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PROJECT TEAM MEMBERS: Christian Glusa (1465)
Marta D'Elia (1444)
Bart Van Bloemen Waanders (1463)
Chester Weiss (8861)

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A FAST SOLVER FOR THE FRACTIONAL HELMHOLTZ EQUATION*

HARBIR ANTIL†, MARTA D'ELIA‡, CHRISTIAN GLUSA§, BART VAN BLOEMEN WAANDERS¶, AND CHESTER J. WEISS‖

Abstract. The purpose of this paper is to study a Helmholtz problem with a spectral fractional Laplacian, instead of the standard Laplacian. Recently, it has been established that such a fractional Helmholtz problem better captures the underlying behavior in Geophysical Electromagnetics. We establish the well-posedness and regularity of this problem. We introduce a hybrid finite element-spectral approach to discretize it and show well-posedness of the discrete system. In addition, we derive a priori discretization error estimates. Finally, we introduce an efficient solver that scales as well as the best possible solver for the classical integer-order Helmholtz equation. We conclude with several illustrative examples that confirm our theoretical findings.

Key words. fractional Helmholtz equation, spectral fractional Laplacian

AMS subject classifications. 65N12, 65N22, 65N30, 65N38, 65N55

1. Introduction.

Recently, starting from the Maxwell's equations, the article [25] has derived the scalar fractional Helmholtz equation. It has also established existence of fractional (anomalous) behavior for the Magnetotelluric Problem in geophysical electromagnetics by showing a direct qualitative match between numerical tests and actual datum. Motivated by these results, the goal of this paper is to take a step towards rigorous mathematical foundation of the fractional Helmholtz equation. In particular, we show its well-posedness, introduce a new hybrid (spectral-finite element) approach for its discretization, establish a priori error estimates, and introduce an efficient solver that scales as well as the best solver in the classical (integer-order) case.

Let Ω be a bounded open domain in $\mathbb{R}^d$. We consider the fractional-order Helmholtz problem

$$\text{(fH)} \qquad \begin{cases} (-\Delta)^s u(\vec{x}) - k^{2s} u(\vec{x}) &= f(\vec{x}), \qquad \vec{x} \in \Omega, \\ u(\vec{x}) &= 0, \qquad \vec{x} \in \partial\Omega \end{cases}$$

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†Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA. (hantil@gmu.edu)

‡Center for Computing Research, Sandia National Laboratories, Albuquerque, NM 87185, USA, (mdelia@sandia.gov).

§Center for Computing Research, Sandia National Laboratories, Albuquerque, NM 87185, USA, (caglusa@sandia.gov).

¶Center for Computing Research, Sandia National Laboratories, Albuquerque, NM 87185, USA, (bartv@sandia.gov).

‖Geophysics Department, Sandia National Laboratories, Albuquerque, NM 87185, USA, (cjweiss@sandia.gov).

with a given wave number $k \in \mathbb{C}$ and right-hand side data f. We restrict ourselves to the case of homogeneous Dirichlet boundary conditions. Non-homogeneous conditions can be incorporated by solving an auxiliary local problem with homogeneous right-hand side, see for instance [3, 4]. For $s \in (0,1)$, $(-\Delta)^s$ denotes the fractional powers of the realization in $L^2(\Omega)$ of the classical Laplacian $(-\Delta)$ supplemented with zero Dirichlet boundary conditions. For a rigorous definition of $(-\Delta)^s$, see Section 2. For completeness, we mention that the spectral Laplacian in (fH) is not the only choice for fractional Laplacian, other popular choice is the so-called *integral fractional Laplacian*. The two definitions, coincide when $\Omega = \mathbb{R}^d$ but are different when Ω is bounded [21]. Towards this end, we emphasize that our choice to use the spectral fractional Laplacian is directly motivated by the fact that the article [25] has derived the fractional Helmholtz equation with spectral fractional Laplacian using the "first principles".

In order for problem (fH) to be well-posed, we require that

$$\lambda_m \neq k^2, \quad m \in \mathbb{N}. \tag{1.1}$$

where $\lambda_m > 0$ are the eigenvalues of standard Laplacian with zero Dirichlet boundary conditions, see (Eig).

The article [25] solved the nonlocal operator $(-\Delta)^s$ using the so-called Kato or Balakrishnan formula [13]; the use of this formula in the context of fractional Poisson equation was first proposed in [7]. However, in this work, we use the so-called extension approach that stems from probability literature [17], but was pioneered by L. Caffarelli and collaborators [9, 23]. The extension approach says that $(-\Delta)^s$ is the Dirichlet-to-Neumann map for a harmonic extension of the solution. The key advantage of this is the fact that the extension problem is local, albeit it is posed on a semi-infinite domain, $\Omega \times (0,\infty) \subset \mathbb{R}^{d+1}$, with one additional space dimension. This fact introduces computational challenges. In order to create finite element based numerical approximation, the article [18], in case of Poisson equation, introduced a truncation approach so that the resulting domain is bounded. On the other hand, [1] introduced a different approach where no such truncation is needed. Our hybrid spectral-finite element discretization of (fH) is motivated by the latter. We establish well-posedness of both continuous and discrete problems and derive a priori error estimates. We also introduce an efficient solver, which is as good as the best solver for the classical (integer-order) Helmholtz equation. We present numerical results in three dimensions.

The remainder of the work is structured as follows: Section 2 introduces the necessary notation and spaces. In Section 3, we show well-posedness of (fH). In Section 4 we introduce the extension problem and derive properties of its eigenfunctions. Section 5 deals with the hybrid finite element - spectral discretization of the problem and a priori error estimates. In Section 6 we discuss the solver of the resulting linear system. We conclude by showing numerical examples in Section 7.

Remark 1.1. The choice of the coefficient $-k^{2s}$ in (fH) might appear non-intuitive at first. In [25], the fractional Helmholtz problem is stated as

$$\begin{cases} (-\Delta)^s u(\vec{x}) - \kappa^2 u(\vec{x}) &= f(\vec{x}), \qquad \vec{x} \in \Omega, \\ u(\vec{x}) &= 0, \qquad \vec{x} \in \partial\Omega \end{cases}$$

with wave number $\kappa \in \mathbb{C}$. Clearly, this is just a matter of notation, and $k = \kappa^{1/s}$. We prefer the coefficient $-k^{2s}$, because this choice leads to weaker restrictions on the mesh size h when solving fractional Helmholtz problem for fixed wave number k and different values of the fractional order s. We also notice that, with the proposed formulation, we need to solve the classical, integer-order, Helmholtz problem with wave number k. Nevertheless, everything that follows also holds if we use $\kappa^{1/s}$ instead of k.

2. Notation. The purpose of this section is to introduce relevant notation and preliminary results. The content of this section is well-known. Unless otherwise stated, Ω will be a bounded Lipschitz domain in $\mathbb{R}^d$. To this end, we define the fractional-order Sobolev (Hilbert) space as

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \mid \|u\|_{H^s(\Omega)} < \infty \right\}, \tag{2.1}$$

equipped with the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \int_\Omega d\vec{x} \int_\Omega d\vec{y} \frac{|u(\vec{x}) - u(\vec{y})|^2}{|\vec{x} - \vec{y}|^{d+2s}} \right)^{\frac{1}{2}}.$$

Next, we define the spectral fractional Laplacian $(-\Delta)^s$. Let $0 < \lambda_0 \leq \lambda_1 \leq \ldots$ and $\phi_0, \phi_1, \ldots$ be the eigenvalues and eigenfunctions of the standard Laplacian, i.e.

$$\begin{cases} -\Delta\phi_m(\vec{x}) &= \lambda_m \phi_m(\vec{x}), \qquad \vec{x} \in \Omega, \\ \phi_m(\vec{x}) &= 0, \qquad \vec{x} \in \partial\Omega, \end{cases} \tag{Eig}$$

normalized so that $\|\phi_m\|_{L^2(\Omega)} = 1$. Then $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(\vec{x}) = \sum_{m=0}^{\infty} u_m \lambda_m^s \phi_m(\vec{x}), \quad \text{with} \quad u_m = (u, \phi)_{L^2}.$$

Notice that the eigenfunctions $\{\phi_m\}_{m=0}^{\infty}$ form a complete orthonormal basis of $L^2(\Omega)$.

Using the spectrum of the Laplacian $\{(\lambda_m, \phi_m)\}_{m\in\mathbb{N}}$, we define yet another fractional-order Sobolev space [24], [8, Appendix B]

$$\widetilde{H}^s(\Omega) = \left\{ u \in L^2(\Omega) \mid |u|_{\widetilde{H}^s(\Omega)} < \infty \right\}, \tag{2.2}$$

where the norm is given by

$$|u|_{\widetilde{H}^s(\Omega)} = \left(\sum_{m=0}^{\infty} |u_m|^2 \lambda_m^s \right)^{\frac{1}{2}},$$

and $u_m = (u, \phi_m)_{L^2(\Omega)}$. The two spaces in (2.1) and (2.2) are related to each other. Indeed, for $s > 1/2$, $\widetilde{H}^s(\Omega)$ coincides with the space $H_0^s(\Omega)$ defined to be the closure of $C_0^\infty(\Omega)$ with respect to the $H^s(\Omega)$-norm, whilst for $s < 1/2$, $\widetilde{H}^s(\Omega)$ is identical to $H^s(\Omega)$. In the critical case $s = 1/2$, $\widetilde{H}^s(\Omega) \subset H_0^s(\Omega)$, and the inclusion is strict, $\widetilde{H}^{\frac{1}{2}}(\Omega)$ is known as the Lions-Magenes space. (See for example [14, Chapter 3].) We denote the dual space of $\widetilde{H}^s(\Omega)$ by $\widetilde{H}^{-s}(\Omega)$ and use $\langle \cdot, \cdot \rangle_{\widetilde{H}^s(\Omega), \widetilde{H}^{-s}(\Omega)}$ to denote the duality pairings. For simplicity we drop the subscripts from the duality pairings when it is clear from the context.

The spaces $\widetilde{H}^s(\Omega)$ are useful to describe the properties of the spectral fractional Laplacian. For instance, suppose $f \in \widetilde{H}^r(\Omega)$, $r \geq -s$, and $f = \sum_{m=0}^{\infty} f_m \phi_m(\vec{x})$ with

$f_m = \langle f, \phi_m \rangle$; then, the solution u to the fractional Poisson problem of order s with right-hand side f

$$\text{(fP)} \qquad \begin{cases} (-\Delta)^s u(\vec{x}) &= f(\vec{x}), & \vec{x} \in \Omega, \\ u(\vec{x}) &= 0, & \vec{x} \in \partial\Omega \end{cases}$$

is given by

$$u = \sum_{m=0}^{\infty} u_m \phi_m(\vec{x}), \qquad u_m = f_m \lambda_m^{-s}, \tag{2.3}$$

and hence $u \in \widetilde{H}^{r+2s}(\Omega)$. Notice that no additional smoothness on the domain Ω is needed to get this higher regularity. Nevertheless to establish an equivalence between $\widetilde{H}^{r+2s}$, when $r + 2s > 1$, with higher order Sobolev spaces additional smoothness on the domain Ω is needed. A more detailed regularity theory for spectral Poisson problems can be found in [11].

In principle one could use the expression (2.3) to compute u. However, the cost of pre-computing the unknown eigenvalues *and* eigenfunctions makes this an expensive task. To overcome this hurdle, as mentioned in the Introduction, we follow the approach of Stinga and Torrea [23].

