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H. J. Ren, X. Q. Xu

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H. Ren*
*CAS Key Laboratory of Geospace Environment and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, P. R. China*

X. Q. Xu
*Lawrence Livermore National Laboratory, Livermore, California 94550, USA*

Collisionless damping of geodesic acoustic mode (GAM) is analytically investigated by considering finite-orbit-width (FOW) resonance effect to the 3-rd order in the gyro-kinetic equations. A concise and transparent expression for the damping rate is presented for the first time. Good agreement is found between the analytical damping rate and the previous TEMPEST simulation result[Phys. Rev. Lett. **100**, 215001 (2008)] for systematic $q$ scans. Our result also shows that it is of sufficient accuracy and has to take into account the FOW effect to the 3-rd order.

The geodesic acoustic mode (GAM)[1] is a well observed experimental phenomenon in tokamak plasmas[2, 3]. It is basically an electrostatic acoustic mode occurring on the magnetic surfaces with a radially local structure. The mode involves $m = 1$ pressure and density disturbance and $m = 0$ poloidal perturbed flow with toroidally symmetric spatial distribution. It has attracted much attention during last decades (see, for example, Refs. 4–8 and references therein) due to its important role in modulating the plasma turbulence and turbulent transport by affecting the radial electric field $E_r$ and resulting $\vec{E}\times\vec{B}$ poloidal flow[9–11]. The physical interpretation of GAM is that the radial drift of the perturbed particles distribution due to the geodesic curvature effect compensates the polarization drift and an oscillation with a frequency around the ion transit frequency is excited. The passing particles dominant the GAM dispersion relation, leading to the remarkable difference between the damping of GAM and zonal flows (ZFs). The GAM can be damped by both the collisionless wave-particle resonances and collisions, while ZFs are damped only by collisions in which the trapped particles plays a crucial role.

By keeping terms to the 1-st order finite-orbit-width (FOW) effect of passing particles $\delta_i \sim q\rho_i$, where $q$ is the safety factor of tokamaks and $\rho_i$ is the ion gyroradius, the classical damping rate of GAM is found to be independent of $\delta_i$. However, the following theoretical analysis[12, 13], numerical evaluation [14], and simulation date [15] all indicate that the high-order FOW effect plays a key role in large $q$, where the resonant damping rate is sensitive to $k_r\rho_i$, where $k_r$ is the radial wave number. The present Letter is to analytically perform the collisionless damping rate of GAM in the presence of FOW effect to the 3-rd order resonance effect for the first time.

The gyro-kinetic equation determining the perturbed distribution function in an isotropic non-rotating tokamak plasma with a basically electrostatic disturbance is given as $\delta F = q\frac{\partial F_0}{\partial E}\delta\phi + J_0(k_r v_\perp/\Omega)\delta h$, in which $\delta h$ is governed by

$$(\vec{v}_0\cdot\nabla - i\omega)\delta h = ieJ_0(k_r v_\perp/\Omega)Q\delta\phi. \tag{1}$$

Since the plasma is non-rotating with an isotropic equilibrium, the particles unperturbed distribution function $F_0$ has a standard Maxwellian form with a temperature $T^j(\psi)$ ($j = i, e$ for species ions and electrons, respectively) and an unperturbed number density $n_0(\psi)$, leading to the independence of the magnetic moment. Here, $\delta\phi$ is the perturbed electrostatic potential, $E = mv^2/2$ is the particle kinetic energy, $Q = \omega\partial F_0/\partial E + \vec{k}\times\vec{B}\cdot\nabla F_0/(eB^2)$, $\vec{v}_0 = v_\parallel\vec{B}/B + \vec{v}_D$, and $\vec{v}_D$ is the zeroth-order drift velocity defined as $\vec{v}_D = [(v_\parallel^2 + v_\perp^2/2)/\omega_c]\vec{B}\times\vec{\mathcal{K}}/B$ with the parallel velocity $v_\parallel$ and perpendicular velocity $\vec{v}_\perp$, and $\omega_c = eB/m$ is the particle gyro frequency. The individual variables are $(\vec{X}, \mu, E)$, where $\vec{X}$ is the guiding-center position and $\mu$ is the magnetic moment, leading to $\nabla F_0 = \nabla F_0|_{\mu,E}$. Here, the wave vector $\vec{k} = k_r\nabla\psi/(\partial\psi/\partial r)$ has only the radial component for GAMs in particular and $\vec{\mathcal{K}}$ is the magnetic field curvature. Considering a large-aspect-ratio tokamak plasma with a toroidally symmetric magnetic field $\vec{B} = I\nabla\zeta + \nabla\zeta\times\nabla\psi$ and after some transparent calculation, we can rearrange the gyro-kinetic equation as

