SANDIA REPORT

SAND2019-10445
Unclassified Unlimited Release
Printed September, 2019

Evidence Theory Representations for Loss of Assured Safety in Weak Link/Strong Link Systems

Jon C. Helton, Dusty M. Brooks, John L. Darby

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

Issued by Sandia National Laboratories, operated for the United States Department of Energy by National Technology and Engineering Solutions of Sandia, LLC.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone: (865) 576-8401
Facsimile: (865) 576-5728
E-Mail: reports@osti.gov
Online ordering: http://www.osti.gov/scitech

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5301 Shawnee Rd
Alexandria, VA 22312

Telephone: (800) 553-6847
Facsimile: (703) 605-6900
E-Mail: orders@ntis.gov
Online order: https://classic.ntis.gov/help/order-methods/

SAND2019-10445
Printed September, 2019
Unclassified Unlimited Release

Evidence Theory Representations for Loss of Assured Safety in Weak Link/Strong Link Systems

Jon C. Helton
Dept. of Mathematics and Statistics, Arizona State University
Dusty M. Brooks
Structural and Thermal Analysis Dept., Sandia National Laboratories
John L. Darby
Weapon Surety Engineering II Dept., Sandia National Laboratories

Sandia National Laboratories
P. O. Box 5800
Albuquerque, New Mexico 87185-MS-0748

Abstract

The following topics are considered in this presentation: (i) Overview of evidence theory, (ii) Representation of loss of assured safety (LOAS) with evidence theory for a 1 SL, 1 WL system, (iii) Description of 2 SLs and 1 WL used for illustration, (iv) Plausibility and belief for LOAS and associated sampling-based verification calculations for a 2 SL, 1 WL system, (iv) Plausibility and belief for margins associated with LOAS for a 2 SL, 1 WL system, (v) Plausibility and belief for LOAS for a 2 SL, 2 WL system, (vi) Incorporation of evidence spaces for link temperature curves into LOAS calculations, (vii) Plausibility and belief for LOAS for WL/SL systems with SL subsystems, and (viii) Sampling-based procedures for the estimation of plausibility and belief.
.

ACKNOWLEDGMENTS

The authors appreciate the support of the Surety Analysis Group at Sandia National Laboratories for funding this effort. The peer review by Benjamin Schroeder, Sandia, is also much appreciated.

Table of Contents

FIGURES

TABLES

NOMENCLATURE

Abbreviation	Definition
BPA	Basic probability assignment
CBF	Cumulative belief function
CCBF	Complementary cumulative belief function
CCDF	Complementary cumulative distribution function
CCPF	Complementary cumulative plausibility function
CDF	Cumulative distribution function
CPF	Cumulative plausibility function
LOAS	Loss of assured safety
SL	Strong link
WL	Weak link

1. Introduction

As summarized in Ref. [1], the need for an appropriate representation of uncertainty as part of an analysis that supports an important decision is almost universally recognized [2-11]. Traditionally, probability theory has provided the language and mathematical structure for the representation of uncertainty [12-19]. More recently, other languages and mathematical structures for the representation of uncertainty have been introduced, including evidence theory [20-29], possibility theory [30-36], and interval analysis [37-42]. A number of comparative discussions of different approaches to the representation of uncertainty are available [1; 43-49].

The uncertainty to be characterized in the analysis of a complex system is often divided into aleatory uncertainty and epistemic uncertainty, with aleatory uncertainty arising from an inherent randomness in the future performance of the system and epistemic uncertainty arising from a lack of knowledge about the appropriate value to use for an input to the analysis that has a fixed but poorly known value [11; 18; 19; 50-58]. Traditionally, probability has been used to characterize both aleatory uncertainty and epistemic uncertainty in analyses for complex systems (e.g., as in the NUREG-1150 nuclear reactor probabilistic risk assessments [59-61] and the performance assessments for the Waste Isolation Pilot Plant [62] and Yucca Mountain [63] radioactive waste disposal facilities).

However, there is a growing recognition that the use of probability to represent epistemic uncertainty can lead to a characterization of epistemic uncertainty that implies a greater level of knowledge about the uncertainty being represented than is really the case. This concern arises because a probability distribution characterizing epistemic uncertainty defined on an interval $[a, b]$ implies that a probability is also known for every subinterval $[u, v]$ of $[a, b]$ no matter how small the interval $[u, v]$ is. As an example, if available information only indicates that the correct value for the quantity under consideration is equally likely to be in the intervals $[a, c]$ and $[c, b]$ for a known value c, then assigning a probability to every subinterval $[u, v]$ of $[a, b]$ is an extreme over representation of what is actually known.

Different from the assignment of a probability distribution to the indicated interval $[a, b]$, evidence theory provides a mathematical structure that retains the information that $[a, c]$ and $[c, b]$ are equally likely to contain the appropriate value for the quantity under consideration without the introduction of any additional resolution with respect to where this value is potentially located within $[a, b]$. For this reason, evidence theory is becoming a popular alternative to probability theory for the representation of epistemic uncertainty when limited information is available for the characterization of where the correct value for an epistemically uncertain quantity is potentially located.

The purpose of the following presentation is to introduce and illustrate the use of evidence theory in representing the epistemic uncertainty present in the results of analyses of the failure of weak link (WL)/strong link (SL) systems.

Weak link (WL)/strong link (SL) systems are important parts of the overall operational design of high-consequence systems [64-69]. In such designs, the SL system is very robust and is intended to permit operation of the entire system under, and only under, intended conditions (e.g., by transmitting a command to activate the system). In contrast, the WL system is intended to fail in a predictable and irreversible manner under accident conditions (e.g., in the event of a fire) and render the entire system inoperable before an accidental operation of the SL system. Given an accident, failure of the WL system to deactivate the entire system before the SL system fails (i.e., degrades into a condition that could allow an accidental operation of the entire system) is referred to as loss of assured safety (LOAS). The descriptor LOAS is used because failure of the WL system places the entire system in an inoperable condition while failure of the SL system, although undesirable, does not necessarily result in an unintended operation of the entire system. Thus, safety is assured by failure of the WL system.

The following topics are considered in this presentation: (i) Overview of evidence theory (Sect. 2), (ii) Representation of loss of assured safety with evidence theory for a 1 SL, 1 WL system (Sect. 3), (iii) Description of 2 SLs and 2 WLs used for illustration (Sect. 4), (iv) Plausibility and belief for LOAS and associated sampling-based verification of LOAS calculations for a 2 SL, 1 WL system (Sects. 5 and 6), (iv) Plausibility and belief for margins associated with LOAS for a 2 SL, 1 WL system (Sects. 7-13), (v) Plausibility and belief for LOAS for a 2 SL, 2 WL system (Sect. 14), (vi) Incorporation of evidence spaces for link temperature curves into LOAS calculations, (Sect. 15), (vii) Plausibility and belief for LOAS for WL/SL systems with SL subsystems (Sect. 16), and (viii) Sampling-based procedures for the estimation of plausibility and belief (Sect. 17).

2. Evidence Theory

This section provides an introduction to evidence theory. The following areas related to evidence theory are addressed: (i) definition of an evidence space (Sect. 2.1), (ii) definition of belief and plausibility associated with an evidence space (Sect. 2.2), (iii) cumulative and complementary cumulative summaries for belief and plausibility (Sect. 2.3), (iv) functions defined on evidence spaces (Sect. 2.4), and (v) product evidence spaces (Sect. 2.5). With respect to terminology, evidence theory is sometimes referred to as Dempster-Shafer theory in recognition of the work of A.P. Dempster and G. Shafer in the early development of what is now generally referred to as evidence theory [20-23].

2.1 Evidence Space Definition

Evidence theory and probability theory are actually closely related. As will be described, an evidence theory representation for uncertainty corresponds to the use of an incompletely defined probabilistic representation for uncertainty. For this reason, a natural starting point in a discussion of evidence theory is an explanation of the relationship between (i) evidence theory representations for uncertainty and (ii) the better-known probability theory representations for uncertainty. As now described, a probability theory representation for uncertainty is formally based on a probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$, and an evidence theory representation for uncertainty is formally based on an evidence space $(\mathcal{X}_E, \mathbb{X}_E, m_{EX})$. As will become apparent, the components of a probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ and the components of an evidence space $(\mathcal{X}_E, \mathbb{X}_E, m_{EX})$ have many things in common.

As indicated by the notation $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$, the formal definition of a probability space involves three components:

— A set $\mathcal{X}_P$ that contains everything that could potentially occur in the particular "universe" under consideration,

— A set $\mathbb{X}_P$ of subsets of $\mathcal{X}_P$ with the properties that (i) if $\mathcal{E} \in \mathbb{X}_P$, then $\mathcal{E}^c \in \mathbb{X}_P$, where $\mathcal{E}^c$ denotes the complement of $\mathcal{E}$, and (ii) if $\{\mathcal{E}_i\}$ is a countable collection of elements of $\mathbb{X}_P$, then $\cup_i \mathcal{E}_i \in \mathbb{X}_P$ and $\cap_i \mathcal{E}_i \in \mathbb{X}_P$,

— A function m_{PX} defined for elements of $\mathbb{X}_P$ with the properties that (i) $m_{PX}(\mathcal{X}_P) = 1.0$, (ii) if $\mathcal{E} \in \mathbb{X}_P$, then $0 \le m_{PX}(\mathcal{E}) \le 1.0$, and (iii) if $\{\mathcal{E}_i\}$ is a countable collection of disjoint elements of $\mathbb{X}_P$, then $m_{PX}(\cup_i \mathcal{E}_i) = \sum_i m_{PX}(\mathcal{E}_i)$.

With respect to terminology, (i) the set $\mathcal{X}_P$ is called the sample space or universal set, (ii) the elements of $\mathcal{X}_P$ are called elementary events, (iii) the elements of $\mathbb{X}_P$ are called events, and (iv) the function m_{PX} is called a probability measure and defines the probability $m_{PX}(\mathcal{E})$ for each element $\mathcal{E}$ of $\mathbb{X}_P$. In computational implementation, $m_{PX}(\mathcal{E})$ is usually replaced by a density function *d* with the property that

$$m_P(\mathcal{E}) = \int_{\mathcal{E}} d(e)\mathrm{d}e. \tag{2.1}$$

An important take away point here is that probability is defined for sets (i.e., for subsets of the associated sample space $\mathcal{X}_P$ contained in $\mathbb{X}_P$). It is convenient to think of $\mathbb{X}_P$ as containing all possible subsets of $\mathcal{X}_P$; however, for certain theoretical reasons, the subsets of $\mathcal{X}_P$ contained in $\mathbb{X}_P$ must be restricted as indicated above.

As indicated by the notation $(\mathcal{X}_E, \mathbb{X}_E, m_{EX})$, the formal definition of an evidence space also involves three components:

– A set $\mathcal{X}_E$ that contains everything that could potentially occur in the particular "universe" under consideration,
– A countable set $\mathbb{X}_E$ of subsets of $\mathcal{X}_E$,
– A function m_{EX} defined for elements of $\mathbb{X}_E$ with the properties that (i) $m_{EX}(\mathcal{E}) > 0$ for $\mathcal{E} \in \mathbb{X}_E$, (i) $m_{EX}(\mathcal{E}) = 0$ for $\mathcal{E} \subseteq \mathcal{X}_E$ and $\mathcal{E} \notin \mathbb{X}_E$, and (iii) $\sum_{\mathcal{E} \in \mathbb{X}_E} m_{EX}(\mathcal{E}) = 1.0$.

With respect to terminology, (i) the set $\mathcal{X}_E$ is called the sample space or universal set, (ii) the elements of $\mathcal{X}_E$ are called elementary events, (iii) the elements of $\mathbb{X}_E$ are called focal elements rather than events as is the case for probability spaces, and (iv) the function m_{EX} is called a belief measure rather than a probability measure and defines the basic probability assignment (BPA) $m_{PX}(\mathcal{E})$ for each element $\mathcal{E}$ of $\mathbb{X}_E$.

As examination of the preceding definitions for a probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ and an evidence space $(\mathcal{X}_E, \mathbb{X}_E, m_{EX})$ shows, these two spaces have much in common. However, they differ significantly in the resolution at which uncertainty is represented. Specifically, a probability space represents uncertainty by the probabilities defined for the events contained in the set $\mathbb{X}_P$. For continuous distributions (e.g., uniform, triangular, normal, …), $\mathbb{X}_P$ will contain an uncountably infinite number of events (e.g., subintervals of an interval $[a,b]$ for a uniform or triangular distribution defined on $[a,b]$). In contrast, $\mathbb{X}_E$ will contain (i) at most a countable number of focal elements and (ii) only a finite number of focal elements in a typical analysis.

The role of the BPA $m_{EX}(\mathcal{E})$ for a focal element $\mathcal{E}$ of an evidence space $(\mathcal{X}_E, \mathbb{X}_E, m_{EX})$ is to define the amount of probability or credence that can be assigned to the possible occurrence or truth, as appropriate, of $\mathcal{E}$ with no additional specification of likelihood for the individual subsets of $\mathcal{E}$. Or, put another way, $m_{EX}(\mathcal{E})$ is the amount of probability assigned to the set $\mathcal{E}$ but with no specification of how this probability is spread over subsets of $\mathcal{E}$. As indicated above, the assignment of BPAs for individual focal elements is made subject to the restriction $\sum_{\mathcal{E} \in \mathbb{X}_E} m_{EX}(\mathcal{E}) = 1.0$.

The definition of focal elements and associated BPAs is illustrated with a notional evidence space $(\mathcal{T}_E, \mathbb{T}_E, m_{ET})$ for link failure temperature, with the notational use of "T" selected to be suggestive of "temperature". For this evidence space,

$$\mathcal{T}_E = \{T : 450 \le T \le 950 \text{ °C}\} = [450, 950 \text{ °C}], \tag{2.2}$$

$$\mathbb{T}_E = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\} \tag{2.3}$$

and

$$\begin{gathered} \mathcal{T}_1 = [450, 750 \text{ °C}], \mathcal{T}_2 = [550, 700 \text{ °C}], \mathcal{T}_3 = [600, 725 \text{ °C}], \\ \mathcal{T}_4 = [650, 850 \text{ °C}], \mathcal{T}_5 = [800, 950 \text{ °C}]. \end{gathered} \tag{2.4}$$

Further, the focal elements (i.e., $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$) are assumed to correspond to ranges of link failure temperatures obtained in each of five separate analyses. To numerically specify the "credibility" or "relevance" of these results for use in later analyses, a BPA needs to be assigned to each of these focal elements. If all analyses were felt to be equally credible, then the assignment

$$m_{ET}(\mathcal{T}_i) = 1/5 \text{ for } i = 1,2,3,4,5 \tag{2.5}$$

would be appropriate. However, if the results of the individual analyses were not felt to be of equal quality or relevance, then a different assignment of BPAs would be appropriate. For example, the assignments

$$m_{ET}(\mathcal{T}_1) = 1/10, m_{ET}(\mathcal{T}_2) = 1/5, m_{ET}(\mathcal{T}_3) = 2/5, m_{ET}(\mathcal{T}_4) = 1/5, m_{ET}(\mathcal{T}_5) = 1/10 \tag{2.6}$$

might be made based on an assessment of the quality or relevance of the individual analyses.

Another possibility is that the focal elements in Eq. (2.4) are the outcome of an expert review process with the temperature range associated with each focal element supplied by a different "expert". If the individual experts are felt to be equally creditable, then the assignment of BPAs as in Eq. (2.5) would be appropriate. If the individual experts were not felt to be equally creditable, then different BPAs could be assigned to the individual focal elements to incorporate the assessed credibility of the individual experts. However, ranking experts is a difficult and potentially risky undertaking.

The evidence space $(\mathcal{T}_E, \mathbb{T}_E, m_{ET})$ with the properties summarized in Fig. 2.1 as defined in Eqs. (2.2), (2.3), (2.4) and (2.6) is used in several following examples. However, an arbitrary evidence space $(\mathcal{X}_E, \mathbb{X}_E, m_{ET})$ will continue to be used in the general definitions of evidence space properties.

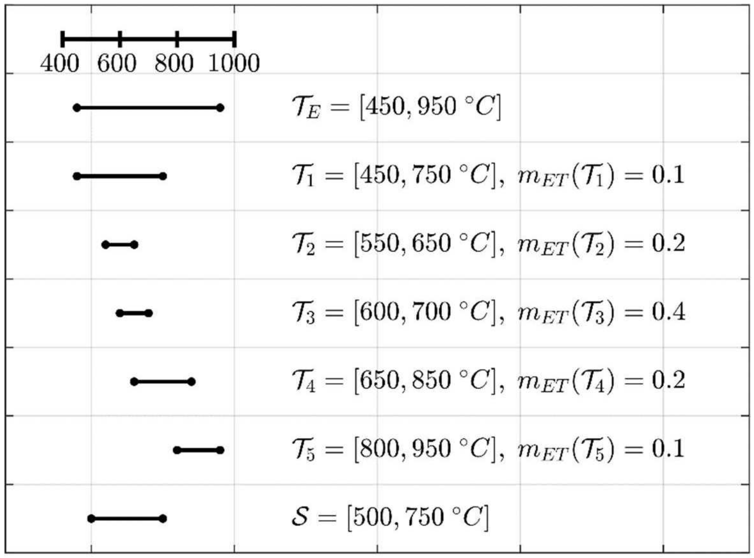


Fig. 2.1 Summary of evidence space $(\mathcal{T}_E, \mathbb{T}_E, m_{ET})$ used to illustrate the definitions of belief and plausibility (Note: In the simplified notation used in later sections, the evidence space $(\mathcal{T}_E, \mathbb{T}_E, m_{ET})$ will be represented by $(\mathcal{T}, \mathbb{T}, m_T)$).

