053210 (2018).
[8] For more information, don't miss the talk by I. Y. Dodin in session MS43 next Thursday.

DW–ZF interactions have mainly been studied using the quasilinear approximation. Can we go beyond quasilinear?

- QL theory is asymptotically valid in the limit of strong ZFs and weak DWs.
- QL theory breaks down when the system is externally driven stronger.[9]
- Weak turbulence theory (WTT) *perturbatively* captures nonlinear DW collisions.
- In WTT, the nonlinear collisions are described by a wave–wave scattering operator.



We recently obtained a WKE-based model that includes nonlinear DW–ZF interactions, as well as turbulent wave–wave scattering.

[9] S. M. Tobias and J. B. Marston, Phys. Rev. Lett. **110**, 104502 (2013).

Physical model

- Our derivation is based on the general Hasegawa–Mima equation, which describes electrostatic 2D turbulence in magnetized plasmas:[10]

$$\partial_t w + \mathbf{v} \cdot \boldsymbol{\nabla} w + \beta\, \partial_x \psi = Q,$$

where

$$\begin{aligned}
\psi(t,\mathbf{x}) &: \quad \text{electric potential,} \\
w(t,\mathbf{x}) &: \quad \text{generalized vorticity } [w = (\nabla^2 - L_\mathrm{D}^{-2} \widehat{\kappa})\psi], \\
\mathbf{v}(t,\mathbf{x}) &: \quad \text{fluid velocity } [\mathbf{v} \doteq \mathbf{e}_z \times \boldsymbol{\nabla}\psi], \\
Q(t,\mathbf{x}) &: \quad \text{external forcing and dissipation,} \\
\beta &: \quad \text{measure of the background density gradient,} \\
L_\mathrm{D} &: \quad \text{plasma sound radius.}
\end{aligned}$$

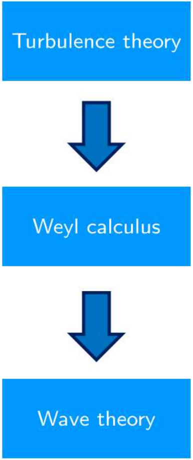
- $\widehat{\kappa}$ is an operator such that $\widehat{\kappa} = 1$ in parts of the spectrum corresponding to DWs and $\widehat{\kappa} = 0$ in those corresponding to ZFs.

- In isolated systems ($Q = 0$), this equation conserves enstrophy $\mathcal{Z}$ and (free) energy $\mathcal{E}$:

$$\mathcal{Z}(t) \doteq \frac{1}{2} \int \mathrm{d}^2\mathbf{x}\, w^2, \qquad \mathcal{E}(t) \doteq -\frac{1}{2} \int \mathrm{d}^2\mathbf{x}\, w\psi.$$

[10] J. A. Krommes and C.-B. Kim, Phys. Rev. E **62**, 8508 (2000); A. I. Smolyakov and P. H. Diamond, Phys. Plasmas **6**, 4410 (1999).

Our approach is based on three main theoretical pillars.



- The fields are separated into their **mean** and **fluctuating** parts.
- We obtain the eqs. for the ZF velocity and the DW **correlation operator**.
- The **quasinormal approx.** is used to statistically close the equations.

- The **Weyl symbol calculus** is used to project the closed operator equations into the ray phase space $(t, \mathbf{x}, \omega, \mathbf{k})$.
- This leads to the **Wigner–Moyal** formulation of DW–ZF interactions with nonlinear DW collisions.

- We use the **geometrical-optics** parameter $\epsilon_{\text{go}} = \max\left(\frac{\lambda_{\text{dw}}}{L_{\text{zf}}}, \frac{L_{\text{D}}}{L_{\text{zf}}}\right) \ll 1$.
- Based on the GO ordering, the **Wigner–Moyal** eqs. can be simplified.
- This leads to the **WKE** model of DW–ZF interactions with nonlinear wave–wave scattering.

WKE model with nonlinear wave–wave scattering included[11]

$$\underbrace{\frac{\partial}{\partial t}U+\mu_{\rm zf}U}_{\text{ZF dynamics}}=\underbrace{\frac{\partial}{\partial y}\int\frac{{\rm d}^2\mathbf{k}}{(2\pi)^2}\,\frac{k_xk_y}{k_{\rm D}^4}J(t,y,\mathbf{k})}_{\text{Ponderomotive driver by DWs}},\qquad \underbrace{\partial_tJ+\{J,\Omega\}}_{\text{DW Hamiltonian dynamics}}=\underbrace{2\Gamma J}_{\text{DW dissipation due to ZFs}}-\underbrace{2\mu_{\rm dw}J}_{\text{External dissipation}}+\underbrace{S_{\rm ext}}_{\text{External forcing}}+\underbrace{C[J,J]}_{\text{Wave—wave collision operator}}.$$

NEW!

- Here $J(t,y,\mathbf{k})$ is the wave-action density for the DWs, and $U(t,y)$ is the ZF velocity.
- The DW wave frequency $\Omega(t,y,\mathbf{k})$ serves as a Hamiltonian for the wave dynamics

$$\Omega(t,y,\mathbf{k})\doteq-\beta k_x/k_{\rm D}^2+k_xU+k_xU''/k_{\rm D}^2.$$

- The dissipation coefficient $\Gamma(t,y,\mathbf{k})$ limits the transfer of enstrophy from DWs to ZFs:

$$\Gamma(t,y,\mathbf{k})\doteq-k_xk_yU'''/k_{\rm D}^4.$$

- The **wave–wave scattering operator** $C[J,J]$ is given by

$$C[J,J]\doteq\underbrace{S_{\rm nl}[J,J]}_{\text{Nonlinear source}}-\underbrace{2\gamma_{\rm nl}[J]J}_{\text{Nonlinear dissipation}}.$$

[11] D. E. Ruiz, M. E. Glinsky, and I. Y. Dodin, under review; arXiv:1803.10817 (2018); D. E. Ruiz, M. E. Glinsky, and I. Y. Dodin, in preparation.

$C[J, J]$ describes nonlinear wave scattering.

- The nonlinear dissipation coeff. $\gamma_{\rm nl}[J]$ and the nonlinear source term $S_{\rm nl}[J, J]$ are

$$\gamma_{\rm nl}[J](t, y, \mathbf{k}) \doteq \int \frac{\mathrm{d}^2\mathbf{p}\,\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\,\delta^2(\mathbf{k}-\mathbf{p}-\mathbf{q})\,\Theta(t, y, \mathbf{k}, \mathbf{p}, \mathbf{q})\,M(\mathbf{p}, \mathbf{q})M(\mathbf{p}, \mathbf{k})\,J(t, y, \mathbf{p}),$$

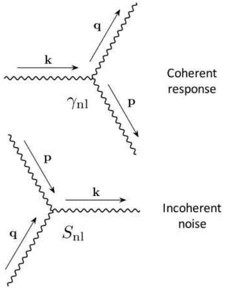
$$S_{\rm nl}[J, J](t, y, \mathbf{k}) \doteq \int \frac{\mathrm{d}^2\mathbf{p}\,\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\,\delta^2(\mathbf{k}-\mathbf{p}-\mathbf{q})\,\Theta(t, y, \mathbf{k}, \mathbf{p}, \mathbf{q})\,|M(\mathbf{p}, \mathbf{q})|^2\,J(t, y, \mathbf{p})J(t, y, \mathbf{q}).$$

- Here $\Theta(t, y, \mathbf{k}, \mathbf{p}, \mathbf{q}) \doteq \pi\delta(\Delta\Omega)$ is the frequency resonance condition, where

$$\Delta\Omega(t, y, \mathbf{k}, \mathbf{p}, \mathbf{q}) \doteq \Omega(t, y, \mathbf{k}) - \Omega(t, y, \mathbf{p}) - \Omega(t, y, \mathbf{q}).$$

- $M(\mathbf{p}, \mathbf{q})$ is a scattering cross section

$$M(\mathbf{p}, \mathbf{q}) \doteq \mathbf{e}_z \cdot (\mathbf{p}\times\mathbf{q})\left(q_{\rm D}^{-2} - p_{\rm D}^{-2}\right).$$



- The present model conserves both total enstrophy $\mathcal{Z}$ and total (free) energy $\mathcal{E}$!

$$\mathcal{Z} = \frac{1}{2}\int \frac{\mathrm{d}y\,\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\,J + \frac{1}{2}\int \mathrm{d}y\,(U')^2, \qquad \mathcal{E} = \frac{1}{2}\int \frac{\mathrm{d}y\,\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\,\frac{J}{k_{\rm D}^2} + \frac{1}{2}\int \mathrm{d}y\,U^2.$$

How does this WKE-based model compare to previous work?