We first introduce some notation. We define the weighted norms on a generic domain $\mathcal{D}$ for a non-negative weight function ω by

$$\|u\|_{L^2_\omega(\mathcal{D})} = \left(\int_{\mathcal{D}} \omega(\vec{x}) |u(\vec{x})|^2 \, d\vec{x} \right)^{\frac{1}{2}}, \qquad |u|_{H^1_\omega(\mathcal{D})} = \left(\int_{\mathcal{D}} \omega(\vec{x}) |\nabla u(\vec{x})|^2 \, d\vec{x} \right)^{\frac{1}{2}},$$
$$\|u\|_{H^1_\omega(\mathcal{D})} = \left(\|u\|^2_{L^2_\omega(\mathcal{D})} + |u|^2_{H^1_\omega(\mathcal{D})} \right)^{\frac{1}{2}},$$

along with the associated weighted spaces

$$L^2_\omega(\mathcal{D}) = \left\{ u \text{ measurable } \mid \|u\|_{L^2_\omega} < \infty \right\}, \quad H^1_\omega(\mathcal{D}) = \left\{ u \in L^2_\omega(\mathcal{D}) \mid \|u\|_{H^1_\omega} < \infty \right\}.$$

In what follows, we use C to denote a generic constant that could change from line to line but is independent of the mesh size h and the wave number k. We will also drop the differential in the integrand when the integration variable is clear from the context.

3. Well-posedness of Fractional Helmholtz Equation. The main goal of this section is to establish existence and uniqueness of the solution to the fractional Helmholtz equation (fH).

We first state the notion of weak solutions.

Definition 3.1. *Given $f \in \widetilde{H}^{-s}(\Omega)$ we say that $u \in \widetilde{H}^s(\Omega)$ is a weak solution to* (fH) *if*

$$a(u, v) = \langle f, v \rangle, \quad \textit{for all } v \in \widetilde{H}^s(\Omega), \tag{3.1}$$

where

$$a(u, v) := \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s} \right) \int_\Omega u \phi_m \int_\Omega \overline{v} \phi_m. \tag{3.2}$$

Next we shall establish the uniqueness of solution to (3.1). We operate under the condition that $k \in \mathbb{C}$ is a constant.

LEMMA 3.2. *Let $f \in \widetilde{H}^{-s}(\Omega)$ be given and $k \in \mathbb{C}$ be a given constant. Assume that (1.1) holds. Then every $u \in \widetilde{H}^s(\Omega)$ solving (fH), according to the Definition 3.1, is unique.*

Proof. It is sufficient to show that when the data $f \equiv 0$ then u solving (3.1) is identically zero. By setting $v = u_\ell \phi_\ell$ (where $\ell \in \mathbb{N}$ is arbitrary) in (3.1), using the orthonormality of $\{\phi_m\}_{m=0}^\infty$, and the assumption that k is a constant, we obtain that

$$a(u, u_\ell\phi_\ell) = \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right) \int_\Omega u\phi_m \int_\Omega \overline{u}_\ell \phi_\ell \phi_m = \left(\lambda_\ell^s - k^{2s}\right) |u_\ell|^2 = 0.$$

Since according to (1.1) $\lambda_\ell^s \neq k^{2s}$, we obtain that $u_\ell = 0$, i.e., $\int_\Omega u\phi_\ell = 0$. Since ℓ was arbitrary, we obtain that $u = 0$ a.e. in Ω. The proof is complete. ☐

LEMMA 3.3 (Gårding's (in)equality). *Let $u \in \widetilde{H}^s(\Omega)$ solve (fH) according to the Definition 3.1 and $k \in \mathbb{C}$ be a given constant, then*

$$a(u, u) + k^{2s}\|u\|^2_{L^2(\Omega)} = \|u\|^2_{\widetilde{H}^s(\Omega)}$$

Proof. From the definition of $a(\cdot,\cdot)$ in (3.2) we obtain that

$$a(u, u) = \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right) |u_m|^2 = \|u\|^2_{\widetilde{H}^s(\Omega)} - k^{2s}\|u\|^2_{L^2(\Omega)}.$$

By rearranging terms in the above equality, we obtain the desired result. ☐

THEOREM 3.4. *Let $f \in \widetilde{H}^{-s}(\Omega)$ be given and $k \in \mathbb{C}$ be a given constant. Assume that (1.1) holds. Then there exists a unique $u \in \widetilde{H}^s(\Omega)$ solving (fH) according to Definition 3.1.*

Proof. Lemma 3.2 and Lemma 3.3 in conjunction with Fredholm alternative give the asserted result. We refer to [6, Theorem 3.3] for similar arguments in case of standard Laplacian. ☐

The next result establishes regularity of solutions of the fractional Helmholtz equation.

PROPOSITION 3.5 (Regularity). *If $f \in \widetilde{H}^r(\Omega)$, $r \geq -s$, then the solution (according to Definition 3.1) to the fractional Helmholtz problem $u \in \widetilde{H}^{r+2s}(\Omega)$.*

Proof. Assume that $u \in \widetilde{H}^\alpha(\Omega)$ for some $\alpha \geq s$. Then $(-\Delta)^s u = f + k^{2s}u \in \widetilde{H}^{\min\{r,\alpha\}}$. By the regularity result for the fractional Poisson problem, we obtain that $u \in \widetilde{H}^{\min\{r,\alpha\}+2s}$. Since $u \in \widetilde{H}^s(\Omega)$, we obtain the desired result by iteration. ☐

4. The Extension Problem. By using [9, 23] we can equivalently cast the fractional Helmholtz problem (fP) as a problem over the extruded domain $\mathcal{C} = \Omega \times [0, \infty)$:

$$\text{(Ext)} \qquad \begin{cases} -\nabla \cdot \omega(y)\nabla U(\vec{x}, y) &= 0, \qquad (\vec{x}, y) \in \mathcal{C}, \\ U(\vec{x}, y) &= 0, \qquad (\vec{x}, y) \in \partial_L\mathcal{C} := \partial\Omega \times [0, \infty), \\ \dfrac{\partial U}{\partial \nu^\omega}(\vec{x}) - k^{2s}U(\vec{x}, 0) &= f(\vec{x}), \qquad \vec{x} \in \Omega, \end{cases}$$

where $\omega(y) = y^\alpha/d_s$, $\alpha = 1-2s$, $d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$, and

$$\frac{\partial U}{\partial \nu^\omega}(\vec{x}) = -\lim_{y\to 0^+} \omega(y)\frac{\partial U}{\partial y}(\vec{x}, y) = (-\Delta)^s U(\vec{x}, 0). \tag{4.1}$$

The solution to (fH) is then recovered by taking the trace of U on Ω, i.e. $u = \operatorname{tr}_\Omega U$.

We define the solution space $\mathcal{H}^1_\omega$ on the semi-infinite cylinder $\mathcal{C}$ as

$$\mathcal{H}^1_\omega = \left\{ V \in H^1_\omega(\mathcal{C}) \mid V = 0 \text{ on } \partial_L \mathcal{C} \right\}.$$

and we denote its dual by $(\mathcal{H}^1_\omega)^*$. Notice that

$$\operatorname{tr}_\Omega \mathcal{H}^1_\omega \equiv \widetilde{H}^s(\Omega),$$

where tr_Ω denotes the Ω-trace operator. Moreover, due to the Poincaré inequality in the weighted Sobolev spaces, we have that the seminorm $|\cdot|_{H^1_\omega}$ is a norm on $\mathcal{H}^1_\omega$, and we write $\|\cdot\|_{\mathcal{H}^1_\omega} := |\cdot|_{H^1_\omega}$. We refer to [10] for details.

The weak formulation of the extension problem (Ext) consists of seeking $U \in \mathcal{H}^1_\omega$ such that:

$$\mathcal{A}(U, V) = \langle f, V \rangle_\Omega \quad \forall V \in \mathcal{H}^1_\omega, \tag{wExt}$$

where we have

$$\mathcal{A}(U, V) = \int_{\mathcal{C}} \omega \nabla U \cdot \nabla \overline{V} - k^{2s} \int_\Omega U\overline{V} \quad \text{and} \quad \langle f, V \rangle_\Omega := \left\langle f, \operatorname{tr}_\Omega \overline{V} \right\rangle.$$

We will also frequently use the shorthand

$$\|V\|_{L^2(\Omega)} := \|\operatorname{tr}_\Omega V\|_{L^2(\Omega)}.$$

We seek a solution of the extension problem using classical separation of variables: $U(\vec{x}, y) = \Phi(\vec{x})\Psi(y)$. Then

$$\frac{-\Delta_{\vec{x}}\Phi}{\Phi} = \frac{\partial_y(\omega(y)\partial_y\Psi)}{\omega(y)\Psi} = A,$$

where A is a constant that is independent of $\vec{x}$ and y. Thanks to (Eig), the boundary condition on the lateral face of the cylinder $\mathcal{C}$, shows that $\Phi = \phi_m$ and $A = \lambda_m$ for $m \in \mathbb{N}$. The associated solution $\Psi = \psi_m$ in the extension direction must therefore satisfy

$$\partial_y(\omega(y)\partial_y\psi_m) = \lambda_m \omega(y)\psi_m. \tag{4.2}$$

Notice that $\psi_m(0) = 1$, moreover using (4.1) we obtain that

$$\frac{\partial \psi_m}{\partial \nu^\omega} = \lambda_m^s. \tag{4.3}$$

By applying integration by parts to (4.2) and using (4.3) we obtain that

$$\lambda_m \int_0^\infty \omega\psi_m\psi_n + \int_0^\infty \omega\psi'_m\psi'_n = \lambda_m^s, \tag{4.4}$$

which is uniquely solvable when we impose $\psi_m(+\infty) = 0$. Subtracting the same identity with indices m and n interchanged results in

$$\int_0^\infty \omega\psi_m\psi_n = \begin{cases} \frac{\lambda_m^s-\lambda_n^s}{\lambda_m-\lambda_n} & \text{if } m \neq n, \\ s\lambda_m^{s-1} & \text{if } m = n, \end{cases} \tag{4.5}$$

and

$$\int_0^\infty \omega\psi_m'\psi_n' = \begin{cases} \frac{\lambda_m\lambda_n^s-\lambda_n\lambda_m^s}{\lambda_m-\lambda_n} & \text{if } m \neq n, \\ (1-s)\lambda_m^s & \text{if } m = n, \end{cases} \tag{4.6}$$

where the identities for $m = n$ are obtained by taking the limit as $\lambda_n \to \lambda_m$. The solution to the extension problem (Ext) is then given by

$$U(\vec{x}, y) = \sum_{m=0}^{\infty} u_m\phi_m(\vec{x})\psi_m(y) \quad \text{where } u_m = \left(\lambda_m^s - k^{2s}\right)^{-1} f_m, \tag{4.7}$$

whilst $u(\vec{x}) = \sum_{m=0}^{\infty} u_m\phi_m(\vec{x})$ as in (2.3). The separable solution (4.7) forms the basis for our choice of discretization of the extension problem to be described in the next section. The main advantage of this approach is that the extension problem involves only integer-order derivatives but comes at the price of having to deal with a degenerate weight $\omega(y)$.

We conclude this section with the following well-posedness result for (wExt).