$$\partial_\theta\delta h - ik_r\delta_i\sin\theta\delta h - i\frac{\omega}{\omega_t}\delta h = ieJ_0\frac{\omega}{\omega_t}\frac{\partial F_0}{\partial E}\delta\phi, \tag{2}$$

Here, $\omega_t = v_\parallel/(qR)$ is the transit frequency and $\delta_i$ is the particle orbit width defined as $v_d/\omega_t$ with $v_d = (v_\parallel^2 + v_\perp^2/2)/(R\omega_c)$. In order to obtain $\delta h$, $\partial F_0/\partial E$ is calculated with fixed $\mu$, and then one make a change of variables from $(E, \mu)$ to $(v_\parallel, v_\perp)$. So the derivation $\partial_\theta\delta h$ is done with $v_\parallel$ fixed. That means the trapped particle effect is not considered here, which requires $q \gg 1$ or a sufficiently large aspect ratio $\epsilon^{-1}$[14]. While a gyrokinetic simulation shows that the trapped electrons play a significant role for high $q$ with a realistic $\epsilon = 0.2$[16]. A recent theoretical study [17] confirmed that the collisionless damping rate is enhanced by trapped electrons

*Electronic address: hjren@ustc.edu.cn

for high $q$. To simplify the algebra and focus on the high-order resonance of FOW, we have to disregard the trapping effects here. All perturbations are assumed to have the form $\delta f = \sum_m \delta f_m(\psi,\theta)e^{im\theta}$, where $f$ denotes the electrostatic potential $\phi$, distribution function $F$, and ions and electrons number density $n_i$ and $n_e$. By making the transform $\delta h = e^{i\frac{\omega}{\omega_t}\theta - ik_r\delta_i\cos\theta}\delta g$, using $e^{ik_r\delta_i\cos\theta} = \sum_m i^m J_m(k_r\delta_i)e^{im\theta}$ and with the help of $\sum_k J_{k+n}J_k = 0(n\neq 0)$ or $1(n=0)$, $\delta F$ is solved as[12, 18]

$$\delta F = (1-J_0^2)\frac{\partial F_0}{\partial E}e\delta\phi + eJ_0^2\frac{\partial F_0}{\partial E}\sum i^{n-k}J_{n+l-k}J_l \times \frac{l-k}{l-k+\omega/\omega_t}\delta\phi_n e^{ik\theta}. \quad (3)$$

Next, in order to obtain an analytic result containing the high-order FOW resonance effect, we need one assumption $k_r\delta_i \ll 1$ to expand the Bessel function $J_m(k_r\delta_i)$, which requires actually $qk_r\rho_i \ll 1$ by recalling $\omega$ and the ions transit frequency $\omega_{ti}$ have the same order. Here, $\rho_i = v_{Ti}/\Omega$ is the ion gyroradius. Let us define $\lambda \equiv k_r\rho_i$ for simplicity of notation. Since $q\lambda \ll 1$, it is naturally to find $\lambda \ll 1$ due to the fact that $q$ is about $2\sim 8$ in a realistic tokamak plasma. Furthermore, we may have $q^2\lambda \ll 1$, indicating sufficiently long wave length perturbations, or the wave length is short enough so that $q^2\lambda \gtrsim 1$ is satisfied. There is no restriction on the order of $q^2\lambda$ term when deriving the analytical dispersion relation.

