AFC Algorithms in Continuous FE Schemes for HD & MHD

S. Mabuza[1], J.N. Shadid[1], D. Kuzmin[2], E.C. Cyr[1], T.M. Smith[1]

[1]Center for Computing Research, Sandia National Laboratories
[2]Institute of Applied Mathematics (LS III), TU Dortmund University

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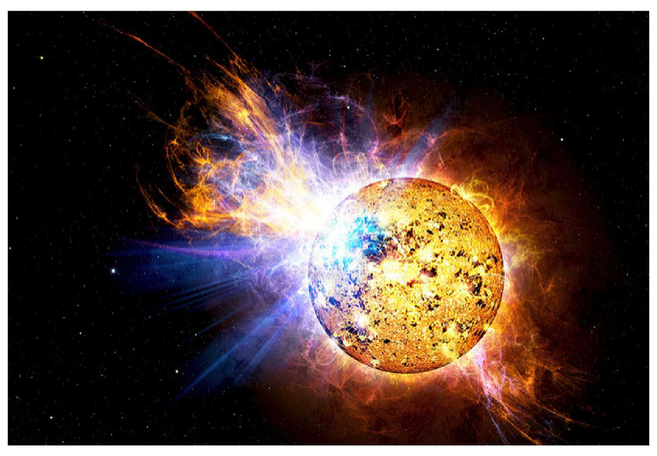
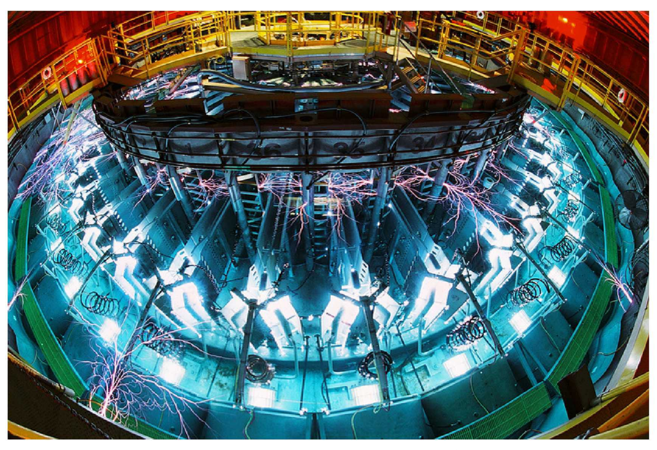
Motivation

Solution to strongly coupled multiphysics problems, often involving strong shocks:

- be positivity preserving for ρ, pressure etc. (e.g. LED schemes)
- be applicable to widely varying timescales
- allow flexible time integrator usage (e.g. RK-IMEX schemes)
- be on general unstructured meshes, high spatial and temporal order
- start with ideal MHD as an effort towards multi-fluid plasma



Astrophysical plasma (Casey Reed/NASA)



Laboratory plasma (SNL)

HD & MHD Systems

Ideal non-viscous,non-resistive MHD:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{v}\\ \rho E\\ \boldsymbol{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\boldsymbol{v}\\ \rho\boldsymbol{v}\otimes\boldsymbol{v}+pI-\boldsymbol{T}_M\\ \boldsymbol{v}\cdot((\rho E+p)I-\boldsymbol{T}_M)\\ \boldsymbol{B}\otimes\boldsymbol{v}-\boldsymbol{v}\otimes\boldsymbol{B}\end{bmatrix} = \mathbf{0},\ \nabla\cdot\boldsymbol{B}=0.$$

where,

$$\rho E = \frac{p}{\gamma-1}+\frac{1}{2}\rho\|\boldsymbol{v}\|^2+\frac{1}{2\mu_0}\|\boldsymbol{B}\|^2,\ \gamma>1.$$

and

$$\boldsymbol{T}_M = \frac{1}{\mu_0}\left[\boldsymbol{B}\otimes\boldsymbol{B}-\frac{1}{2}\|\boldsymbol{B}\|^2 I\right].$$

Solution strategy:

- Do continuous finite element discretization in space (P^1/Q^1).
- Introduce algebraic stabilization (low order diffusion + limiters).
- Do divergence cleaning for MHD (hyperbolic/parabolic/mixed).