PROPOSITION 4.1. *Let Ω be a bounded Lipschitz domain and $f \in \widetilde{H}^{-s}(\Omega)$. Then there exists $U \in \mathcal{H}_\omega^1$ solving (wExt), and such solution depends continuously on the data:*

$$\|U\|_{\mathcal{H}_\omega^1} \leq C_d(k)\,\|f\|_{\widetilde{H}^{-s}(\Omega)}\,.$$

Proof. The proof follows along the lines of Theorem 3.4, i.e., we need to show uniqueness of U and prove Gårding's inequality. Then, the result will follow from Fredholm alternative. Construction of a unique solution using separation of variables is given above. Gårding's inequality can be shown as follows. We have

$$\begin{aligned} \|U\|_{\mathcal{H}_\omega^1}^2 &= \mathcal{A}(U,U) + k^{2s}\left(\operatorname{tr}_\Omega U, \operatorname{tr}_\Omega U\right)_{L^2(\Omega)} \\ &= \mathcal{A}(U,U) + k^{2s}\left\langle \mathcal{T}\operatorname{tr}_\Omega U, \operatorname{tr}_\Omega U\right\rangle_{\widetilde{H}^{-s}(\Omega),\widetilde{H}^s(\Omega)} \\ &= \mathcal{A}(U,U) + k^{2s}\left\langle \left(\operatorname{tr}_\Omega^\star \mathcal{T}\operatorname{tr}_\Omega\right)U, U\right\rangle_{(\mathcal{H}_\omega^1)^*,\mathcal{H}_\omega^1} \end{aligned} \tag{4.8}$$

where in the second equality we have used the existence of a compact operator $\mathcal{T} : \widetilde{H}^s(\Omega) \to \widetilde{H}^{-s}(\Omega)$. Moreover, in the last equality we have used that the trace operator $\operatorname{tr}_\Omega : \mathcal{H}_\omega^1 \to \widetilde{H}^s(\Omega)$ is bounded linear and thus its adjoint $\operatorname{tr}_\Omega^\star : \widetilde{H}^{-s}(\Omega) \to (\mathcal{H}_\omega^1)^*$ is well-defined. Notice that the operator $\operatorname{tr}_\Omega^\star \mathcal{T}\operatorname{tr}_\Omega : \mathcal{H}_\omega^1 \to (\mathcal{H}_\omega^1)^*$ is compact (composition of bounded and compact operators), thus we have shown Gårding's (in)equality [19, Remark 2.1.58] and the proof is complete. ∎

5. Discretization of the Extension Problem and A Priori Error Bounds. For the remainder of the paper, we assume that Ω is sufficiently smooth so that $\widetilde{H}^s(\Omega)$ can be associated with the classical fractional-order Sobolev space $H^s(\Omega)$ (this is needed when $s > 1$). We propose to approximate the variational problem (wExt) using a Galerkin scheme with the subspace consisting of standard low order nodal finite elements of order $p \geq 1$ in the $\vec{x}$-variable and a spectral method in the y-direction. To this end, we let $\mathcal{T}_h$ be a shape regular, globally quasi-uniform triangulation of Ω, and let

$$S_h = \left\{ v_h \in C^0\left(\overline{\Omega}\right) \mid v_h\big|_K \in \mathbb{P}_p\left(K\right) \ \forall K \in \mathcal{T}_h \right\}.$$

In the y-direction, ideally, we would like to use y-basis functions $\{\psi_m\}$ given in the previous section. Unfortunately, this requires knowledge of the true eigenvalues λ_m of $(-\Delta)$ over Ω. Therefore, we use approximations $\widetilde{\lambda}_m \approx \lambda_m$ in place of the true eigenvalues in (4.5) and (4.6).

The Galerkin subspace for the extension problem is then taken to be

$$\mathcal{V}_h = \left\{ V_h = \sum_{m=0}^{M-1} v_{h,m}\left(\vec{x}\right) \widetilde{\psi}_m\left(y\right) \mid v_{h,m} \in S_h \text{ and } \widetilde{\psi}_m \text{ solves (4.4) with } \widetilde{\lambda}_m \right\} \subset \mathcal{H}^1_\omega.$$

Notice that, we do not need an analytic expression for the basis functions $\{\widetilde{\psi}_m\}$, and it is sufficient to know mass and stiffness matrices (4.5) and (4.6). The spectral expansion order M will depend on s, h and the regularity of the solution. The efficient approach to find approximations $\widetilde{\lambda}_m$ is discussed in [1]. We further emphasize that $\mathcal{O}\left(|\log h|\right)$ eigenvalue approximations are sufficient to get "good approximation" properties.

The Galerkin approximation of (wExt) seeks $U_h \in \mathcal{V}_h$ such that

$$\text{(wExt}_\text{h}) \qquad \mathcal{A}\left(U_h, V_h\right) = \langle f, V_h \rangle_\Omega \quad \forall V_h \in \mathcal{V}_h,$$

with the approximation of the fractional Helmholtz problem given by

$$u_h := \operatorname{tr}_\Omega U_h.$$

Having introduced the discrete problem, our next goal is to obtain an estimate for the error $u - u_h$. The trace inequality in [10, Proposition 2.1], see also [18], implies that

$$\left\| u - u_h \right\|_{\widetilde{H}^s(\Omega)} \leq C \left\| U - U_h \right\|_{\mathcal{H}^1_\omega},$$

where the constant is independent of p, M and h. We also refer to [5, Theorem 2.3] for a more general trace inequality. Hence, in order to bound $u - u_h$, it suffices to bound $\|U - U_h\|_{\mathcal{H}^1_\omega}$, the discretization error of the extension problem (wExt$_\text{h}$).

Define the norm

$$\|V\|^2 := \|V\|^2_{\mathcal{H}^1_\omega} + |k|^{2s} \|V\|^2_{L^2(\Omega)}, \qquad V \in \mathcal{H}^1_\omega.$$

Using the trace inequality in [10, Proposition 2.1], see also [18], we find that $\mathcal{A}$ is continuous

$$|\mathcal{A}\left(U, V\right)| \leq \|U\|_{\mathcal{H}^1_\omega} \|V\|_{\mathcal{H}^1_\omega} + |k|^{2s} \|U\|_{L^2(\Omega)} \|V\|_{L^2(\Omega)} \leq C \|U\| \, \|V\| \quad \forall U, V \in \mathcal{H}^1_\omega$$

and immediately satisfies the Gårding type (in)equality

$$\mathcal{A}\left(U, U\right) + k^{2s} \|U\|^2_{L^2(\Omega)} = \|U\|^2_{\mathcal{H}^1_\omega} \qquad \forall U \in \mathcal{H}^1_\omega.$$

Define the solution operator $\mathcal{S}_k : \widetilde{H}^{-s}(\Omega) \to \mathcal{H}^1_\omega$ via

$$\mathcal{A}(\mathcal{S}_k f, V) = \langle f, V\rangle_\Omega \qquad\qquad \forall V \in \mathcal{H}^1_\omega$$

and the adjoint solution operator $\mathcal{S}^*_k : \widetilde{H}^{-s}(\Omega) \to \mathcal{H}^1_\omega$ via

$$\mathcal{A}(W, \mathcal{S}^*_k f) = \langle f, W\rangle_\Omega \qquad\qquad \forall W \in \mathcal{H}^1_\omega.$$

The two operators are can be expressed in terms of each other as

$$\mathcal{S}^*_k = \mathcal{S}_{\overline{k}}.$$

Moreover, let

$$\eta := \sup_{f\in L^2(\Omega)\setminus\{0\}} \inf_{V_h\in\mathcal{V}_h} \frac{\| \mathcal{S}^*_k f - V_h \|}{\|f\|_{L^2(\Omega)}}.$$

The following two results, Theorems 5.1 and 5.2, closely mimick the ideas developed in [16, 15] for the integer-order case. We refer to Appendix A for their respective proofs.

Theorem 5.1. *Assume that*

$$\eta\,|k|^s \le \gamma$$

for small enough constant γ that is independent of h and k. Then $\mathcal{A}$ satisfies the discrete inf-sup condition

$$\inf_{U_h\in\mathcal{V}_h} \sup_{V_h\in\mathcal{V}_h} \frac{|\mathcal{A}(U_h, V_h)|}{\|U_h\|\,\|V_h\|} \ge \frac{1 - C\gamma}{1 + 2(C_d(k) + \eta)\,|k|^s}.$$

Theorem 5.2. *Let $U \in \mathcal{H}^1_\omega$ be the solution of* (wExt) *and $U_h \in \mathcal{V}_h$ the solution of* ($\mathrm{wExt_h}$). *Assume that*

$$\eta\,|k|^s \le \gamma \tag{5.1}$$

for small enough constant γ that is independent of h and k. Then

$$\|U - U_h\| \le C \inf_{V_h\in\mathcal{V}_h} \|U - V_h\|,$$
$$\|U - U_h\|_{L^2(\Omega)} \le C\eta\,\|U - U_h\|,$$

where the constants are independent of h and k.

Before we turn our attention to the approximation results, we state the required assumptions on the approximation space $\mathcal{V}_h$ and the eigenvalue approximations $\{\widetilde{\lambda}_m\}$, parameterized by a parameter t that will be linked to the solution regularity.

Assumption 5.3. *Given $t \ge s$, assume that the following hold:*

- *M is large enough such that $\lambda_M^{(s-t)/2} \sim h^{\min\{p,t-s\}}$.*
- *For $0 \le m \le M-1$ it holds that*

$$\left(\frac{\widetilde{\lambda}_m}{\lambda_m}\right)^s, \left(\frac{\lambda_m}{\widetilde{\lambda}_m}\right)^{1-s} \le c_\sigma^2 \tag{5.2}$$

  with a positive constant c_σ that is independent of h.

- *For $0 \le m \le M-1$ it holds that*

$$g\left(s, \widetilde{\lambda}_m/\lambda_m\right) \le \lambda_m^{t-s} h^{2\min\{p,t-s\}}, \tag{5.3}$$

where

$$g(s,\rho) = 1 - \frac{1}{(1-s)\rho^s + s\rho^{s-1}}.$$

We refer the reader to [1] for a discussion on how these requirements can be achieved in practice using the asymptotic behavior of the eigenvalues and by finite element discretization. The assumptions (5.4) and (5.5) in the following theorem are also discussed in [1] in more detail.

THEOREM 5.4. *Let $s \le t \le p+1$ and assume that Assumption 5.3 holds for t. Moreover, assume that there exist positive constants C_0, C_1 independent of h such that the following two inequalities hold for any $\vec{\gamma} \in \mathbb{R}^M$:*

$$\sum_{m,n=0}^{M-1} \gamma_m\gamma_n \int_\Omega (\phi_m - \pi_h\phi_m)(\phi_n - \pi_h\phi_n) \le C_0 \log(\lambda_M) \sum_{m=0}^{M-1} \gamma_m^2 \|\phi_m - \pi_h\phi_m\|_{L^2(\Omega)}^2, \tag{5.4}$$

$$\sum_{m,n=0}^{M-1} \gamma_m\gamma_n \int_\Omega \nabla(\phi_m - \pi_h\phi_m)\cdot\nabla(\phi_n - \pi_h\phi_n) \le C_1 \log(\lambda_M) \sum_{m=0}^{M-1} \gamma_m^2 \|\nabla(\phi_m - \pi_h\phi_m)\|_{L^2(\Omega)}^2, \tag{5.5}$$

where π_h is the Scott-Zhang interpolant [20].

Let U satisfy the variational equality (wExt) and assume that $u = \operatorname{tr}_\Omega U \in \widetilde{H}^q(\Omega)$, for $s \le q \le t$. Then

$$\inf_{V_h\in\mathcal{V}_h} \|U - V_h\| \le C\,|u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|}\left\{h^{\min\{p,t-s\}\frac{q-s}{t-s}} + |k|^s\, h^{\min\{p,t-s\}\min\{q,2t-2s\}/(t-s)}\right\},$$

where C is independent of h and k.