- In weak turbulence theory (WTT), there exists many studies on nonlinear DW scattering.[12]
  - These works treat both the DW and ZF components of the fields as incoherent.
- The present model makes a distinction between the statistics of the ZFs and the DWs.
  - The DW component of the vorticity consists of an **incoherent wave bath**.
  - In contrast, the ZFs are treated as **coherent structures**.
- Our model is more complex than the naive way of writing the WKE with the linear part taken from quasilinear theory and the collision operator taken from homogeneous WTT.
  - The frequency resonance condition includes nonlinearities due to the ZFs:

$$\Theta(t, y, \mathbf{k}, \mathbf{p}, \mathbf{q}) = \frac{\pi}{|\beta - U''(t,y)|} \, \delta\left(\frac{k_x}{k_{\rm D}^2} - \frac{p_x}{p_{\rm D}^2} - \frac{q_x}{q_{\rm D}^2} \right) .$$

  - $|\beta - U''|$ is related to the Rayleigh–Kuo criterion, which marks the onset of the tertiary instability.[13]
  - The present theory seems to break down in regions where $\beta - U'' = 0$.

[12] See, for example: C. Connaughton, S. Nazarenko, and B. Quinn, Phys. Rep. **604**, 1 (2015).
[13] H.-L. Kuo, J. Met. **6**, 105 (1949); H. Zhu, Y. Zhou, D. E. Ruiz, and I. Y. Dodin, Phys. Rev. E **97**, 053210 (2018).

Conclusions

- Starting from the gHME, we systematically derived a WKE describing the interaction between DWs and ZFs with wave–wave collisions included.
- Our approach is based on three main theoretical pillars.
  - We applied a statistical closure based on the **quasinormal approximation**.
  - Using the **Weyl calculus**, we projected the operator eqs. to the ray phase space.
  - Arguments based on **geometrical optics** were used to simplify the equations to get the WKE model.
- (Not in this talk) Our procedure was generalized to obtain a WKE for a field satisfying[11]

$$(\widehat{\mathcal{D}}\psi)(t, \mathbf{x}) = (\widehat{\alpha}\psi)(t, \mathbf{x})\,(\widehat{\beta}\psi)(t, \mathbf{x}) + \mathcal{S}(t, \mathbf{x}).$$

[11] D. E. Ruiz, M. E. Glinsky, and I. Y. Dodin, under review; arXiv:1803.10817 (2018)

Future work

1. Investigate the effects of wave scattering in the WKE for Hasegawa–Mima system.
   - How will the spontaneous emergence of ZFs be modified in the presence of wave collisions?
   - How will the ZF saturation state be modified?
   - How will the Kolmogorov–Zakharov spectrum change in the presence of ZFs?

2. Develop a general theory for nonlinear multicomponent waves.
   - Hasegawa–Wakatani equation for DW turbulence
   - Raman instability of white light in plasmas
   - Dynamo and Magneto–Rayleigh–Taylor instability in magnetohydrodynamics

3. Develop new phase-space models for describing strong turbulence.
   - For modeling strong turbulence, there are other statistical closures that have been proposed; e.g., DIA, RMC, and MSR.
   - Can we marry the phase-space techniques presented here with these more advanced statistical closures?

Backup slides
Deriving the WKE

Dynamics of the mean and fluctuating components

- Let us separate ψ into a large-amplitude, low-frequency, coherent component $\bar{\psi}$ and a small-amplitude, high-frequency, fluctuating component $\tilde{\psi}$:

$$\psi(x) = \bar{\psi}(x) + \epsilon\,\widetilde{\psi}(x),$$

where $\epsilon \ll 1$ is a small parameter characterizing the amplitude of the fluctuations.

- For any field g, the mean part is defined as $\bar{g} \doteq \int \mathrm{d}x_1 \, \langle\!\langle g \rangle\!\rangle / L_x$, where L_x is the system length along the x_1 axis. $\langle\!\langle \ldots \rangle\!\rangle$ denotes some statistical average.

- A simple calculation gives

$$\partial_t U + \mu_{\mathrm{zf}} U + \epsilon^2 \partial_y \langle\!\langle \widetilde{v}_x \widetilde{v}_y \rangle\!\rangle = 0,$$

$$\partial_t \widetilde{w} + U \partial_x \widetilde{w} + [\beta - (\partial_y^2 U)] \partial_x \widetilde{\psi} + \mu_{\mathrm{dw}} \widetilde{w} + \epsilon f_{\mathrm{eddy}} = \epsilon \widetilde{\xi},$$

where $U(t, y) = -\partial_y \bar{\psi}$ is the zonal-flow velocity and

$$f_{\mathrm{eddy}}[\widetilde{w}, \widetilde{w}] \doteq \widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla} \widetilde{w} - \langle\!\langle \widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla} \widetilde{w} \rangle\!\rangle.$$

- Here, $\widetilde{\xi}$ is some external stochastic forcing with zero mean.

- The coefficients $\mu_{\mathrm{dw,zf}}$ represent dissipation caused by the external environment.

Dirac abstract representation

- $\widetilde{w}$ will be described in a Hilbert space $\mathbf{L^2}(\mathbb{R}^3)$ of wave states with inner product[6]

$$\langle \Phi \mid \Psi \rangle = \int_{\mathbb{R}^3} \underbrace{\mathrm{d}^3 x}_{=\mathrm{d}t\,\mathrm{d}^2\mathbf{x}} \, \Phi^*(x)\, \Psi(x).$$

- Here $\widetilde{w}(x) = \langle x \mid \widetilde{w} \rangle$, where $| \, x \, \rangle$ are the eigenstates of the coordinate operators:

$$\langle x \mid \widehat{t} \mid x' \rangle = t\, \delta^3(x - x'), \qquad \langle x \mid \widehat{\mathbf{x}} \mid x' \rangle = \mathbf{x}\, \delta^3(x - x').$$

- The frequency operator $\widehat{\omega}$ and the wavevector operator $\widehat{\mathbf{k}}$ are

$$\langle x \mid \widehat{\omega} \mid \widetilde{w} \rangle = i\partial_t \widetilde{w}(x), \qquad \langle x \mid \widehat{\mathbf{k}} \mid \widetilde{w} \rangle = -i\partial_{\mathbf{x}} \widetilde{w}(x).$$

- We introduce the correlation operator for the fluctuating fields

$$\widehat{\mathcal{W}} \doteq \overline{|\, \widetilde{w} \,\rangle \langle\, \widetilde{w} \,|}.$$

- The correlation operator is related to the correlation function used in CE2 theories:[5]

$$\mathcal{C}\left(t, \frac{y + y'}{2}, \mathbf{x} - \mathbf{x}'\right) \doteq \overline{\langle\, t, \mathbf{x} \mid \widetilde{w} \,\rangle \langle\, \widetilde{w} \mid t, \mathbf{x}' \,\rangle} = \langle\, t, \mathbf{x} \mid \widehat{\mathcal{W}} \mid t, \mathbf{x}' \,\rangle\,,$$

[6] I. Y. Dodin, Phys. Lett. A **378**, 1598 (2014).