Previous work[14] shows that $\delta\phi_{\pm n} \sim \mathcal{O}(\lambda^n)\delta\phi_0$ for $n = 0, \pm 1, ....$ As a result, in order to take into account the ions FOW effects, we need consider the contribution of $\delta\phi_{\pm n}$ ($n = \pm 2, \pm 3$) harmonics, namely, the 2-nd and 3-rd FOW resonance in the dispersion relation. It has been initially confirmed by Ref. [12] and then re-studied by Ref. [14] two years later that the effect of FOW to the 2-nd order can enhance the collisionless damping of GAM. Ref. [13] used different expanding method to look into the 2-nd resonance effect by considering a special case $q^2\lambda \gg 1$. The collisionless damping of GAM with more higher-order FOW effects were only examined by numerical evaluation [14] and simulation[15]. The purpose of this Letter is to derive a transparent analytical damping rate for the first time by considering the FOW resonance effects on the GAM to the 3-rd order.

Now we keep terms to the order of $\mathcal{O}(\lambda^6)$ and neglect all the higher-order terms to obtain the harmonics of $\delta F^i$. It should be very carefully to expand the Bessel function to the appropriate orders when deriving $\delta F^i_m$. For ions, the finite-Larmor-radius (FLR) effect is taken into account ($\lambda \ll 1$). While for electrons, the FLR and FOW effects are both ignorable since $k_r\rho_e \simeq 0$. Then the electrons perturbed distribution function is simplified to $\delta F^e = \frac{e}{T_e}(\delta\phi - \delta\phi_0)$. Using the charge quasi-neutrality condition, $\delta n_i = \delta n_e$, namely, integrating $\delta F^j_m$ over the velocity space leads to the dispersion relation of GAM. It is easy to find $\delta\phi_{-m} = (-1)^m\delta\phi_m$ for $m = 1, 2, 3$. The algebraic manipulation is highly intricate and the expressions are redundant and complicated. Let us skip them to the general dispersion relation of GAM by using $\delta n^i_0 = 0$ and $\delta n^i_m = \delta n^e_m$,

$$\mathcal{D}(\zeta, \mathcal{Z}_n(\zeta), \mathcal{Z}_n(\zeta/2), \mathcal{Z}_n(\zeta/3)) = 0, \quad (4)$$

where $\zeta$ is short for $q\Omega$ with $\Omega$ being the GAM frequency normalized by $R/v_{Ti}$ and we have defined the following function

$$\mathcal{Z}_n(\zeta) = \frac{1}{\sqrt{\pi}}\int_L \frac{x^n e^{-x^2}}{x-\zeta}dx, \quad (5)$$

and $\mathcal{Z}_0$ is abbreviated as $\mathcal{Z}$, which is the so-called plasma dispersion function. The original expression for $\mathcal{D}$ is so involved that no useful information can be obtained directly from the dispersion relation. Here we try to derive a transparent analytical expression for the GAM frequency and collisionless damping rate. In general case, there is no analytical solution to the dispersion relation. Fortunately, we can restrict ourselves to the GAM with $\zeta \gg 1$ so that we can find an asymptotic solution, which requires large safety factor. Hence, analytical results in this Letter are expected to be accurate enough only when $q$ is sufficiently high, while actually, previous study (with only 1-st order FOW, see, for example, in Ref. 14) shows that the asymptotic analytical solution agrees well with the numerical result not only in the large safety factor limit. Generally for $q \gtrsim 2$, the discrepancy between them is almost ignorable. Now we asymptotically expand the plasma dispersion function $\mathcal{Z}(\zeta) = i\sqrt{\pi}\exp(-\zeta^2) - \zeta^{-1}(1+\zeta^{-2}/2+3\zeta^{-4}/4+\cdots)$ as usual and neglect all terms of order higher than $\mathcal{O}(\zeta^{-4})$, and then we arrive at

