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A Vertex Coarse Space for BDDC in Three Dimensions

Clark Dohrmann & Kendall Pierson

Computational Solid Mechanics &
Structural Dynamics Department
Sandia National Laboratories

joint work with Olof Widlund

Professor Emeritus, Courant Institute

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Outline

- **Background:**
  - Some history
  - Motivating example
- **Approach:**
  - Inexact solver for BDDC coarse problem
  - Vertex-based coarse space
- **Theory:**
  - Basic outline
  - Condition number bounds
- **Numerical Results:**
  - Material property jumps
  - Unstructured meshes & decompositions
  - Performance improvements

Background

- **Some history:**
  - FETI-DP[1] came on scene around 2000
  - BDDC[2] followed shortly after
  - Vertex coarse space performance was disappointing
    - $C(H/h)(1+\log(H/h))^2$ condition number bound (KWD02)
    - Subsequent focus on edge and face-based coarse spaces
  - Recent small coarse space work for overlapping Schwarz
    - Based on subdomain vertices (DW17)
    - Significant reduction in coarse space dimensions for GDSW[3]

- **Some questions:**
  - Can we do better with vertex coarse spaces for BDDC?
  - Can recent work for GDSW be adapted to this end?
  - Is it worth the effort?

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[1] Finite Element Tearing and Interconnecting – Dual Primal

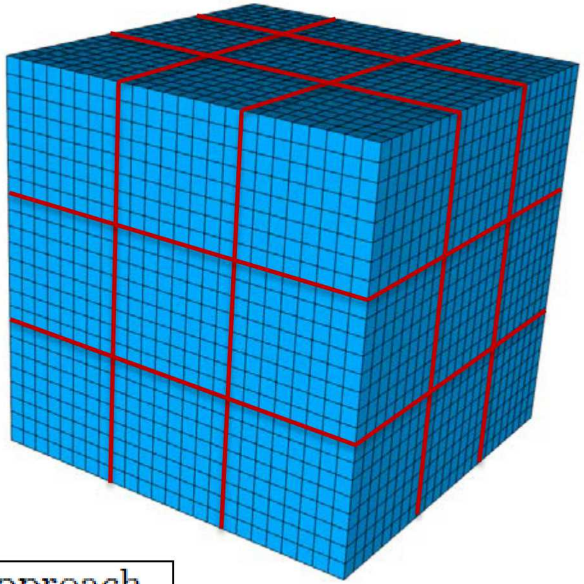
[2] Balancing Domain Decomposition by Constraints

[3] Generalized Dryja Smith Widlund

Background

- **Motivating example:**
  - Unit cube, Dirichlet BCs on one side, random load, 10^{-8} solver tolerance, constant material properties, Poisson equation, 27 subdomains, H/h = 8 shown to right

	standard approach				proposed approach	
	vertices		edges			
H/h	iter	cond	iter	cond	iter	cond
4	28	27.1	12	2.36	14	2.50
8	38	75.2	14	2.93	16	3.13
12	45	132	16	3.37	18	3.59
16	47	195	17	3.73	19	3.97



Increasing number of subdomains

	standard approach						proposed approach		
	vertices			edges					
N	n_c	iter	cond	n_c	iter	cond	n_c	iter	cond
64	27	55	74.5	108	15	2.98	27	17	3.25
216	125	70	73.7	450	15	2.94	125	17	3.26
512	343	74	73.6	1176	15	2.95	343	17	3.30
1000	729	75	73.6	2430	15	2.95	729	17	3.32

N: number of subdomains
n_c: coarse space dimension

Approach

- **Basic idea:**
  - Use inexact solver (preconditioner) for coarse problem
    - Original coarse problem based on either edge or face constraints
    - 3 and multilevel methods essentially do this (e.g. T07)
    - studies for FETI-DP also looked into this (e.g. KR07)
  - Standard BDDC preconditioner:

$$M^{-1} = M_{local}^{-1} + \Phi_D \boxed{K_c^{-1}} \Phi_D^T,$$

K_c: coarse matrix

Φ_D: weighted interpolation matrix

  - Approximate BDDC preconditioner:

$$M_a^{-1} := M_{local}^{-1} + \Phi_D \boxed{M_c^{-1}} \Phi_D^T$$

M_c^{-1}: preconditioner for K_c

Takeaway: Simply replace direct solver for K_c with a preconditioner

Approach

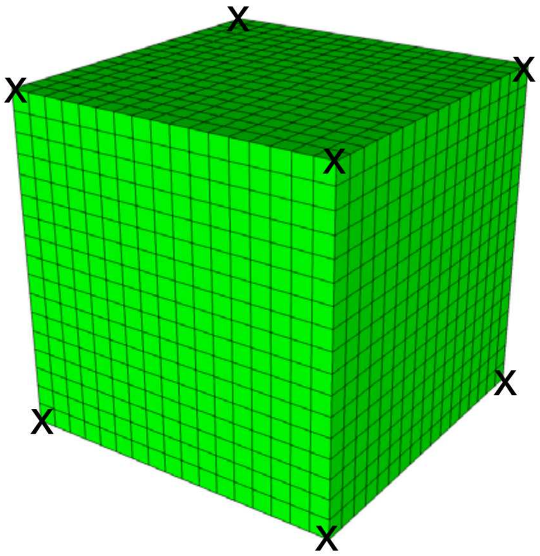
- **Coarse problem preconditioner:**
  - Standard 2-level "overlapping" Schwarz

$$M_c^{-1} = \Psi K_{cr}^{-1} \Psi^T + \mathrm{diag}(K_c)^{-1}$$

$$K_{cr} := \Psi^T K_c \Psi$$

rows of Ψ: orig coarse space

columns of Ψ: sub vertices



  - Coarse correction
    - Restricts original coarse residual to vertex coarse space
    - Solves smaller vertex-based coarse problem
    - Prolongates solution back to original coarse space
  - Local correction
    - Simple Jacobi "smoothing" (no sparse direct solvers)
    - Gauss-Seidel in implementation

Theory

- **Basic outline:**
  - Coarse problem preconditioner assumed to satisfy

$$\beta_1 u_c^T K_c^{-1} u_c \leq u_c^T M_c^{-1} u_c \leq \beta_2 u_c^T K_c^{-1} u_c \quad \forall u_c,$$

  - We then have (κ is condition number)

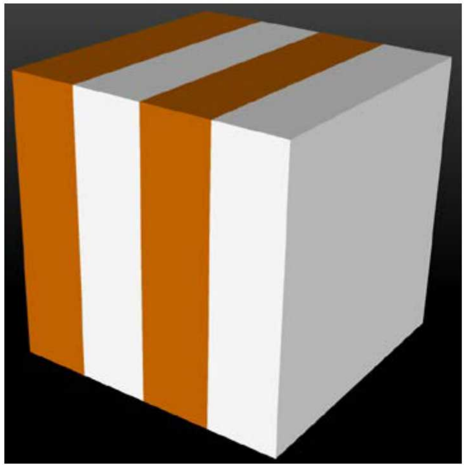
$$\kappa(M_a^{-1}K) \leq \frac{\max(1,\beta_2)}{\min(1,\beta_1)}\kappa(M^{-1}K)$$

  - Getting estimate for β_2 is easy, challenge is for β_1
  - Standard Schwarz analysis
    - Similar to recent small GDSW coarse space analysis, but different
    - Working with coarse BDDC elements rather than finite elements
    - Work by Tu07 helpful for edge-based coarse spaces
    - New result needed for face-based coarse spaces
    - Korn inequalities for linear elasticity

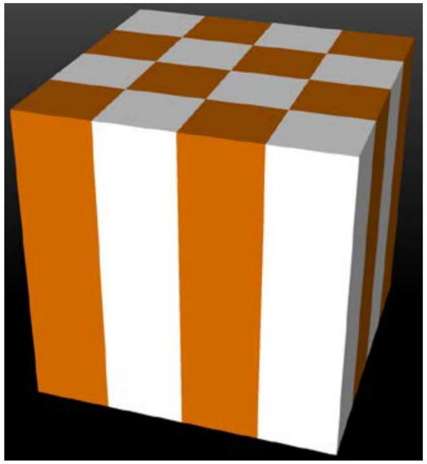
Theory

- **Material property assumptions:**
  - Edge-based coarse spaces:
    - Quasi-monotone edge connected paths for scalar elliptic problems
    - Quasi-monotone face connected paths for linear elasticity
  - Face-based coarse spaces:
    - Quasi-monotone face connected paths (scalar case)*
    - Quasi-monotone face connected paths (linear elasticity)