Proof. Since U is a solution to (wExt), we can expand it as

$$U = \sum_{m=0}^{\infty} u_m\phi_m(\vec{x})\,\psi_m(y).$$

We choose $V_M \in \mathcal{H}_\omega^1$ and $V_h \in \mathcal{V}_h$ to be

$$V_M = \sum_{m=0}^{M-1} \alpha_m u_m \phi_m(\vec{x})\,\widetilde{\psi}_m(y), \qquad V_h = \sum_{m=0}^{M-1} \alpha_m u_m (\pi_h\phi_m)(\vec{x})\,\widetilde{\psi}_m(y),$$

where $\alpha_m \in \mathbb{R}$ will be determined below and π_h is the Scott-Zhang interpolant [20]. The triangle inequality gives

$$\|U - V_h\|_{\mathcal{H}_\omega^1} \le \|U - V_M\|_{\mathcal{H}_\omega^1} + \|V_M - V_h\|_{\mathcal{H}_\omega^1}.$$

Direct computation gives

$$\|U - V_M\|_{\mathcal{H}_\omega^1}^2 = \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} u_m\overline{u}_n \left\langle \phi_m\left(\psi_m - \alpha_m\widetilde{\psi}_m\right), \phi_n\left(\psi_n - \alpha_n\widetilde{\psi}_n\right)\right\rangle_{\mathcal{H}_\omega^1}$$

$$+ 2 \sum_{m=0}^{M-1} \sum_{n=M}^{\infty} u_m \overline{u}_n \left\langle \phi_m \left(\psi_m - \alpha_m \widetilde{\psi}_m \right), \phi_n \psi_n \right\rangle_{\mathcal{H}^1_\omega}$$
$$+ \sum_{m=M}^{\infty} \sum_{n=M}^{\infty} u_m \overline{u}_n \left\langle \phi_m \psi_m, \phi_n \psi_n \right\rangle_{\mathcal{H}^1_\omega}.$$

To deal with the first term, we observe that for arbitrary smooth functions h_1 and h_2 there holds

$$\begin{aligned}\left\langle \phi_m(\vec{x}) h_1(y), \phi_n(\vec{x}) h_2(y) \right\rangle_{\mathcal{H}^1_\omega} &= \int_{\mathcal{C}} \omega(y) \nabla \left[\phi_m(\vec{x}) h_1(y) \right] \cdot \nabla \left[\phi_n(\vec{x}) \overline{h_2}(y) \right] \\ &= \int_\Omega \phi_m \phi_n \int_0^\infty \omega(y) h_1' \overline{h_2'} + \int_\Omega \nabla_{\vec{x}} \phi_m \cdot \nabla_{\vec{x}} \phi_n \int_0^\infty \omega(y) h_1 \overline{h_2} \\ &= \delta_{nm} \left(h_1, h_2 \right)_m\end{aligned}$$

where the inner product in the final equality is defined to be

$$(h_1, h_2)_m = \int_0^\infty \omega(y) h_1' \overline{h_2'} + \lambda_m \int_0^\infty \omega(y) h_1 \overline{h_2},$$

with the induced norm denoted by $\|\cdot\|_m = \sqrt{(\cdot,\cdot)_m}$. In particular, from (4.5) and (4.6) we obtain $\|\psi_m\|_m^2 = \lambda_m^s$. Therefore

$$\|U - V_M\|^2_{\mathcal{H}^1_\omega} = \sum_{m=0}^{M-1} |u_m|^2 \left\| \psi_m - \alpha_m \widetilde{\psi}_m \right\|_m^2 + \sum_{m=M}^{\infty} |u_m|^2 \|\psi_m\|_m^2.$$

The coefficients $\{\alpha_m\}$ are chosen to minimize the right-hand side. A simple computation reveals that the optimal choice is $\alpha_m = \dfrac{\|\psi_m\|_m}{\left\|\widetilde{\psi}_m\right\|_m} \cos\theta_m$, where

$$\cos\theta_m = \frac{\left(\psi_m, \widetilde{\psi}_m\right)_m}{\|\psi_m\|_m \left\|\widetilde{\psi}_m\right\|_m} = \sqrt{1 - g\left(s, \widetilde{\lambda}_m/\lambda_m\right)},$$

so that

$$\left\| \psi_m - \alpha_m \widetilde{\psi}_m \right\|_m^2 = \|\psi_m\|_m^2 \sin^2\theta_m = \lambda_m^s \sin^2\theta_m = \lambda_m^s g\left(s, \widetilde{\lambda}_m/\lambda_m\right).$$

The first term is then easily estimated thanks to (5.3):

$$\begin{aligned}\|U - V_M\|^2_{\mathcal{H}^1_\omega} &= \sum_{m=0}^{M-1} |u_m|^2 \lambda_m^s g\left(s, \widetilde{\lambda}_m/\lambda_m\right) + \sum_{m=M}^{\infty} |u_m|^2 \lambda_m^s \\ &\le h^{2\min\{p,t-s\}} \sum_{m=0}^{M-1} |u_m|^2 \lambda_m^{t-q} \lambda_m^q + \lambda_M^{s-q} \sum_{m=M}^{\infty} |u_m|^2 \lambda_m^q \\ &\le h^{2\min\{p,t-s\}} \lambda_M^{t-q} \sum_{m=0}^{M-1} |u_m|^2 \lambda_m^q + \lambda_M^{s-q} \sum_{m=M}^{\infty} |u_m|^2 \lambda_m^q \\ &\le h^{2\min\{p,t-s\}\frac{q-s}{t-s}} |u|^2_{\widetilde{H}^q(\Omega)}, \end{aligned} \tag{5.6}$$

where we recall M is chosen large enough such that $\lambda_M^{(s-t)/2} \sim h^{\min\{p,t-s\}}$.

In the second term, elementary manipulation gives

$$
\begin{aligned}
&\|V_M - V_h\|_{\mathcal{H}^1_\omega}^2 \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} \alpha_m\alpha_n u_m \overline{u_n} \int_{\mathcal{C}} \omega \nabla \left[(\phi_m - \pi_h\phi_m)\,\widetilde{\psi}_m\right] \cdot \nabla \left[(\phi_n - \pi_h\phi_n)\,\widetilde{\psi}_n\right] \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} \alpha_m\alpha_n u_m \overline{u_n} \left\{ \int_\Omega \nabla(\phi_m - \pi_h\phi_m)\cdot\nabla(\phi_n - \pi_h\phi_n) \int_0^\infty \omega \widetilde{\psi}_m\widetilde{\psi}_n \right. \\
&\qquad \left. + \int_\Omega (\phi_m - \pi_h\phi_m)(\phi_n - \pi_h\phi_n) \int_0^\infty \omega \widetilde{\psi}_m'\widetilde{\psi}_n' \right\} \\
&\leq \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} \alpha_m\alpha_n u_m \overline{u_n} \left\{ \int_\Omega \nabla(\phi_m - \pi_h\phi_m)\cdot\nabla(\phi_n - \pi_h\phi_n) \sqrt{\int_0^\infty \omega \widetilde{\psi}_m^2}\sqrt{\int_0^\infty \omega \widetilde{\psi}_n^2} \right. \\
&\qquad \left. + \int_\Omega (\phi_m - \pi_h\phi_m)(\phi_n - \pi_h\phi_n) \sqrt{\int_0^\infty \omega \left(\widetilde{\psi}_m'\right)^2}\sqrt{\int_0^\infty \omega \left(\widetilde{\psi}_n'\right)^2} \right\} \\
&\leq \log(\lambda_M) \sum_{m=0}^{M-1} \alpha_m^2 |u_m|^2 \left\{ C_1 \|\nabla\phi_m - \nabla\pi_h\phi_m\|_{L^2(\Omega)}^2 \int_0^\infty \omega\widetilde{\psi}_m^2 \right. \\
&\qquad\qquad \left. + C_0 \|\phi_m - \pi_h\phi_m\|_{L^2(\Omega)}^2 \int_0^\infty \omega \left(\widetilde{\psi}_m'\right)^2 \right\} \\
&\leq \max\{C_0, C_1\}\log(\lambda_M) \sum_{m=0}^{M-1} |u_m|^2 \left\{ \|\nabla\phi_m - \nabla\pi_h\phi_m\|_{L^2(\Omega)}^2 \int_0^\infty \omega\widetilde{\psi}_m^2 \right. \\
&\qquad\qquad \left. + \|\phi_m - \pi_h\phi_m\|_{L^2(\Omega)}^2 \int_0^\infty \omega \left(\widetilde{\psi}_m'\right)^2 \right\},
\end{aligned}
$$

where we used (5.4), (5.5), and that $\alpha_m^2 \leq 1$. Standard properties of the Scott-Zhang interpolant give

$$
\begin{aligned}
\|\nabla\phi_m - \nabla\pi_h\phi_m\|_{L^2(\Omega)} &\leq Ch^p |\phi_m|_{H^{p+1}} \leq Ch^p \lambda_m^{(p+1)/2}, \\
\|\phi_m - \pi_h\phi_m\|_{L^2(\Omega)} &\leq Ch^{p+1} |\phi_m|_{H^{p+1}} \leq Ch^{p+1} \lambda_m^{(p+1)/2},
\end{aligned}
$$

while, from (4.5) and (4.6),

$$
\int_0^\infty \omega\widetilde{\psi}_m^2 = s\widetilde{\lambda}_m^{s-1}, \qquad \text{and} \qquad \int_0^\infty \omega\left(\widetilde{\psi}_m'\right)^2 = (1-s)\widetilde{\lambda}_m^s.
$$

Hence,

$$
\begin{aligned}
\|V_M - V_h\|_{\mathcal{H}^1_\omega}^2 &\leq C\log(\lambda_M)h^{2p}\sum_{m=0}^{M-1}|u_m|^2\lambda_m^{p+1}\widetilde{\lambda}_m^{s-1} + C\log(M)h^{2p+2}\sum_{m=0}^{M-1}|u_m|^2\lambda_m^{p+1}\widetilde{\lambda}_m^s \\
&\leq C|u|_{\widetilde{H}^q(\Omega)}^2 |\log h| \left[h^{2p} \max_{m=0,\dots,M-1} \lambda_m^{p-(q-s)} \left(\frac{\lambda_m}{\widetilde{\lambda}_m}\right)^{1-s} \right. \\
&\qquad \left. + h^{2p+2} \max_{m=0,\dots,M-1} \lambda_m^{p+1-(q-s)} \left(\frac{\widetilde{\lambda}_m}{\lambda_m}\right)^s \right],
\end{aligned}
$$

where we used the fact that $\log(\lambda_M) \sim |\log h|$. Thanks to assumption (5.2), we obtain

$$\|V_M - V_h\|^2_{\mathcal{H}^1_\omega} \leq C\, |u|^2_{\widetilde{H}^q(\Omega)}\, |\log h| \begin{cases} \lambda_{M-1}^{s-q}\left(h^{2p}\lambda_{M-1}^{p} + h^{2p+2}\lambda_{M-1}^{p+1}\right) & \text{if } 0 \leq q-s \leq p, \\ h^{2p}\left(1 + h^2\lambda_{M-1}^{p+1-(q-s)}\right) & \text{if } p \leq q-s \leq p+1, \\ h^{2p} & \text{if } q-s \geq p+1. \end{cases}$$

Recalling that M is chosen such that $\lambda_M^{(s-t)/2} \sim h^{\min\{p,t-s\}}$, we obtain

$$\|V_M - V_h\|_{\mathcal{H}^1_\omega} \leq C\, |u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|}\, h^{\min\{p,t-s\}\frac{q-s}{t-s}}. \tag{5.7}$$

Finally, by combining (5.6) and (5.7), we deduce that

$$\|U - V_h\|_{\mathcal{H}^1_\omega} \leq C\, |u|_{\widetilde{H}^q(\Omega)}\, h^{\min\{p,t-s\}\frac{q-s}{t-s}} \sqrt{|\log h|}.$$

In a similar fashion, we estimate

$$\|U - V_h\|_{L^2(\Omega)} \leq \|U - V_M\|_{L^2(\Omega)} + \|V_M - V_h\|_{L^2(\Omega)}.$$

Since

$$1 - \alpha_m = 1 - \frac{\|\psi_m\|_m}{\left\|\widetilde{\psi}_m\right\|_m} \cos\theta_m = g\left(s, \widetilde{\lambda}_m/\lambda_m\right)$$