Achievements so far:

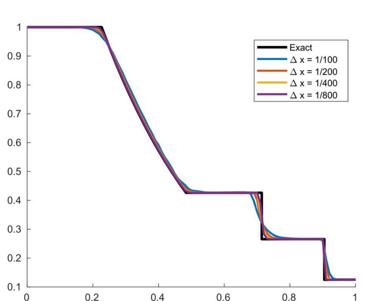
- Extensive studies for Euler equations (different limiter designs)
- Applications to steady hydrodynamics problems (Scramjet)
- Applications to 1D/2D magnetohydrodynamics problems
- Applied explicit and implicit time stepping (explicit RK4 & Crank-Nicolson)

Numerical results HD

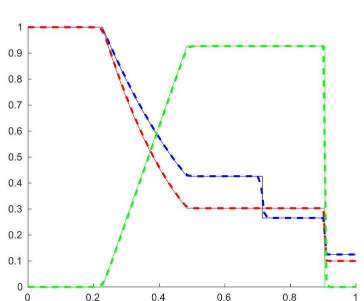
1D results

Sod shock tube

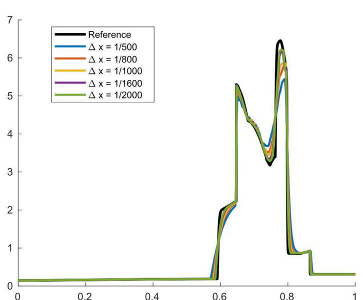
Woodward-Colella



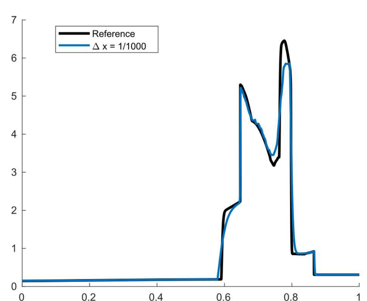
ρ vs x



prim. vars vs x, for $\Delta x = \frac{1}{800}$



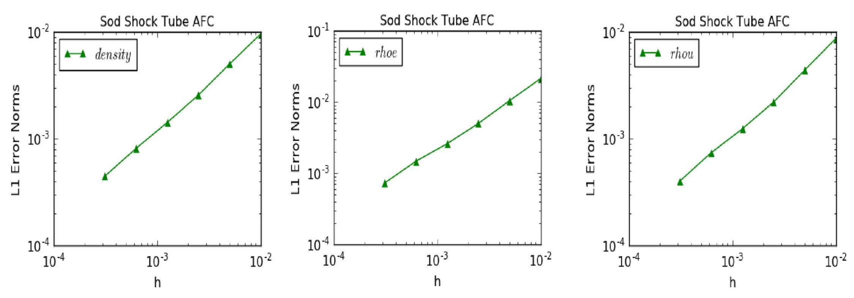
ρ vs x



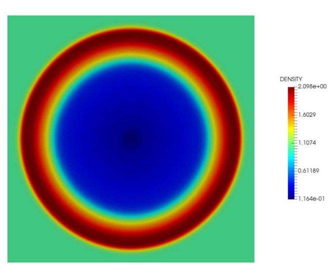
ρ vs x at $\Delta x = \frac{1}{1000}$

Sod ShockTube – Mesh Convergence 100-3200 cells

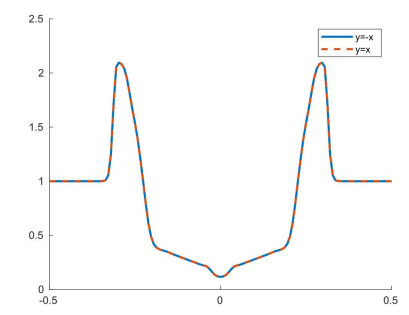
	L1 Slope (100-200)	L1 Slope (200-400)	L1 Slope (400-800)	L1 Slope (800-1600)	L1 Slope (1600-3200)
rho	9.36689e-01	9.65261e-01	8.52469e-01	8.06609e-01	8.60413e-01
rhoE	1.03158e+00	1.06352e+00	9.39890e-01	8.16929e-01	1.01285e+00
rhou	9.87064e-01	9.94115e-01	8.30446e-01	7.41841e-01	8.76973e-01



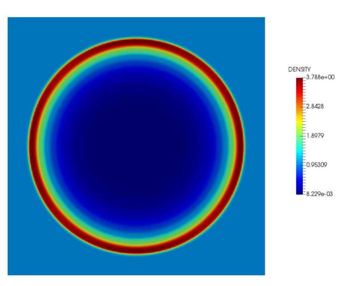
2D results; Top: Radial Riemann, Bottom: Sedov point blast