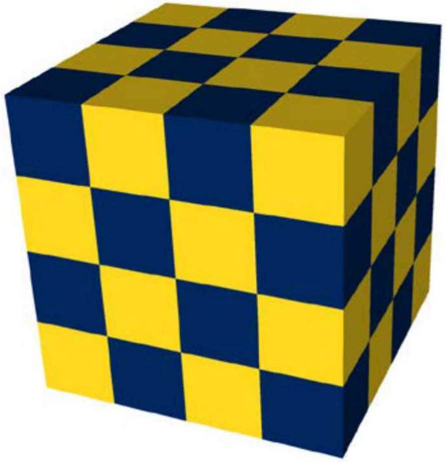
face connected



edge connected



checkerboard



*extra factor of $(1 + \log(H/h))$ in estimates if only edge-connected paths

Theory

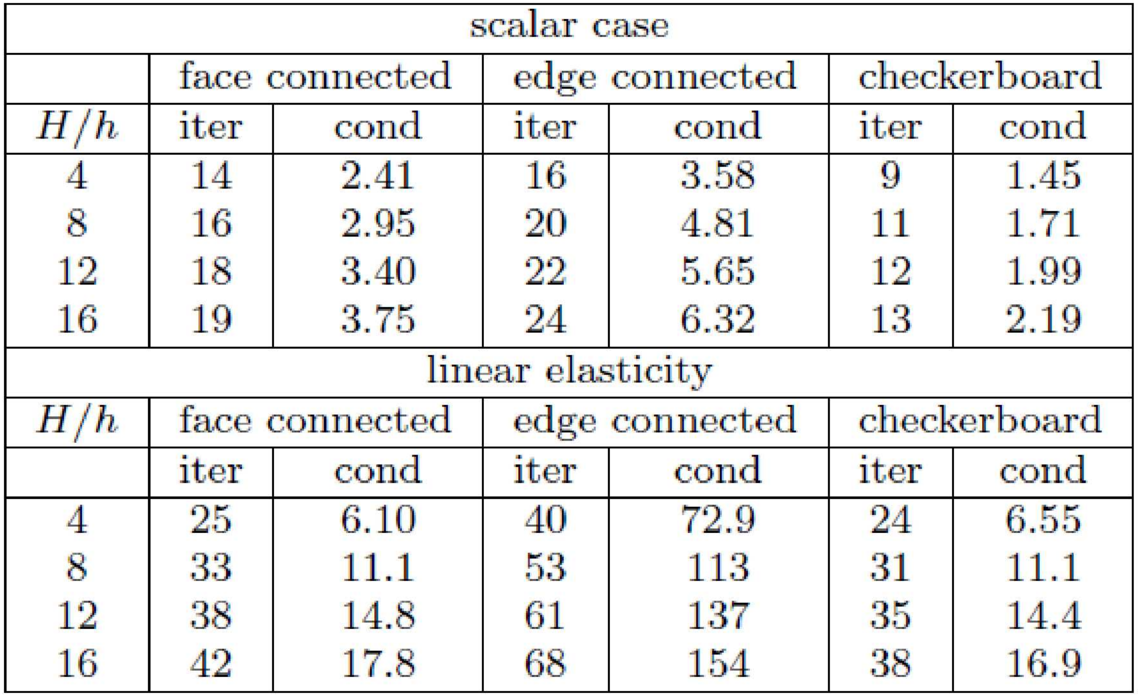
- **Condition number bounds:**
  - We obtain a uniform lower bound on β_1 under the stated assumptions for
    - Scalar elliptic and linear elasticity problems
    - Edge-based and face-based coarse spaces
  - Using existing BDDC results for $\kappa(M^{-1}K)$ then gives us the standard estimate

$$\kappa(M_a^{-1}K) \leq C(1 + \log(H/h))^2$$

Numerical Results

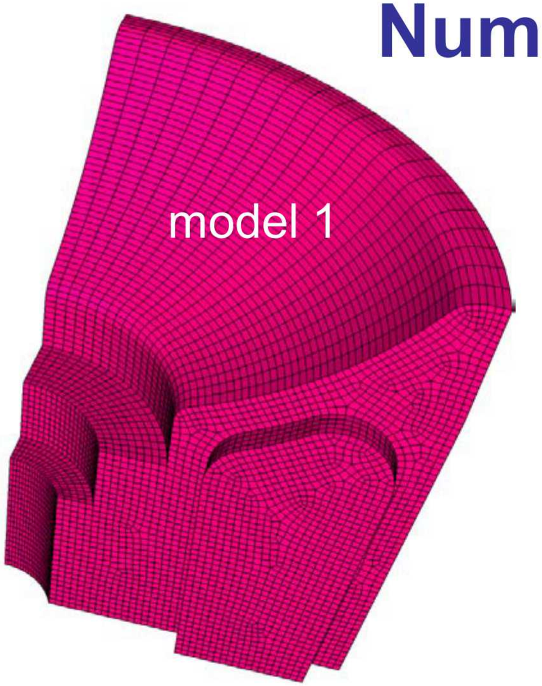
- **Material property jumps:**
  - The material properties in white and yellow regions have $\rho = 1$ for the scalar case and $E = 1$, $\nu = 0.3$ for elasticity. Likewise, orange and blue regions have $\rho = 10^3$, $E = 10^3$, $\nu = 0.3$