$$\begin{aligned}
&1-\lambda^2\left(\frac{3}{8}-\frac{\tau}{2}\right) - \frac{1}{\Omega^2}\left[\frac{7}{4}+\tau+\frac{\tau}{2q^2}+\lambda^2\left(\frac{\tau^2}{4}+\frac{11}{16}\tau\right.\right.\\
&\left.\left.-\frac{13}{8}\right)\right] - \frac{1}{\Omega^4}\left[\frac{23}{8q^2}+\frac{9\tau}{8q^2}+\lambda^2\left(\frac{747}{64}+\frac{285}{64}\tau+\frac{7}{16}\tau^2\right)\right]\\
&+i\sqrt{\pi}e^{-q^2\Omega^2}q^5\Omega^3 + i\sqrt{\pi}e^{-\frac{q^2}{4}\Omega^2}\lambda^2\frac{q^7\Omega^3}{512}\left(\frac{15}{2}\tau+8+q^2\Omega^2\right)\\
&+i\sqrt{\pi}e^{-\frac{q^2}{9}\Omega^2}\lambda^4 q^{11}\Omega^5\frac{54+845\tau+(2+36\tau)q^2\Omega^2}{2519424} = 0.
\end{aligned} \quad (6)$$

Here, $\tau$ is defined as $T_e/T_i$, which should be much greater than $m_e/m_i$. In the real part (assuming $\Omega$ is real temporarily ), we disregard terms proportional to $\lambda^4$ due to the fact that the corrections of FOW to the GAM real frequency is small. While for the imaginary part, we should keep terms to the order of $\lambda^4$ and ignoring all terms with higher orders. Strictly speaking, terms on the order of $\mathcal{O}(\lambda^6)$ should be disregarded for self-consistency since only terms to the order of $\lambda^4$ in $\delta\phi_{\pm 2}$, terms to the order of $\lambda^3$ in $\delta\phi_{\pm 3}$ and terms to the order of $\lambda^5$ in $\delta\phi_{\pm 1}$ are kept. That indicates the equation above is not precise on the order of $\lambda^6$. We also disregard terms of the order $\mathcal{O}(\lambda^2 q^{-2})$ by recalling $\lambda \ll q^{-1}$ since terms proportional to $1/q^4$ are neglected as a sacrifice for the analytical expression. We point out that the simplified dispersion relation seems not identical with the previous one (see, for example, Refs. [19] and [20]) due to the different ways of asymptotic expansion[8].

The real part of the dispersion relation (6) yields the frequency of GAM to the order of $1/q^2$ and $\lambda^2$ as

$$\Omega_{\mathrm{G}}^2 = \left(\frac{7}{4}+\tau\right)\left[1+\frac{1}{q^2}\frac{46+32\tau+8\tau^2}{(7+4\tau)^2} - \lambda^2\frac{\tau^2-\frac{3}{4}\tau+\frac{31}{8}}{7+4\tau}+\lambda^2\frac{747+285\tau+28\tau^2}{(14+8\tau)^2}\right]. \quad (7)$$

One can see the that the FOW modifies the GAM frequency by introducing terms on the order of $\mathcal{O}(\lambda^2)$ as shown in the previous study[12–14, 19, 20], which is much less than the terms proportional to $1/q^2$. That is, not surprising and brand new, the effect of FOW on the GAM frequency is very slight. On the other hand, the formula above indicates that the order relation $q^2\lambda \gg 1$ should be satisfied for self-consistency since the terms of $\lambda^2$ are kept whereas the terms on the order of $\mathcal{O}(1/q^4)$ are neglected. Or else, the FOW terms should be removed.

Different with the GAM oscillation frequency, the FOW has significant effect on the collisionless damping rate of the GAM. For simplicity of notation, we define$\gamma_c = -\frac{\sqrt{\pi}}{2}q^5\frac{\Omega_{\mathrm{G}}^6}{7/4+\tau}e^{-q^2\Omega_{\mathrm{G}}^2}$ as the classical collisionless damping rate[12, 14]. Then the Landau damping rate is changed to

$$\gamma_d = \gamma_c\left[1+\frac{\lambda^2q^2}{512}e^{\frac{3}{4}q^2\Omega_{\mathrm{G}}^2}\left(\frac{15}{2}\tau+8+q^2\Omega_{\mathrm{G}}^2\right) + e^{\frac{8}{9}q^2\Omega_G^2}\lambda^4q^6\Omega^2\frac{54+845\tau+(2+36\tau)q^2\Omega_G^2}{2519424}\right], \quad (8)$$