Abstract representation

- The next step is to write dynamics equations in the abstract representation:

$$\frac{\partial}{\partial t}U + \mu_{\rm zf}U = \epsilon^2 \frac{\partial}{\partial y}\langle\, x \mid \hat{k}_{\rm D}^{-2}\hat{k}_x \widehat{\mathcal{W}} \hat{k}_y \hat{k}_{\rm D}^{-2} \mid x\,\rangle\,,$$

$$\widehat{\mathcal{D}}\,|\,\widetilde{w}\,\rangle = \frac{i\epsilon}{2}\,|\, f_{\rm nl}[\widetilde{w},\widetilde{w}]\,\rangle - \frac{i\epsilon}{2}\,|\,\overline{f_{\rm nl}[\widetilde{w},\widetilde{w}]}\,\rangle + i\epsilon\,|\,\widetilde{\xi}\,\rangle\,,$$

where

$$\widehat{\mathcal{D}} \doteq \widehat{\omega} - \widehat{U}\widehat{k}_x + (\beta - \widehat{U}'')\widehat{k}_x\widehat{k}_{\rm D}^{-2} + i\mu_{\rm dw}, \qquad \widehat{k}_{\rm D}^2 \doteq \widehat{k}^2 + L_{\rm D}^{-2}.$$

- Additionally, the ket $|\, f_{\rm nl}[\phi,\psi]\,\rangle$ is given by

$$|\, f_{\rm nl}[\phi,\psi]\,\rangle \doteq \int {\rm d}^3x\ |\,x\,\rangle\langle\,\phi\,|\,\widehat{\mathcal{K}}(x)\,|\,\psi\,\rangle\,, \qquad |\,\overline{f_{\rm nl}[\phi,\psi]}\,\rangle \doteq \int {\rm d}^3x\ |\,x\,\rangle\overline{\langle\,\phi\,|\,\widehat{\mathcal{K}}(x)\,|\,\psi\,\rangle},$$

where

$$\widehat{\mathcal{K}}(t,\mathbf{x}) \doteq \widehat{\alpha}^j\,|\,t,\mathbf{x}\,\rangle\langle\,t,\mathbf{x}\,|\,\widehat{\beta}^j + \widehat{\beta}^j\,|\,t,\mathbf{x}\,\rangle\langle\,t,\mathbf{x}\,|\,\widehat{\alpha}^j.$$

Here we make use of the Einstein convention to denote summation among the repeated indices. The operators $\widehat{\alpha}^j$ and $\widehat{\beta}^j$ are given by

$$\widehat{\alpha}^j \doteq (\mathbf{e}_z \times \widehat{\mathbf{k}})^j k_{\rm D}^{-2}, \qquad \widehat{\beta}^j \doteq \widehat{\mathbf{k}}^j.$$

Abstract representation

- The ket $|\, f_{\mathrm{nl}}[\widetilde{w},\widetilde{w}]\,\rangle$ is a bilinear mapping that represents the nonlinear coupling:

$$
\begin{aligned}
\langle\, t,\mathbf{x} \mid f_{\mathrm{nl}}[\widetilde{w},\widetilde{w}]\,\rangle &= \langle\, \widetilde{w} \mid \widehat{\mathcal{K}}(x) \mid \widetilde{w}\,\rangle \\
&= \langle\, \widetilde{w} \mid \widehat{k}_{\mathrm{D}}^{-2}(\mathbf{e}_z \times \widehat{\mathbf{k}})^j \mid x\,\rangle \langle\, x \mid \widehat{\mathbf{k}}^j \mid \widetilde{w}\,\rangle + \langle\, \widetilde{w} \mid \widehat{\mathbf{k}}^j \mid x\,\rangle \langle\, x \mid (\mathbf{e}_z \times \widehat{\mathbf{k}})^j \widehat{k}_{\mathrm{D}}^{-2} \mid \widetilde{w}\,\rangle \\
&= -(\langle\, x \mid (\mathbf{e}_z \times \widehat{\mathbf{k}})^j \mid \widetilde{\psi}\,\rangle)^* \langle\, x \mid \widehat{\mathbf{k}}^j \mid \widetilde{w}\,\rangle - (\langle\, x \mid \widehat{\mathbf{k}}^j \mid \widetilde{w}\,\rangle)^* \langle\, x \mid (\mathbf{e}_z \times \widehat{\mathbf{k}})^j \mid \widetilde{\psi}\,\rangle \\
&= -(\mathbf{e}_z \times -i\boldsymbol{\nabla}\widetilde{\psi})^* \cdot (-i\boldsymbol{\nabla}\widetilde{w}) - (-i\boldsymbol{\nabla}\widetilde{w})^* \cdot (\mathbf{e}_z \times -i\boldsymbol{\nabla}\widetilde{\psi}) \\
&= -2(\mathbf{e}_z \times \boldsymbol{\nabla}\widetilde{\psi}) \cdot \boldsymbol{\nabla}\widetilde{w} \\
&= -2\widetilde{\mathbf{v}} \cdot \boldsymbol{\nabla}\widetilde{w}.
\end{aligned}
$$

- Here we used $|\,\widetilde{\psi}\,\rangle = -\hat{k}_{\mathrm{D}}^{-2}\,|\,\widetilde{w}\,\rangle$, which comes from the relation $\widetilde{w} = (\boldsymbol{\nabla}^2 - L_{\mathrm{D}}^{-2})\widetilde{\psi}$.

Statistical closure problem

- We then project the eq. for the fluctuations by $\langle \widetilde{w} \,|$ from the right and average. Substracting with its Hermitian conjugate gives the following set of equations:

$$\partial_t U + \mu_{\rm zf} U = \epsilon^2 \partial_y \langle x \mid \hat{k}_{\rm D}^{-2} \hat{k}_x \widehat{\mathcal{W}} \hat{k}_y \hat{k}_{\rm D}^{-2} \mid x \rangle \,,$$

$$[\widehat{\mathcal{D}}_{\rm H}, \widehat{\mathcal{W}}]_- + i[\widehat{\mathcal{D}}_{\rm A}, \widehat{\mathcal{W}}]_+ = i\epsilon \left[\overline{\mid f_{\rm nl}[\widetilde{w}, \widetilde{w}] \,\rangle \langle\, \widetilde{w} \mid}\right]_{\rm H} + 2i\epsilon \left[\overline{\mid \widetilde{\xi} \,\rangle \langle\, \widetilde{w} \mid}\right]_{\rm H}.$$

- Here the dispersion operator $\widehat{\mathcal{D}}$ is decomposed into its Hermitian and anti-Hermitian parts:

$$\widehat{\mathcal{D}}_{\rm H} = \widehat{\omega} - \widehat{U}\widehat{k}_x + \beta \widehat{k}_{\rm D}^{-2} - [\widehat{U}'', \widehat{k}_x \widehat{k}_{\rm D}^{-2}]_+/2,$$

$$\widehat{\mathcal{D}}_{\rm A} = -[\widehat{U}'', \widehat{k}_x \widehat{k}_{\rm D}^{-2}]_-/(2i) + \mu_{\rm dw}.$$

- Also, $[\cdot, \cdot]_\mp$ respectively denote the commutators and anticommutators:

$$[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_- = \widehat{\mathcal{A}}\widehat{\mathcal{B}} - \widehat{\mathcal{B}}\widehat{\mathcal{A}}, \qquad [\widehat{\mathcal{A}}, \widehat{\mathcal{B}}]_+ = \widehat{\mathcal{A}}\widehat{\mathcal{B}} + \widehat{\mathcal{B}}\widehat{\mathcal{A}}.$$