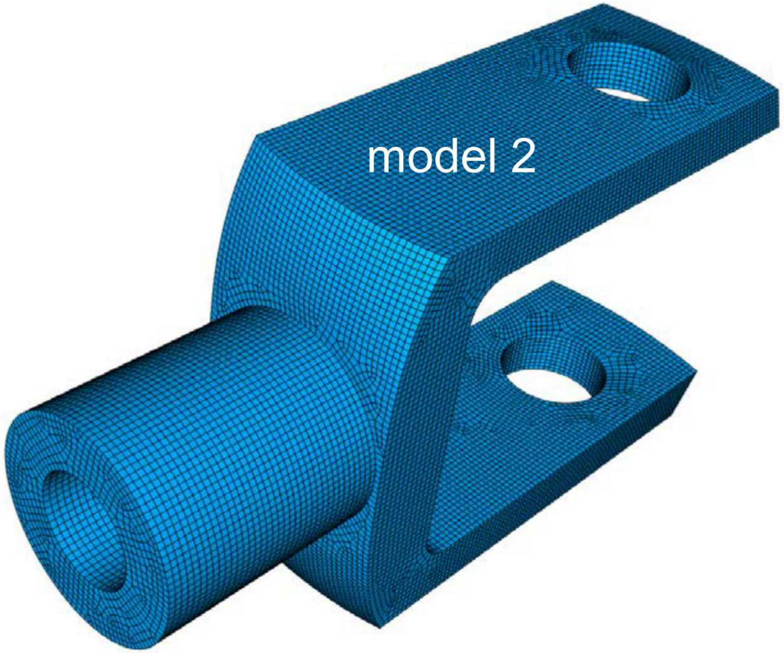
Results for edge-based coarse spaces



scalar case						
	face connected		edge connected		checkerboard	
H/h	iter	cond	iter	cond	iter	cond
4	14	2.41	16	3.58	9	1.45
8	16	2.95	20	4.81	11	1.71
12	18	3.40	22	5.65	12	1.99
16	19	3.75	24	6.32	13	2.19
linear elasticity						
H/h	face connected		edge connected		checkerboard	
	iter	cond	iter	cond	iter	cond
4	25	6.10	40	72.9	24	6.55
8	33	11.1	53	113	31	11.1
12	38	14.8	61	137	35	14.4
16	42	17.8	68	154	38	16.9

Numerical Results





Scalar elliptic results for edge-based coarse spaces

			standard approach			**proposed approach**		
model	N	#elem	n_c	iter	cond	n_c	iter	cond
1	100	47,887	555	25	8.1	236	30	14
1	805	363,024	6101	28	7.7	2909	34	14
2	201	97,316	564	29	13	269	31	15
2	903	450,661	5166	22	4.4	2201	23	4.9

Numerical Results

- **Performance improvements:**
  - Unit cube decomposed into smaller cubic subdomains. Results for scalar case & edge-based coarse spaces.

Speedups for proposed approach and ratios of coarse problem dimensions

# subdomains	Initialization	Solve	ratio n_c
13,824	1.8	1.0	0.32
21,952	2.1	1.1	0.32
32,768	2.5	1.2	0.32
46,656	3.0	1.4	0.32

Note: best times used from 4 runs on compute server.

Recap

- New approach similar to using standard BDDC vertex coarse space, but with much better convergence rates
  - Combines vertex coarse space efficiency with attractive convergence rates of edge or face-based approaches
- Can delay need for additional coarse levels
  - Significantly smaller coarse space dimensions compared to standard edge or face-based approaches
  - Reduced number of synchronization points
- Theory developed for both scalar elliptic and linear elasticity
  - BDDC coarse space may originate from either edge or face-based approaches

Extra Slides