Subsequent sections will deal primarily with evidence spaces. Therefore, to simplify notation and with limited risk of confusion, the subscript "*E*" will be omitted from the representations for evidence spaces in these sections. Also, focal elements for an evidence space will, when practical, be represented by the letter used to represent the sample space with integer subscripts used to identify the individual focal elements (e.g., the focal elements for an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ will be represented by $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_{nX}$ with *nX* corresponding to the number of focal elements).

2.2 Definition of Belief and Plausibility

The assignment of BPAs in the development of an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ is not the final step in an evidence theory representation of uncertainty. Rather, this assignment provides the basis for the determination of belief and plausibility for subsets of the sample space $\mathcal{X}$. Specifically, belief and plausibility for a subset $\mathcal{S}$ of $\mathcal{X}$ are defined by

$$Bel(\mathcal{S}) = \sum_{\mathcal{X}_i \in \mathbb{X} \text{ and } \mathcal{X}_i \subseteq \mathcal{S}} m_X(\mathcal{X}_i) \tag{2.7}$$

and

$$Pl(\mathcal{S}) = \sum_{\mathcal{X}_i \in \mathbb{X} \text{ and } \varnothing \neq \mathcal{X}_i \cap \mathcal{S}} m_X(\mathcal{X}_i), \tag{2.8}$$

respectively, with $\mathbb{X}=\{\mathcal{X}_1,\mathcal{X}_2,...\mathcal{X}_{nX}\}$. In words, $Bel(\mathcal{S})$ is the sum of the BPAs for all focal elements that are subsets of $\mathcal{S}$, and $Pl(\mathcal{S})$ is the sum of the BPAs for all focal elements that intersect $\mathcal{S}$. Given their definitions, (i) $Bel(\mathcal{S})$ provides a measure of the extent to which the available information (i.e., the focal elements and their BPAs) fully supports the proposition that $\mathcal{S}$ contains the element x of $\mathcal{X}$ of intertest (e.g., the failure temperature for a specific link in a system of WLs and SLs), and (ii) $Pl(\mathcal{S})$ provides a measure of the extent to which the available information supports the weaker proposition that $\mathcal{S}$ could (i.e., might) contain the element x of $\mathcal{X}$ of interest. Belief and plausibility are the basic uncertainty measures used to express the outcomes of evidence theory analyses.

An illustration of belief and plausibility is provided by the evidence space $(\mathcal{T},\mathbb{T},m_T)$ summarized in Fig. 2.1. Only focal elements $\mathcal{T}_2$ and $\mathcal{T}_3$ are subsets of the set $\mathcal{S}=[500,750\text{ °C}]$ indicated in Fig. 2.1, with the result that

$$Bel(\mathcal{S})=\sum_{\mathcal{T}_i\in\mathbb{T}\text{ and }\mathcal{T}_i\subseteq\mathcal{S}} m_T(\mathcal{T}_i)=m_T(\mathcal{T}_2)+m_T(\mathcal{T}_3)=0.2+0.4=0.6. \tag{2.9}$$

Focal elements $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ intersect $\mathcal{S}$ and focal element $\mathcal{T}_5$ does not intersect $\mathcal{S}$, with the result that

$$Pl(\mathcal{S})=\sum_{\mathcal{T}_i\in\mathbb{T}\text{ and }\varnothing\neq\mathcal{T}_i\cap\mathcal{S}} m_T(\mathcal{T}_i)=\sum_{i=1}^{4} m_T(\mathcal{T}_i)=0.1+0.2+0.4+0.2=0.9. \tag{2.10}$$

Similarly, only $\mathcal{T}_5$ is a subset of $\mathcal{S}^c=[450,500)\cup(750,950]$ and only $\mathcal{T}_1$, $\mathcal{T}_4$ and $\mathcal{T}_5$ intersect $\mathcal{S}^c$, with the result that

$$Bel(\mathcal{S}^c)=m_T(\mathcal{T}_5)=0.1 \tag{2.11}$$

and

$$Pl(\mathcal{S}^c)=m_T(\mathcal{T}_1)+m_T(\mathcal{T}_4)+m_T(\mathcal{T}_5)=0.1+0.2+0.1=0.4. \tag{2.12}$$

The indicated subset and intersection properties involving the $\mathcal{T}_i$'s, $\mathcal{S}$ and $\mathcal{S}^c$ can be easily seen in Fig. 2.1.

As noted earlier, evidence spaces and probability spaces have certain similar characteristics. In fact, an evidence space is actually an incompletely defined probability space. The conversion of an evidence space $(\mathcal{X},\mathbb{X},m_X)$ with focal elements $\mathcal{X}_1,\mathcal{X}_2,...,\mathcal{X}_{nX}$ into an associated probability space $(\mathcal{X}_P,\mathbb{X}_P,m_{PX})$ can be performed by defining density functions $d_i(x), i=1,2,...,nX$, on $\mathcal{X}$ with the properties that

$$\int_{\mathcal{X}_i} d_i(x)\mathrm{d}x = 1.0 \text{ and } d_i(x) = 0 \text{ for } x \notin \mathcal{X}_i. \tag{2.13}$$

Then, a density function for a probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ consistent with the original evidence space is defined by

$$d(x) = \sum_{i=1}^{n} m_X(\mathcal{X}_i) d(x_i). \tag{2.14}$$

With completion, the following relation

$$Bel(\mathcal{S}) \le prob(\mathcal{S}) = m_{PX}(\mathcal{S}) = \int_{\mathcal{S}} d(x)\mathrm{d}x \le Pl(\mathcal{S}) \tag{2.15}$$

holds for $\mathcal{S} \in \mathbb{X}_P$, which in normal situations simply means that $\mathcal{S}$ is a subset of $\mathcal{X}_P = \mathcal{X}$. The preceding is a very important property of evidence spaces. Specifically, evidence spaces have no specified uncertainty structure internal to individual focal elements. However, if a probabilistic structure is added internal to the individual focal elements of an evidence space as indicated in Eqs. (2.13) and (2.14), the result will be a probability space in which resultant set probabilities are bounded below and above by corresponding beliefs and plausibilities.

A widely-used procedure is to assume a uniform distribution over a set of values for a quantity when no distribution is known or specified. As an example, the imposition of a uniform distribution over each focal element of the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1 is illustrated. This evidence space has five focal elements (i.e., $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$). Corresponding density functions $d_i(T), i = 1, 2, ..., 5,$ that are uniform over the five focal elements and equal to zero elsewhere are defined by

$$d_i(T) = \begin{cases} 1/[750\ °\mathrm{C} - 450\ °\mathrm{C}] = 3.33 \times 10^{-3}\ °\mathrm{C}^{-1} & \text{for } i = 1 \text{ and } T \in \mathcal{T}_1 = [450, 750\ °\mathrm{C}] \\ 1/[700\ °\mathrm{C} - 550\ °\mathrm{C}] = 6.67 \times 10^{-3}\ °\mathrm{C}^{-1} & \text{for } i = 2 \text{ and } T \in \mathcal{T}_2 = [550, 700\ °\mathrm{C}] \\ 1/[725\ °\mathrm{C} - 600\ °\mathrm{C}] = 8.00 \times 10^{-3}\ °\mathrm{C}^{-1} & \text{for } i = 3 \text{ and } T \in \mathcal{T}_3 = [600, 725\ °\mathrm{C}] \\ 1/[850\ °\mathrm{C} - 650\ °\mathrm{C}] = 5.00 \times 10^{-3}\ °\mathrm{C}^{-1} & \text{for } i = 4 \text{ and } T \in \mathcal{T}_4 = [650, 850\ °\mathrm{C}] \\ 1/[950\ °\mathrm{C} - 800\ °\mathrm{C}] = 6.67 \times 10^{-3}\ °\mathrm{C}^{-1} & \text{for } i = 5 \text{ and } T \in \mathcal{T}_5 = [800, 950\ °\mathrm{C}] \end{cases} \tag{2.16}$$

and

$$d_i(T) = 0 \text{ if } T \notin \mathcal{T}_i \text{ for } i = 1, 2, 3, 4, 5. \tag{2.17}$$

As a reminder, the density function for a uniform distribution over an interval $a \le x \le b$ is $d(x) = 1/(b - a)$. Given the preceding definitions for the density functions $d_i(T), i = 1, 2, ..., 5,$ the resultant density function defining a piecewise uniform distribution over $\mathcal{T} = [450, 950\ °\mathrm{C}]$ is defined by

$$\begin{aligned} d(T) &= \sum_{i=1}^{5} m_X(\mathcal{T}_i) d_i(T) \\ &= (1/10)d_1(T) + (1/5)d_2(T) + (2/5)d_3(T) + (1/5)d_4(T) + (1/10)d_5(T) \\ &= (2.86 \times 10^{-4}\ °\mathrm{C}^{-1})\delta_1(T) + (1.33 \times 10^{-3}\ °\mathrm{C}^{-1})\delta_2(T) + (3.20 \times 10^{-3}\ °\mathrm{C}^{-1})\delta_3(T) \\ &\quad + (1.00 \times 10^{-3}\ °\mathrm{C}^{-1})\delta_4(T) + (6.67 \times 10^{-4}\ °\mathrm{C}^{-1})\delta_5(T) \end{aligned} \tag{2.18}$$

with

$$\delta_i(T) = \begin{cases} 1 \text{ if } T \in \mathcal{T}_i \\ 0 \text{ if } T \notin \mathcal{T}_i \end{cases} \tag{2.19}$$

for $i = 1, 2, ..., 5$. In turn, the probability $p(\mathcal{S})$ for the set $\mathcal{S} = [500, 750\ \mathrm{C}]$ considered in Eqs. (2.9) and (2.10) is given by

$$p(\mathcal{S}) = \int_{\mathcal{S}} d(T)\mathrm{d}T = \int_{500}^{750} d(T)\mathrm{d}T = 0.78. \tag{2.20}$$

As should be the case,

$$0.6 = Bel(\mathcal{S}) \le p(\mathcal{S}) = 0.78 \le Pl(\mathcal{S}) = 0.9 \tag{2.21}$$

with $Bel(\mathcal{S}) = 0.6$ and $Pl(\mathcal{S}) = 0.9$ determined in Eqs. (2.9) and (2.10).

The sample space $\mathcal{X}$ and focal elements for an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ are usually related by

$$\mathcal{X} = \bigcup_{\mathcal{X}_i \in \mathbb{X}} \mathcal{X}_i \tag{2.22}$$

as is the case with evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ for which

$$[450, 950\ °\mathrm{C}] = \mathcal{T} = \bigcup_{\mathcal{T}_i \in \mathbb{T}} \mathcal{T}_i = \bigcup_{i=1}^{5} \mathcal{T}_i = [450, 950\ °\mathrm{C}]. \tag{2.23}$$

However, there is no gain or loss in uncertainty information if the union of focal elements is a proper subset of $\mathcal{X}_E$ because

$$Bel(\mathcal{S}) = Pl(\mathcal{S}) = 0 \text{ for } \mathcal{S} \subseteq \mathcal{X} \text{ with } \mathcal{S} \cap \mathcal{X}_i = \varnothing \text{ for } \mathcal{X}_i \in \mathbb{X}. \tag{2.24}$$

As an example, the sample space $\mathcal{T}_E = [450, 950\ °\text{C}]$ for the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1 could be redefined as $\mathcal{T} = [400, 1000\ °\text{C}]$ with no change in the supplied uncertainty information because

$$Bel(\mathcal{S}) = Pl(\mathcal{S}) = 0 \ \text{ for } \ \mathcal{S} \subseteq [400, 450\ °\text{C}) \ \text{ or } \ \mathcal{S} \subseteq (950, 1000\ °\text{C}] \tag{2.25}$$

after this redefinition of $\mathcal{T}$.

The basic relational properties of belief and plausibility for an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ are

$$Bel(\mathcal{S}) + Pl(\mathcal{S}^c) = 1, \tag{2.26}$$

$$Bel(\mathcal{S}) + Bel(\mathcal{S}^c) \leq 1, \tag{2.27}$$

$$Pl(\mathcal{S}) + Pl(\mathcal{S}^c) \geq 1, \tag{2.28}$$

and

$$Bel(\mathcal{S}) \leq Pl(\mathcal{S}) \tag{2.29}$$

for $\mathcal{S} \subseteq \mathcal{X}$ and $\mathcal{S}^c$ denoting the complement or $\mathcal{S}$. The set $\mathcal{S}$ defined in Fig. 2.1 and the beliefs and plausibilities

$$Bel(\mathcal{S}) = 0.6, Pl(\mathcal{S}) = 0.9, Bel(\mathcal{S}^c) = 0.1, Pl(\mathcal{S}^c) = 0.4 \tag{2.30}$$

obtained in Eqs. (2.10)-(2.12) provide the following examples of the general results in Eqs. (2.26) -(2.29):

$$Bel(\mathcal{S}) + Pl(\mathcal{S}^c) = 0.6 + 0.4 = 1, \tag{2.31}$$

$$Bel(\mathcal{S}) + Bel(\mathcal{S}^c) = 0.6 + 0.1 = 0.7 \leq 1, \tag{2.32}$$

$$Pl(\mathcal{S}) + Pl(\mathcal{S}^c) = 0.9 + 0.4 = 1.3 \geq 1, \tag{2.33}$$

and

$$0.6 = Bel(\mathcal{S}) \leq Pl(\mathcal{S}) = 0.9. \tag{2.34}$$

In contrast to the relationships in Eqs. (2.26)-(2.29),

$$p(\mathcal{S}) + p(\mathcal{S}^c) = 1 \tag{2.35}$$

is the basic relational property for probability. As indicated by the relationships in Eqs. (2.26)-(2.35), belief and plausibility provide a more nuanced, but less detailed, representation of uncertainty than is provided by probability. Specifically, if the probability of a set $\mathcal{S}$ is specified, then the probability of $\mathcal{S}^c$ is also deemed to be known as indicated in Eq. (2.35). In contrast, belief $Bel(\mathcal{S})$ for a set $\mathcal{S}$ places a bound $0 \le Bel(\mathcal{S}^c) \le 1 - Bel(\mathcal{S})$ on possible values for $Bel(\mathcal{S}^c)$ as indicated in Eq. (2.27) but does not completely determine $Bel(\mathcal{S}^c)$. Similarly, plausibility $Pl(\mathcal{S})$ for a set $\mathcal{S}$ places a bound $1 - Pl(\mathcal{S}) \le Pl(\mathcal{S}^c) \le 1$ on possible values for $Pl(\mathcal{S}^c)$ but does not completely determine $Pl(\mathcal{S}^c)$ as indicated in Eq. (2.28).

2.3 Cumulative and Complementary Cumulative Summaries for Belief and Plausibility

The examples for belief and plausibility in Eqs. (2.9) and (2.10) are for a single subset of the sample space $\mathcal{T}$. For evidence spaces in which the sample space $\mathcal{X}$ is an interval $[x_{mn}, x_{mx}]$ of real numbers, plots of cumulative belief functions (CBFs), cumulative plausibility functions (CPFs), complementary cumulative belief functions (CCBFs) and complementary cumulative plausibility functions (CCPFs) provide detailed summaries of the beliefs and plausibilities associated with the evidence space. Specifically, CBFs, CPFs, CCBFs and CCPFs are defined by plots for $x_{mn} \le x \le x_{mx}$ of the points

$$\left(x, Bel([x_{mn}, x])\right) \text{ for CBFs,} \tag{2.36}$$

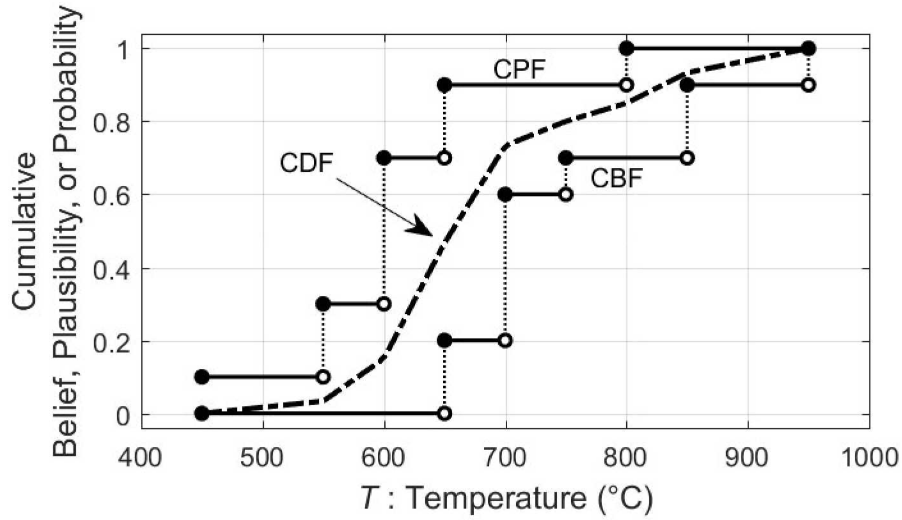
$$\left(x, Pl([x_{mn}, x])\right) \text{ for CPFs,} \tag{2.37}$$

$$\left(x, Bel((x, x_{mx}])\right) \text{ for CCBFs,} \tag{2.38}$$

and

$$\left(x, Pl((x, x_{mx}])\right) \text{ for CCPFs.} \tag{2.39}$$

As examples, the CBF, CPF, CCBF and CCPF for the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1 are (i) defined in Eqs. (2.36)-(2.39) and (ii) shown as plots in Fig. 2.2.