The equations are not closed. $\Rightarrow$ Statistical closure problem

Quasilinear approximation (revisited)

- In the **quasilinear approximation**, one neglects the nonlinear interactions in the equation for the fluctuations:[4]

$$\widehat{\mathcal{D}}\,|\,\widetilde{w}\,\rangle = \frac{i\epsilon}{2}\left\{\,\cancel{|\,f_{\rm nl}[\widetilde{w},\widetilde{w}]\,\rangle - \overline{|\,f_{\rm nl}[\widetilde{w},\widetilde{w}]\,\rangle}}\,\right\} + i\epsilon\,|\,\widetilde{\xi}\,\rangle\,.$$

- Thus, $\widehat{\mathcal{D}}\widehat{\mathcal{W}} = i\epsilon\overline{|\,\widetilde{S}\,\rangle\,\langle\,\xi\,|}$. Using $|\,\widetilde{w}\,\rangle \simeq i\epsilon\widehat{\mathcal{D}}^{-1}\,|\,\widetilde{\xi}\,\rangle$, we have

$$\boxed{\begin{aligned}\partial_t U + \mu_{\rm zf} U &= \epsilon^2 \partial_y \,\langle\, x\,|\,\hat{k}_{\rm D}^{-2}\hat{k}_x\widehat{\mathcal{W}}\hat{k}_y\hat{k}_{\rm D}^{-2}\,|\,x\,\rangle\,,\\ [\widehat{\mathcal{D}}_{\rm H},\widehat{\mathcal{W}}]_- + i[\widehat{\mathcal{D}}_{\rm A},\widehat{\mathcal{W}}]_+ &= 2i\epsilon^2\left[\widehat{\mathcal{S}}(\widehat{\mathcal{D}}^{-1})^\dagger\right]_{\rm A},\end{aligned}}$$

where $\widehat{\mathcal{S}}$ is density operator associated to the source fluctuations

$$\widehat{\mathcal{S}} \doteq \overline{|\,\widetilde{\xi}\,\rangle\,\langle\,\widetilde{\xi}\,|}.$$

- The equations above are the abstract representation for all quasilinear theories, such as the CE2 and the Wigner–Moyal formalisms.

- Since nonlinear interactions are omitted, this quasilinear approximation does not develop a Kolmogorov–Zakharov spectrum for the waves.

A statistical closure beyond the quasilinear approximation

- We shall perturbatively incorporate nonlinear effects into the equation for the fluctuating field. Let us separate $\widetilde{w}$ into two parts:

$$| \, \widetilde{w} \, \rangle = | \, \widetilde{w}_0 \, \rangle + \epsilon \, | \, \widetilde{w}_1 \, \rangle \, .$$

These fields satisfy

$$\widehat{\mathcal{D}} \, | \, \widetilde{w}_0 \, \rangle = 0,$$

$$\widehat{\mathcal{D}} \, | \, \widetilde{w}_1 \, \rangle = \frac{i}{2} \underbrace{\left\{ | \, f_{\rm nl}[\widetilde{w}_0, \widetilde{w}_0] \, \rangle - \overline{| \, f_{\rm nl}[\widetilde{w}_0, \widetilde{w}_0] \, \rangle} \right\}}_{\doteq \widehat{\mathcal{D}} | \, \widetilde{\phi} \, \rangle} + i \, | \, \widetilde{\xi} \, \rangle \, .$$

- The fluctuations in $\widetilde{w}_0 \sim \mathcal{O}(1)$ are due to random initial conditions, whose statistics are considered to be uncorrelated to those of $\widetilde{\xi}$.
- When substituting into the equation for the fluctuations, one finds

$$[\widehat{\mathcal{D}}_{\rm H}, \widehat{\mathcal{W}}]_- + i[\widehat{\mathcal{D}}_{\rm A}, \widehat{\mathcal{W}}]_+ = i\epsilon \left\{ \overline{| \, f_{\rm nl}[\widetilde{w}_0, \widetilde{w}_0] \, \rangle \langle \, \widetilde{w}_0 \, |} \right\}_{\rm A} + 2i\epsilon^2 \left\{ \overline{| \, f_{\rm nl}[\widetilde{\phi}, \widetilde{w}_0] \, \rangle \langle \, \widetilde{w}_0 \, |} \right\}_{\rm A}$$
$$+ \, i\epsilon^2 \left\{ \overline{| \, f_{\rm nl}[\widetilde{w}_0, \widetilde{w}_0] \, \rangle \langle \, \widetilde{\phi} \, |} \right\}_{\rm A} + 2i\epsilon^2 \left[\widehat{\mathcal{S}} (\widehat{\mathcal{D}}^{-1})^\dagger \right]_{\rm A} + \mathcal{O}(\epsilon^3).$$

Quasinormal approximation

- In the **quasinormal approximation**, one assumes that the statistics of $\widetilde{w}_0$ is approximately given by a normal distribution with zero mean.[7] Hence,

$$\overline{\widetilde{w}_0} = 0, \qquad \overline{\widetilde{w}_0(x_1)\widetilde{w}_0(x_2)} = \langle\, x_1 \mid \widehat{\mathcal{W}}_0 \mid x_2 \,\rangle\,, \qquad \overline{\widetilde{w}_0(x_1)\widetilde{w}_0(x_2)\widetilde{w}_0(x_3)} = 0,$$

$$\overline{\widetilde{w}_0(x_1)\widetilde{w}_0(x_2)\widetilde{w}_0(x_3)\widetilde{w}_0(x_4)} = \langle\, x_1 \mid \widehat{\mathcal{W}}_0 \mid x_2 \,\rangle \langle\, x_3 \mid \widehat{\mathcal{W}}_0 \mid x_4 \,\rangle + \langle\, x_1 \mid \widehat{\mathcal{W}}_0 \mid x_3 \,\rangle \langle\, x_2 \mid \widehat{\mathcal{W}}_0 \mid x_4 \,\rangle + \langle\, x_1 \mid \widehat{\mathcal{W}}_0 \mid x_4 \,\rangle \langle\, x_2 \mid \widehat{\mathcal{W}}_0 \mid x_3 \,\rangle\,.$$

- With this approximation, a direct calculation leads to

$$\overline{|\, f_{\mathrm{nl}}[\widetilde{w}_0, \widetilde{w}_0]\,\rangle \langle\, \widetilde{w}_0 \,|} = 0,$$

$$\overline{|\, f_{\mathrm{nl}}[\widetilde{\phi}, \widetilde{w}_0]\,\rangle \langle\, \widetilde{w}_0 \,|} = \int \mathrm{d}^3x\, \mathrm{d}^3y \;|\, x\,\rangle \langle\, y \,|\, (\widehat{\mathcal{D}}^{-1})^\dagger\, \widehat{\mathcal{K}}(x)\, \widehat{\mathcal{W}}_0\, \widehat{\mathcal{K}}^\dagger(y)\, \widehat{\mathcal{W}}_0,$$

$$\overline{|\, f_{\mathrm{nl}}[\widetilde{w}_0, \widetilde{w}_0]\,\rangle \langle\, \widetilde{\phi} \,|} = \int \mathrm{d}^3x\, \mathrm{d}^3y \;|\, x\,\rangle \langle\, y \,|\, (\widehat{\mathcal{D}}^{-1})^\dagger\, \mathrm{Tr}[\, \widehat{\mathcal{K}}(x)\, \widehat{\mathcal{W}}_0\, \widehat{\mathcal{K}}^\dagger(y)\, \widehat{\mathcal{W}}_0\,],$$

where

$$\widehat{\mathcal{W}}_0 \doteq \overline{|\, \widetilde{w}_0 \,\rangle \langle\, \widetilde{w}_0 \,|}.$$

[7] M. Millionstchikov, C. R. Acad. Sci. U.S.S.R. **32**, 615 (1941).