by the presence of FOW to the 3-rd order resonance. Let us define the second term in the bracket as $\chi_1$ and the last term as $\chi_2$, which represent the 2-nd resonance and 3-rd resonance, respectively. Now one can look into a simple case in which $q = 4$, $\lambda = 0.1375$ and $\tau = 0$, one finds that $\chi_1 = 3.83 \times 10^8$ and $\chi_2 = 1.71 \times 10^8$. Apparently, the 2-nd order and 3-rd order FOW resonance dramatically increases the Landau damping rate of GAM[14, 15]. Since $\chi_1$ and $\chi_2$ increases with $q$ while $\gamma_c$ decreases as $q$ increases, it is predictable that $\gamma_d$ decreases first as $q$ increases and then increases with $q$. As $q$ continues to climb up, the damping rate decreases again as $q$ increases. For sufficiently large $q$, the damping rate is predicted to climb up again and then deceases due to the higher-order FOW resonance. Previous numerical evaluation [14] and TEMPEST simulation [15] both illustrated this phenomenon. Fig. 1 confirms this point. There is an obvious fluctuations on the curve of the dependence of negative $\gamma_d$ on the safety factor $q$. Let us focus on the curve representing the analytical damping rate for $\tau = 0$. The fluctuation occurs at about $q \simeq 2$ with the peak at about $q \simeq 2.9$. Apparently, it is introduced by $\chi_1$, the 2-nd FOW resonance since in this regime, the 3-rd resonance effect is not remarkable yet. Theoretically, $\chi_2$ may introduce a second fluctuation when $q$ is sufficiently large. But this one is not apparent although it indeed dramatically lifts up the curve at about $q \simeq 5.5$.

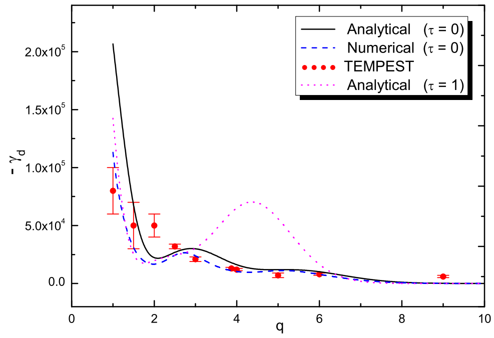


FIG. 1: The negative collisionless damping rate versus the safety factor $q$ with the major radius $R_0 = 1.71$m, ions temperature $T_i = 3$ kev with deuterium ions. $\lambda$ is assumed to be 0.1375 as the same with Ref. [15]. The numerical curve (dashed one) is plotted according to the exact numerical solution of $\mathcal{D} = 0$ for $\tau = 0$, namely, Eq. (9) below.

The general dispersion relation to the order of $\mathcal{O}(\lambda^2)$ is also presented in Ref. [14], in which the authors did not give simplified and transparent expressions for the GAM frequency and collisionless damping rate. To the order of $\lambda^4$, only numerical evaluation was carried out according to the general dispersion relation. In order to obtain a transparent expression for the damping rate in the presence of FOW with 2-nd resonance, Ref. [13] analytically derived the GAM frequency and damping rate by using different expanding method. However, The GAM frequency they obtained is not the same as the one in Eq. (7), and the key different is the collisionless damping rate. Their result is still a monotonic function about $q$. As $q$ increases from 2 to 10, the damping rate curve shows no fluctuation, while the previous numerical evaluation and simulation both proved such a fluctuation. As claimed by the authors, their analytical result is justified only for $q^2\lambda \gg 1$. Our analytical damping rate (8) agrees better with the simulation and numerical results, and requires fewer ordering restrains. That makes our analytical damping rate (8) has a wider range of applicability, even restricted to the order of $\lambda^2$.

