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A Memory Efficient BDDC Algorithm for Higher Order Elements

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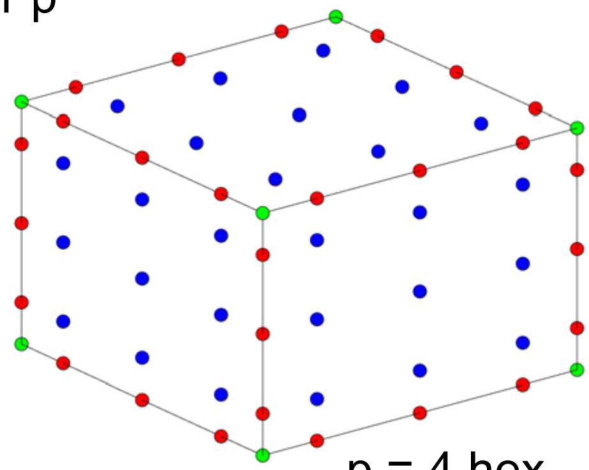
Outline

- **Background:**
  - Some basics
  - A case for BDDC
  - Related work
- **Approach:**
  - Avoid dense matrices
  - Allow for inexact solvers
  - Work and memory estimates
- **Numerical Results**
- **Recap**

Background

- **Some basics:**
  - Focus on 3D elements (degree p)
    - $O(p^3)$ degrees of freedom, $O(p^6)$ matrix entries
    - Matrix storage may be prohibitive for larger p
      - Interest in matrix-free approaches
  - Simple approach:
    - $O(p^9)$ operations for element matrix
    - $O(p^6)$ for matrix-vector products
  - Optimized approaches:
    - Sum factorization method (Orszag, 1980)
      - See refs for tensor product and simplicial elements
    - $O(p^6)$ operations for matrix (optimal)
    - $O(p^4)$ for matrix-vector products



p = 4 hex element

Background

- **A case for BDDC:**
  - Treat each higher order element as a subdomain
    - Challenging computations isolated to individual elements
  - Coarse problem involves assembly of lower order elements
    - We know how to solve this problem
  - Existing theory available and promising numerical results
- **So what's the challenge?**
  - Dirichlet & Neumann problems at subdomain level involve large dense matrices, leading to memory & computational concerns
- **What to do?**
  - Develop BDDC preconditioner for easier problem that avoids large dense matrices & use in Krylov method for original problem

Background

- **Related work*:**
  - Pavarino (2007):
    - Theory, 2D examples, exact solvers, single element subdomains
  - Klawonn, Pavarino, Rheinbach (2008):
    - Theory, 2D focus, exact & inexact FETI-DP solvers
  - Pavarino, Widlund, Zampini (2010)
    - Theory, 3D focus, almost incompressible elasticity, exact solvers
  - Isogeometric analysis: BCPS13, BPSWZ14
  - Bertoluzza, Pennacchio, Prada (2017)
    - Virtual element method, 2D only, theory for triangular elements
  - Discontinuous Galerkin
- **Talk today:**
  - Focus on memory efficiency & inexact solvers

*see Chapter 7 of TW05 for a nice discussion of earlier non-BDDC work

Approach

- **Let's get our feet wet**
  - Exact solvers for dense element matrices (not memory friendly)
  - BDDC coarse space based on edge averages

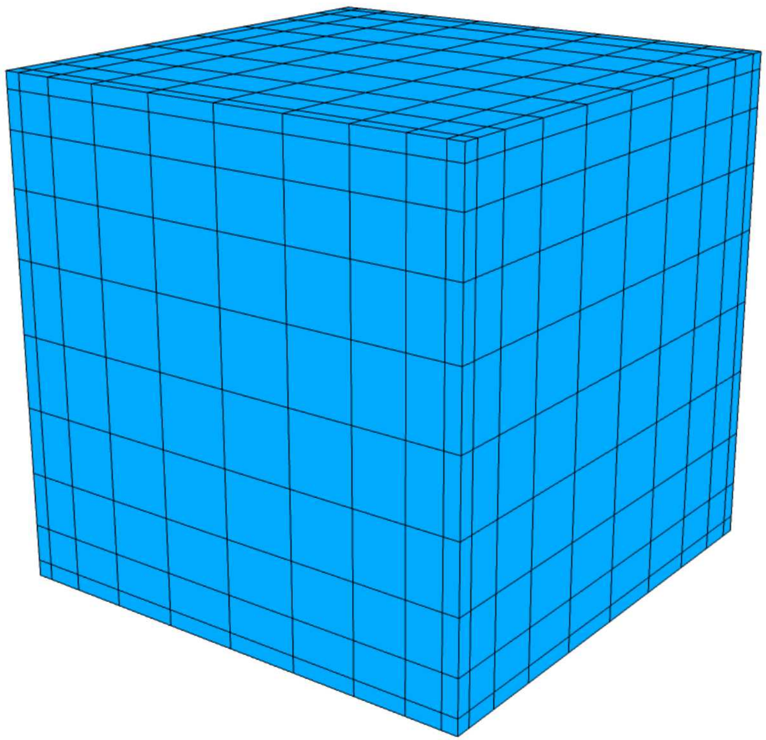
p	4x4x4 hex mesh		6x6x6 hex mesh		8x8x8 hex mesh	
	iter	cond	iter	cond	iter	cond
2	10	2.0	11	2.1	11	2.2
3	9	1.7	10	1.8	10	1.8
4	12	2.4	13	2.5	13	2.6
5	13	2.7	14	2.8	14	2.8