The abstract representation is useful for these calculations.

- The abstract representation results is very convenient for the calculation of the previous terms. As an example,

$$\overline{|\, f_{\rm nl}[\widetilde{w}_0, \widetilde{w}_0]\,\rangle\,\langle\, \widetilde{\phi}\,|} = \int \mathrm{d}^3x\, |\, x\,\rangle\, \overline{\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}(x) \,|\, \widetilde{w}_0\,\rangle\,\langle\, \widetilde{\phi}\,|}$$

$$= \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\, |\, x\,\rangle\,\langle\, y\,|\,(\widehat{\mathcal{D}}^{-1})^\dagger\, \overline{\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}(x) \,|\, \widetilde{w}_0\,\rangle \left[\,\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle - \overline{\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle}\,\right]}\,.$$

- Using the quasinormal approximation, we obtain

$$\overline{\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}(x) \,|\, \widetilde{w}_0\,\rangle \left[\,\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle - \overline{\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle}\,\right]}$$

$$= \langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}(x) \,|\, \widetilde{w}_0\,\rangle\,\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle + \langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}(x) \,|\, \widetilde{w}_0\,\rangle\,\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle$$

$$= 2\,\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}(x) \,|\, \widetilde{w}_0\,\rangle\,\langle\, \widetilde{w}_0 \,|\, \widehat{\mathcal{K}}^\dagger(y) \,|\, \widetilde{w}_0\,\rangle$$

$$= 2\,\mathrm{Tr}[\,\widehat{\mathcal{K}}(x)\,\overline{|\, \widetilde{w}_0\,\rangle\,\langle\, \widetilde{w}_0\,|}\,\widehat{\mathcal{K}}^\dagger(y)\,\overline{|\, \widetilde{w}_0\,\rangle\,\langle\, \widetilde{w}_0\,|}\,]$$

$$= 2\,\mathrm{Tr}[\,\widehat{\mathcal{K}}(x)\,\widehat{\mathcal{W}}_0\,\widehat{\mathcal{K}}^\dagger(y)\,\widehat{\mathcal{W}}_0\,].$$

- Hence, one obtains

$$\overline{|\, f_{\rm nl}[\widetilde{w}_0, \widetilde{w}_0]\,\rangle\,\langle\, \widetilde{\phi}\,|} = \int \mathrm{d}^3x\,\mathrm{d}^3y\, |\, x\,\rangle\,\langle\, y\,|\,(\widehat{\mathcal{D}}^{-1})^\dagger\,\mathrm{Tr}[\,\widehat{\mathcal{K}}(x)\,\widehat{\mathcal{W}}_0\,\widehat{\mathcal{K}}^\dagger(y)\,\widehat{\mathcal{W}}_0\,].$$

Closed equations in the abstract representation

- Upon gathering all the terms, one obtains the set of closed equations:

$$\partial_t U + \mu_{\rm zf} U = \epsilon^2 \partial_y \langle x \mid \hat{k}_{\rm D}^{-2} \hat{k}_x \widehat{\mathcal{W}} \hat{k}_y \hat{k}_{\rm D}^{-2} \mid x \rangle ,$$
$$[\widehat{\mathcal{D}}_{\rm H}, \widehat{\mathcal{W}}]_- + i[\widehat{\mathcal{D}}_{\rm A}, \widehat{\mathcal{W}}]_+ = 2i\epsilon^2 \left[\widehat{\mathcal{F}}\, (\widehat{\mathcal{D}}^{-1})^\dagger\right]_{\rm A} - 2i\epsilon^2 \left[\widehat{\eta}\, \widehat{\mathcal{W}}\right]_{\rm A} + 2i\epsilon^2 \left[\widehat{\mathcal{S}}(\widehat{\mathcal{D}}^{-1})^\dagger\right]_{\rm A},$$

where

$$\widehat{\eta} \doteq - \int \mathrm{d}^3 x\, \mathrm{d}^3 y \; |\, x \,\rangle \langle \, y \,|\, (\widehat{\mathcal{D}}^{-1})^\dagger \widehat{\mathcal{K}}(x)\, \widehat{\mathcal{W}}\, \widehat{\mathcal{K}}^\dagger(y),$$
$$\widehat{\mathcal{F}} \doteq \frac{1}{2} \int \mathrm{d}^3 x\, \mathrm{d}^3 y \; |\, x \,\rangle \langle \, y \,|\, \mathrm{Tr}[\, \widehat{\mathcal{K}}(x)\, \widehat{\mathcal{W}}\, \widehat{\mathcal{K}}^\dagger(y)\, \widehat{\mathcal{W}}\,].$$

- These equations are accurate upto $\mathcal{O}(\epsilon^3)$. Hence, they could potentially be adequate for modeling weak wave turbulence with background sheared flows.
- These equations are independent of the coordinates used.
- To do calculations, a representation is needed. $\Longrightarrow$ **Weyl phase-space representation**

The Weyl transformation maps linear differential operators to functions of phase space.

$$\mathsf{W}[\widehat{\mathcal{D}}] \doteq \int \mathrm{d}\tau\, \mathrm{d}^3\mathbf{s}\, e^{i\omega\tau - i\mathbf{k}\cdot\mathbf{s}} \langle\, t + \tfrac{1}{2}\tau, \mathbf{x} + \tfrac{1}{2}\mathbf{s} \mid \widehat{\mathcal{D}} \mid t - \tfrac{1}{2}\tau, \mathbf{x} - \tfrac{1}{2}\mathbf{s}\,\rangle$$

$$\widehat{\mathcal{D}} = \mathcal{D}(x, i\partial) \qquad D(t, \mathbf{x}, \omega, \mathbf{k})$$

$$\mathsf{W}^{-1}[D(x,k)] \doteq \frac{1}{(2\pi)^4} \int \mathrm{d}^4x\, \mathrm{d}^4k\, \mathrm{d}^4s\, D(x,k)\, |\, x - s/2\,\rangle \langle\, x + s/2\,|$$

- Some examples of this transformation are

$$f(\hat{t}, \hat{\mathbf{x}}) \Longleftrightarrow f(t, \mathbf{x}), \quad g(\hat{\omega}, \hat{\mathbf{k}}) \Longleftrightarrow g(\omega, \mathbf{k}), \quad \hat{\mathbf{x}}^i \hat{\mathbf{k}}_j \Longleftrightarrow \mathbf{x}^i \mathbf{k}_j + \frac{i}{2}\delta^i_j.$$

- The Weyl symbol of $\widehat{\mathcal{C}} = \widehat{\mathcal{A}}\widehat{\mathcal{B}}$ is given by the Moyal product

$$\widehat{\mathcal{A}}\widehat{\mathcal{B}} \Longleftrightarrow A(t, \mathbf{x}, \omega, \mathbf{k}) \star B(t, \mathbf{x}, \omega, \mathbf{k}) \doteq A \exp\left(\frac{i}{2}\widehat{\mathcal{L}}\right) B,$$

$$\widehat{\mathcal{L}} \doteq \overleftarrow{\partial_\omega} \cdot \overrightarrow{\partial_t} - \overleftarrow{\partial_t} \cdot \overrightarrow{\partial_\omega} + \overleftarrow{\partial_\mathbf{x}} \cdot \overrightarrow{\partial_\mathbf{p}} - \overleftarrow{\partial_\mathbf{k}} \cdot \overrightarrow{\partial_\mathbf{x}} = \{\,\cdot\,,\,\cdot\,\}_8.$$

Hermann Weyl

(1885-1955)