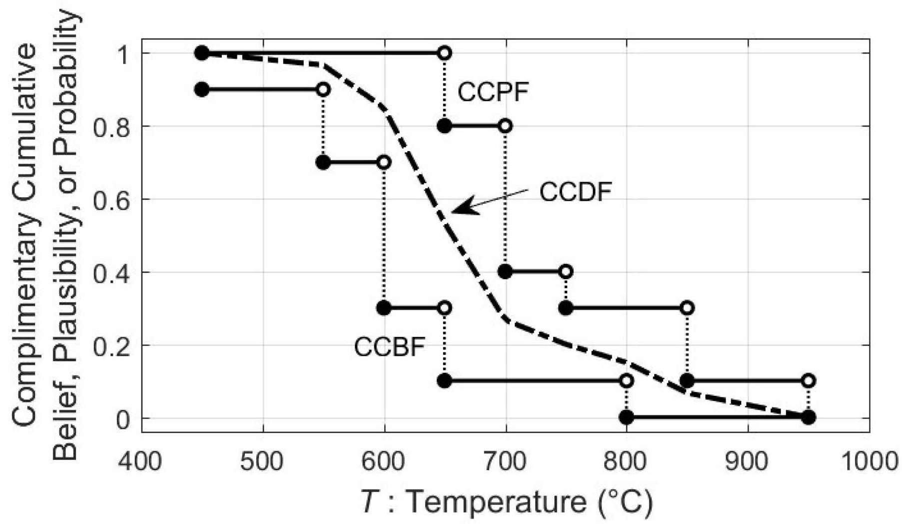


Fig. 2.2 Example CBF, CPF, CCBF and CCPF plots for the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1.

Further, the CBF, CPF, CCBF and CCPF in Fig. 2.2 are formally defined by

$$CBF(T) = Bel\{[450, T]\} \text{ for } 450 \le T \le 950$$

$$= \begin{cases} 0 \text{ for } 450 \le T < 650 \\ m_{TM}(\mathcal{T}_2) = 0.2 \text{ for } 650 \le T < 700 \\ \sum_{i=2}^{3} m_T(\mathcal{T}_i) = 0.6 \text{ for } 700 \le T < 750 \\ \sum_{i=1}^{3} m_{TM}(\mathcal{T}_i) = 0.7 \text{ for } 750 \le t < 850 \\ \sum_{i=1}^{4} m_{TM}(\mathcal{T}_i) = 0.9 \text{ for } 850 \le T < 950 \\ \sum_{i=1}^{5} m_{TM}(\mathcal{T}_i) = 1.0 \text{ for } T = 950, \end{cases} \tag{2.40}$$

$$CPF(T) = Pl\{[450, T]\} \text{ for } 450 \le T \le 950$$

$$= \begin{cases} m_{TM}(\mathcal{T}_1) = 0.1 \text{ for } 450 \le T < 550 \\ \sum_{i=1}^{2} m_T(\mathcal{T}_i) = 0.3 \text{ for } 550 \le T < 600 \\ \sum_{i=1}^{3} m_T(\mathcal{T}_i) = 0.7 \text{ for } 600 \le T < 650 \\ \sum_{i=1}^{4} m_T(\mathcal{T}_i) = 0.9 \text{ for } 650 \le t < 800 \\ \sum_{i=1}^{5} m_T(\mathcal{T}_i) = 1.0 \text{ for } 800 \le T \le 950, \end{cases} \tag{2.41}$$

$$CCBF(T) = Bel\{(T, 950]\} \text{ for } 450 \le T \le 950$$

$$= \begin{cases} \sum_{i=2}^{5} m_{TM}(\mathcal{T}_i) = 0.9 \text{ for } 450 \le T < 550 \\ \sum_{i=3}^{5} m_{TM}(\mathcal{TM}_i) = 0.7 \text{ for } 550 \le T < 600 \\ \sum_{i=5}^{5} m_{TM}(\mathcal{TM}_i) = 0.3 \text{ for } 600 \le t < 650 \\ \sum_{i=5}^{5} m_{TM}(\mathcal{TM}_i) = 0.1 \text{ for } 650 \le t < 800 \\ 0 \text{ for } 800 \le T \le 950, \end{cases} \tag{2.42}$$

and

$$CCPF(T) = Pl\{(T, 950]\} \text{ for } 450 \le T < 950$$

$$= \begin{cases} \sum_{i=1}^{5} m_{TM}(\mathcal{T}_i) = 1.0 \text{ for } 450 \le T < 650 \\ \sum_{i=1, i \ne 2}^{5} m_T(\mathcal{T}_i) = 0.8 \text{ for } 650 \le T < 700 \\ m_T(\mathcal{T}_1) + \sum_{i=4}^{5} m_T(\mathcal{T}_i) = 0.4 \text{ for } 700 \le t < 750 \\ \sum_{i=4}^{5} m_T(\mathcal{T}_i) = 0.3 \text{ for } 750 \le t < 850 \\ \sum_{i=5}^{5} m_T(\mathcal{T}_i) = 0.1 \text{ for } 850 \le T < 950. \end{cases} \tag{2.43}$$

For plotting simplicity, the included and excluded points associated with the inequalities in Eqs.(2.40)-(2.43) are not explicitly shown in Fig. 2.2.

Just as belief and plausibility bound probability for a probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ that is obtained by completing an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ as illustrated in Eq. (2.21), a similar bounding occurs for cumulative distribution functions (CDFs) and complementary cumulative distribution functions (CCDFs) that summarize a probability space obtained by completing an evidence space. Specifically, the probability space CDF will fall between the evidence space CPF and CBF, and the probability space CCDF will fall between the evidence space CCBF and CCPF. This pattern is illustrated in Fig. 2.2 for the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1 and the associated probability space defined in Eqs. (2.16)-(2.21) with

$$CDF(T) = \int_{450}^{T} d(T)\mathrm{d}T \text{ and } CCDF(T) = \int_{T}^{950} d(T)\mathrm{d}T \tag{2.44}$$

for $450\ {}^\circ\mathrm{C} \le T \le 950\ {}^\circ\mathrm{C}$ and $d(T)$ defined in Eq. (2.18).

From a computational perspective, generation of CBFs, CPFs, CCBFs and CCPFs is the same as the generation of CDFs and CCDFs for discrete probability distributions. For an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ with focal elements $\mathcal{X}_i, i = 1, 2, ..., nX$, (i) the CBF associated with $(\mathcal{X}, \mathbb{X}, m_X)$ corresponds to the discrete CDF defined by the pairs

$$\left[\max(\mathcal{X}_i), m_X(\mathcal{X}_i)\right], i = 1, 2, ..., nX, \tag{2.45}$$

(ii) the CPF associated with $(\mathcal{X}, \mathbb{X}, m_X)$ corresponds to the discrete CDF defined by the pairs

$$\left[\min(\mathcal{X}_i), m_X(\mathcal{X}_i)\right], i = 1, 2, ..., nX, \tag{2.46}$$

(iii) the CCBF associated with $(\mathcal{X}, \mathbb{X}, m_X)$ corresponds to the discrete CCDF defined by the pairs

$$\left[\min(\mathcal{X}_i), m_X(\mathcal{X}_i)\right], i = 1, 2, ..., nX, \tag{2.47}$$

and (iv) the CCPF associated with $(\mathcal{X}, \mathbb{X}, m_X)$ corresponds to the discrete CCDF defined by the pairs

$$\left[\max(\mathcal{X}_i), m_X(\mathcal{X}_i)\right], i = 1, 2, ..., nX. \tag{2.48}$$

Discrete CDFs and CCDFs of the form indicated can be generated with standard plotting tools.

The indicated relationships in Eqs. (2.45)-(2.48) are now elaborated on. Supposed an evidence space with focal elements $\mathcal{X}_i, i = 1, 2, ..., n,$ is under consideration. To facilitate the following development of representations for CBFs, CCBFs, CPFs and CCPFs, the focal elements $\mathcal{X}_i$ are reordered into the sequences

$$\underline{\mathcal{X}}_i, i = 1, 2, \ldots n, \text{ with } \underline{m}_i = m_X(\underline{\mathcal{X}}_i),\ \underline{x}_i = \min(\underline{\mathcal{X}}_i), \text{ and the } \underline{\mathcal{X}}_i \text{ ordered so that } \underline{x}_i < \underline{x}_{i+1} \tag{2.49}$$

and

$$\bar{\mathcal{X}}_i, i = 1, 2, \ldots n, \text{ with } \bar{m}_i = m_X(\bar{\mathcal{X}}_i),\ \bar{x}_i = \max(\bar{\mathcal{X}}_i) \text{ and the } \bar{\mathcal{X}}_i \text{ ordered so that } \bar{x}_i < \bar{x}_{i+1}. \tag{2.50}$$

Given the preceding, CBFs, CPFs, CCBFs and CCPFs are, in effect, equivalent to step functions corresponding to discrete probability distributions. Specifically, CBFs, CPFs, CCBFs and CCPFs are defined by the following discrete probability distributions:

$$\text{CBF}\sim\left[\bar{x}_i, m_X(\bar{\mathcal{X}}_i)\right], i = 1, 2, \ldots n, \text{ with } \left[\bar{x}_i, CBF(\bar{x}_i)\right] = \left[\bar{x}_i, \sum_{j=1}^{i} m_X(\bar{\mathcal{X}}_j)\right], \tag{2.51}$$

$$\text{CPF}\sim\left[\underline{x}_i, m_X(\underline{\mathcal{X}}_i)\right], i = 1, 2, \ldots n, \text{ with } \left[\underline{x}_i, CPF(\underline{x}_i)\right] = \left[\underline{x}_i, \sum_{j=1}^{i} m_X(\underline{\mathcal{X}}_j)\right], \tag{2.52}$$

$$\text{CCBF}\sim\left[\underline{x}_i, m_X(\underline{\mathcal{X}}_i)\right], i = 1, 2, \ldots n, \text{ with } \left[\underline{x}_i, CCBF(\underline{x}_i)\right] = \left[\underline{x}_i, \sum_{j=i+1}^{n} m_X(\underline{\mathcal{X}}_j)\right], \tag{2.53}$$

and

$$\text{CCPF}\sim\left[\bar{x}_i, m_X(\bar{\mathcal{X}}_i)\right], i = 1, 2, \ldots n, \text{ with } \left[\bar{x}_i, CCPF(\bar{x}_i)\right] = \left[\bar{x}_i, \sum_{j=i+1}^{n} m_X(\bar{\mathcal{X}}_j)\right]. \tag{2.54}$$

On a technical note, if the values for several $\underline{x}_i$ are the same, then these values must be combined into a single value by adding their corresponding values for $m_X(\underline{\mathcal{X}}_i)$. A similar requirement holds for the $\bar{x}_i$.

Eqs. (2.55)-(2.57) below provide an illustration of the determination of CBFs, CPFs, CCBFs and CCPFs as indicated in Eqs. (2.51)-(2.54):

$$
\begin{array}{ll}
\mathcal{X}_1 & \qquad\qquad\qquad \bullet\text{-----}\bullet \\
\mathcal{X}_2 & \quad \bullet\text{--------------}\bullet \\
\mathcal{X}_3 & \qquad\qquad \bullet\text{--------------}\bullet \\
\mathcal{X}_4 & \qquad \bullet\text{------}\bullet \\
\mathcal{X}_5 & \bullet\text{--------}\bullet
\end{array}
\tag{2.55}
$$

$$
\left.
\begin{array}{ll}
\underline{\mathcal{X}}_1 = \mathcal{X}_5 & \underline{x}_1 \rightarrow \bullet\text{--------}\bullet \\
\underline{\mathcal{X}}_2 = \mathcal{X}_2 & \quad \underline{x}_2 \rightarrow \bullet\text{--------------}\bullet \\
\underline{\mathcal{X}}_3 = \mathcal{X}_4 & \qquad \underline{x}_3 \rightarrow \bullet\text{------}\bullet \\
\underline{\mathcal{X}}_4 = \mathcal{X}_3 & \qquad\qquad \underline{x}_4 \rightarrow \bullet\text{--------------}\bullet \\
\underline{\mathcal{X}}_5 = \mathcal{X}_1 & \qquad\qquad\quad \underline{x}_5 \rightarrow \bullet\text{-----}\bullet
\end{array}
\right\} \Rightarrow
\begin{cases}
\left[\underline{x}_i, CPF(\underline{x}_i)\right] = \left[\underline{x}_i, \sum_{j=1}^{i} m_X(\underline{\mathcal{X}}_j)\right] \\
\left[\underline{x}_i, CCBF(\underline{x}_i)\right] = \left[\underline{x}_i, \sum_{j=i+1}^{5} m_X(\underline{\mathcal{X}}_j)\right]
\end{cases}
\tag{2.56}
$$

$$
\left.
\begin{array}{ll}
\overline{\mathcal{X}}_1 = \mathcal{X}_5 & \bullet\text{--------}\bullet \leftarrow \overline{x}_1 \\
\overline{\mathcal{X}}_2 = \mathcal{X}_4 & \qquad \bullet\text{------}\bullet \leftarrow \overline{x}_2 \\
\overline{\mathcal{X}}_3 = \mathcal{X}_2 & \quad \bullet\text{--------------}\bullet \leftarrow \overline{x}_3 \\
\overline{\mathcal{X}}_4 = \mathcal{X}_1 & \qquad\qquad\qquad \bullet\text{-----}\bullet \leftarrow \overline{x}_4 \\
\overline{\mathcal{X}}_5 = \mathcal{X}_3 & \qquad\qquad \bullet\text{--------------}\bullet \leftarrow \overline{x}_5
\end{array}
\right\} \Rightarrow
\begin{cases}
\left[\overline{x}_i, CBF(\overline{x}_i)\right] = \left[\overline{x}_i, \sum_{j=1}^{i} m_X(\overline{\mathcal{X}}_j)\right] \\
\left[\overline{x}_i, CCPF(\overline{x}_i)\right] = \left[\overline{x}_i, \sum_{j=i+1}^{5} m_X(\overline{\mathcal{X}}_j)\right].
\end{cases}
\tag{2.57}
$$

Eq. (2.55) shows the five focal elements associated with an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$. Then, Eq. (2.56) illustrates the reordering of focal elements as described in Eq. (2.49) and the definition of "probabilities" to be used in the construction of CPFs and CCBFs as indicated in Eqs. (2.52) and (2.53). Similarly, Eq. (2.57) illustrates the reordering of focal elements as described in Eq. (2.50) and the definition of "probabilities" to be used in the construction of CBFs and CCPFs as indicated in Eqs. (2.51) and (2.54). As noted after Eq. (2.54), if the values for several $\underline{x}_i$ are the same, then these values must be combined into a single value by adding their corresponding values for $m_X(\underline{\mathcal{X}}_i)$. A similar requirement holds for the $\overline{x}_i$.

The relationships defining CPFs, CCBFs, CBFs and CCPFs in Eqs. (2.56)-(2.57) are easy to implement in a DO loop structure but do not have a form that is intuitively suggestive of what is being determined. As indicated below, the core relationships in Eqs. (2.56)-(2.57) are cumulative and complementary cumulative distribution functions for discrete probability distributions defined by integrals of density functions that are linear combinations of Dirac delta functions. As described in Sect. 5.7 of Ref. [70], a Dirac delta function $\delta(x-a)$ is defined by

$$
\int_u^v \delta(x-a)\mathrm{d}x = \begin{cases} 1 & \text{for } u < a < v \\ 0 & \text{for } a < u \text{ or } v < a \end{cases} \quad \text{for any } u < v
\tag{2.58}
$$

and provides a way to symbolically enter a discontinuity into the value of an integral.

For generality in the following representations for CPFs, CCBFs, CBFs and CCPFs, the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ under consideration is assumed to have n focal elements that have been ordered as notationally indicated in Eqs. (2.56)-(2.57) after the previously indicated BPA adjustments have been made in the event that some focal elements have shared end points. In practice, this could result in different numbers lower and upper focal end points. For notational simplicity, the number of potentially reordered focal element end points is assumed to be n for both lower and upper endpoints. In addition, the CPF, CCBF, CBF and CCPF representations are determined for an interval $[x_{mn} x_{mx}]$ with $x_{mn} < \underline{x}_1 < \overline{x}_n < x_{mx}$.