When only the 2-nd resonance is taken into account, a recent NEMORB simulation performed a good agreement with the analytical result to the 2-nd FOW effects for $1 \leq q \leq 4$[21]. It is also shown that for $q > 3.5$, the discrepancy between the theoretical result and the simulation one becomes remarkable[15]. Xu *et al.* suggested that the high-order resonance should be considered and they also numerically found that the damping rate with

4-th order resonance is almost the same as the rate with 10-th resonance[15]. Meanwhile, the numerical evaluation by Gao *et al.* [14] shows that the damping rate with 3-rd resonance and the one with 4-th resonance have only slight discrepancy when $q$ is about greater than 7. As a result, as illustrated in Fig. 1, our analytical result with 3-rd FOW resonance effect agrees well with TEMPEST result for $q > 2.5$. From the aspect of analytical study, it is of sufficient accuracy to consider the FOW effects only to the 3-rd order. Physically speaking, the resonance effects contain two parts. One is related to the interaction between particles and wave through $\delta\phi_n$, represented by the finite temperature ratio $\tau$ and the other is introduced by the FLR and FOW effects on ions mainly via $\delta\phi_0$. Analytical damping rate (8) contains both of them. The curves between the case of $\tau = 1$ and the case of $\tau = 0$ have similar shape. It can be concluded that $\tau$ plays a quantitative rather than a qualitative role in the resonance damping process. According to TEMPEST simulation result[15], the damping rate numerical calculated for $\tau = 0$ is better fitted with the date. Fig. 1 confirms this point and shows our analytical damping rate (8) agrees well with TEMPEST result for $\tau = 0$. There is a significant discrepancy between the $\tau = 1$ curve and TEMPEST dates. Comparing Fig. 1 and Fig. 3(a) in Ref. [22], we see that the analytical damping rate in Eq. (8) is not precise for $\tau \sim \mathcal{O}(1)$. More large $\tau$ is, more significant the error value between the analytical solution and the exact numerical solution is due to the order approximation. The case of $\tau = 0.1$ is plotted in Fig. 2. One can see that our numerical solution of Eq. (4) matches very well with the COGENT date[22]. A sight discrepancy appears between the analytical damping rate and the simulation one when $q < 2$. According to Figs. 1 and 2, the general dispersion relation (4) agrees very well with the simulation results, whereas the analytical result Eq. (8) agrees the simulation ones and numerical solution of Eq. (4) only for $\tau \ll 1$ and $q > 2$.

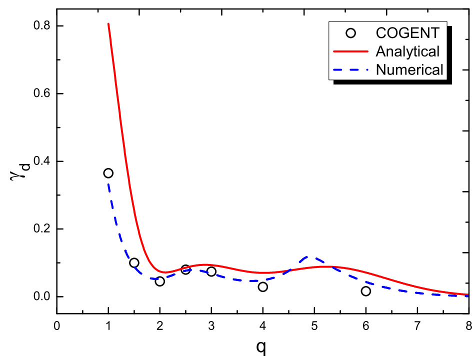


FIG. 2: The $q$ scan of negative normalized collisionless damping rate for analytical result (red line), numerical one (dashed), and COGENT simulation data(circle)[22]. $\lambda$ is still assumed to be 0.1375 and $\tau = 0.1$ is used.

Then we can simplify the discussion by restricting ourselves to the case of $\tau = 0$ to find out how the FOW acts on the GAM damping. The resonance condition is the passing ions parallel velocity $v_\parallel \sim q\omega R/n$. The $n$-th order resonance shows slight consequence on the real oscillation frequency of GAM by inducing terms proportional to $\lambda^{2(n-1)}$. While the resonance damping is significant by introducing terms proportional to $e^{-\zeta^2/n^2}\lambda^{2(n-1)}$. Due to the quick decay of exponential function, this term becomes dominant in the high $q$ limit. The exponential factor, of course, represents the $n-$th order resonance damping, and meanwhile, the factor $\lambda^{2(n-1)}$ is brought by $J_n^2/J_1^2$. For $\tau = 0$, the general dispersion relation $\mathcal{D}$ can be simplified to