The first term is given by

$$\begin{aligned} \|U - V_M\|^2_{L^2(\Omega)} &= \sum_{m=0}^{M-1} |1-\alpha_m|^2 |u_m|^2 + \sum_{m=M}^{\infty} |u_m|^2 \\ &\leq \sum_{m=0}^{M-1} g\left(s, \widetilde{\lambda}_m/\lambda_m\right)^2 |u_m|^2 + \lambda_M^{-q} \sum_{m=M}^{\infty} |u_m|^2 \lambda_m^q \\ &\leq C h^{4\min\{p,t-s\}} \sum_{m=0}^{M-1} \lambda_m^{2t-2s} |u_m|^2 + \lambda_M^{-q} \sum_{m=M}^{\infty} |u_m|^2 \lambda_m^q \\ &\leq C\, |u|^2_{\widetilde{H}^q(\Omega)} \left\{ h^{4\min\{p,t-s\}} \max_{m=0,\ldots,M-1} \lambda_m^{2t-2s-q} + \lambda_M^{-q} \right\}. \end{aligned}$$

Hence, the first term can be estimated by

$$\|U - V_M\|^2_{L^2(\Omega)} \leq C\, |u|^2_{\widetilde{H}^q(\Omega)} \begin{cases} h^{4\min\{p,t-s\}} \lambda_M^{2t-2s-q} + \lambda_M^{-q} & \text{if } q \leq 2t-2s, \\ h^{4\min\{p,t-s\}} + \lambda_M^{-q} & \text{if } q \geq 2t-2s \end{cases}$$

Since $\lambda_M^{(s-t)/2} \sim h^{\min\{p,t-s\}}$, we find

$$\begin{aligned} \|U - V_M\|^2_{L^2(\Omega)} &\leq C\, |u|^2_{\widetilde{H}^q(\Omega)} \begin{cases} h^{2\min\{p,t-s\}\frac{q}{t-s}} + h^{2\min\{p,t-s\}\frac{q}{t-s}} & \text{if } q \leq 2t-2s, \\ h^{4\min\{p,t-s\}} + h^{2\min\{p,t-s\}\frac{q}{t-s}} & \text{if } q \geq 2t-2s \end{cases} \\ &\leq C\, |u|^2_{\widetilde{H}^q(\Omega)} \begin{cases} h^{2\min\{p,t-s\}\frac{q}{t-s}} & \text{if } q \leq 2t-2s, \\ h^{4\min\{p,t-s\}} & \text{if } q \geq 2t-2s \end{cases} \end{aligned}$$

Hence

$$\|U - V_M\|_{L^2(\Omega)} \le C\,|u|_{\widetilde{H}^q(\Omega)}\, h^{\min\{p,t-s\}\min\{q,2t-2s\}/(t-s)}.$$

The second term can be estimated by

$$\begin{aligned}\|V_M - V_h\|^2_{L^2(\Omega)} &= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} \alpha_m u_m \alpha_m \overline{u_m} \int_\Omega (\phi_m - \pi_h\phi_m)(\phi_n - \pi_h\phi_n) \\ &\le C_0 \log(\lambda_M) \sum_{m=0}^{M-1} |u_m|^2 \|\phi_m - \pi_h\phi_m\|^2_{L^2(\Omega)}.\end{aligned}$$

Again, by properties of the Scott-Zhang interpolant,

$$\begin{aligned}\|V_M - V_h\|^2_{L^2(\Omega)} &\le C\,|\log h|\, h^{2p+2} \sum_{m=0}^{M-1} |u_m|^2 \lambda_m^{p+1} \\ &\le C\,|u|^2_{\widetilde{H}^q(\Omega)}\, |\log h|\, h^{2p+2} \max_{m=0,\dots,M-1} \lambda_m^{p+1-q}.\end{aligned}$$

Hence, since $\lambda_M^{(s-t)/2} \sim h^{\min\{p,t-s\}}$ and $q \le t \le p+1$,

$$\begin{aligned}\|V_M - V_h\|_{L^2(\Omega)} &\le C\,|u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|}\, h^{p+1} \lambda_M^{(p+1-q)/2} \\ &\le C\,|u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|}\, h^{p+1-\min\{p,t-s\}\frac{p+1-q}{t-s}} \\ &\le C\,|u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|}\, h^q.\end{aligned}$$

This means that by combining, we find

$$\|U - V_h\|_{L^2(\Omega)} \le C\,|u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|}\, h^{\min\{p,t-s\}\min\{q,2t-2s\}/(t-s)}$$

where we have used that $\min\{p, t-s\}\min\{q, 2t-2s\}/(t-s) \le q$. Finally, we find

$$\begin{aligned}\|U - V_h\| &\le \|U - V_h\|_{\mathcal{H}^1_\omega} + |k|^s \|U - V_h\|_{L^2(\Omega)} \\ &\le C\,|u|_{\widetilde{H}^q(\Omega)} \sqrt{|\log h|} \left\{ h^{\min\{p,t-s\}\frac{q-s}{t-s}} + |k|^s h^{\min\{p,t-s\}\min\{q,2t-2s\}/(t-s)} \right\}. \quad \square\end{aligned}$$

Remark 5.5. Given a right-hand side function $f \in \widetilde{H}^r(\Omega)$, the regularity result in Proposition 3.5 gives that the solution to the fractional Helmholtz problem has regularity of order $r + 2s$. Using elements of order p, we want to select M and eigenvalue approximations $\widetilde{\lambda}_m$ to satisfy Assumption 5.3 for $t = \max\{0, \min\{r, p-s\}\} + 2s$. Satisfying the conditions for larger values of t will not lead to any improvements in the approximation result. This also shows that the method cannot take advantage of right-hand side regularity $r \ge p - s$.

THEOREM 5.6. *Let $f \in \widetilde{H}^r(\Omega)$, $r \ge -s$. Assume that Assumption 5.3 is satisfied for $t = \max\{0, \min\{r, p-s\}\} + 2s$ and that the conditions of Theorem 5.4 hold, and that*

$$\sqrt{|\log h|}\left[(h\,|k|)^s + (h\,|k|)^{2s}\right] \le \gamma \tag{5.8}$$

for small enough constant γ that is independent of h and k. Let $U \in \mathcal{H}^1_\omega$ be the solution of (wExt) *and $U_h \in \mathcal{V}_h$ the solution of* ($\mathrm{wExt_h}$) *and u and u_h be their respective traces on Ω. Then*

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq C \|U - U_h\|_{\mathcal{H}^1_\omega} \leq C \|\!|U - U_h|\!\| \leq C h^{\min\{p, r+s\}} \sqrt{|\log h|}\, |f|_{\widetilde{H}^r(\Omega)},$$
$$\|u - u_h\|_{L^2(\Omega)} \leq C h^{\min\{p+s, r+2s\}} \sqrt{|\log h|}\, |f|_{\widetilde{H}^r(\Omega)},$$

where the constants are independent of h and k.

Proof. Without loss of generality, assume that $r+s \leq p$. Let $g \in L^2(\Omega)$. Then the solution $\mathcal{S}^*_k g$ has regularity $q = 2s \leq \max\{0, r\} + 2s = t$. Since $t - s = \max\{0, r\} + s \leq p$, we find $\min\{p, t-s\} \frac{q-s}{t-s} = q - s = s$, and

$$\min\{p, t-s\} \min\{q, 2t-2s\}/(t-s) = \min\{q, 2t-2s\} = \min\{2s, 2\max\{0, r\} + 2s\} = 2s.$$

Hence, by applying Theorem 5.4, we have

$$\eta \leq C \sqrt{|\log h|} h^s \left[1 + (h\,|k|)^s\right].$$

Combining the latter with Theorem 5.1 and (5.8), we obtain that the discrete inf-sup condition holds.

Now, let $f \in \widetilde{H}^r(\Omega)$. Then the solution of the fractional Helmholtz problem is in $\widetilde{H}^{r+2s}(\Omega)$ and hence, applying Theorem 5.4 with $q = r + 2s \leq t$, we obtain that

$$\begin{aligned}\inf_{V_h \in \mathcal{V}_h} \|\!|U - V_h|\!\| &\leq C\,|u|_{\widetilde{H}^{r+2s}(\Omega)} \sqrt{|\log h|} h^{r+s} \left[1 + (h\,|k|)^s\right] \\ &= C\,|f|_{\widetilde{H}^r(\Omega)} \sqrt{|\log h|} h^{r+s} \left[1 + (h\,|k|)^s\right].\end{aligned}$$

Combining with Theorem 5.2 and (5.8), we obtain the estimates

$$\begin{aligned}\|\!|U - U_h|\!\| &\leq C h^{\min\{p, r+s\}} \sqrt{|\log h|}\, |f|_{\widetilde{H}^r(\Omega)}, \\ \|U - U_h\|_{L^2(\Omega)} &\leq C h^{\min\{p+s, r+2s\}} \sqrt{|\log h|}\, |f|_{\widetilde{H}^r(\Omega)}.\end{aligned}$$

We conclude by noting that due to the trace inequality

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq C \|U - U_h\|_{\mathcal{H}^1_\omega} \leq C \|\!|U - U_h|\!\|. \qquad \square$$

6. Solution of the Linear System. Let $\{\Phi_i\}_{i=1}^n$ denote the nodal basis functions of the finite element solution space S_h, then the solution of the discretized fractional Helmholtz problem can be written as $u_h(\vec{x}) = \sum_{i=1}^n u_{h;i} \Phi_i(\vec{x}) = \vec{u}_h \cdot \vec{\Phi}(\vec{x})$.

Here, for the ease of notation, we have assumed that the eigenvalue approximations $\widetilde{\lambda}_m$, $m = 0, \ldots, \widetilde{M} - 1$, are all distinct. We refer to [1] for the procedure that selects $\widetilde{M}$ distinct eigenvalue approximations, $\widetilde{M} \leq M$.

By expanding the finite element functions as linear combinations with respect to the basis functions Φ_i, the solution $U_h(\vec{x}, y)$ of the extension problem ($\mathrm{wExt_h}$) can be written in the form

$$U_h(\vec{x}, y) = \sum_{m=0}^{\widetilde{M}-1} \sum_{i=1}^{n} c_{i,m} \Phi_i(\vec{x})\, \widetilde{\psi}_m(y) \in \mathcal{V}_h$$

with the coefficients $(c_{i,m}) = \vec{U}_h$ obtained by solving the linear system

$$\left[\boldsymbol{M}_{FE} \otimes \left(\boldsymbol{S}_\sigma - k^{2s}\boldsymbol{B}_\sigma\right) + \boldsymbol{S}_{FE} \otimes \boldsymbol{M}_\sigma\right] \vec{U}_h = \vec{F}_h, \tag{6.1}$$

where

$$\begin{aligned}
\boldsymbol{M}_{FE} &= \left(\int_\Omega \Phi_i \Phi_j\right), & \boldsymbol{S}_{FE} &= \left(\int_\Omega \nabla\Phi_i \nabla\Phi_j\right), \\
\boldsymbol{M}_\sigma &= \left(\int_0^\infty \omega \widetilde{\psi}_m \widetilde{\psi}_n\right), & \boldsymbol{S}_\sigma &= \left(\int_0^\infty \omega \widetilde{\psi}'_m \widetilde{\psi}'_n\right), \\
\boldsymbol{B}_\sigma &= \vec{1}_{\widetilde{M}} \otimes \vec{1}^T_{\widetilde{M}}, & & \\
\vec{F}_h &= \vec{f}_h \otimes \vec{1}_{\widetilde{M}}, & \vec{f}_h &= \left(\langle f, \Phi_i \rangle\right).
\end{aligned}$$

Here, $\vec{1}_{\widetilde{M}}$ is the vector of ones of length $\widetilde{M}$. The approximation to the solution of the fractional Helmholtz problem is then obtained by taking the trace of U_h on Ω:

$$u_h = \operatorname{tr}_\Omega U_h = \sum_{i=1}^{n} \left(\sum_{m=0}^{\widetilde{M}-1} c_{i,m}\right) \Phi_i\left(\vec{x}\right), \tag{6.2}$$

where we recall the normalization $\widetilde{\psi}_m(0) = 1$. In matrix form, the trace operator is given by $\boldsymbol{I} \otimes \vec{1}^T_{\widetilde{M}} \in \mathbb{R}^{n \times \mathcal{N}}$, so that $\vec{u}_h = \left[\boldsymbol{I} \otimes \vec{1}^T_{\widetilde{M}}\right] \vec{U}_h$.