For the CPF and CCBF definitions in Eq. (2.56), the underlying density function is defined by

$$d_1(x) = \sum_{j=1}^{n} m_X(\underline{\mathcal{X}}_j)\delta(x - \underline{x}_j). \tag{2.59}$$

In turn, the corresponding CPF and CCBF are formally defined by

$$\begin{aligned}
[\tilde{x}, CPF(\tilde{x})] &= \left[\tilde{x}, \int_{x_{mn}}^{\tilde{x}} d_1(\mathrm{x})\mathrm{d}x\right] \\
&= \left[\tilde{x}, \int_{x_{mn}}^{\tilde{x}} \left(\sum_{j=1}^{n} m_X(\underline{\mathcal{X}}_j)\delta(x - \underline{x}_j)\right)\mathrm{d}x\right] \\
&= \begin{cases} [\tilde{x}, 0] \text{ for } x_{mn} \le \tilde{x} < \underline{x}_1 \\ \left[\tilde{x}, \sum_{j=1}^{i} m_X(\underline{\mathcal{X}}_j)\right] \text{ for } \underline{x}_i \le \tilde{x} < \underline{x}_{i+1} \text{ and } i = 1, 2, ..., n-1 \\ [\tilde{x}, 1.0] \text{ for } \underline{x}_{n-1} \le \tilde{x} \le x_{mx} \end{cases}
\end{aligned} \tag{2.60}$$

and

$$\begin{aligned}
[\tilde{x}, CCBF(\tilde{x})] &= \left[\tilde{x}, \int_{\tilde{x}}^{x_{mx}} d_1(\mathrm{x})\mathrm{d}x\right] \\
&= \left[\tilde{x}, \int_{\tilde{x}}^{x_{mx}} \left(\sum_{j=1}^{n} m_X(\underline{\mathcal{X}}_j)\delta(x - \underline{x}_j)\right)\mathrm{d}x\right] \\
&= \begin{cases} [\tilde{x}, 1.0] \text{ for } x_{mn} \le \tilde{x} < \underline{x}_1 \\ \left[\tilde{x}, \sum_{j=i+1}^{n} m_X(\underline{\mathcal{X}}_j)\right] \text{ for } \underline{x}_i \le \tilde{x} < \underline{x}_{i+1} \text{ and } i = 1, 2, ..., n-1 \\ [\tilde{x}, 0] \text{ for } \underline{x}_n \le \tilde{x} \le x_{mx}. \end{cases}
\end{aligned} \tag{2.61}$$

For the definitions of CBFs and CCPFs in Eq. (2.57), the underlying density function is defined by

$$d_2(x) = \sum_{j=1}^{n} m_X(\bar{\mathcal{X}}_j)\delta(x - \bar{x}_j). \tag{2.62}$$

In turn, the corresponding CBF and CCPF are formally defined by

$$\begin{aligned}
\left[\tilde{x}, CBF(\tilde{x})\right] &= \left[\tilde{x}, \int_{x_{mn}}^{\tilde{x}} d_2(\mathrm{x})\mathrm{d}x\right] \\
&= \left[\tilde{x}, \int_{x_{mn}}^{\tilde{x}} \left(\sum\nolimits_{j=1}^{n} m_X(\bar{\mathcal{X}}_j)\delta(x - \bar{x}_j)\right)\mathrm{d}x\right] \\
&= \begin{cases} [\tilde{x}, 0] \text{ for } x_{mn} \le \tilde{x} < \bar{x}_1 \\ \left[\tilde{x}, \sum\nolimits_{j=1}^{i} m_X(\bar{\mathcal{X}}_j)\right] \text{ for } \bar{x}_i \le \tilde{x} < \bar{x}_{i+1} \text{ and } i = 1, 2, ..., n-1 \\ [\tilde{x}, 1.0] \text{ for } \bar{x}_n \le \tilde{x} \le x_{mn} \end{cases}
\end{aligned} \tag{2.63}$$

and

$$\begin{aligned}
\left[\tilde{x}, CCPF(\tilde{x})\right] &= \left[\tilde{x}, \int_{\tilde{x}}^{x_{mx}} d_2(\mathrm{x})\mathrm{d}x\right] \\
&= \left[\tilde{x}, \int_{\tilde{x}}^{x_{mx}} \left(\sum\nolimits_{j=1}^{n} m_X(\bar{\mathcal{X}}_j)\delta(x - \bar{x}_j)\right)\mathrm{d}x\right] \\
&= \begin{cases} [\tilde{x}, 1.0] \text{ for } x_{mn} \le \tilde{x} < \bar{x}_1 \\ \left[\tilde{x}, \sum\nolimits_{j=i+1}^{n} m_X(\bar{\mathcal{X}}_j)\right] \text{ for } \bar{x}_i \le \tilde{x} < \bar{x}_{i+1} \text{ and } i = 1, 2, ..., n-1 \\ [\tilde{x}, 0] \text{ for } \bar{x}_n \le \tilde{x} \le x_{mn}. \end{cases}
\end{aligned} \tag{2.64}$$

Two insights with respect to evidence spaces can be obtained from the development leading to Eqs. (2.59)-(2.64).

First, for a given evidence space $(\mathcal{X}, \mathbb{X}, m_X)$, probability spaces $(\mathcal{X}_{P1}, \mathbb{X}_{P1}, m_{P1})$ and $(\mathcal{X}_{P2}, \mathbb{X}_{P2}, m_{P2})$ corresponding to the density functions $d_1(x)$ and $d_2(x)$ can be defined such that (i) the CDF and CCDF for the probability space $(\mathcal{X}_{P1}, \mathbb{X}_{P1}, m_{P1})$ exactly match the CPF and CCBF for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ and (ii) the CDF and CCDF for the probability space $(\mathcal{X}_{P2}, \mathbb{X}_{P2}, m_{P2})$ exactly match the CBF and CCPF for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$.

Second, an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ with

$$\mathbb{X} = \{\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n\} \tag{2.65}$$

can be viewed as adaptation of a discrete probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ with

$$\mathcal{X}_P = \{\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n\}. \tag{2.66}$$

Given its definition, $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ may appear to be an unusual probability space in that its elementary events are sets (i.e., $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ belonging to $\mathcal{X}_P$) rather than numbers or vectors as is more commonly the case. However, probability spaces of this form are commonly the outcome of a review process intended to assess the epistemic uncertainty associated with a quantity used as an input to a complex analysis with $m_{PX}(\mathcal{X}_i)$ equal to the assessed probability that the set $\mathcal{X}_i$ contains the correct value for the quantity under consideration. The evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ derives from the probability space $(\mathcal{X}_P, \mathbb{X}_P, m_{PX})$ through the definitions

$$\mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_i, \quad \mathbb{X} = \{\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n)\}, \quad m_X(\mathcal{X}_i) = m_{PX}(\mathcal{X}_i) \text{ for } i = 1, 2, ..., n \tag{2.67}$$

and the introduction of belief and plausibility to measure the implications of (i) the probabilities $m_X(\mathcal{X}_i) = m_{PX}(\mathcal{X}_i)$ assigned to the individual sets $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ and (ii) the extent to which these sets intersect subsets of $\mathcal{X}$ that are important to the analysis being performed.

2.4 Functions Defined on Evidence Spaces

A common and important analysis situation involves (i) an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$, (ii) a function $f(x)$ defined for $x \in \mathcal{X}$ that maps $\mathcal{X}$ into the set

$$\mathcal{Y} = \{y : y = f(x) \text{ for } x \in \mathcal{X}\}, \tag{2.68}$$

(iii) a subset $\mathcal{S}$ of $\mathcal{Y}$ of particular interest, and (iv) a desire to know the belief $Bel(\mathcal{S})$ and plausibility $Pl(\mathcal{S})$ for $\mathcal{S}$ that derives from the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ and the function $f(x)$. There are two approaches to obtaining $Bel(\mathcal{S})$ and $Pl(\mathcal{S})$, with both approaches producing the same value for $Bel(\mathcal{S})$ and $Pl(\mathcal{S})$. Both approaches are based on the assumption that the belief and plausibility for a set of function evaluations should be the same as the belief and plausibility for the subset of the function's domain that results in these evaluations.

The first approach (i.e., Approach 1) involves using the function $f(x)$ to map $(\mathcal{X}, \mathbb{X}, m_X)$ into a new evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$ with each focal element $\mathcal{Y}_i$ and associated BPA $m_Y(\mathcal{Y}_i)$ defined by

$$\mathcal{Y}_i = f(\mathcal{X}_i) = \{y : y = f(x) \text{ for } x \in \mathcal{X}_i\} \text{ with } m_Y(\mathcal{Y}_i) = m_X(\mathcal{X}_i) \tag{2.69}$$

for $\mathbb{X} = \{\mathcal{X}_i, i = 1, 2, ..., nX\}$. Once the new evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$ is defined, $Bel(\mathcal{S})$ and $Pl(\mathcal{S})$ can be determined as shown in Eqs. (2.7) and (2.8).

The second approach (i.e., Approach 2) involves using the inverse of the function $f(x)$ to map $\mathcal{S}$ into the subset $\mathcal{X}_S$ of $\mathcal{X}$ defined by

$$\mathcal{X}_S = \{x : x \in \mathcal{X} \text{ and } f(x) \in \mathcal{S}\} = \{x : x \in f^{-1}(\mathcal{S})\}. \tag{2.70}$$

Then, $Bel(\mathcal{S})$ and $Pl(\mathcal{S})$ are defined by

$$Bel(\mathcal{S}) = Bel(\mathcal{X}_S) \text{ and } Pl(\mathcal{S}) = Pl(\mathcal{X}_S) \tag{2.71}$$

with $Bel(\mathcal{X}_S)$ and $Pl(\mathcal{X}_S)$ defined with respect to the original evidence space $(\mathcal{X}, \mathbb{X}, m_X)$.

In the following, Approaches 1 and 2 are illustrated with (i) the evidence space $(\mathcal{T}_E, \mathbb{T}_E, m_{TE})$ for link failure temperatures summarized in Fig. 2.1 and (ii) a notional link with time-dependent temperature defined by

$$T(t) = \frac{T_0 T_\infty}{T_0 + (T_\infty - T_0)\exp(-rt)} \tag{2.72}$$

with $0 \le t \le 200$ min, $T(0) = T_0 = 225$ °C, $T_\infty = 900$ °C, and $r = 0.022$ (see Fig. 2.3). The function $f(T)$ is defined by

$$\begin{aligned} f(T) &= \text{time } t \text{ at which link fails for failure temperature } T \\ &= \begin{cases} (-1/r)\ln[T_0(T_\infty - T)/T(T_\infty - T_0)] & \text{for } 450 \le T \le 900 \text{ °C} \\ t_\infty & \text{for } 900 \text{ °C} < T, \end{cases} \end{aligned} \tag{2.73}$$

with the symbolic term t_∞ introduced as a way to record that the link does not fail for $900 \text{ °C} < T$ (see Fig. 2.3).

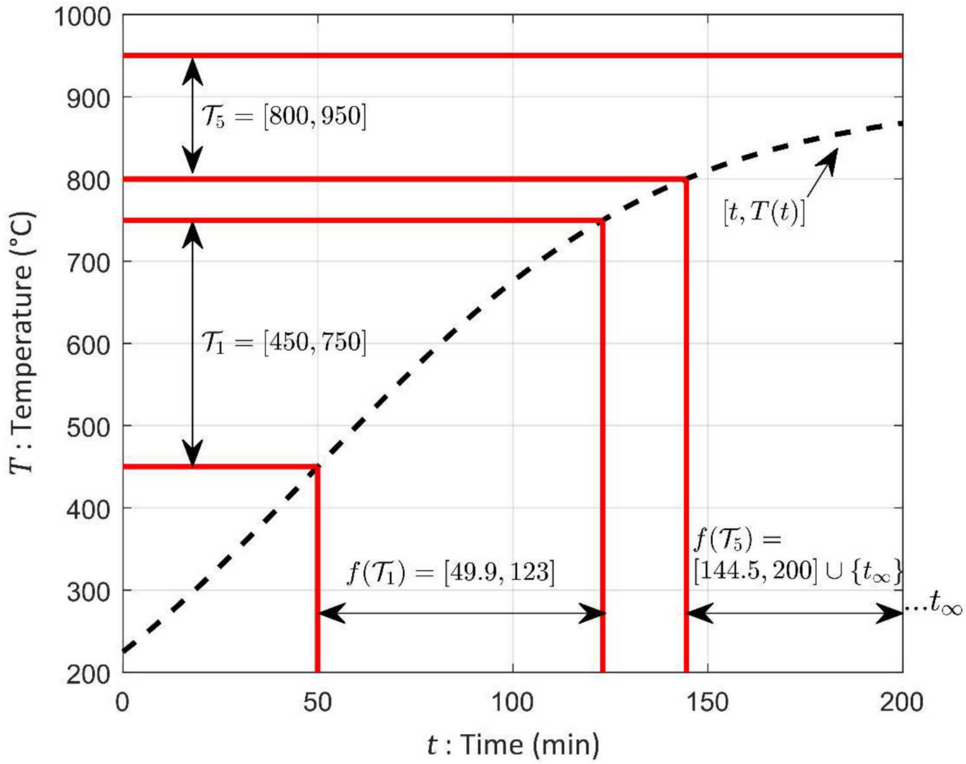


Fig. 2.3 Components of example used to illustrate the determination of belief and plausibility for subsets of the range of a function: (i) time-dependent link temperature $T(t)$ defined in Eq. (2.72) and (ii) function $f(T)$ defined in Eq. (2.73) mapping link temperature to link failure time.

For Approach 1, a new evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time with "TM" used as a mnemonic for time is obtained from (i) the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ and (ii) the function $f(T)$ with resultant values of

$$\mathcal{TM} = f(\mathcal{T}) = \{t : t = f(T) \text{ for } T \in \mathcal{T}\} = \{t : 18 \le t \le 110\} \cup \{t_\infty\}, \tag{2.74}$$

$$\mathbb{TM} = \{\mathcal{TM}_1, \mathcal{TM}_2, \mathcal{TM}_3, \mathcal{TM}_4, \mathcal{TM}_5\}, \tag{2.75}$$

and

$$\mathcal{TM}_i = \begin{cases} \mathcal{TM}_1 = f(\mathcal{T}_1) = [49.9, 123.1] \text{ with } m_{TM}(\mathcal{TM}_1) = m_T(\mathcal{T}_1) = 1/10 \text{ for } i = 1 \\ \mathcal{TM}_2 = f(\mathcal{T}_2) = [70.5, 93.4] \text{ with } m_{TM}(\mathcal{TM}_2) = m_T(\mathcal{T}_2) = 1/5 \text{ for } i = 2 \\ \mathcal{TM}_3 = f(\mathcal{T}_3) = [81.4, 106.9] \text{ with } m_{TM}(\mathcal{TM}_3) = m_T(\mathcal{T}_3) = 2/5 \text{ for } i = 3 \\ \mathcal{TM}_4 = f(\mathcal{T}_4) = [93.4, 178.7] \text{ with } m_{TM}(\mathcal{TM}_4) = m_T(\mathcal{T}_4) = 1/5 \text{ for } i = 4 \\ \mathcal{TM}_5 = f(\mathcal{T}_5) = [144.5, 200] \cup \{t_\infty\} \text{ with } m_{TM}(\mathcal{TM}_5) = m_T(\mathcal{T}_5) = 1/10 \text{ for } i = 5. \end{cases} \tag{2.76}$$

As illustrated in Fig. 2.3, the focal element $\mathcal{T}_i = [T_{mn,i}, T_{mx,i}]$ of the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ is transformed into the focal element

$$\mathcal{TM}_i = [f(T_{mn,i}), f(T_{mx,i})] = [t_{mn,i} t_{mx,i}] \tag{2.77}$$

of the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for $i = 1,2,3,4$. Specifically, this transformation corresponds to drawing horizontal lines from $T_{mn,i}$ and $T_{mx,i}$ on the ordinate of Fig. 2.3 to the curve $T(t)$ and then drawing vertical lines to the abscissa to obtain $t_{mn,i}$ and $t_{mx,i}$. The construction of $\mathcal{TM}_5$ is similar but slightly more complicated due to the need to include t_∞ to account for nonfailures for $900\ °\mathrm{C} < T$.

Once the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ is constructed, belief and plausibility for subsets of $\mathcal{TM}$ can be obtained as shown in Eqs. (2.9) and (2.10). As an example, belief and plausibility for the set

$$\begin{aligned}\mathcal{FTM} &= \{t : \text{link potentially fails at time } t\}\\ &= \{t : 49.9 \le t \le 200\}\\ &= [49.9, 200]\end{aligned} \tag{2.78}$$

are used for illustration. Specifically,

$$\begin{aligned}Bel(\mathcal{FTM}) &= \sum_{\mathcal{TM}_i \in \mathbb{TM} \text{ and } \mathcal{TM}_i \subseteq \mathcal{FTM}} m_{TM}(\mathcal{TM}_i)\\ &= m_{TM}(\mathcal{TM}_1) + m_{TM}(\mathcal{TM}_2) + m_{TM}(\mathcal{TM}_3) + m_{TM}(\mathcal{TM}_4)\\ &= 0.1 + 0.2 + 0.4 + 0.2\\ &= 0.9\end{aligned} \tag{2.79}$$

results because (i) $\mathcal{TM}_1, \mathcal{TM}_2, \mathcal{TM}_3$ and $\mathcal{TM}_4$ are subsets of $\mathcal{FTM}$ and (ii) $\mathcal{TM}_5$ is not a subset of $\mathcal{FTM}$ due to the inclusion of t_∞ in $\mathcal{TM}_5$. Similarly,

$$\begin{aligned}Pl(\mathcal{FTM}) &= \sum_{\mathcal{TM}_i \in \mathbb{TM} \text{ and } \varnothing \ne \mathcal{TM}_i \cap \mathcal{FTM}} m_{TM}(\mathcal{TM})\\ &= \sum_{i=1}^{5} m_{TM}(\mathcal{TM}_i)\\ &= 0.1 + 0.2 + 0.4 + 0.2 + 0.1\\ &= 1.0\end{aligned} \tag{2.80}$$

as consequence of every focal element intersecting $\mathcal{FTM}$.