Weyl phase-space representation of the fluctuations

- Applying the Weyl transformation to the equation for the fluctuations leads to

$$[\widehat{\mathcal{D}}_{\mathrm{H}}, \widehat{\mathcal{W}}]_- + i[\widehat{\mathcal{D}}_{\mathrm{A}}, \widehat{\mathcal{W}}]_+ = 2i\epsilon^2 \left[\widehat{\mathcal{F}}\,(\widehat{\mathcal{D}}^{-1})^\dagger\right]_{\mathrm{A}} - 2i\epsilon^2 \left[\widehat{\eta}\,\widehat{\mathcal{W}}\right]_{\mathrm{A}} + 2i\epsilon^2 \left[\widehat{\mathcal{S}}(\widehat{\mathcal{D}}^{-1})^\dagger\right]_{\mathrm{A}}$$

$$\{\{D_{\mathrm{H}}, W\}\} + [[D_{\mathrm{A}}, W]] = 2\epsilon^2\,\mathrm{Im}\{F \star [D^{-1}]^*\} - 2\epsilon^2\,\mathrm{Im}\,(\eta \star W) + 2\epsilon^2\,\mathrm{Im}\{S \star [D^{-1}]^*\}.$$

- Here $W(t, y, \omega, \mathbf{k})$ is the **Wigner function** of the fluctuation wave field

$$W(t, y, \omega, \mathbf{k}) \doteq \int \mathrm{d}\tau\,\mathrm{d}^2\mathbf{s}\, e^{i\omega\tau - i\mathbf{k}\cdot\mathbf{s}}\, \overline{\widetilde{w}(t + \tfrac{1}{2}\tau, \mathbf{x} + \tfrac{1}{2}\mathbf{s})\,\widetilde{w}(t - \tfrac{1}{2}\tau, \mathbf{x} - \tfrac{1}{2}\mathbf{s})}\,.$$

- Also, $\{\{A, B\}\}$ and $[[A, B]]$ are the Moyal brackets

$$\{\{A, B\}\} \doteq -i\,(A \star B - B \star A) = 2A \sin\left(\overleftrightarrow{\mathcal{L}}/2\right) B,$$

$$[[A, B]] \doteq A \star B + B \star A = 2A \cos\left(\overleftrightarrow{\mathcal{L}}/2\right) B.$$

Difficulty: This is an infinite-order PDE in the eight-dimensional phase space !

Assumptions of the wave kinetic equation

- Let $(\tau_{\rm dw}, \lambda_{\rm dw})$ and $(\tau_{\rm zf}, \lambda_{\rm zf})$ be the characteristic wavelengths and timescales for DWs and ZFs, respectively. We then introduce the **geometrical optics** ordering:[8]

$$\epsilon_{\rm go} \doteq \max\left(\frac{\tau_{\rm dw}}{\tau_{\rm zf}}, \frac{\lambda_{\rm dw}}{\lambda_{\rm zf}}, \frac{L_{\rm D}}{\lambda_{\rm zf}}\right) \ll 1.$$

- Hence, the following estimates will be adopted:[8]

$$\begin{aligned} &\partial_t W \backsim \tau_{\rm zf}^{-1} W, \quad \partial_\omega W \backsim \tau_{\rm dw} W, \qquad \partial_t D \backsim \tau_{\rm zf}^{-1} D, \\ &\partial_y W \backsim \lambda_{\rm zf}^{-1} W, \quad \partial_{\mathbf{p}} W \backsim \lambda_{\rm dw} W, \qquad \partial_y D \backsim \lambda_{\rm zf}^{-1} D, \quad \partial_{\mathbf{p}} D \backsim L_{\rm D} D. \end{aligned}$$

- Thus, the Moyal products and brackets are approximated by[9]

$$\begin{aligned} A \star B &= AB + \tfrac{i}{2}\epsilon_{\rm go}\{A, B\}_6 + \mathcal{O}(\epsilon_{\rm go}^2), \\ \{\{A, B\}\} &= \epsilon_{\rm go}\{A, B\}_6 + \mathcal{O}(\epsilon_{\rm go}^3), \\ [[A, B]] &= 2AB + \mathcal{O}(\epsilon_{\rm go}^2). \end{aligned}$$

- Also assume that the medium is only slightly dissipative so

$$D_{\rm A} \sim \mathcal{O}(\epsilon_{\rm go}).$$

- Finally, let the nonlinearity and GO parameters scale as $\epsilon_{\rm go} \sim \epsilon^2$.

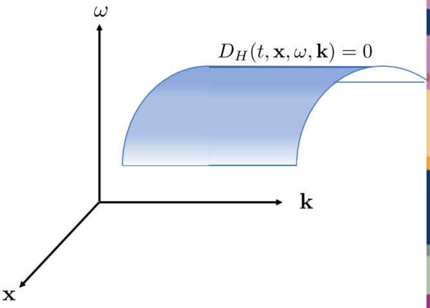
[13] G. B. Whitham, Linear and Nonlinear Waves (John Wiley & Sons, Inc., Hoboken, NJ, USA, 1999).
[9] S. W. McDonald and A. N. Kaufman, Phys. Rev. A **32**, 1708 (1985).

The dispersion manifold

- $W(t, \mathbf{x}, \omega, \mathbf{k})$ describes a density of wave quanta in the 6-dim. phase space.[10]

- By GO considerations, the wave quanta should lie on the dispersion manifold:

$$\begin{aligned} 0 &= D_{\mathrm{H}} \\ &= \omega - U k_x + \beta k_{\mathrm{D}}^{-2} - [[U'', k_x k_{\mathrm{D}}^{-2}]]/2 \\ &\simeq \omega - \underbrace{[-\beta k_x / k_{\mathrm{D}}^2 + U k_x + U'' k_x / k_{\mathrm{D}}^2]}_{\Omega(t,y,\mathbf{k})}. \end{aligned}$$



- $\Omega(t, y, \mathbf{k})$ is the **wave frequency**

$$\Omega(t, y, \mathbf{k}) \doteq -\beta k_x / k_{\mathrm{D}}^2 + U k_x + U'' k_x / k_{\mathrm{D}}^2.$$

- Hence, we propose as ansatz:

$$W = 2\pi \delta(D_H) J(t, \mathbf{x}, \mathbf{k}),$$

where $J(t, \mathbf{x}, \mathbf{k})$ is the wave-action density.

[10] E. R. Tracy, A. J. Brizard, A. S. Richardson, and A. N. Kaufman, *"Ray Tracing and Beyond: Phase Space Methods in Plasma Wave Theory,"* (Cambridge University Press, New York, 2014).

Deriving the WKE: zonal-flow equation

- In the abstract representation, the equation for the zonal flow is

$$\partial_t U + \mu_{\mathrm{zf}} U = \epsilon^2 \partial_y \langle x \mid \hat{k}_{\mathrm{D}}^{-2} \hat{k}_x \widehat{\mathbb{W}} \hat{k}_y \hat{k}_{\mathrm{D}}^{-2} \mid x \rangle .$$

- For any operator $\widehat{\mathcal{A}}$, one has

$$\langle x \mid \widehat{\mathcal{A}} \mid x \rangle = \int \frac{\mathrm{d}\omega \, \mathrm{d}\mathbf{k}}{(2\pi)^3} A(t, \mathbf{x}, \omega, \mathbf{k}).$$