Proposition 6.1. *There exist weights w_m and shift coefficients μ_m such that*

$$\vec{u}_h = \sum_{m=0}^{\widetilde{M}-1} w_m \left[\boldsymbol{M}_{FE}\mu_m + \boldsymbol{S}_{FE}\right]^{-1} \vec{f}_h. \tag{6.3}$$

When k^{2s} is real, all μ_m are real and at most one coefficient μ_m is negative.

Proof. We consider the following generalized eigenvalue problem:

$$\left(\boldsymbol{S}_\sigma - k^{2s}\boldsymbol{B}_\sigma\right)\boldsymbol{Q} = \boldsymbol{M}_\sigma \boldsymbol{Q}\boldsymbol{\mu}, \tag{6.4}$$

with the normalization $\boldsymbol{Q}^H \boldsymbol{M}_\sigma \boldsymbol{Q} = \boldsymbol{I}$ and $\boldsymbol{\mu}$ a diagonal matrix with entries μ_m. If k^{2s} is real, then all μ_m are real. Then

$$\left(\boldsymbol{I} \otimes \boldsymbol{Q}^H\right)\left[\boldsymbol{M}_{FE} \otimes \left(\boldsymbol{S}_\sigma - k^{2s}\boldsymbol{B}_\sigma\right) + \boldsymbol{S}_{FE} \otimes \boldsymbol{M}_\sigma\right]\left(\boldsymbol{I} \otimes \boldsymbol{Q}\right) = \boldsymbol{M}_{FE} \otimes \boldsymbol{\mu} + \boldsymbol{S}_{FE} \otimes \boldsymbol{I}.$$

Hence, we have

$$\left[\boldsymbol{M}_{FE} \otimes \left(\boldsymbol{S}_\sigma - k^{2s}\boldsymbol{B}_\sigma\right) + \boldsymbol{S}_{FE} \otimes \boldsymbol{M}_\sigma\right]^{-1} = \left(\boldsymbol{I} \otimes \boldsymbol{Q}\right)\left[\boldsymbol{M}_{FE} \otimes \boldsymbol{\mu} + \boldsymbol{S}_{FE} \otimes \boldsymbol{I}\right]^{-1}\left(\boldsymbol{I} \otimes \boldsymbol{Q}^H\right).$$

Since $\vec{F}_h = \vec{f}_h \otimes \vec{1}_{\widetilde{M}}$ and $\vec{u}_h = \left[\boldsymbol{I} \otimes \vec{1}^T_{\widetilde{M}}\right] \vec{U}_h$, we obtain

$$\vec{u}_h = \sum_{m=0}^{\widetilde{M}-1} \underbrace{\left(\boldsymbol{Q}^H \vec{1}_{\widetilde{M}}\right)^2_m}_{=:w_m} \left(\boldsymbol{M}_{FE}\mu_m + \boldsymbol{S}_{FE}\right)^{-1} f_h.$$

Both the spectral mass matrix $\boldsymbol{M}_\sigma$ and the spectral stiffness matrix $\boldsymbol{S}_\sigma$ are real-valued, symmetric and non-negative, and so we know that the eigenvalues $\mu_m^{(0)}$ of the related problem

$$\boldsymbol{S}_\sigma \boldsymbol{Q}_0 = \boldsymbol{M}_\sigma \boldsymbol{Q}_0 \boldsymbol{\mu}^0, \qquad\qquad \boldsymbol{Q}_0^T \boldsymbol{M}_\sigma \boldsymbol{Q}_0 = \boldsymbol{I},$$

are all real and non-negative. Here, the entries of the diagonal matrix $\boldsymbol{\mu}^0$ are $\mu_m^{(0)}$. Without loss of generality, we assume that $0 \leq \mu_0^{(0)} \leq \mu_1^{(0)} \leq \cdots \leq \mu_{\widetilde{M}-1}^{(0)}$. The eigenvalues μ_m, in turn, satisfy the characteristic equation

$$\begin{aligned} 0 &= \det\left(\boldsymbol{S}_\sigma - k^{2s}\boldsymbol{B}_\sigma - \mu \boldsymbol{M}_\sigma\right) \\ &= \det\left(\boldsymbol{Q}_0\right)^{-2} \det\left(\boldsymbol{\mu}^0 - k^{2s}\vec{z} \otimes \vec{z}^T - \mu \boldsymbol{I}\right), \end{aligned}$$

where $\vec{z} = \boldsymbol{Q}_0^T \vec{1}_{\widetilde{M}}$. Here, we have exploited the tensor product structure of $\boldsymbol{B}_\sigma$. This means that we are interested in the impact on the spectrum of a rank one perturbation of a diagonal matrix. The eigenvalues of the rank one perturbation are $-k^{2s}\left\|z\right\|^2$ with multiplicity one and 0 with multiplicity $\widetilde{M}-1$. If k^{2s} is real, then $\boldsymbol{\mu}^0 - k^{2s}\vec{z}\otimes\vec{z}^T$ is Hermitian and all μ_m are real. We assume without loss of generality that $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{\widetilde{M}-1}$. Applying Weyl's theorem [12, Theorem 4.3.7], one can show then that

$$\begin{aligned} \mu_0^{(0)} - k^{2s}\left\|z\right\|^2 \leq \mu_0 &\leq \mu_0^{(0)}, \\ \mu_{m-1}^{(0)} \leq \mu_m &\leq \mu_m^{(0)}, \qquad\qquad \text{for } m \geq 1. \end{aligned}$$

Since all $\mu_m^{(0)}$ are non-negative, we can conclude that at most one eigenvalue μ_m is negative. □

The above proposition shows that we need to solve a sequence of systems with matrix of the form

$$\boldsymbol{M}_{FE}\mu + \boldsymbol{S}_{FE}, \quad \mu \in \mathbb{C}.$$

Depending on μ, we use different solver strategies:

- μ is real and non-negative (this corresponds to a classical, integer-order reaction-diffusion problem):
  We employ a conjugate gradient solver with standard geometric multigrid preconditioner.
- μ is real and negative (this corresponds to an integer-order Helmholtz problem):
  We use GMRES preconditioned by geometric multigrid which has been constructed from the shifted system $\boldsymbol{S}_{FE} + (1+i\beta)\mu\boldsymbol{M}_{FE}$ with $\beta = 0.5$.
- μ is complex, $\operatorname{Re}\mu$ is non-negative:
  We use GMRES preconditioned by standard geometric multigrid.
- μ is complex, $\operatorname{Re}\mu$ is negative:
  Let $\mu =: -\nu(1+i\alpha)$ with $\nu \in \mathbb{R}_{\geq 0}$ and $\alpha \in \mathbb{R}$. We use GMRES preconditioned by geometric multigrid which has been constructed from the shifted system $\boldsymbol{S}_{FE} - \nu\left(1+i\beta\right)\boldsymbol{M}_{FE}$ with $\beta = 0.5$.

We note that this solution approach exposes a significant amount of parallelism. The solution of the $\widetilde{M}$ decoupled problems is embarrassingly parallel, and each of the integer-order problems can be performed in parallel. We also note that these

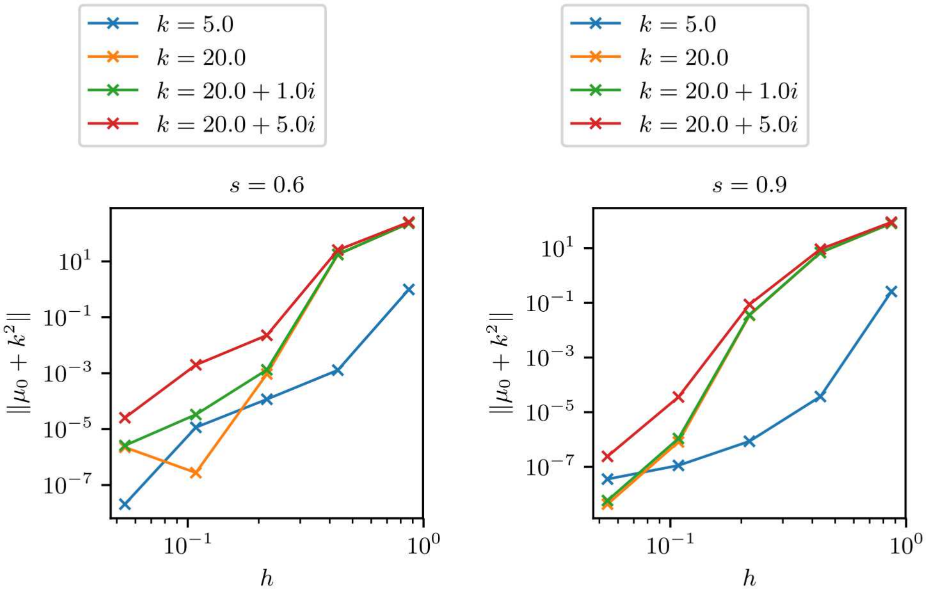

FIG. 6.1. *Convergence of the shift coefficient μ closest to $-k^2$.*

solvers merely reuse existing solver technologies. In particular, this implies that any improvements that can be made for the (potentially costly) solution of the integer-order Helmholtz equation will benefit the solution of its fractional equivalent.

In practice, we observe that one of the shift coefficients $\{\mu_m\}$ approaches $-k^2$ as $h \to 0$. In Figure 6.1, we plot the distance between this shift coefficient and $-k^2$ for $\Omega = [0,1]^3$, $s \in \{0.6, 0.9\}$ and different wave numbers k.

6.1. Comparison with the integer-order case. When k^{2s} is real, only a single μ_m is negative according to Proposition 6.1. The above observation entails that the single integer-order Helmholtz problem that needs to be solved has wave number (very close to) k. This permits to compare the solution complexity of the classical integer-order Helmholtz problem to the fractional case. The fractional-order case differs in that we need to

- compute eigenvalue approximations $\widetilde{\lambda}_m$, $m = 0, \dots, \widetilde{M}$,
- solve a generalized eigenvalue problem to obtain shifts μ_m and weights w_m, $m = 0, \dots, \widetilde{M}$,
- solve $\widetilde{M} - 1$ reaction-diffusion type systems.

The generalized eigenvalue problem (6.4) can be solved in $\mathcal{O}\left(\widetilde{M}^3\right)$ operations, and the weights w_m can be computed in $\mathcal{O}\left(\widetilde{M}^2\right)$ operations. Since finding the eigenvalue approximations is also an inexpensive operation (cf. [1]), the computations are entirely dominated by the the linear solves. Solving an integer-order Helmholtz problem can be significantly more costly than solving reaction-diffusion problems, especially when the wave number k is large. Therefore, we expect that the overall cost of solving the fractional Helmholtz problem is comparable to the classical integer-order case.

6.2. Solving sequences of problems with variable fractional order. If the eigenvalue approximations are chosen such that they satisfy Assumption 5.3 for a range of fractional orders $[s_{\min}, s_{\max}] \subset (0,1)$, the resulting solver can be used to solve fractional Helmholtz problems of order $s_{\min} \leq s \leq s_{\max}$ without rediscretization. This is quite beneficial since the exact value of the fractional exponent s is generally unknown and needs to be determined through repeated linear solves with varying s in the framework of an inverse problem. See for instance, [22, 2], and [5], where the exponent s is spatially dependent. We do not explore the property further in the context of the present work.