For Approach 2, the belief and plausibility of a subset $\mathcal{S}$ of $\mathcal{TM}$ are obtained by mapping $\mathcal{S}$ back to the set $\mathcal{T}$ of temperature values that resulted in failure times in $\mathcal{S}$. Specifically,

$$\mathcal{T}_S = f^{-1}(\mathcal{S}) = T(\mathcal{S}) = \{T : T = T(t) \text{ for } t \in \mathcal{S}\} \tag{2.81}$$

in consistency with the definitions of $f(T)$ and $T(t)$ in Eqs. (2.73) and (2.72) Then,

$$Bel(\mathcal{S}) = Bel(\mathcal{T}_S) \text{ and } Pl(\mathcal{S}) = Pl(\mathcal{T}_S). \tag{2.82}$$

The set $\mathcal{FTM} = [49.9, 200]$ of possible failure times is again used as an example and results in the set

$$\mathcal{T}_{FTM} = f^{-1}(\mathcal{FTM}) = T(\mathcal{FTM}) = \{T : T = T(t) \text{ for } t \in \mathcal{FTM}\} = [450, 900\ °\text{C}]. \tag{2.83}$$

In turn,

$$\begin{aligned} Bel(\mathcal{FTM}) &= Bel(\mathcal{T}_{FTM}) \\ &= \sum_{\mathcal{T}_i \in \mathbb{T} \text{ and } \mathcal{T}_i \subseteq \mathcal{T}_{FTM}} m_T(\mathcal{T}_i) \\ &= m_T(\mathcal{T}_1) + m_T(\mathcal{T}_2) + m_T(\mathcal{T}_3) + m_T(\mathcal{T}_4) \\ &= 0.1 + 0.2 + 0.4 + 0.2 \\ &= 0.9 \end{aligned} \tag{2.84}$$

because (i) $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$ are subsets of $\mathcal{T}_{FTM} = [450, 900\ °\text{C}]$ and (ii) $\mathcal{T}_5 = [800, 950\ °\text{C}]$ is not a subset of $\mathcal{FT}$. Similarly,

$$\begin{aligned} Pl(\mathcal{FTM}) &= Pl(\mathcal{T}_{FTM}) \\ &= \sum_{\mathcal{T}_i \in \mathbb{T} \text{ and } \varnothing \neq \mathcal{T}_i \cap \mathcal{T}_{FTM}} m_T(\mathcal{T}_i) \\ &= \sum_{i=1}^{5} m_T(\mathcal{T}_i) \\ &= 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\ &= 1.0 \end{aligned} \tag{2.85}$$

as consequence of every focal element intersecting $\mathcal{T}_{FTM}$.

Both approaches when implemented correctly produce the same values for belief and plausibility as illustrated for $Bel(\mathcal{FTM})$ and $Pl(\mathcal{FTM})$. Which approach is easiest to use in practice can depend on the properties of individual problems.

Once either Approach 1 or Approach 2 has been used to obtain beliefs and plausibilities for subsets of the set $\mathcal{Y}$ indicated in Eq. (2.68), the CBF, CPF, CCBF and CCPF for the function evaluations contained in $\mathcal{Y}$ can be defined as described in Sect. 2.3. As an example, the CBF, CPF, CCBF and CCPF for the link failure times contained in the set $\mathcal{FTM}$ defined in Eq. (2.78) are shown in Fig. 2.4 and derive from the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1 and the function $T(t)$ defined in Eq. (2.72) and illustrated in Fig. 2.3.

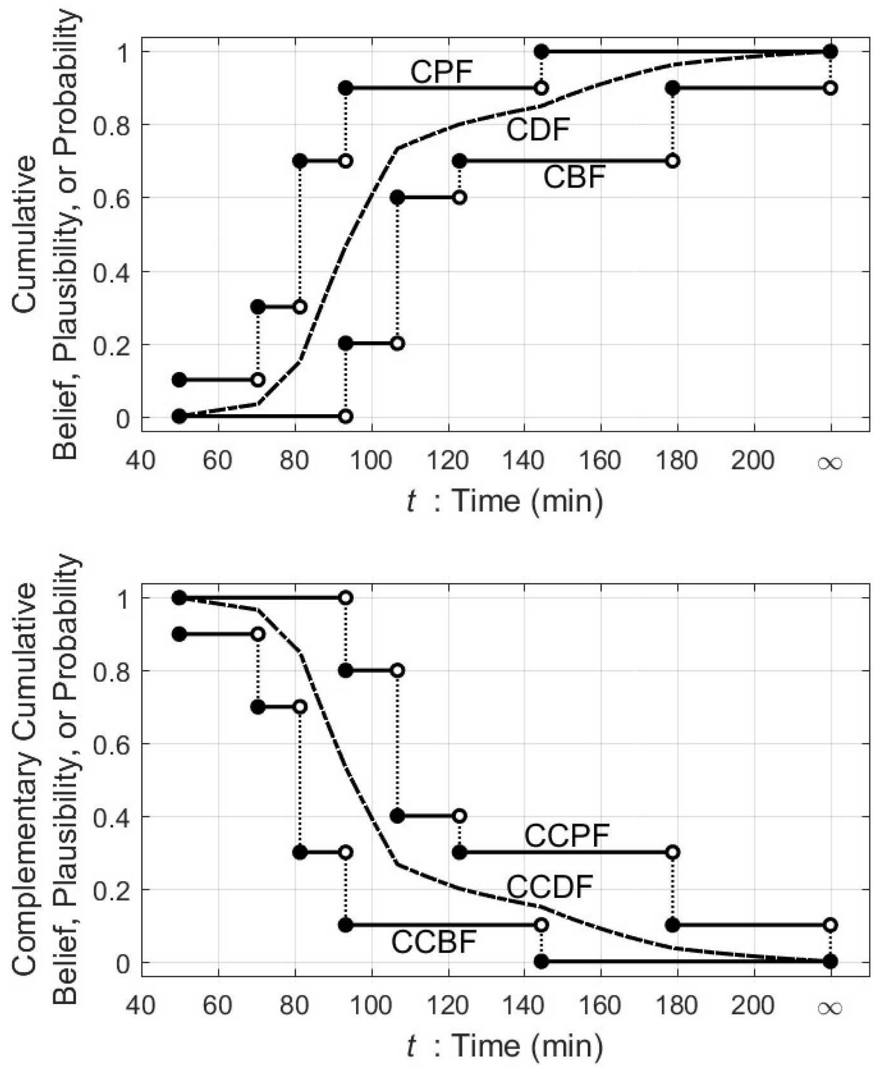


Fig. 2.4 Illustration of the CBF, CPF, CCBF and CCPF for the link failure times contained in the set $\mathcal{FTM}$ defined in Eq. (2.78), with the associated beliefs and plausibilities deriving from the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ summarized in Fig. 2.1 and the function $T(t)$ defined in Eq. (2.72) and illustrated in Fig. 2.3.

Further, the CBF, CPF, CCBF and CCPF in Fig. 2.4 are formally defined by

$$CBF(t) = Bel\{[49.9, t]\} \text{ for } 49.9 \le t \le 200$$

$$= \begin{cases} 0 \text{ for } 49.9 \le t < 93.4 \\ m_{TM}(\mathcal{TM}_2) = 0.2 \text{ for } 93.4 \le t < 106.9 \\ \sum_{i=2}^{3} m_{TM}(\mathcal{TM}_i) = 0.6 \text{ for } 106.9 \le t < 123.1 \\ \sum_{i=1}^{3} m_{TM}(\mathcal{TM}_i) = 0.7 \text{ for } 123.1 \le t < 178.7 \\ \sum_{i=1}^{4} m_{TM}(\mathcal{TM}_i) = 0.9 \text{ for } 178.7 \le t \le 200, \end{cases} \tag{2.86}$$

$$CPF(t) = Pl\{[49.9, t]\} \text{ for } 49.9 \le t \le 200$$

$$= \begin{cases} m_{TM}(\mathcal{TM}_1) = 0.1 \text{ for } 49.9 \le t < 70.5 \\ \sum_{i=1}^{2} m_{TM}(\mathcal{TM}_i) = 0.3 \text{ for } 70.5 \le t < 81.4 \\ \sum_{i=1}^{3} m_{TM}(\mathcal{TM}_i) = 0.7 \text{ for } 81.4 \le t < 93.4 \\ \sum_{i=1}^{4} m_{TM}(\mathcal{TM}_i) = 0.9 \text{ for } 93.4 \le t < 144.5 \\ \sum_{i=1}^{5} m_{TM}(\mathcal{TM}_i) = 1.0 \text{ for } 144.5 \le t \le 200, \end{cases} \tag{2.87}$$

$$CCBF(t) = Bel\{(t, 200]\} \text{ for } 49.9 \le t \le 200$$

$$= \begin{cases} \sum_{i=2}^{4} m_{TM}(\mathcal{TM}_i) = 0.8 \text{ for } 49.9 \le t < 70.5 \\ \sum_{i=3}^{4} m_{TM}(\mathcal{TM}_i) = 0.6 \text{ for } 70.5 \le t < 81.4 \\ \sum_{i=4}^{4} m_{TM}(\mathcal{TM}_i) = 0.2 \text{ for } 81.4 \le t < 144.5 \\ 0 \text{ for } 144.5 \le t \le 200 \end{cases} \tag{2.88}$$

and

$$CCPF(t) = Pl\{(t, 200]\} \text{ for } 49.9 \le t \le 200$$

$$= \begin{cases} \sum_{i=1}^{5} m_{TM}(\mathcal{TM}_i) = 1.0 \text{ for } 49.9 \le t < 93.4 \\ \sum_{i=1, i \ne 2}^{5} m_{TM}(\mathcal{TM}_i) = 0.8 \text{ for } 93.4 \le t < 106.9 \\ m_{TM}(\mathcal{TM}_1) + \sum_{i=4}^{5} m_{TM}(\mathcal{TM}_i) = 0.4 \text{ for } 106.9 \le t < 123.1 \\ \sum_{i=4}^{5} m_{TM}(\mathcal{TM}_i) = 0.3 \text{ for } 123.1 \le t < 178.7 \\ m_{TM}(\mathcal{TM}_5) = 0.1 \text{ for } 178.7 \le t \le 200. \end{cases} \tag{2.89}$$

The illustrations in Eqs. (2.86)-(2.89) stop at $t = 200$ min because (i) the example calculation ends at $t = 200$ min and (ii) $t_\infty = \infty$ is a place holder indicating that link failure did not occur. The included and excluded points associated with the inequalities in Eqs. (2.86)-(2.89) are indicated in Fig. 2.4 by solid and open circles.

As described in conjunction with Eqs. (2.13)-(2.15), a probability space $(\mathcal{X}_P, \mathbb{X}_P, m_P)$ consistent with an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ can be obtained by defining the density function $d(x)$ for the probability space in a manner that incorporates the focal element BPAs associated with the evidence space. When this is done and a real-valued function $f(x)$ is defined for $x \in \mathcal{X}$ with the resultant set $\mathcal{Y}$ of function evaluations, then (i) the CDF associated with $f(x)$ falls between the CPF and CBF associated with $f(x)$ and (ii) the CCDF associated with $f(x)$ falls between the CCBF and CCPF associated with $f(x)$. Specifically, if $y \in \mathcal{Y} = [y_{mn}, y_{mx}]$, then

$$CDF(y) = \int_{\mathcal{X}} \underline{\delta}_y[f(x)]d(x)\mathrm{d}x \text{ with } \underline{\delta}_y[f(x)] = \begin{cases} 1 \text{ if } f(x) \le y \\ 0 \text{ otherwise,} \end{cases} \tag{2.90}$$

$$CCDF(y) = \int_{\mathcal{X}_E} \overline{\delta}_y[f(x)]d(x)\mathrm{d}x \text{ with } \overline{\delta}_y[f(x)] = \begin{cases} 1 \text{ if } y < f(x) \\ 0 \text{ otherwise,} \end{cases} \tag{2.91}$$

$$Bel([y_{mn}, y]) \le CDF(y) = \int_{\mathcal{X}} \underline{\delta}_y[f(x)]d(x)\mathrm{d}x \le Pl([y_{mn}, y]) \tag{2.92}$$

and

$$Bel((y, y_{mx}]) \le CCDF(y) = \int_{\mathcal{X}} \overline{\delta}_y[f(x)]d(x)\mathrm{d}x \le Pl((y, y_{mx}]). \tag{2.93}$$

The relations in Eqs. (2.90)-(2.93) also hold if $\mathcal{Y} = [y_{mn}, y_{mx}]$ is a subset of $f(\mathcal{X})$. For computational implementation, *CDF*(y) and *CCDF*(y) can be approximated by

$$CDF(y) \cong \sum_{r=1}^{nR} \underline{\delta}_y[f(x_r)]/nR \text{ and } CDF(y) \cong \sum_{r=1}^{nR} \overline{\delta}_y[f(x_r)]/nR \tag{2.94}$$

where $x_1, x_2, ..., x_{nR}$ is a random sample from $\mathcal{X}$ generated in consistency with the density function $d(x)$.

As an example, the CDF and CCDF for the link failure times contained in the set $\mathcal{FTM}$ defined in Eq. (2.78) are shown in Fig. 2.4 in addition to the corresponding CBF, CPF, CCBF and CCPF. The indicated CDF and CCDF are (i) defined with the density function $d(T)$ in Eq. (2.18) to produce a probability space consistent with the evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ and (ii) approximated as indicated in Eq. (2.94) with a random sample of size $nR = 10^7$. As should be the case, the resultant CDF falls between the corresponding CBF and CPF, and the resultant CCDF falls between the corresponding CCBF and CCPF.

2.5 Maximum Time for Link Failure

For a failure temperature focal element $\mathcal{T}_i$ and corresponding failure time focal element $\mathcal{TM}_i$, the maximum time $\bar{t}_{Fi}$ at which link failure actually occurs may not be the same as the maximum time $\bar{t}_i$ contained in $\mathcal{TM}_i$ due to the possible inclusion of $t_\infty = \infty$ in $\mathcal{TM}_i$ to indicate that link failure did not occur for one or more of the link failure times in $\mathcal{T}_i$. For example, this is the case for focal elements $\mathcal{T}_5$ and $\mathcal{TM}_5$ in Fig. 2.3, with (i) with the maximum link failure time contained in $\mathcal{TM}_5$ defined by $\bar{t}_{F5} = 200$ min and (ii) the maximum time contained in $\mathcal{TM}_5$ defined by $\bar{t}_i = \infty$. The indicator $\bar{t}_i = \infty$ is included in $\mathcal{TM}_5$ to signify that some of the failure temperatures contained in $\mathcal{T}_5$ did not result in link failure.

If (i) the link temperature function $T(t)$ is continuous and increasing on $[t_{mn}, t_{mx}]$, (ii) $\mathcal{T} = \{T : \underline{T} \le T \le \bar{T}\}$ with $\underline{T} < \bar{T}$ is a focal element for link failure temperature, and (iii) $T(t_{mn}) < \underline{T} < T(t_{mx})$, then the maximum link failure temperature $\bar{t}_F$ associated with $T(t)$ and $\mathcal{T}$ is defined by

$$\bar{t}_F = \begin{cases} T^{-1}(\bar{T}) & \text{for } \bar{T} \le T(t_{mx}) \\ t_{mx} & \text{for } T(t_{mx}) \le \bar{T}. \end{cases} \tag{2.95}$$

Without the requirement $\underline{T} \le T(t_{mx})$, there is no link failure and the focal element for link failure time contains only the indicator time $t_\infty = \infty$ (i.e., $\mathcal{TM} = \{t_\infty\} = \{\infty\}$). Further, if $\underline{T} = T(t_{mx})$, then the corresponding focal element for link failure time is $\mathcal{TM} = \{t_{mx}, \infty\}$.

As an example, the link failure time focal elements $\mathcal{T}_i, i = 1, 2, ..., 5$, defined in Fig. 2.1 and the functions $T(t)$ and $f(T) = T^{-1}(T)$ defined in Eqs. (2.72) and (2.73) result in the following values for $\bar{t}_{Fi}, i = 1, 2, ..., 5$:

$$\bar{t}_{Fi} = \begin{cases} \bar{t}_{F1} = T^{-1}(750) = f(750) = 123.1 & \text{for } 750 = \bar{T}_1 < 868.0 = T(t_{mx}) = T(200) \\ \bar{t}_{F2} = T^{-1}(650) = f(650) = 93.4 & \text{for } 650 = \bar{T}_2 < 868.0 = T(t_{mx}) = T(200) \\ \bar{t}_{F3} = T^{-1}(700) = f(700) = 106.9 & \text{for } 700 = \bar{T}_3 < 868.0 = T(t_{mx}) = T(200) \\ \bar{t}_{F4} = T^{-1}(850) = f(850) = 178.7 & \text{for } 850 = \bar{T}_4 < 868.0 = T(t_{mx}) = T(200) \\ \bar{t}_{F5} = t_{mx} = 200 & \text{for } 868.0 = T(t_{mx}) = T(200) < \bar{T}_5 = 950 \end{cases} \tag{2.96}$$

with $\bar{t}_{Fi}, i = 1, 2, 3, 4,$ defined as indicated in the first line of Eq. (2.95) and $\bar{t}_{F5}$ defined as indicated in the second line of Eq. (2.95).