$$\begin{aligned}
&\underbrace{\frac{1}{2}}_{\mathrm{FLR}_1} + \underbrace{\frac{q^2}{2\zeta}I_c}_{\mathrm{FOW}_1} - \underbrace{\frac{3\lambda^2}{16}}_{\mathrm{FLR}_2} - \underbrace{\frac{q^2\lambda^2}{4\zeta}\mathcal{R}_3(\zeta)}_{\mathrm{FLR}_1-\mathrm{FOW}_1} + \underbrace{\frac{3}{64\zeta}q^2\lambda^4\mathcal{R}_4(\zeta)}_{\mathrm{FLR}_2-\mathrm{FOW}_1} \\
&-\underbrace{\frac{q^4\lambda^2}{8\zeta^3}[\mathcal{R}_1(\zeta) - 2\mathcal{R}_1(\zeta/2)]}_{\mathrm{FOW}_2} + \underbrace{\frac{q^4\lambda^4}{16\zeta^3}[\mathcal{R}_2(\zeta) - 2\mathcal{R}_2(\zeta/2)]}_{\mathrm{FLR}_1-\mathrm{FOW}_2} \\
&+\underbrace{\frac{q^6\lambda^4}{384\zeta^5}[5\mathcal{R}_0(\zeta) + 81\mathcal{R}_0(\zeta/3) - 64\mathcal{R}_0(\zeta/2)]}_{\mathrm{FOW}_3} = 0. \qquad (9)
\end{aligned}$$

in which

$$\begin{aligned}
I_c(\zeta) &= \mathcal{Z}_4(\zeta) + \mathcal{Z}_2(\zeta) + \mathcal{Z}(\zeta)/2, \\
\mathcal{R}_0(\zeta) &= \mathcal{Z}_{12}(\zeta) + 3\mathcal{Z}_{10}(\zeta) + 15\mathcal{Z}_8(\zeta)/2 + 15\mathcal{Z}_6(\zeta) \\
&\quad + 45\mathcal{Z}_4(\zeta)/2 + 45\mathcal{Z}_2(\zeta)/2 + 45\mathcal{Z}(\zeta)/4, \\
\mathcal{R}_1(\zeta) &= \mathcal{Z}_8(\zeta) + 2\mathcal{Z}_6(\zeta) + 3\mathcal{Z}_4(\zeta) + 3\mathcal{Z}_2(\zeta) + 3\mathcal{Z}(\zeta)/2, \\
\mathcal{R}_2(\zeta) &= \mathcal{Z}_8(\zeta) + 4\mathcal{Z}_6(\zeta) + 9\mathcal{Z}_4(\zeta) + 12\mathcal{Z}_2(\zeta) + 15\mathcal{Z}(\zeta)/2, \\
\mathcal{R}_3(\zeta) &= \mathcal{Z}_4(\zeta) + 2\mathcal{Z}_2(\zeta) + 3\mathcal{Z}(\zeta)/2, \\
\mathcal{R}_4(\zeta) &= 2\mathcal{Z}_4(\zeta) + 6\mathcal{Z}_2(\zeta) + 6\mathcal{Z}(\zeta). \qquad (10)
\end{aligned}$$

The subscript of FLR and FOW indicates the order, for example, $\mathrm{FLR}_1$ means the 1-st order FLR effect. One should note that the $\mathrm{FLR}_3$ term is neglected since it contributes only to the real part of GAM frequency by introducing modification on the order of $\mathcal{O}(\lambda^4)$. On the other hand, due to the Landau damping, the $\mathrm{FOW}_3$ term affects the damping rate dramatically. Asymptotic expanding of $\mathcal{Z}(\zeta/3)$ requires actually $q \gg 3$ for self-consistency. It is naturally to ask that the damping rate (8) is precise or not when $q$ is not so much large. Fortunately, $\chi_2$ is much less than unit when $q$ is small. This explains why there is a slight discrepancy between the analytic curve and the numerical curve for $q \in (3, 4)$ in Fig. 1.

In summary, we analytically solved the gyro-kinetic equation containing the FOW effects to the 3-rd order. The collisionless damping rate of GAM is presented with a transparent and concise expression. Our result is of sufficient accuracy for the case of $\tau \ll 1$ and $q > 2$. In this case, the analytical damping rate agrees well with TEMPEST simulation result but has a slight discrepancy with the COGENT simulation. Meanwhile, our numerical result is found to agree well with the simulations. The analytical damping rate presented here can provide a quick and convenient estimation to the GAM decay applying to relative tokamak experiments.

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