7. Numerical examples. Let $\Omega = [0,1]^d$. We solve the fractional-order Helmholtz equation

$$\begin{cases} (-\Delta)^s u(\vec{x}) - k^{2s} u &= f(\vec{x}), \qquad \vec{x} \in \Omega, \\ u(\vec{x}) &= 0, \qquad \vec{x} \in \partial\Omega. \end{cases}$$

In order to evaluate the error convergence rates, we prescribe the analytic solution $u = C \prod_{i=1}^{d} [x_i(1-x_i)]^{r+2s-1/2}$, with a given right-hand side regularity of index r and C chosen to normalize u. We obtain an approximation of the corresponding right-hand side function f via the discrete sine transform. We observe that

$$f \in \widetilde{H}^r(\Omega), \qquad\qquad u \in \begin{cases} C^\infty\left(\overline{\Omega}\right) & \text{for } r+2s \in \mathbb{N}+3/2, \\ \widetilde{H}^{r+2s}(\Omega) & \text{else.} \end{cases}$$

We have to resort to this approach, since we are not aware of any non-trivial analytic pairs of right-hand side and solution for the fractional Helmholtz problem that reflect the regularity properties of the equation. We also note that prescribing the solution, and finding an approximation to the right-hand side function f, instead of the other way around, permits us to compute the L^2-error as follows:

$$\|u - u_h\|_{L^2(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 - 2(u, u_h)_{L^2(\Omega)} + \|u_h\|_{L^2(\Omega)}^2.$$

Here, the first term can be evaluated analytically and the second and third term can be evaluated using quadrature of sufficiently high order. The $\mathcal{H}_\omega^1$-error is given by

$$\|U - U_h\|_{\mathcal{H}_\omega^1}^2 = (f, u)_{L^2(\Omega)} - (f, u_h)_{L^2(\Omega)} + k^{2s}\|u - u_h\|_{L^2(\Omega)}^2.$$

Here, we have used the variational formulation given in (wExt). Since the expansion coefficients u_m and the exact eigenvalues λ_m are know, we can use the expansion

$$(f, u)_{L^2(\Omega)} = \sum_{m=0}^{\infty} \left(\lambda_m^s - k^{2s}\right)^{-1} |u|_m^2$$

to approximate the true value by truncating the sum to a sufficient number of terms, as long as we make sure that the truncation error is dominated by the discretization error. As stated in Theorem 5.6 the $\widetilde{H}^s$-error is bounded by the $\mathcal{H}_\omega^1$-error.

It is important to note that the fact that the domains Ω are hypercubes is exploited only to obtain good approximations for pairs of solutions and right-hand sides in order to compute error norms. The discretization of the problem as well as the solver are entirely oblivious to this fact and do not take advantage of it.

In what follows, we solve the above problem for the d-hypercube, $d = 3$, fractional order $s \in \{0.6, \ldots, 0.9\}$ and

I a low regularity case ($r = 1/2$), using piecewise linear ($p = 1$) elements and real-valued wave number $k \in \{5, 20\}$, II a low regularity case ($r = 1/2$), using piecewise quadratic ($p = 2$) elements and real-valued wave number $k \in \{5, 20\}$,

III a high regularity case ($r = 2$), using piecewise quadratic ($p = 2$) elements and real-valued wave number $k \in \{5, 20\}$,

IV a low regularity case ($r = 1/2$), using piecewise linear ($p = 1$) elements and complex-valued wave number $k = 20 + 5i$.

In all settings, we use quasi-uniform meshes.

In Figure 7.1, we display the solution errors measured in $\mathcal{H}^1_\omega$- and in $L^2(\Omega)$-norm for the first two test cases, I and II. In Figure 7.2, the convergence results of the latter two test cases, III and IV, are shown. As predicted by Theorem 5.6, order $h^{\min\{p,r+s\}}$ convergence is observed in $\mathcal{H}^1_\omega$-norm. For the L^2-error, convergence of order $h^{\min\{p+1,r+2s\}}$ is observed. This is better than the rate of $h^{\min\{p+s,r+2s\}}$ predicted by Theorem 5.6. We notice an apparent slowdown of convergence in the first two plots of Figure 7.2. This is an artifact of the inefficient way of obtaining the right-hand side function f using the discrete sine transform. Given the limited regularity of the function, we need to use many Fourier terms to obtain an accurate representation of f. The slowdown is due to the fact that we cannot increase the number of terms any further without exceeding the available memory of our machine. We point out however that this is merely a drawback of how we obtain pairs of known solutions and right-hand sides, but not of the presented method.

All computations are performed on a dual socket Intel Xeon E5-2650V3, 2.30GHz, 20 core workstation. In Figure 7.3 we display timings for the solution of the linear problems. We display both the total solve time as well as the cumulative time for all reaction-diffusion type solves. We observe that as expected, the integer-order Helmholtz solve dominates the overall solution time for larger wave number k. This shows that for high wave number k solution of the fractional problem and its integer-order equivalent have very comparable cost. We also observe that for fixed k, the solution time scales roughly linearly with the number of degrees of freedom $n = \dim S_h$ of the finite element space.

8. Anticipated outcomes and impact. An immediate application of the fractional Helmholtz equation is the acoustic/electromagnetic interrogation of fractured media - weapon system components; rocks abused by hydrofracking, earthquakes or weapons testing.

In this context, a natural next step would be to learn the a priori unknown fractional order s from available experimental data. The presented solver was designed with this application in mind and supports solving sequences of fractional Helmholtz problems without rediscretization. Therefore, our work enables the use of high-fidelity fractional-order models in optimal control and uncertainty quantification which rely on repeated and efficient solution of the forward problem. A key demonstration for the presented algorithm development would be to compare results (efficiency, accuracy, etc) against existing algorithms HiFEM and FrachNet.

The presented solver and its theoretical underpinning also lend themselves naturally to be generalized to other types of fractional-order equations. In the context of geophysics, a natural extension would be to consider not a fractional Helmholtz equation, but the full set of time-dependent Maxwell's equations, without making any of the simplifying assumptions. Other Sandia application areas that could benefit from improved linear solve times are nuclear waste disposal and subsurface damage caused by carbon sequestration, where fractional-order equations provide improved predictive capability.

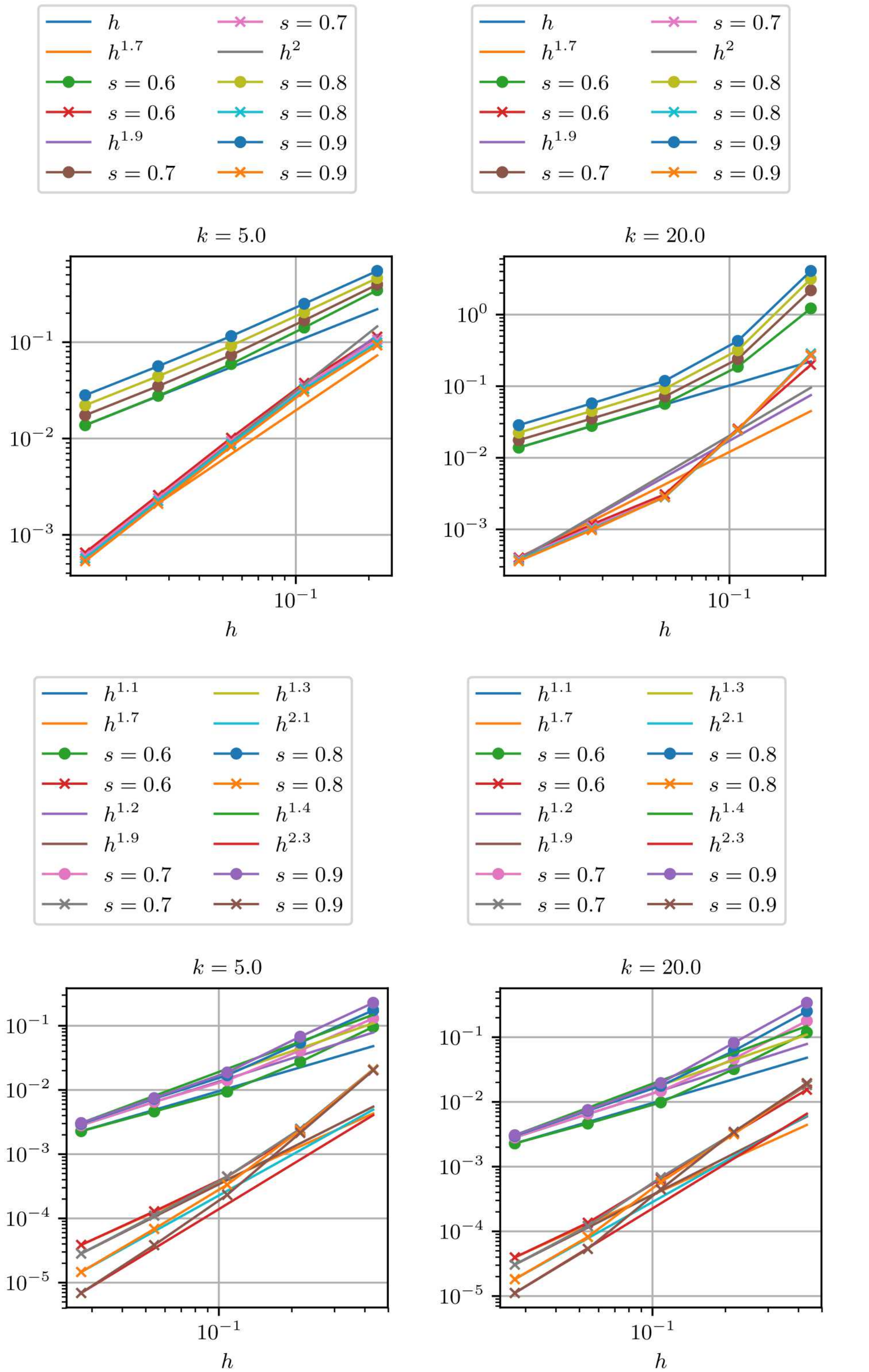

FIG. 7.1. *L^2-errors ($\times$) and $\mathcal{H}^1_\omega$-errors ($\bullet$) for the solution of the fractional Helmholtz problem on the unit cube for test cases I and II: wave number $k = 5$ (left) and $k = 20$ (right), fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$ and piecewise linear ($p = 1$, top) and piecewise quadratic finite elements ($p = 2$, bottom) for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 1/2$.*