In words for the link failure temperature focal element $\mathcal{T}_1 = [\underline{T}_1, \overline{T}_1] = [450, 750\,°C]$, the first line in Eq. (2.96) indicates that (i) $\overline{T}_1$ results in link failure before the end of the calculation at time $t_{mx} = 200\,\text{min}$ (i.e., in math: $750\,°\text{C} = \overline{T}_1 < 868.0\,°\text{C} = T(t_{mx}) = T(200\,\text{min})$) and (ii) as a result, the last time at which link failure can occur for focal element $\mathcal{T}_1$ is the time at which link failure occurs at the temperature $\overline{T}_1 = 750\,°\text{C}$ (i.e., in math: $\overline{t}_{F1} = T^{-1}(750\,°C) = f(750\,°C) = 123.1\,\text{min}$). The representations for $\overline{t}_{F2}$, $\overline{t}_{F3}$ and $\overline{t}_{F4}$ in lines 2, 3 and 4 of Eq. (2.96) are defined in the manner. In contrast, line 5 in Eq. (2.96) for link failure temperature focal element $\mathcal{T}_5 = [\underline{T}_5, \overline{T}_5] = [800, 950\,°C]$ indicates that (i) $\overline{T}_5 = 950\,°C$ does not result in link failure prior to $t_{mx} = 200\,\text{min}$ (i.e., in math: $868.0\,°C = T(t_{mx}) = T(200\,\text{min}) < \overline{T}_5 = 950\,°C$) and (ii) as a result, $t_{mx} = 200\,\text{min}$ is the last time at which a link failure temperature in $\mathcal{T}_5$ can result in link failure (i.e., in math: $\overline{t}_{F5} = t_{mx} = 200\,\text{min}$). As a reminder, the representations for $\overline{t}_{Fi}, i = 1,2,3,4,5,$ in Eq. (2.96) are predicated on the assumption that $T(t)$ is a continuous, increasing function.

The graphical determination of $\overline{t}_{F4}$ and $\overline{t}_{F5}$ is illustrated in Fig. 2.3. Further, the set of actual link failure times is the closed interval $[\underline{t}, \overline{t}_F]$, where $\underline{t} = T^{-1}(\underline{T})$ is the earliest time at which link failure can occur provided (i) the link temperature function $T(t)$ is continuous and increasing on $[t_{mn}, t_{mx}]$, (ii) the focal element $\mathcal{T} = [\underline{T}, \overline{T}]$ for link failure time is a closed interval, and (ii) $T(t_{mn}) < \underline{T} \le T(t_{mx})$. Intervals of the form $[\underline{t}, \overline{t}_F]$ are illustrated in Fig. 2.3 and Eq. (2.76).

The definition of the maximum link failure time $\overline{t}_F$ in Eq. (2.95) is predicated on the assumptions that (i) the link temperature function $T(t)$ is continuous and increasing on $[t_{mn}, t_{mx}]$, (ii) $\mathcal{T} = \{T : \underline{T} \le T \le \overline{T}\}$ with $\underline{T} < \overline{T}$ is a focal element for link failure temperature, and (iii) $T(t_{mn}) < \underline{T} < T(t_{mx})$. If the preceding assumptions are modified by eliminating the requirement that $T(t)$ is increasing on $[t_{mn}, t_{mx}]$, then the definition for $\overline{t}_F$ becomes

$$\overline{t}_F = \begin{cases} \min\{t : t \in [t_{mn}, t_{mx}] \text{ and } t = T^{-1}(\overline{T})\} \\ \quad \text{if } \overline{T} \le \max\{T : t \in [t_{mn}, t_{mx}] \text{ and } T = T(t)\} \\ \min\{t : t \in [t_{mn}, t_{mx}] \text{ and } t = T^{-1}(\overline{T}_F)\} \\ \quad \text{if } \overline{T}_F = \max\{T : t \in [t_{mn}, t_{mx}] \text{ and } T = T(t)\} \le \overline{T}. \end{cases} \tag{2.97}$$

2.6 Product Evidence Spaces

The definition of an evidence space allows for a variety of possibilities for what the elements of the sample space could be. However, the two most widely-employed possibilities are probably sample spaces consisting of real numbers or vectors of real numbers. Further, evidence spaces involving vectors of real numbers are often developed from multiple evidence spaces involving real numbers. This situation arises when n evidence spaces

$$(\mathcal{X}_i, \mathbb{X}_i, m_{Xi}), i = 1,2,...,n, \text{ with } \mathbb{X}_i = \{\mathcal{X}_{ij}, j = 1,2,...,nX_i\} \tag{2.98}$$

are known and need to be combined into a single evidence space. Provided no correlations or other relationships exist between the individual evidence spaces, this combining of evidence spaces can be performed to produce a product evidence space $(\mathcal{X}, \mathbb{X}_E, m_X)$ with

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \\ &= \{\mathbf{x} : \mathbf{x} = [x_1, x_2, ..., x_n] \text{ with } x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2, ..., x_n \in \mathcal{X}_n\}, \end{aligned} \tag{2.99}$$

$$\begin{aligned} \mathcal{I} &= \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \text{ with } \mathcal{I}_i = \{1,2,...,nX_i\} \text{ for } i = 1,2,...,n, \\ &= \{\mathbf{j} : \mathbf{j} = [j_1, j_2, ..., j_n] \text{ with } j_1 \in \mathcal{I}_1, j_2 \in \mathcal{I}_2, ..., j_n \in \mathcal{I}_n\}, \end{aligned} \tag{2.100}$$

$$\mathbb{X} = \{\mathcal{X}_{\mathbf{j}} : \mathcal{X}_{\mathbf{j}} = \mathcal{X}_{1j_1} \times \mathcal{X}_{2j_2} \times \cdots \times \mathcal{X}_{nj_n} \text{ for } \mathbf{j} = [j_1, j_2, ..., j_n] \in \mathcal{I}\}, \tag{2.101}$$

and

$$m_X(\mathcal{X}_{\mathbf{j}}) = m_{X1}(\mathcal{X}_{1j_1}) \times m_{X2}(\mathcal{X}_{2j_2}) \times \cdots \times m_{Xn}(\mathcal{X}_{nj_n}) \text{ for } \mathbf{j} = [j_1, j_2, ..., j_n] \in \mathcal{I}. \tag{2.102}$$

In the preceding, the set $\mathcal{I}$ of integer vectors of the form $\mathbf{j} = [j_1, j_2, ..., j_n]$ is used to define all possible combinations of the focal elements associated with the individual evidence spaces. Further, the product definition for $m_X(\mathcal{X}_{\mathbf{j}})$ in Eq. (2.102) is predicated on the assumption that there are no correlations or other relationships between the individual evidence spaces.

3. Representation of LOAS with Evidence Theory for a 1 WL and 1 SL System

3.1 Belief and Plausibility for the Occurrence of LOAS

For simplicity of explanation and illustration, this section describes the use of evidence theory to characterize the potential occurrence of LOAS for a WL/SL system with 1 WL and 1 SL. For a 1 WL and 1 SL system, LOAS corresponds to the failure of the SL before failure of the WL. More complex systems will be considered in later sections.

An evidence space $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ for WL failure temperature T_{WL} is introduced and defined by

$$\mathcal{T}_{WL} = [450, 950\ °C],\ \mathbb{T}_{WL} = \{\mathcal{T}_{WL,1}, \mathcal{T}_{WL,2}, \mathcal{T}_{WL,3}\}, \tag{3.1}$$

$$\mathcal{T}_{WL,1} = [450, 650\ °C], \mathcal{T}_{WL,2} = [550, 850\ °C], \mathcal{T}_{WL,3} = [750, 950\ °C], \tag{3.2}$$

and

$$m_{WL,T}(\mathcal{T}_{WL,1}) = 0.5, m_{WL,T}(\mathcal{T}_{WL,2}) = 0.3, m_{WL,T}(\mathcal{T}_{WL,3}) = 0.2. \tag{3.3}$$

Similarly, an evidence space $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ for SL failure temperature T_{SL} is introduced and defined by

$$\mathcal{T}_{SL} = [600, 1050\ °C],\ \mathbb{T}_{SL} = \{\mathcal{T}_{SL,1}, \mathcal{T}_{SL,2}, \mathcal{T}_{SL,3}\}, \tag{3.4}$$

$$\mathcal{T}_{SL,1} = [600, 850\ °C], \mathcal{T}_{SL,2} = [700, 1000\ °C], \mathcal{T}_{SL,3} = [950, 1050\ °C], \tag{3.5}$$

and

$$m_{SL,T}(\mathcal{T}_{SL,1}) = 0.2, m_{SL,T}(\mathcal{T}_{SL,2}) = 0.3, m_{SL,T}(\mathcal{T}_{SL,3}) = 0.5. \tag{3.6}$$

An example with small numbers of focal elements for the evidence spaces $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ is chosen so that the resultant product space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time will have a sufficiently small number of focal elements (i.e., 9) to permit a display and discussion of all focal elements for this evidence space.

As in Eq. (2.72), time-dependent link temperatures are defined by (i)

$$\begin{aligned} T_{WL}(t) &= \text{WL temperature in °C at time } t \\ &= \frac{T_0 T_\infty}{T_0 + (T_\infty - T_0)\exp(-rt)} \end{aligned} \tag{3.7}$$

with $T(0) = T_0 = 225$ °C, $T_\infty = 1000$ °C, and $r = 0.065$, and (ii)

$$\begin{aligned} T_{SL}(t) &= SL \text{ temperature in °C at time } t \\ &= \frac{T_0 T_\infty}{T_0 + (T_\infty - T_0)\exp(-rt)} \end{aligned} \tag{3.8}$$

with $T(0) = T_0 = 225$ °C, $T_\infty = 1100$ °C, and $r = 0.08$ (see Fig. 3.1).

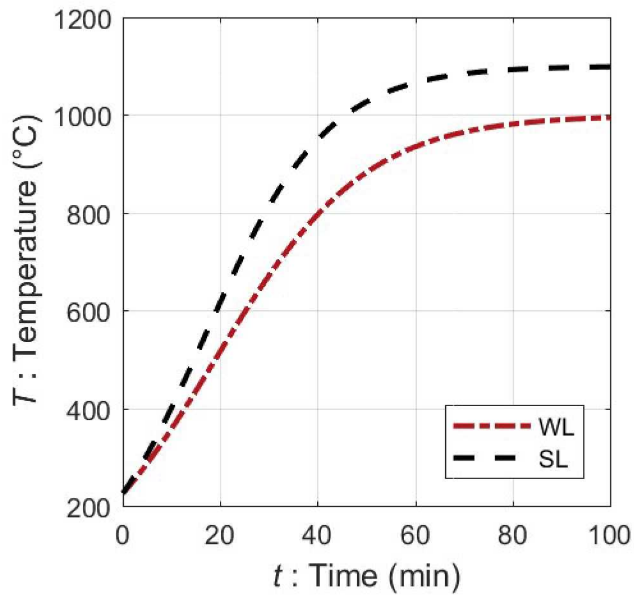


Fig. 3.1 Time dependent link temperatures for example link system with 1 WL and 1 SL.

To assess the uncertainty associated with the potential occurrence of LOAS, it is necessary to know the potential times at which the individual links fail and the uncertainty associated with these times. As described for Approach 1 in Sect. 2.4, this can be accomplished by using the function $f(T)$ defined in Eq. (2.73) to map the evidence spaces $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ for link failure temperatures into evidence spaces $(\mathcal{TM}_{WL}, \mathbb{TM}_{WL}, m_{WL,t})$ and $(\mathcal{TM}_{SL}, \mathbb{TM}_{SL}, m_{SL,t})$ for link failure times t_{WL} and t_{SL}. Specifically, $(\mathcal{TM}_{WL}, \mathbb{TM}_{WL}, m_{WL,t})$ is defined by

$$\mathcal{TM}_{WL} = [15, 65 \text{ min}],\ \mathbb{TM}_{WL} = \{\mathcal{TM}_{WL,1}, \mathcal{TM}_{WL,2}, \mathcal{TM}_{WL,3}\}, \tag{3.9}$$

$$\mathcal{TM}_{WL,i} = \begin{cases} \mathcal{TM}_{WL,1} = f(\mathcal{T}_{WL,1}) = [15, 28 \text{ min}] & \text{for } i = 1 \\ \mathcal{TM}_{WL,2} = f(\mathcal{T}_{WL,2}) = [22, 45 \text{ min}] & \text{for } i = 2 \\ \mathcal{TM}_{WL,3} = f(\mathcal{T}_{WL,3}) = [36, 65 \text{ min}] & \text{for } i = 3, \end{cases} \tag{3.10}$$

and

$$m_{WL,t}(\mathcal{TM}_{WL,i}) = \begin{cases} m_{WL,t}(\mathcal{TM}_{WL,i}) = m_{WL,T}(\mathcal{T}_{WL,1}) = 0.5 \text{ for } i = 1 \\ m_{WL,t}(\mathcal{TM}_{WL,2}) = m_{WL,T}(\mathcal{T}_{WL,2}) = 0.3 \text{ for } i = 2 \\ m_{WL,t}(\mathcal{TM}_{WL,3}) = m_{WL,T}(\mathcal{T}_{WL,3}) = 0.2 \text{ for } i = 3. \end{cases} \quad (3.11)$$

Similarly, $(\mathcal{TM}_{SL}, \mathbb{TM}_{SL}, m_{SL,t})$ is defined by

$$\mathcal{TM}_{SL} = [19, 55 \text{ min}], \mathbb{TM}_{SL} = \{\mathcal{TM}_{SL,1}, \mathcal{TM}_{SL,2}, \mathcal{TM}_{SL,3}\}, \quad (3.12)$$

$$\mathcal{TM}_{SL,i} = \begin{cases} \mathcal{TM}_{SL,1} = f(\mathcal{T}_{SL,1}) = [19, 33 \text{ min}] \text{ for } i = 1 \\ \mathcal{TM}_{SL,2} = f(\mathcal{T}_{SL,2}) = [24, 46 \text{ min}] \text{ for } i = 2 \\ \mathcal{TM}_{SL,3} = f(\mathcal{T}_{SL,3}) = [40, 55 \text{ min}] \text{ for } i = 3, \end{cases} \quad (3.13)$$

and

$$m_{SL,t}(\mathcal{TM}_{SL,i}) = \begin{cases} m_{SL,t}(\mathcal{TM}_{SL,i}) = m_{SL,T}(\mathcal{T}_{SL,1}) = 0.2 \text{ for } i = 1 \\ m_{SL,t}(\mathcal{TM}_{SL,2}) = m_{SL,T}(\mathcal{T}_{SL,2}) = 0.3 \text{ for } i = 2 \\ m_{SL,t}(\mathcal{TM}_{SL,3}) = m_{SL,T}(\mathcal{T}_{SL,3}) = 0.5 \text{ for } i = 3. \end{cases} \quad (3.14)$$

As examples, resultant focal elements $\mathcal{TM}_{WL,1}, \mathcal{TM}_{WL,3}, \mathcal{TM}_{SL,1}$ and $\mathcal{TM}_{SL,3}$ for the evidence spaces $(\mathcal{TM}_{WL}, \mathbb{TM}_{WL}, m_{WL,t})$ and $(\mathcal{TM}_{SL}, \mathbb{TM}_{SL}, m_{SL,t})$ are illustrated in Fig. 3.2.

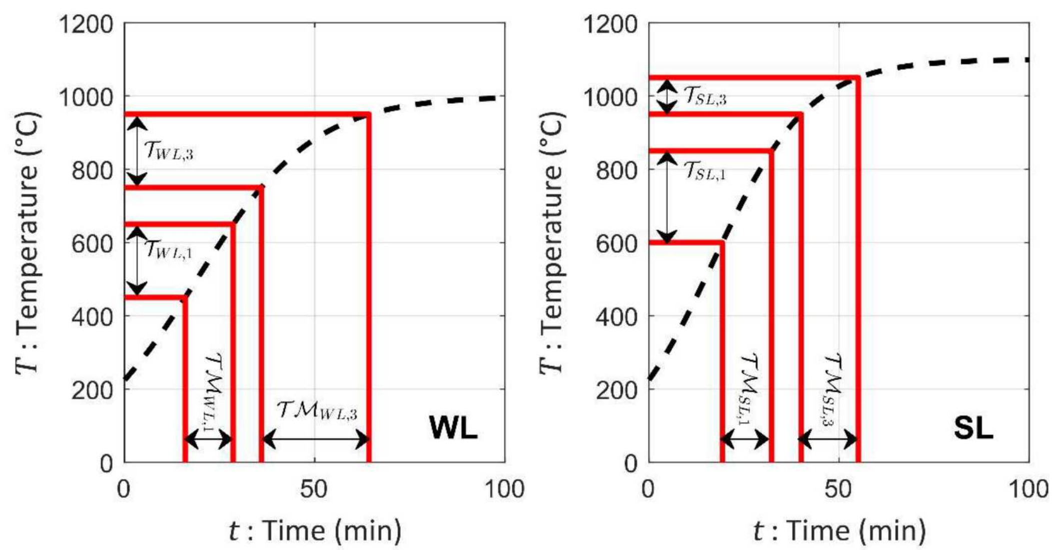


Fig. 3.2 Illustration of focal elements $\mathcal{TM}_{WL,1}, \mathcal{TM}_{WL,3}, \mathcal{TM}_{SL,1}$ and $\mathcal{TM}_{SL,3}$ for the evidence spaces $(\mathcal{TM}_{WL}, \mathbb{TM}_{WL}, m_{WL,t})$ and $(\mathcal{TM}_{SL}, \mathbb{TM}_{SL}, m_{SL,t})$.