p	4x4x4 tet mesh		6x6x6 tet mesh		8x8x8 tet mesh	
	Iter	cond	iter	cond	iter	cond
2	12	2.3	13	2.4	13	2.5
3	20	8.5	21	8.5	21	8.5
4	29	15.2	29	15.1	29	15.1
5	35	21.3	35	21.3	36	21.3

Takeaway: Hex mesh results look great $\Rightarrow$ focus on them for remainder of talk

Approach

- **How to avoid dense matrices?**
  - Use old idea from Orszag (1980)
  - See Chapter 7 of TW05 for discussion and other applications



	K_p (dense)	K_h (sparse)
Memory	$O(p^6)$	$O(p^3)$
Factorization work	$O(p^9)$	$O(p^6)$
Factorization memory	$O(p^6)$	$O(p^{4/3})$
AMG initialization		$O(p^3)$
AMG memory		$O(p^3)$

Recall optimized matrix-vector products require $O(p^4)$ work

- It's been said "A good preconditioner of a good preconditioner remains a good preconditioner"

Approach

- **Just how good is this preconditioner?**
  - Calculate min and max eigenvalues of $K_h x = \lambda K_p x$
  - Gauss-Lobatto integration for K_p
  - Gauss-Lobatto integration for each constituent linear element of K_h gave better results than standard Gauss quadrature

p	λ_{min}	λ_{max}	$\lambda_{max}/\lambda_{min}$
2	0.42	2.25	5.3
4	0.50	2.51	5.01
6	0.49	2.64	5.35
8	0.48	2.69	5.60
10	0.47	2.72	5.79

Takeaway: Pretty good approximation, but no need for super accurate inexact solvers

Approach

- **Overview:**
  - Construct BDDC preconditioner M^{-1} for A_h
    - A_h obtained from assembly of K_h matrices
    - Each K_h matrix treated as its own subdomain
  - Use either sparse direct solver or AMG preconditioner for local Dirichlet and Neumann problems of BDDC
    - Either way, no longer solving an interface problem
  - Care needed for inexact solution of Neumann problems
    - Neumann problems used to construct coarse matrices
    - Inexact solvers should satisfy a null space property (see D07)
  - Solve linear system $A_p x = b$ using a Krylov method with M^{-1} as the preconditioner

Numerical Results

- **First step: How well does M^{-1} precondition A_h?**

Unit cube, 64 subdomains, DBC on one side, solver tol 10^{-6}, edge-based coarse space, sparse direct subdomain solvers

p	iterations	condition #
2	7	1.4
3	8	1.7
4	10	2.2
5	12	2.7
6	14	3.2
7	15	3.8
10	19	5.6

- **How well should M^{-1} precondition A_p?**
  - Corollary C.2 of TW05: $\kappa\left(M^{-1}A_p\right) \leq \kappa(M^{-1}A_h)\kappa(A_h^{-1}A_p)$

Numerical Results

Same problem but now BDDC preconditioner for original higher order problem (sparse direct or AMG subdomain solvers*)

	sparse direct		AMG	
p	iterations	condition #	iterations	condition #
2	17	5.6	17	5.7
3	15	5.1	18	5.7
4	17	7.0	20	7.4
5	21	9.9	24	10.8
6	23	13.5	27	14.3
7	25	17.8	31	18.6

*V-cycle, single pre/post smoothing step, max coarse size 10, semi-coarsening

Recap

- Effective BDDC preconditioner for higher order elements
  - Challenging computations isolated to elements
    - – Problem looks like standard one after first level of coarsening
  - Memory:
    - – $O(p^4)$ for sparse direct subdomain solvers
    - – $O(p^3)$ memory for AMG subdomain preconditioners
    - – Note: there are order p^3 unknowns in the problem
  - Work (per iteration):
    - – sparse direct subdomain solvers
      - » $O(p^6)$ initialization phase, $O(p^4)$ solve phase
    - – AMG subdomain preconditioners
      - » $O(p^3)$ for both initialization and solve phases
  - Looks promising for direct solvers & AMG preconditioners

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