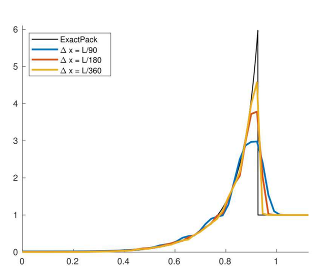
$\Delta x = \Delta y = 1/128$ and $\Delta t = 1\times10^{-3}$



ρ vs x trace at $x=y$ and $y=-x$

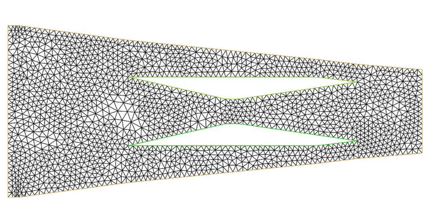


$\Delta x = \Delta y = 0.0125$ and $\Delta t = 1\times10^{-5}$.

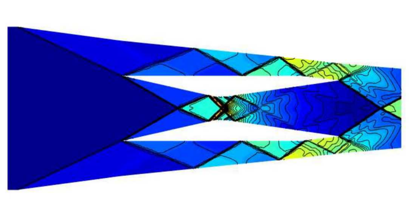


ρ vs x, $L = 2.25$ and $t = 0.24$.

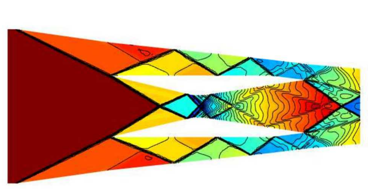
Steady 2D $Ma = 3$ supersonic combustion ramjet engine



Coarse unstructured mesh



Density



Mach

65256 elements and 33859 degrees of freedom per variable.

Stabilized CG scheme

Continuous problem $\Omega\subset\mathbb{R}^d$:

$$\mathbf{u}_t+\nabla\cdot\mathbf{f}(\mathbf{u})=\mathbf{0},\ (\boldsymbol{x},t)\in\Omega\times\mathbb{R}_+,\ \mathbf{u}(\boldsymbol{x},0)=\mathbf{u}_0(\boldsymbol{x}),$$
$$\mathbf{u}:(\boldsymbol{x},t)\mapsto\mathbb{R}^m,\ \mathbf{f}:\mathbb{R}^m\to\mathbb{R}^{m\times d}.$$

High order semi-discrete scheme in FE space $V^h=P^1$ or Q^1 with N_h degrees of freedom:

$$\mathbf{M}_L\frac{d\boldsymbol{U}}{dt}=\boldsymbol{K}(\boldsymbol{U})+\mathbf{D}(\boldsymbol{U})\boldsymbol{U}+\boldsymbol{B}(\boldsymbol{U})+\boldsymbol{F}(\boldsymbol{U}).$$

$$\mathbf{M}_L=\mathrm{diag}\{m_i\}_{i=1}^{N_h}I_{m\times m},\ m_i=\sum_j m_{ij},\ \mathbf{M}_C=\{M_{ij}\}_{i,j=1}^{N_h},\ M_{ij}=m_{ij}I_{m\times m},\ m_{ij}=\int_\Omega\phi_i\phi_j d\boldsymbol{x}.$$

$$\boldsymbol{K}(\boldsymbol{U})=\{\boldsymbol{k}_i\}_{i=1}^{N_h},\ \boldsymbol{k}_i=\sum_{e=1}^{N_K}\int_{K_e}\nabla\phi_i\cdot\mathbf{f}(\mathbf{u}_h)d\boldsymbol{x},\ \boldsymbol{B}(\boldsymbol{U})=\{\boldsymbol{b}_i\}_{i=1}^{N_h},\ \boldsymbol{b}_i=-\sum_{e=1}^{N_K}\int_{\partial K_e\cap\Gamma}\phi_i\mathbf{f}(\mathbf{u}_h)\cdot\boldsymbol{n}d\sigma.$$

Artificial diffusion (Rusanov or Roe) (Kuzmin et al., 2012):

$$\mathbf{D}(\boldsymbol{U})=\{D_{ij}\},\ D_{ij}=\sum_e D_{ij}^{(e)},$$

Antidiffusion:

$$\boldsymbol{F}^{(e)}=(\mathbf{M}_L^{(e)}-\mathbf{M}_C^{(e)})\dot{\boldsymbol{U}}^{(e)}-\mathbf{D}^{(e)}(\boldsymbol{U}^{(e)})\boldsymbol{U}^{(e)}.$$