Assessing the occurrence of LOAS involves determining belief and plausibility for the set

$$\mathcal{L} = \{(t_{SL}, t_{WL}) : t_{SL} \in \mathcal{TM}_{SL}, t_{WL} \in \mathcal{TM}_{WL}, t_{SL} < t_{WL}\}. \tag{3.15}$$

In turn, this determination requires the introduction of the product evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ that results from combining the evidence spaces $(\mathcal{TM}_{WL}, \mathbb{TM}_{WL}, m_{WL,t})$ and $(\mathcal{TM}_{SL}, \mathbb{TM}_{SL}, m_{SL,t})$. Specifically, $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ is defined by

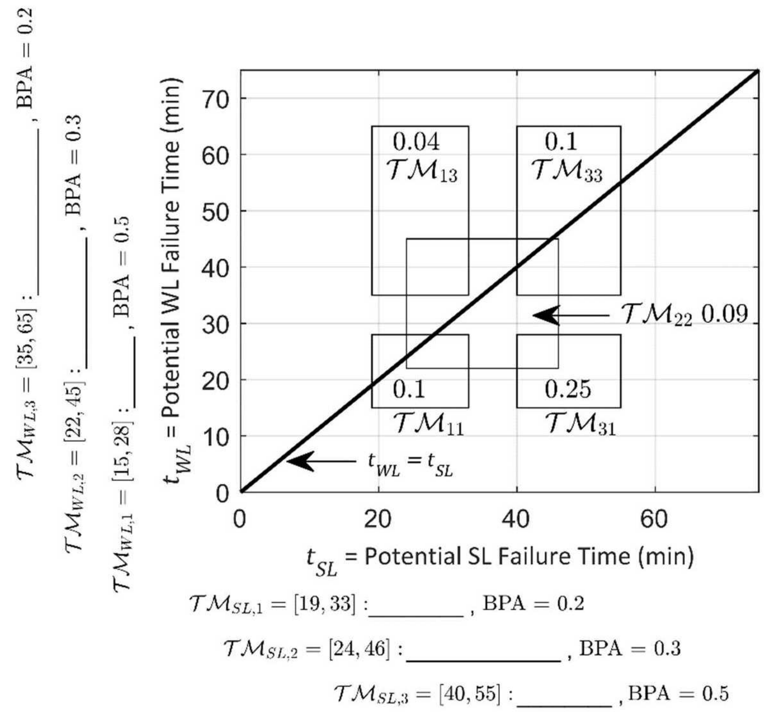
$$\mathcal{TM} = \mathcal{TM}_{SL} \times \mathcal{TM}_{WL} = \{\mathbf{t} : \mathbf{t} = (t_{SL}, t_{WL}) \in [19, 55 \text{ min}] \times [15, 65 \text{ min}]\}, \tag{3.16}$$

$$\mathbb{TM} = \left\{\mathcal{TM}_{ij} = \mathcal{TM}_{SL,i} \times \mathcal{TM}_{WL,j} \ \text{ for } (i, j)\} \in \{1,2,3\} \times \{1,2,3\}\right\}, \tag{3.17}$$

and

$$m_{TM}(\mathcal{TM}_{ij}) = m_{SL,t}(\mathcal{TM}_{SL,i}) m_{WL,t}(\mathcal{TM}_{WL,j}) \ \text{ for } (i, j)\} \in \{1,2,3\} \times \{1,2,3\}. \tag{3.18}$$

The resultant focal elements and associated BPAs for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ are illustrated in Fig. 3.3.



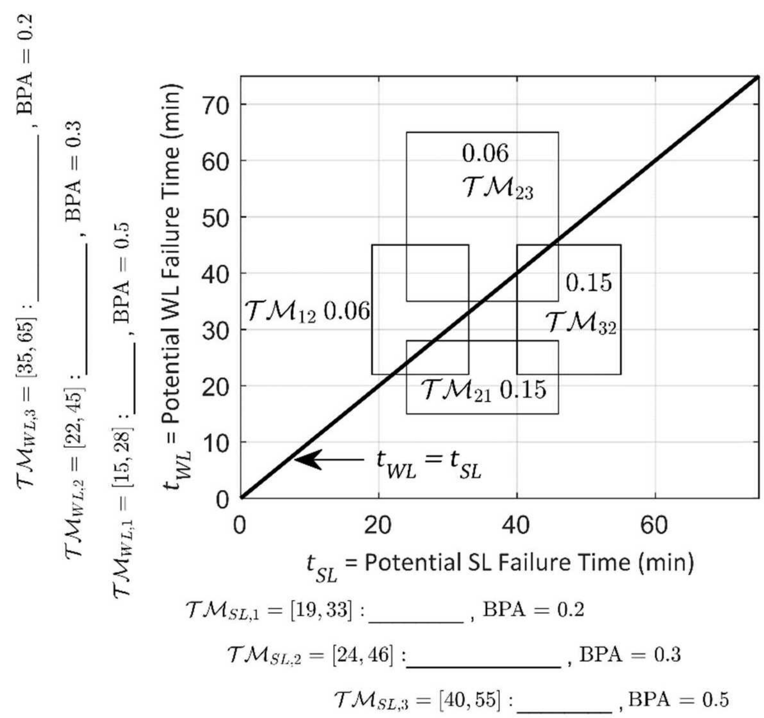


Fig. 3.3 Illustration of focal elements $\mathcal{TM}_{ij}$ and associated BPAs for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ with the number inside the boundaries of each focal element equal to the BPA for that focal element (e.g., $m_{TM}(\mathcal{TM}_{23}) = m_{SL,t}(\mathcal{TM}_{SL,2}) m_{WL,t}(\mathcal{TM}_{WL,3}) = 0.3 \times 0.2 = 0.06$).

The diagonal line in Fig. 3.3 corresponds to $t_{WL} = t_{SL}$. As a consequence of Fig. 3.3 t_{SL} representing t_{WL} on the ordinate and on the abscissa, (i) the inequality $t_{WL} > t_{SL}$ holds for any point (t_{SL}, t_{WL}) above the indicated line $t_{WL} = t_{SL}$, and (ii) the set $\mathcal{L}$ defined in Eq. (3.15) corresponding to the occurrence of LOAS is equal to the intersection of $\mathcal{TM}$ with the points above the line $t_{WL} = t_{SL}$. Thus, a focal element that is located entirely above the line $t_{WL} = t_{SL}$ is a subset of $\mathcal{L}$, and a focal element that intersects the region above the line $t_{WL} = t_{SL}$ also intersects $\mathcal{L}$. Given the preceding,

$$Bel(\mathcal{L}) = \sum_{\mathcal{TM}_{ij} \subseteq \mathcal{L}} m_{TM}(\mathcal{TM}_{ij}) = m_{TM}(\mathcal{TM}_{13}) = 0.04 \tag{3.19}$$

as a result of $\mathcal{TM}_{13}$ being the only focal element that is a subset of $\mathcal{L}$ as indicated by $\mathcal{TM}_{13}$ being the only focal element this located entirely above the line $t_{WL} = t_{SL}$. Similarly,

$$\begin{aligned} Pl(\mathcal{L}) &= \sum_{\mathcal{TM}_{ij} \cap \mathcal{L} \neq \varnothing} m_{TM}(\mathcal{TM}_{ij}) \\ &= m_{TM}(\mathcal{TM}_{11}) + m_{TM}(\mathcal{TM}_{12}) + m_{TM}(\mathcal{TM}_{13}) + m_{TM}(\mathcal{TM}_{21}) \\ &\quad + m_{TM}(\mathcal{TM}_{22}) + m_{TM}(\mathcal{TM}_{23}) + m_{TM}(\mathcal{TM}_{32}) + m_{TM}(\mathcal{TM}_{33}) \\ &= 0.1 + 0.06 + 0.04 + 0.15 + 0.09 + 0.06 + 0.15 + 0.1 \\ &= 0.75 \end{aligned} \tag{3.20}$$

as a result of all focal elements except $\mathcal{TM}_{31}$ intersecting $\mathcal{L}$ as indicated by $\mathcal{TM}_{31}$ being the only focal element that does not intersect the region above the line $t_{WL} = t_{SL}$.

This example is defined for ease of presentation with (i) three focal elements for the evidence spaces $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ and (ii) a resultant nine focal elements for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$. The ease of visual inspection as done with Fig. 3.3 is greatly diminished when $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ have a large number of focal elements. For example, if $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ each have 50 focal elements, then $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ will have 2500 focal elements. Fortunately, a computationally simple procedure can be defined to determine $Bel(\mathcal{L})$ and $Pl(\mathcal{L})$ when $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ have a large number of intervals as focal elements.

To illustrate this procedure, it is assumed that (i) $(\mathcal{T}_{WL}, \mathbb{T}_{WL}, m_{WL,T})$ and $(\mathcal{T}_{SL}, \mathbb{T}_{SL}, m_{SL,T})$ have nWL and nSL interval-valued focal elements, (ii)

$$[tWL_{mn,j}, tWL_{mx,j}], j = 1, 2, ..., nWL, \text{ and } [tSL_{mn,i}, tSL_{mx,i}], i = 1, 2, ..., nSL, \tag{3.21}$$

are the resultant focal elements for the evidence spaces $(\mathcal{TM}_{WL}, \mathbb{TM}_{WL}, m_{WL,t})$ and $(\mathcal{TM}_{SL}, \mathbb{TM}_{SL}, m_{SL,t})$ for link failure times t_{WL} and t_{SL}, and (iii) the sets

$$\mathcal{TM}_{ij} = [tSL_{mn,i}, tSL_{mx,i}] \times [tWL_{mn,j}, tWL_{mx,j}] \text{ for } (i,j) \in \{1,2,...,nSL\} \times \{1,2,...,nWL\} \tag{3.22}$$

are the resultant focal elements for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$. With respect to notation, the use of *i* and *j* is defined so that *i* corresponds to values associated with the abscissa in Fig. 3.3 and *j* corresponds to values associated with the ordinate in Fig. 3.3.

The focal element $\mathcal{TM}_{ij} = [tSL_{mn,i}, tSL_{mx,i}] \times [tWL_{mn,j}, tWL_{mx,j}]$ of WL and SL failure times is located above the line $t_{WL} = t_{SL}$ only if

$$tSL_{mx,i} < tWL_{mn,j} \tag{3.23}$$

as illustrated by $\mathcal{TM}_{13}$ in Fig. 3.3. As a consequence of this property, $Bel(\mathcal{L})$ is given by the summation

$$Bel(\mathcal{L}) = \sum_{\mathcal{TM}_{ij} \subseteq \mathcal{L}} m_{TM}(\mathcal{TM}_{ij}) = \sum_{i=1}^{nSL} \sum_{j=1}^{nWL} \delta_B(\mathcal{TM}_{ij}) m_{TM}(\mathcal{TM}_{ij}) \tag{3.24}$$

with

$$\delta_B(\mathcal{TM}_{ij}) = \begin{cases} 1 & \text{if } tSL_{mx,i} < tWL_{mn,j} \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

Similarly, $\mathcal{TM}_{ij}$ intersects the region above the line $t_{WL} = t_{SL}$ only if

$$tSL_{mn,i} < tWL_{mx,j} \tag{3.26}$$

as illustrated by $\mathcal{TM}_{11}, \mathcal{TM}_{12}, \mathcal{TM}_{13}, \mathcal{TM}_{21}, \mathcal{TM}_{22}, \mathcal{TM}_{23}, \mathcal{TM}_{32}$ and $\mathcal{TM}_{33}$ in Fig. 3.3. As a consequence of this property, $Pl(\mathcal{L})$ is given by the summation

$$Pl(\mathcal{L}) = \sum_{\varnothing \neq \mathcal{TM}_{ij} \cap \mathcal{L}} m_{TM}(\mathcal{TM}_{ij}) = \sum_{i=1}^{nSL} \sum_{j=1}^{nWL} \delta_P(\mathcal{TM}_{ij}) m_{TM}(\mathcal{TM}_{ij}) \tag{3.27}$$

with

$$\delta_P(\mathcal{TM}_{ij}) = \begin{cases} 1 & \text{if } tSL_{mn,i} < tWL_{mx,j} \\ 0 & \text{otherwise.} \end{cases} \tag{3.28}$$

As illustrated in Fig. 3.4, a link temperature function $T(t)$ does not have to be strictly increasing for the relationships defining $Bel(\mathcal{L})$ and $Pl(\mathcal{L})$ in Eqs. (3.23)-(3.28) to be valid. In this example, (i) $\mathcal{T}_i = [500, 900\ °\text{C}]$ is a focal element for an evidence space $(\mathcal{T}, \mathbb{T}, m_T)$ for link failure temperature, (ii) $T(t)$ is a continuous nonlinear function of time defining link temperature, (iii) link failure occurs when link temperature reaches link failure temperature, and (iv) $\mathcal{TM}_i$ is the resultant focal element for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time with

$$\mathcal{TM}_i = \mathcal{TM}_{i1} \cup \mathcal{TM}_{i2} = [20, 40\ \text{min}] \cup (60, 86\ \text{min}]. \tag{3.29}$$

In turn, if $\mathcal{TM}_i$ is a focal element in an analysis of the form in Eqs. (3.23)-(3.28), the times

$$t_{mn,i} = \min\{t : t \in \mathcal{TM}_i\} \text{ and } t_{mx,i} = \max\{t : t \in \mathcal{TM}_i\} \tag{3.30}$$

are used in the same manner in the calculation of $Bel(\mathcal{L})$ and $Pl(\mathcal{L})$ as would be the case if they were the endpoints of a closed interval $[t_{mn,i}, t_{mn,i}]$.

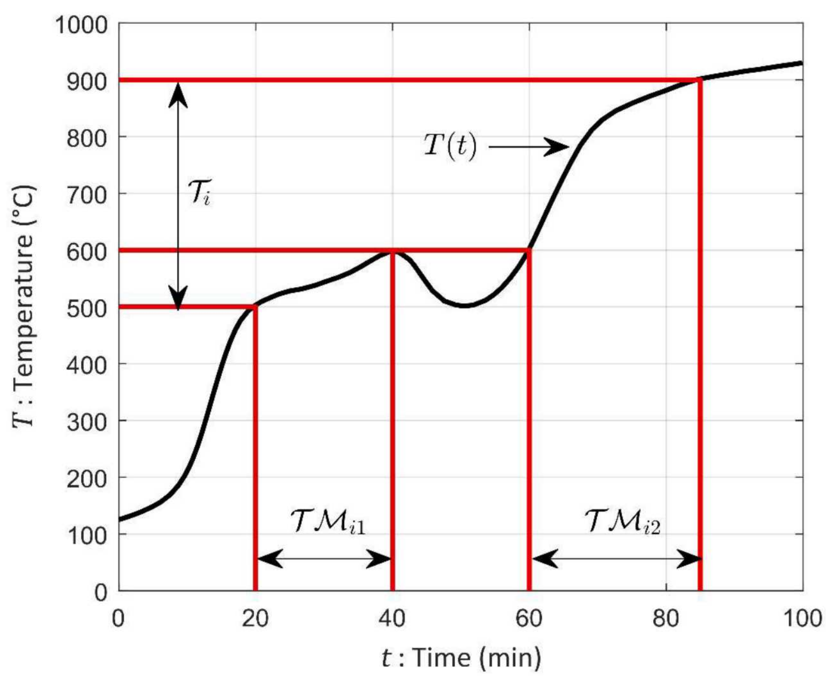


Fig. 3.4 Example of focal element definition for link failure time with link temperature a continuous nonlinear function of time.