- Hence, one obtains

$$\begin{aligned}
\langle x \mid \hat{k}_{\mathrm{D}}^{-2} \hat{k}_x \widehat{\mathbb{W}} \hat{k}_y \hat{k}_{\mathrm{D}}^{-2} \mid x \rangle &= \int \frac{\mathrm{d}\omega \, \mathrm{d}^2\mathbf{k}}{(2\pi)^3} \frac{k_x}{k_{\mathrm{D}}^2} \star W(t, y, \omega, \mathbf{k}) \star \frac{k_y}{k_{\mathrm{D}}^2} \\
&= \int \frac{\mathrm{d}\omega \, \mathrm{d}^2\mathbf{k}}{(2\pi)^3} \frac{k_x}{k_{\mathrm{D}}^2} W(t, y, \omega, \mathbf{k}) \frac{k_y}{k_{\mathrm{D}}^2} + \mathcal{O}(\epsilon_{\mathrm{go}}) \\
&= \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{k_x k_y}{k_{\mathrm{D}}^4} J(t, y, \mathbf{k}) + \mathcal{O}(\epsilon_{\mathrm{go}}).
\end{aligned}$$

Deriving the WKE: linear terms

$$\{\{D_{\mathrm{H}}, W\}\} + [[D_{\mathrm{A}}, W]] = 2\epsilon^2 \operatorname{Im}\{F \star [D^{-1}]^*\} - 2\epsilon^2 \operatorname{Im}(\eta \star W) + 2\epsilon^2 \operatorname{Im}\{S \star [D^{-1}]^*\}.$$

- Regarding the Hamiltonian part, one has

$$\begin{aligned}\{\{D_{\mathrm{H}}, W\}\} &\simeq 2\pi\epsilon_{\mathrm{go}}\{D_{\mathrm{H}}, \delta(D_{\mathrm{H}})J\} \\ &= 2\pi\epsilon_{\mathrm{go}}\delta(D_{\mathrm{H}})\{D_{\mathrm{H}}, J\} \\ &= 2\pi\epsilon_{\mathrm{go}}\delta(D_{\mathrm{H}})\,[\partial_t J + \{J, \Omega\}],\end{aligned}$$

where

$$D_{\mathrm{H}}(t, y, \omega, \mathbf{k}) \simeq \omega - \Omega(t, y, \mathbf{k}), \qquad \Omega(t, y, \mathbf{k}) \doteq -\beta k_x/k_{\mathrm{D}}^2 + U k_x + U'' k_x/k_{\mathrm{D}}^2.$$

- Since dissipation is assumed to be small

$$[[D_{\mathrm{A}}, W]] \simeq 2\epsilon_{\mathrm{go}} D_{\mathrm{A}} W = 4\pi\epsilon_{\mathrm{go}} D_{\mathrm{A}}\delta(D_{\mathrm{H}})J,$$

where

$$D_{\mathrm{A}} = -\{\{U'', k_x k_{\mathrm{D}}^{-2}\}\}/2 + \mu_{\mathrm{dw}} \simeq -\{U'', k_x k_{\mathrm{D}}^{-2}\}/2 + \mu_{\mathrm{dw}} = U''' k_x k_y/k_{\mathrm{D}}^4 + \mu_{\mathrm{dw}}.$$

Deriving the WKE: wave scattering

$$\{\{D_{\mathrm{H}}, W\}\} + [[D_{\mathrm{A}}, W]] = 2\epsilon^2 \operatorname{Im}\{F \star [D^{-1}]^*\} - 2\epsilon^2 \operatorname{Im}(\eta \star W) + 2\epsilon^2 \operatorname{Im}\{S \star [D^{-1}]^*\}.$$

- As a reminder,

$$\widehat{\mathcal{F}} \doteq \frac{1}{2} \int \mathrm{d}^3 x \, \mathrm{d}^3 y \, |\, x \,\rangle \langle \, y \,|\, \mathrm{Tr}[\widehat{\mathcal{K}}(x) \, \widehat{\mathcal{W}} \, \widehat{\mathcal{K}}^\dagger(y) \, \widehat{\mathcal{W}}].$$

- We first note that the trace can be written as

$$\begin{aligned}
\mathrm{Tr}[\widehat{\mathcal{K}}(x)\widehat{\mathcal{W}}\widehat{\mathcal{K}}^\dagger(y)\widehat{\mathcal{W}}] \\
= \quad & \langle \, x \,|\, \widehat{\beta}^j \, \widehat{\mathcal{W}} \, (\widehat{\alpha}^k)^\dagger \,|\, y \,\rangle \langle \, y \,|\, (\widehat{\beta}^k)^\dagger \, \widehat{\mathcal{W}} \, \widehat{\alpha}^j \,|\, x \,\rangle + \langle \, x \,|\, \widehat{\beta}^j \, \widehat{\mathcal{W}} \, (\widehat{\beta}^k)^\dagger \,|\, y \,\rangle \langle \, y \,|\, (\widehat{\alpha}^k)^\dagger \, \widehat{\mathcal{W}} \, \widehat{\alpha}^j_{\mathrm{T}} \,|\, x \,\rangle \\
& + \langle \, x \,|\, \widehat{\alpha}^j \, \widehat{\mathcal{W}} \, (\widehat{\alpha}^k)^\dagger \,|\, y \,\rangle \langle \, y \,|\, (\widehat{\beta}^k)^\dagger \, \widehat{\mathcal{W}} \, \widehat{\beta}^j \,|\, x \,\rangle + \langle \, x \,|\, \widehat{\alpha}^j \, \widehat{\mathcal{W}} \, (\widehat{\beta}^k)^\dagger \,|\, y \,\rangle \langle \, y \,|\, (\widehat{\alpha}^k)^\dagger \, \widehat{\mathcal{W}} \, \widehat{\beta}^j \,|\, x \,\rangle .
\end{aligned}$$

- When calculating the Weyl symbol, terms will appear as follows:

$$\int \mathrm{d}^3 s \, e^{ik\cdot s} \, \langle \, x + \tfrac{1}{2}s \,|\, \widehat{\mathcal{A}} \,|\, x - \tfrac{1}{2}s \,\rangle \langle \, x - \tfrac{1}{2}s \,|\, \widehat{\mathcal{B}} \,|\, x + \tfrac{1}{2}s \,\rangle = \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 q}{(2\pi)^3} \underbrace{\delta^3(k - p - q)}_{\text{very important}} A(x, p) B(x, -q).$$

Deriving the WKE: wave scattering

- Applying the previous result gives

$$F(x,k) = \frac{1}{2}\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 q}{(2\pi)^3} \delta^3(k-p-q) |M(\mathbf{p},\mathbf{q})|^2 W(x,p) W(x,q) + \mathcal{O}(\epsilon_{\mathrm{go}}),$$

where

$$M(\mathbf{p},\mathbf{q}) = \mathbf{e}_z \cdot (\mathbf{p}\times\mathbf{q}) \left(\frac{1}{q_{\mathrm{D}}^2} - \frac{1}{p_{\mathrm{D}}^2}\right).$$

- Inserting $W(t,y,\omega,\mathbf{k}) = 2\pi\delta(D_{\mathrm{H}}) J(t,y,\mathbf{k})$ and integrating gives

$$F \simeq \int \frac{\mathrm{d}^2\mathbf{p}\, \mathrm{d}^2\mathbf{q}}{(2\pi)^2} \delta^2(\mathbf{k}-\mathbf{p}-\mathbf{q})\, \pi\delta\big(\omega - \Omega(t,\mathbf{x},\mathbf{p}) - \Omega(t,\mathbf{x},\mathbf{q})\big) |M(\mathbf{p},\mathbf{q})|^2 J(t,y,\mathbf{p}) J(t,y,\mathbf{q}).$$

- Returning to the WKE, when integrating over ω, one has

$$\int \mathrm{d}\omega\, \mathrm{Im}\{F \star [D^{-1}]^*\} \simeq \int \frac{\mathrm{d}^2\mathbf{p}\, \mathrm{d}^2\mathbf{q}}{(2\pi)^2} \delta^2(\mathbf{k}-\mathbf{p}-\mathbf{q})\, \Theta(t,y,\mathbf{k},\mathbf{p},\mathbf{q}) |M(\mathbf{p},\mathbf{q})|^2 J(t,y,\mathbf{p}) J(t,y,\mathbf{q}).$$