Stabilized semi-discrete scheme:

$$\mathbf{M}_L\frac{d\boldsymbol{U}}{dt}=\boldsymbol{K}(\boldsymbol{U})+\mathbf{D}(\boldsymbol{U})\boldsymbol{U}+\boldsymbol{B}(\boldsymbol{U})+\bar{\boldsymbol{F}},\ \text{where }\bar{\boldsymbol{F}}=\sum_e\alpha_e\boldsymbol{F}^{(e)}$$

Construct a linearity preserving limiter (Kuzmin et al., CMAME 2017):

$$\alpha_e=\min\{\alpha_e^\rho,\alpha_e^{\rho p}\},\ \text{OR}\ \alpha_e=\min\{\alpha_e^\rho,\alpha_e^s\},\ \text{where } s=\log p-\gamma\log\rho.$$

such that

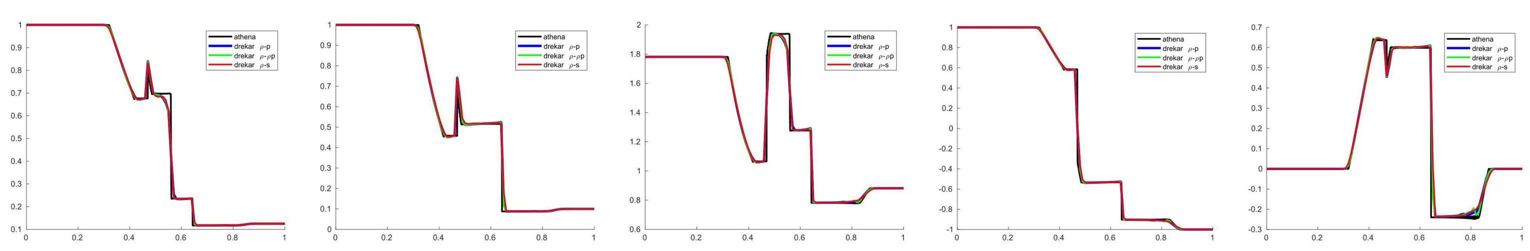
- $\Phi_i(u_i-u_j)$ depends continuously on u_j, $j\in\mathcal{N}(i)$.
- $\Phi_i(u_i-u_j)=0$ at a local maximum and local minimum.
- $\Phi_i(u_i-u_j)=(u_i-u_j)$ if u_h is linear on $\Omega_i=\mathrm{supp}\{\phi_i\}$.

$$\Phi_i^u=1-\left[\frac{\left|\sum_{j\neq i}\beta_{ij}(u_j-u_i)\right|+\epsilon}{\sum_{j\neq i}\beta_{ij}|u_j-u_i|+\epsilon}\right]^q,$$

where $\beta_{ij}\geq0$ such that $\sum_{j\neq i}\beta_{ij}\boldsymbol{g}_i\cdot(\boldsymbol{x}_j-\boldsymbol{x}_i)=0$, $q=10$, and $\epsilon=1\times10^{-16}$.

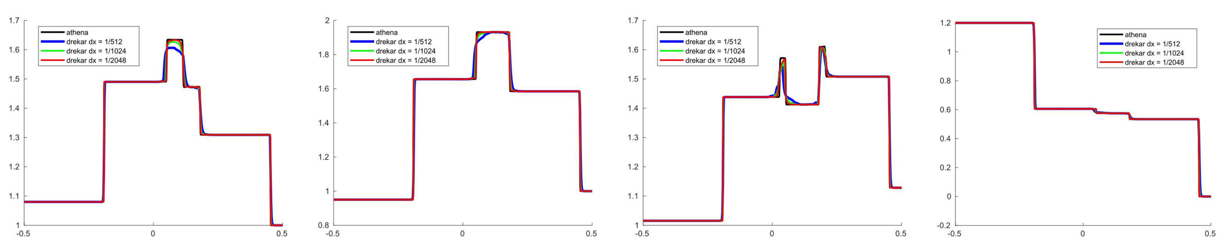
Numerical results MHD

1D Brio-Wu shock tube problem (Brio and Wu, JCP 1988):



ρ p ρE B_y v_x

1D Ryu-Jones problem (Ryu and Jones, Astro J., 1995):