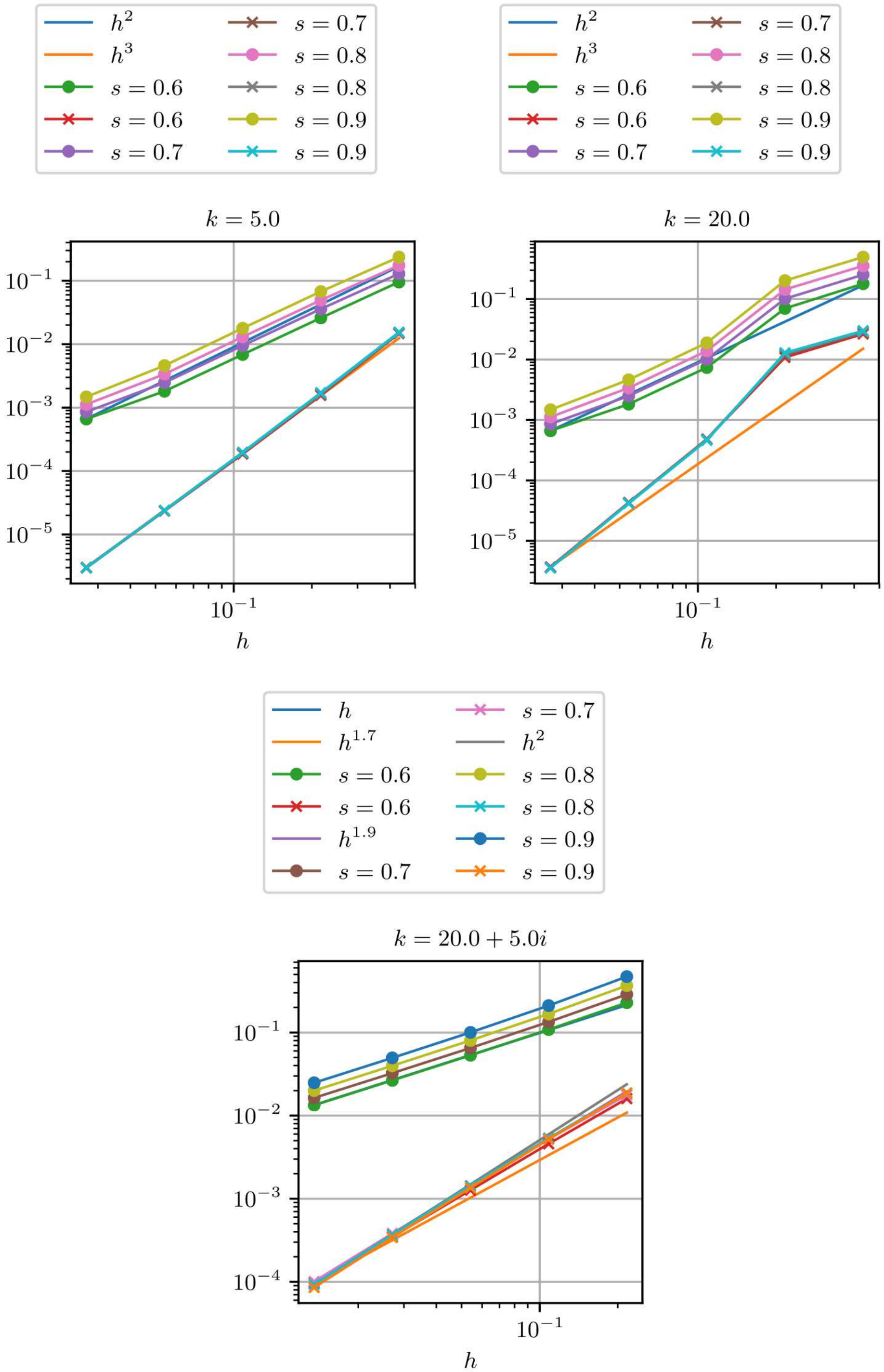

FIG. 7.2. *L^2-errors ($\times$) and $\mathcal{H}^1_\omega$-errors ($\bullet$) for the solution of the fractional Helmholtz problem on the unit cube for test cases III and IV:* Top: *wave number $k = 5$ (*left*) and $k = 20$ (*right*), fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$ and piecewise quadratic finite elements ($p = 2$) for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 2$.* Bottom: *wave number $k = 20 + 5i$, fractional orders $s \in \{0.6, 0.7, 0.8, 0.9\}$ and piecewise linear finite elements ($p = 1$) for a right-hand side $f \in \widetilde{H}^r(\Omega)$, $r = 1/2$.*

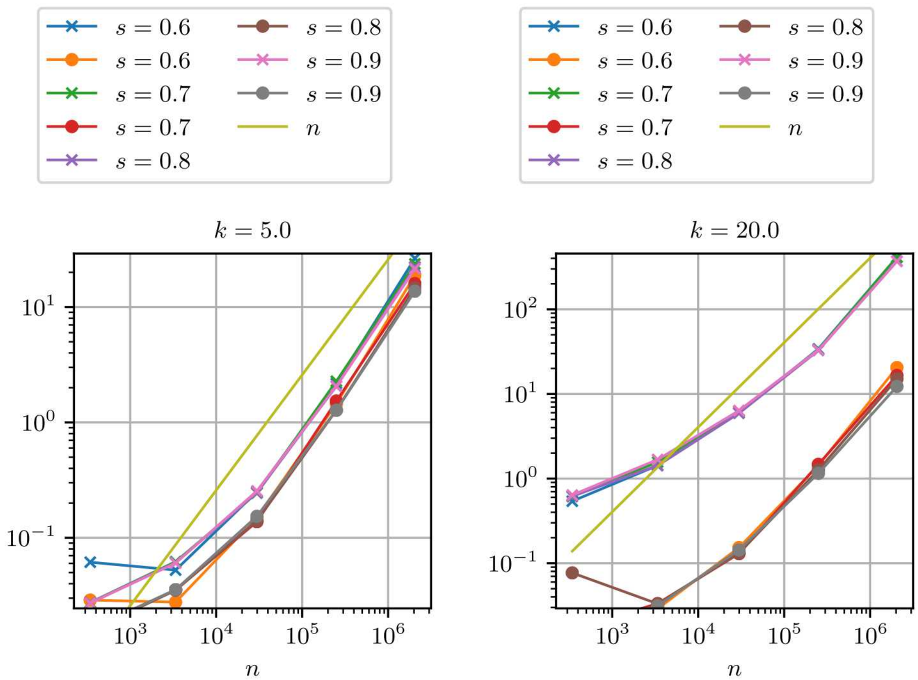

FIG. 7.3. *Solution times for the fractional Helmholtz problem on the unit cube. Total time (×) and time for all reaction-diffusion type sub-problems (•).*

An important follow-up question that needs to be addressed to make fractional order models more widely available to Sandia applications is how to extend to spatially dependent fractional orders instead of assuming a constant exponent s throughout the domain. Neither theoretical nor computational aspects of this question have been exhaustively explored. However, allowing for spatially dependent fractional orders is of uttermost importance for the accurate modeling of the transition between different regimes, e.g. of different material layers in the subsurface.

Another topic, with potentially very wide reaching consequences, is whether there exists a mapping between a class of kernel functions (such as $k(\vec{x}, \vec{y}) = 1/\left|\vec{x}-\vec{y}\right|^{d+2s}$ in the case of the fractional Laplacian) that can be mapped in a similar fashion to an extension problem. The existence of such a mapping would allow to solve a much larger class of nonlocal problems in much the same fashion (and with similar efficiency) as the fractional Helmholtz problem.

Having a fast solver for a class of fractional-order equations at our disposal, we are confident that we could address these next steps in a follow-up Laboratory Directed Research & Development (LDRD) project. Given the breadth of applications of fractional-order models, and the multitude of theoretical and computational issues that we would like to address, we believe that an application in the Computer & Information Sciences (CIS) Investment Area would be appropriate.

9. Conclusion. In this work, we have presented a fractional-order Helmholtz problem. We have discussed well-posedness and convergence of a hybrid finite element - spectral discretization. An efficient solver has been proposed that scales as well as the best possible solver for the integer-order Helmholtz equation, making the, more appropriate, fractional equation a preferable alternative for geophysical electromagnetics modeling. Numerical examples have been used to illustrate the obtained theoretical results.

Appendix A. Proof of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Let $U_h \in \mathcal{V}_h \setminus \{0\}$ be arbitrary and take $Z \in \mathcal{H}^1_\omega$ to be $Z := (k^{2s} + |k|^{2s})\mathcal{S}_k^* \operatorname{tr}_\Omega U_h$. Take $Z_h \in \mathcal{V}_h$ to be the best approximation to Z with respect to the $\|\cdot\|$-norm and set $V_h := U_h + Z_h \in \mathcal{V}_h$. Then, by using the Gårding's (in)equality

$$\begin{aligned}\mathcal{A}(U_h, V_h) &= \mathcal{A}(U_h, U_h) + \mathcal{A}(U_h, Z) + \mathcal{A}(U_h, Z_h - Z)\\ &= \|U_h\|^2_{\mathcal{H}^1_\omega} - k^{2s}\|U_h\|^2_{L^2(\Omega)} + k^{2s}\|U_h\|^2_{L^2(\Omega)} + |k|^{2s}\|U_h\|^2_{L^2(\Omega)} + \mathcal{A}(U_h, Z_h - Z)\\ &= \|U_h\|^2 + \mathcal{A}(U_h, Z_h - Z).\end{aligned}$$

Hence due to the continuity of $\mathcal{A}$

$$|\mathcal{A}(U_h, V_h)| \geq \|U_h\| - C\|U_h\|\,\|Z - Z_h\|. \tag{A.1}$$

Since

$$\|Z - Z_h\| \leq 2\eta |k|^{2s}\|U_h\|_{L^2(\Omega)} \leq 2\eta |k|^s \|U_h\|$$

and

$$\|Z\| \leq 2C_d(k)|k|^{2s}\|U\|_{L^2(\Omega)} \leq 2C_d(k)|k|^s\|U\|$$

we also obtain

$$\begin{aligned}\|V_h\| &\leq \|U_h\| + \|Z\| + \|Z - Z_h\|\\ &\leq \|U_h\| + 2C_d(k)|k|^s\|U_h\| + 2\eta|k|^s\|U_h\|\\ &= (1 + 2(C_d(k) + \eta)|k|^s)\|U_h\|. \end{aligned}\tag{A.2}$$

Combining (A.2), (A.1)

$$\begin{aligned}\frac{|\mathcal{A}(U_h, V_h)|}{\|U_h\|\,\|V_h\|} &\geq \frac{\|U_h\| - C\|Z - Z_h\|}{(1 + 2(C_d(k) + \eta)|k|^s)\|U_h\|}\\ &\geq \frac{1 - C\eta|k|^s}{1 + 2(C_d(k) + \eta)|k|^s} \geq \frac{1 - C\gamma}{1 + 2(C_d(k) + \eta)|k|^s}.\end{aligned}$$

Hence

$$\inf_{U_h \in \mathcal{V}_h} \sup_{V_h \in \mathcal{V}_h} \frac{|\mathcal{A}(U_h, V_h)|}{\|U_h\|\,\|V_h\|} \geq \frac{1 - C\gamma}{1 + 2(C_d(k) + \eta)|k|^s}.$$ □

Proof of Theorem 5.2. Take $Y \in \mathcal{H}^1_\omega$ to be $Y := \mathcal{S}^*(U - U_h)$. Then, due to Galerkin orthogonality and for arbitrary $Y_h \in \mathcal{V}_h$

$$\|U - U_h\|^2_{L^2(\Omega)} = \mathcal{A}(U - U_h, Y) = \mathcal{A}(U - U_h, Y - Y_h) \leq C\|U - U_h\|\,\|Y - Y_h\|.$$

Taking Y_h to be the best approximation to Y, we have

$$\|Y - Y_h\| \leq \eta\|U - U_h\|_{L^2(\Omega)},$$

Hence, we find

$$\|U - U_h\|_{L^2(\Omega)} \leq C\eta \, |||U - U_h||| \, .$$

Due to the Gårding's inequality, Galerkin orthogonality and continuity of $\mathcal{A}$, we have for arbitrary $V_h \in \mathcal{V}_h$

$$\begin{aligned} |||U - U_h||| &\leq |\mathcal{A}(U - U_h, U - U_h)| + 2\,|k|^{2s}\,\|U - U_h\|^2_{L^2(\Omega)} \\ &= |\mathcal{A}(U - U_h, U - V_h)| + 2\,|k|^{2s}\,\|U - U_h\|^2_{L^2(\Omega)} \\ &\leq C\,|||U - U_h|||\,|||U - V_h||| + 2\eta^2\,|k|^{2s}\,|||U - U_h|||^2 . \end{aligned}$$

Therefore, since we assume (5.1),

$$|||U - U_h||| \leq \frac{C}{1 - 2\eta^2\,|k|^{2s}}\,|||U - V_h||| \leq C\,|||U - V_h||| \, .$$

Since this holds for arbitrary $V_h \in \mathcal{V}_h$, we have that

$$|||U - U_h||| \leq C \inf_{V_h \in \mathcal{V}_h} |||U - V_h||| \, . \qquad \square$$

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