- Here $\Theta(t,y,\mathbf{k},\mathbf{p},\mathbf{q}) \doteq \pi\delta(\Delta\Omega)$ is the **frequency resonance condition**, where

$$\Delta\Omega(t,y,\mathbf{k},\mathbf{p},\mathbf{q}) \doteq \Omega(t,y,\mathbf{k}) - \Omega(t,y,\mathbf{p}) - \Omega(t,y,\mathbf{q}).$$

Deriving the WKE: source term

$$\{\{D_{\mathrm{H}}, W\}\} + [[D_{\mathrm{A}}, W]] = 2\epsilon^2\,\mathrm{Im}\{F \star [D^{-1}]^*\} - 2\epsilon^2\,\mathrm{Im}\,(\eta \star W) + 2\epsilon^2\,\mathrm{Im}\{S \star [D^{-1}]^*\}.$$

- We assume that $\widetilde{\xi}$ is white noise, then

$$\langle x \mid \widehat{\mathcal{S}} \mid x \rangle = \overline{\widetilde{\xi}(t,\mathbf{x})\widetilde{\xi}(t',\mathbf{x}')} = \delta(t-t')\ \Xi((y+y')/2, \mathbf{x}-\mathbf{x}').$$

- The Weyl symbol of the zonal-averaged density operator for the stochastic forcing is given by

$$\begin{aligned} S(t,y,\mathbf{k}) &= \int \mathrm{d}\tau\,\mathrm{d}^2\mathbf{s}\,e^{-i\omega\tau + i\mathbf{k}\cdot\mathbf{s}}\overline{\widetilde{\xi}(t+\tfrac{\tau}{2},\mathbf{x}+\tfrac{\mathbf{s}}{2})\widetilde{\xi}(t-\tfrac{\tau}{2},\mathbf{x}-\tfrac{\mathbf{s}}{2})} \\ &= \int \mathrm{d}^2\mathbf{s}\,e^{i\mathbf{k}\cdot\mathbf{s}}\,\Xi(y,\mathbf{s}) \\ &= 2\int \mathrm{d}^2\mathbf{s}\ \Xi(y,\mathbf{s})\cos(\mathbf{p}\cdot\mathbf{s}). \qquad [\,\Xi(y,\mathbf{s}) = \Xi(y,-\mathbf{s})\,] \end{aligned}$$

- Thus, when integrating over ω, one obtains

$$2\epsilon^2\int \mathrm{d}\omega\,\mathrm{Im}\{S \star [D^{-1}]^*\} \simeq 2\epsilon^2\mathrm{Im}\int \mathrm{d}\omega\,S\ \underbrace{\frac{1}{D_{\mathrm{H}} - iD_{\mathrm{A}}}}_{\simeq i\pi\delta(D_{\mathrm{H}}) + \mathcal{P}\frac{1}{D_{\mathrm{H}}}} \simeq 2\pi\epsilon^2 S(t,y,\mathbf{k})$$

Backup slides
General WKE with wave scattering

General WKE with wave scattering

- Let us consider a scalar real wave $\psi(t,\mathbf{x})$ propagating in a medium that can be nonstationary and inhomogeneous and also contains a second-order nonlinearity.

$$\underbrace{(\widehat{\mathcal{D}}_{\text{lin}}\psi)(x)}_{\text{linear propagation}} = \underbrace{(\widehat{\alpha}\psi)(x)\,(\widehat{\beta}\psi)(x)}_{\text{local quadratic nonlinearity}} + \underbrace{S(x)}_{\text{source term}}$$

- Following a more general, but similar, procedure, one obtains the following WKE:

$$\boxed{\partial_t J + \{J,\Omega\} = 2\gamma J + S_{\text{ext}} + C[J,J].}$$

- Here Ω is the wave frequency satisfying: $D_{\text{H}}\big(t,\mathbf{x},\Omega(t,\mathbf{x},\mathbf{k},\mathbf{k}),\mathbf{k}\big)=0$.

- The dissipation term γ and the source term S_{ext} are given by

$$\gamma(t,\mathbf{x},\mathbf{k}) \doteq -\left(\frac{D_{\text{A}}}{\partial D_{\text{H}}/\partial\omega}\right)_{\omega=\Omega}(t,\mathbf{x},\mathbf{k}),$$

$$S_{\text{ext}}(t,\mathbf{x},\mathbf{k}) \doteq \left(\frac{S}{\partial D_{\text{H}}/\partial\omega}\right)_{\omega=\Omega}(t,\mathbf{x},\mathbf{k}).$$

- Finally, $C[J,J]$ represents **wave–wave collisions**

$$C[J,J] \doteq S_{\text{nl}}[J,J] - 2\gamma_{\text{nl}}[J]J.$$

Wave–wave collisions

- The nonlinear dissipation coeff. $\gamma_{\rm nl}[J]$ and the nonlinear source term $S_{\rm nl}[J, J]$ are

$$\gamma_{\rm nl}[J](t,\mathbf{x},\mathbf{k}) \doteq -\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{q})\,\frac{\Theta(t,\mathbf{x},\mathbf{k},\mathbf{p},\mathbf{q})}{\mathcal{N}}\,\mathrm{Re}[\,M(t,\mathbf{x},\mathbf{p},\mathbf{q})M^*(t,\mathbf{x},\mathbf{p},-\mathbf{k})\,]\,J(t,\mathbf{x},\mathbf{p}),$$

$$S_{\rm nl}[J,J](t,\mathbf{x},\mathbf{k}) \doteq \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{q})\,\frac{\Theta(t,\mathbf{x},\mathbf{k},\mathbf{p},\mathbf{q})}{\mathcal{N}}\,|M(t,\mathbf{x},\mathbf{p},\mathbf{q})|^2\,J(t,\mathbf{x},\mathbf{p})J(t,\mathbf{x},\mathbf{q}).$$

- Here $\Theta(t,\mathbf{x},\mathbf{k},\mathbf{p},\mathbf{q}) \doteq \pi\delta(\Delta\Omega)$, and

$$\Delta\Omega(t,\mathbf{x},\mathbf{k},\mathbf{p},\mathbf{q}) \doteq \Omega(t,\mathbf{x},\mathbf{k}) - \Omega(t,\mathbf{x},\mathbf{p}) - \Omega(t,\mathbf{x},\mathbf{q}).$$

- $\mathcal{N}(t,\mathbf{x},\mathbf{k},\mathbf{p},\mathbf{q})$ is a normalization factor:

$$\mathcal{N} \doteq \partial_\omega D_{\rm H}(t,\mathbf{x},\mathbf{k})\,\partial_\omega D_{\rm H}(t,\mathbf{x},\mathbf{p})\,\partial_\omega D_{\rm H}(t,\mathbf{x},\mathbf{q}).$$

- $M(x,p,q)$ is a scattering cross section:

$$M(t,\mathbf{x},\mathbf{p},\mathbf{q}) \doteq M(t,\mathbf{x},p_0,\mathbf{p},q_0,\mathbf{q})|_{p_0=\Omega(t,\mathbf{x},\mathbf{p}),\ q_0=\Omega(t,\mathbf{x},\mathbf{q})},$$

$$M(x,p,q) \doteq \alpha(x,p)\beta(x,q) + \alpha(x,q)\beta(x,p),$$

where α and β are the Weyl symbols of $\widehat{\alpha}$ and $\widehat{\beta}$.

Note: $(\widehat{\mathfrak{D}}_{\rm lin}\psi)(x) = (\widehat{\alpha}\psi)(x)\,(\widehat{\beta}\psi)(x) + S(x)$