ρ p B_y v_x

Divergence cleaning

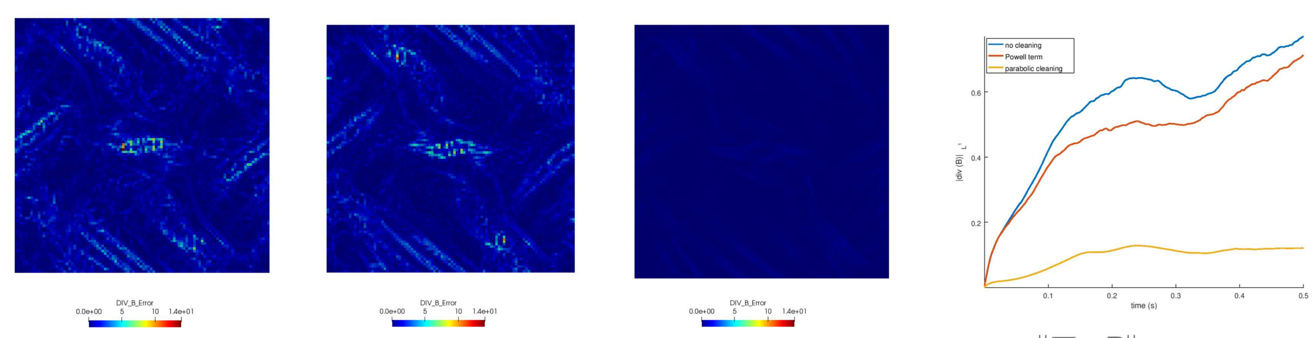
Use of Powell term and parabolic cleaning:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{v}\\ \rho E\\ \boldsymbol{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{v}\\ \rho\boldsymbol{v}\otimes\boldsymbol{v}+pI-\boldsymbol{T}_M\\ \boldsymbol{v}\cdot((\rho E+p)I-\boldsymbol{T}_M)\\ \boldsymbol{B}\otimes\boldsymbol{v}-\boldsymbol{v}\otimes\boldsymbol{B}-c_p^2(\nabla\cdot\boldsymbol{B})I\end{bmatrix}+\begin{bmatrix}0\\ \boldsymbol{B}\\ \boldsymbol{v}\cdot\boldsymbol{B}\\ \boldsymbol{v}\end{bmatrix}(\nabla\cdot\boldsymbol{B})=\mathbf{0}.$$

$\|\nabla\cdot\boldsymbol{B}\|_{L^1(K)}$ range for *no cleaning, Powell term* and *parabolic cleaning*:

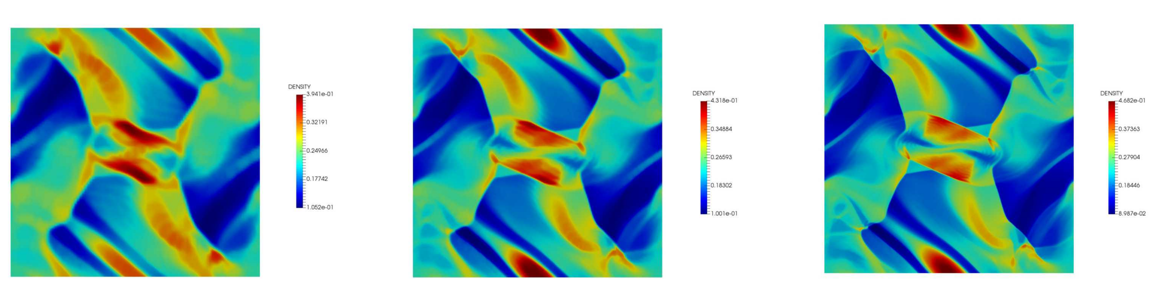
No cleaning	8.7×10^{-3} to 13.955
Powell term	4.802×10^{-3} to 11.0264
Parabolic cleaning	2.6318×10^{-3} to 0.6333

Plots of $\|\nabla\cdot\boldsymbol{B}\|_{L^1(K)}$ *color-scheme* between 0 and 13.955:



No cleaning　Powell term　Parabolic cleaning　$\|\nabla\cdot\boldsymbol{B}\|_{L^1(\Omega)}$ vs. t

Orszag-Tang profile at 3 mesh levels.



ρ for $\Delta x=\Delta y=1/100$　ρ for $\Delta x=\Delta y=1/200$　ρ for $\Delta x=\Delta y=1/400$

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