3.2 Cumulative and Complementary Cumulative Representations of Belief and Plausibility for the Occurrence of LOAS

Cumulative and complementary cumulative representations of belief and plausibility for LOAS occurrence time are considered first. The starting point for this representation is the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in Eqs. (3.16)-(3.18). For $(t_{SL}, t_{WL}) \in \mathcal{TM}$, LOAS occurrence time is defined by

$$TML(t_{SL}, t_{WL}) = \begin{cases} t_{SL} & \text{for } t_{SL} < t_{WL} \\ \infty & \text{for } t_{WL} \le t_{SL}. \end{cases} \tag{3.31}$$

In turn, the evidence space $(\mathcal{TML}, \mathbb{TML}, m_{TML})$ for LOAS occurrence time is defined by

$$\mathcal{TML} = \{t : t = TML(t_{SL}, t_{WL}) \text{ for } (t_{SL}, t_{WL}) \in \mathcal{TM}\}, \tag{3.32}$$

$$\mathcal{TML}_{ij} = \{t : t = TML(t_{SL}, t_{WL}) \text{ for } (t_{SL}, t_{WL}) \in \mathcal{TM}_{ij}\}, \tag{3.33}$$

$$\mathbb{TML} = \left\{\mathcal{TML}_{ij} : (i, j) \in \{1,2,3\} \times \{1,2,3\}\right\}, \tag{3.34}$$

$$m_{TML}(\mathcal{TML}_{ij}) = m_{TM}(\mathcal{TM}_{ij}), \tag{3.35}$$

with

$$\underline{t}_{ij} = \text{lower bound for } \mathcal{TML}_{ij} = \begin{cases} \infty & \text{for } \overline{t}_{WL,j} \le \underline{t}_{SL,i} \\ \underline{t}_{SL,i} & \text{for } \underline{t}_{SL,i} < \overline{t}_{WL,j}, \end{cases} \tag{3.36}$$

$$\overline{t}_{ij} = \text{upper bound for } \mathcal{TML}_{ij} = \begin{cases} \infty & \text{for } \underline{t}_{WL,j} \le \overline{t}_{SL,i} \\ \overline{t}_{SL,i} & \text{for } \overline{t}_{SL,i} < \underline{t}_{WL,j}. \end{cases} \tag{3.37}$$

Given the evidence space $(\mathcal{TML}, \mathbb{TML}, m_{TML})$ and the associated focal element bounds $\underline{t}_{ij}$ and $\overline{t}_{ij}$, cumulative and complementary cumulative representations of belief and plausibility for LOAS occurrence time can be determined as described in Sect. 2.3 and illustrated in Fig. 3.5.

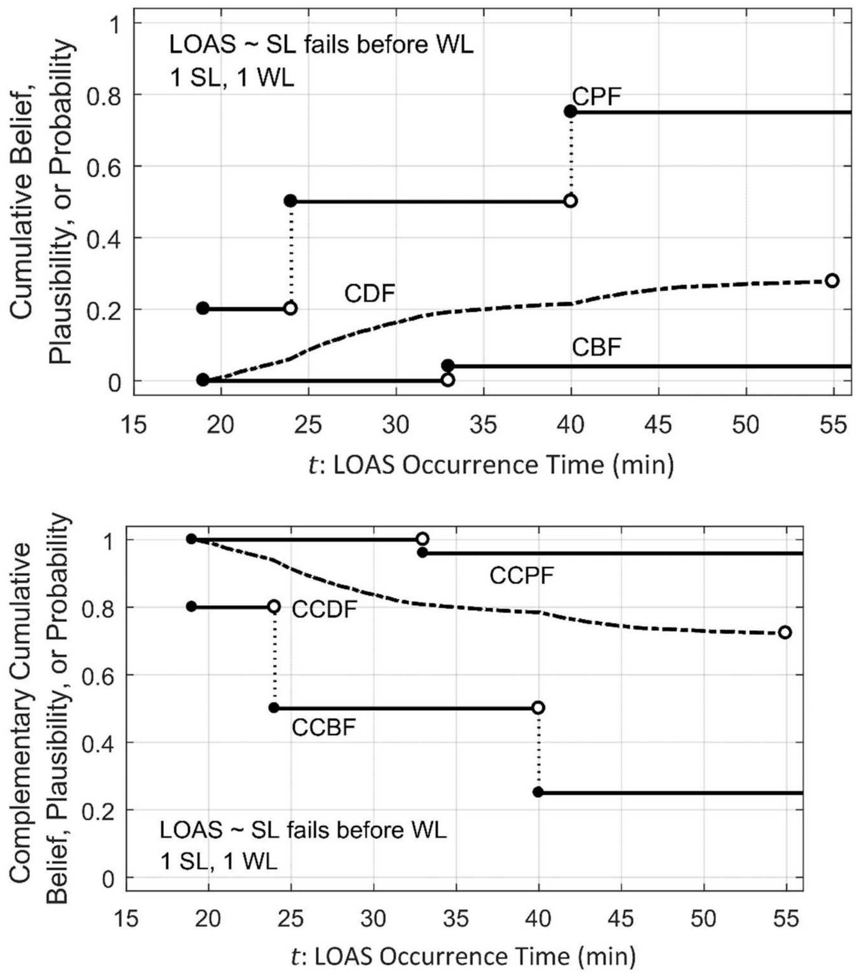


Fig. 3.5 Cumulative and complementary cumulative representations of belief and plausibility for LOAS occurrence time for the evidence space $(\mathcal{TML}, \mathbb{TML}, m_{TML})$ defined in Eqs. (3.32)-(3.35) for a 1 WL, 1 SL system.

Cumulative and complementary cumulative representations of belief and plausibility for LOAS occurrence time margins are now considered. As before, the starting point for this representation is the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in Eqs. (3.16)-(3.18). For $(t_{SL}, t_{WL}) \in \mathcal{TM}$, LOAS occurrence time margin is defined by

$$MTM(t_{SL}, t_{WL}) = \begin{cases} -\infty \;\; \text{for} \;\; t_{SL} < \infty, t_{WL} = \infty \\ t_{SL} - t_{WL} \;\; \text{for} \;\; \max\{t_{SL}, t_{WL}\} < \infty \\ \infty \quad \text{for} \;\; t_{WL} \le t_{SL} = \infty. \end{cases} \tag{3.38}$$

In turn, the evidence space $(\mathcal{MTM}, \mathbb{MTM}, m_{MTM})$ for LOAS occurrence time is defined by

$$\mathcal{MTM} = \{t : t = MTM(t_{SL}, t_{WL}) \text{ for } (t_{SL}, t_{WL}) \in \mathcal{TM}\}, \tag{3.39}$$

$$\mathcal{MTM}_{ij} = \{t : t = MTM(t_{SL}, t_{WL}) \text{ for } (t_{SL}, t_{WL}) \in \mathcal{TM}_{ij}\}, \tag{3.40}$$

$$\mathbb{MTM} = \left\{\mathcal{MTM}_{ij} : (i, j) \in \{1,2,3\} \times \{1,2,3\}\right\}, \tag{3.41}$$

$$m_{MTM}(\mathcal{MTM}_{ij}) = m_{TM}(\mathcal{TM}_{ij}), \tag{3.42}$$

with

$$\underline{t}_{ij} = \text{lower bound for } \mathcal{MTM}_{ij} = \begin{cases} -\infty \text{ for } \overline{t}_{WL,j} = \infty \text{ and } \underline{t}_{SL,i} < \infty \\ \underline{t}_{SL,i} - \overline{t}_{WL,j} \text{ for } \max\{\underline{t}_{SL,i}, \overline{t}_{WL,j}\} < \infty \\ \infty \text{ for } \underline{t}_{SL,i} = \infty, \end{cases} \tag{3.43}$$

$$\overline{t}_{ij} = \text{ upper bound for } \mathcal{MTM}_{ij} = \begin{cases} -\infty \text{ for } \overline{t}_{WL,j} = \infty \text{ and } \overline{t}_{SL,i} < \infty \\ \overline{t}_{SL,i} - \underline{t}_{WL,j} \text{ for } \max\{\underline{t}_{SL,i}, \overline{t}_{WL,j}\} < \infty \\ \infty \text{ for } \overline{t}_{SL,i} = \infty. \end{cases} \tag{3.44}$$

Given the evidence space $(\mathcal{MTM}, \mathbb{MTM}, m_{MTM})$ and the associated focal element bounds $\underline{t}_{ij}$ and $\overline{t}_{ij}$, cumulative and complementary cumulative representations of belief and plausibility for LOAS occurrence time margins can be determined as described in Sect. 2.3 and illustrated in Fig. 3.5.

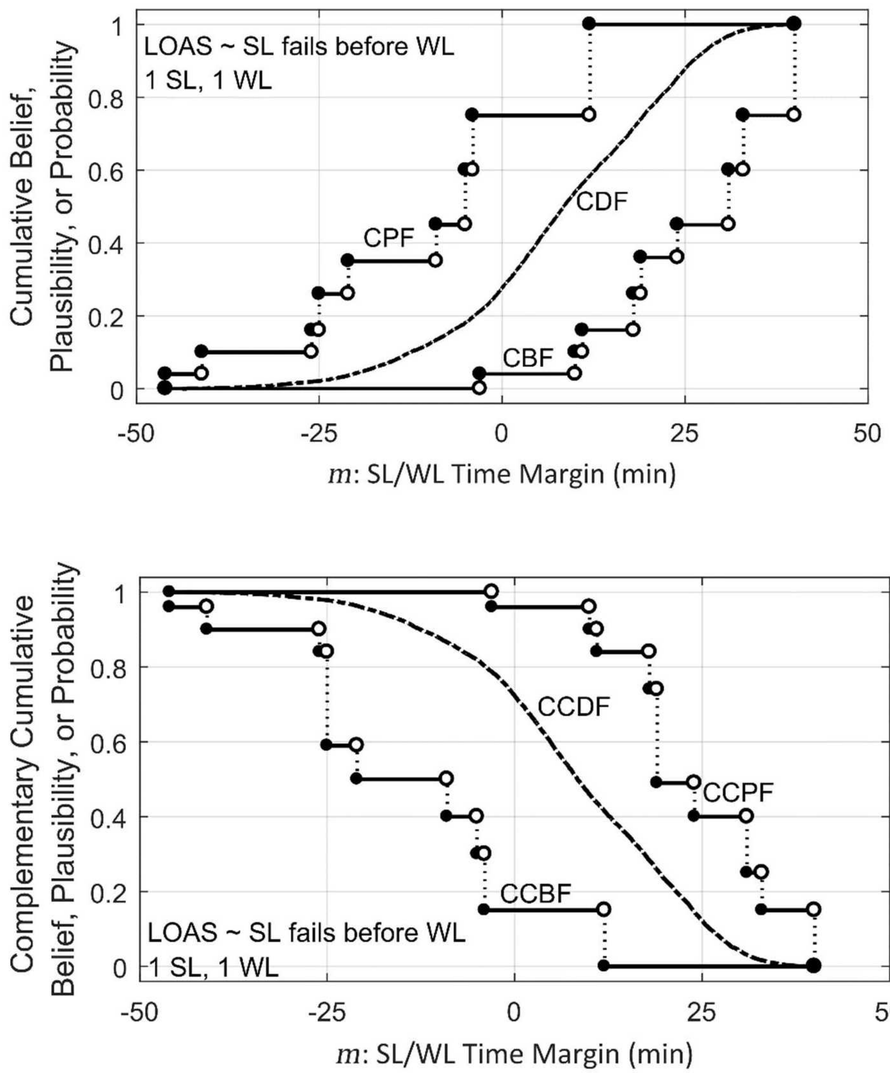


Fig. 3.6 Cumulative and complementary cumulative representations of belief and plausibility for LOAS occurrence time margins for the evidence space $(\mathcal{MTM}, \mathbb{MTM}, m_{MTM})$ defined in Eqs. (3.39)-(3.42) for a 1 WL, 1 SL system

4. Example Links Used for Illustration

The developments in the following sections of this report consider systems involving a variety of combinations of WLs and SLs. The purpose of this section is to define and illustrate 2 SLs and 2 WLs that will be used for examples involving (i) 2 SLs and 1 WL, (ii) 2 SLs, (iii) 2 WLs, and (iv) 2 SLs and 2 WLs. For convenience, the 2 SLs will be referred to as SL 1 and SL 2, and the 2 WLs will be referred to as WL 1 and WL 2.

The following entities underlie the results presented in later sections: (i) evidence spaces $(\mathcal{T}_{SL1}, \mathbb{T}_{SL1}, m_{SL1})$ and $(\mathcal{T}_{SL2}, \mathbb{T}_{SL2}, m_{SL2})$ with $nSL1$ and $nSL2$ focal elements for SL 1 and SL 2 failure temperatures, (ii) evidence spaces $(\mathcal{T}_{WL1}, \mathbb{T}_{WL1}, m_{WL1})$ and $(\mathcal{T}_{WL2}, \mathbb{T}_{WL2}, m_{WL2})$ with $nWL1$ and $nWL2$ focal elements for WL 1 and WL 2 failure temperatures, and (iii) functions $T_{SL1}(t)$, $T_{SL2}(t)$, $T_{WL1}(t)$ and $T_{WL2}(t)$ that define time-dependent link temperatures for SL 1, SL 2. WL 1 and WL 2. In turn, the indicated evidence spaces and link temperature functions result in corresponding evidence spaces $(\mathcal{TM}_{SL1}, \mathbb{TM}_{SL1}, m_{SL1,t})$, $(\mathcal{TM}_{SL2}, \mathbb{TM}_{SL2}, m_{SL2,t})$, $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1,t})$ and $(\mathcal{TM}_{WL2}, \mathbb{TM}_{WL2}, m_{WL2,t})$ for link failure times as discussed in Sect. 3 with:

(i) properties of $(\mathcal{TM}_{SL1}, \mathbb{TM}_{SL1}, m_{SL1,t})$ defined by

$$\mathcal{TM}_{SL1,i} = T_{SL1}^{-1}(\mathcal{T}_{SL1,i}) = \left\{t : t = T_{SL1}^{-1}(T) = \min\{t : T = T_{SL1}(t)\} \text{ for } T \in \mathcal{T}_{SL1,i}\right\}, \tag{4.1}$$

$$m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1}(\mathcal{T}_{SL1,i}) = m_{SL1,i}, \tag{4.2}$$

$$(\underline{t}_{SL1,i}, \overline{t}_{SL1,i}) = \left(\min(\mathcal{TM}_{SL1,i}), \max(\mathcal{TM}_{SL1,i})\right), \tag{4.3}$$

for $\mathcal{T}_{SL1,i} \in \mathbb{T}_{SL1}$, $\mathcal{TM}_{SL1,i} \in \mathbb{TM}_{SL1}$ and $i \in \{1, 2, ..., nSL1\} = \mathcal{I}_{SL1}$,

(ii) properties of $(\mathcal{TM}_{SL2}, \mathbb{TM}_{SL2}, m_{SL2,t})$ defined by

$$\mathcal{TM}_{SL2,j} = T_{SL2}^{-1}(\mathcal{T}_{SL2,j}) = \left\{t : t = T_{SL2}^{-1}(T) = \min\{t : T = T_{SL2}(t)\} \text{ for } T \in \mathcal{T}_{SL2,j}\right\}, \tag{4.4}$$

$$m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2}(\mathcal{T}_{SL2,j}) = m_{SL2,j}, \tag{4.5}$$

$$(\underline{t}_{SL2,j}, \overline{t}_{SL2,j}) = \left(\min(\mathcal{TM}_{SL2,j}), \max(\mathcal{TM}_{SL2,j})\right), \tag{4.6}$$

for $\mathcal{T}_{SL2,j} \in \mathbb{T}_{SL2}$, $\mathcal{TM}_{SL2,j} \in \mathbb{TM}_{SL2}$ and $j \in \{1, 2, ..., nSL2\} = \mathcal{I}_{SL2}$,

(iii) properties of $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1,t})$ defined by

$$\mathcal{TM}_{WL1,k} = T_{WL1}^{-1}(\mathcal{T}_{WL1,k}) = \left\{t : t = T_{WL1}^{-1}(T) = \min\{t : T = T_{WL1}(t)\} \text{ for } T \in \mathcal{T}_{WL1,k}\right\}, \quad (4.7)$$

$$m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1}(\mathcal{T}_{WL1,k}) = m_{WL1,k}, \quad (4.8)$$

$$(\underline{t}_{WL1,k}, \overline{t}_{WL1,k}) = \left(\min(\mathcal{TM}_{WL1,k}), \max(\mathcal{TM}_{WL1,k})\right), \quad (4.9)$$

for $\mathcal{T}_{WL1,k} \in \mathbb{T}_{WL1}$, $\mathcal{TM}_{WL1,k} \in \mathbb{TM}_{WL1}$ and $k \in \{1,2,...,nWL1\} = \mathcal{I}_{WL1}$, and

(iv) properties of $(\mathcal{TM}_{WL2}, \mathbb{TM}_{WL2}, m_{WL2,t})$ defined by

$$\mathcal{TM}_{WL2,k} = T_{WL2}^{-1}(\mathcal{T}_{WL2,k}) = \left\{t : t = T_{WL2}^{-1}(T) = \min\{t : T = T_{WL2}(t)\} \text{ for } T \in \mathcal{T}_{WL2,k}\right\}, \quad (4.10)$$

$$m_{WL2,t}(\mathcal{TM}_{WL2,l}) = m_{WL2}(\mathcal{T}_{WL2,l}) = m_{WL2,l}, \quad (4.11)$$

$$(\underline{t}_{WL2,l}, \overline{t}_{WL2,l}) = \left(\min(\mathcal{TM}_{WL2,l}), \max(\mathcal{TM}_{WL2,l})\right), \quad (4.12)$$

for $\mathcal{T}_{WL2,l} \in \mathbb{T}_{WL2}$, $\mathcal{TM}_{WL2,l} \in \mathbb{TM}_{WL2}$ and $l \in \{1,2,...,nWL2\} = \mathcal{I}_{WL2}$.

With respect to notation, the $\min\{\sim\}$ condition in Eqs.(4.1), (4.4), (4.7) and (4.10) is not needed if the associated temperature function (i.e., $T_{SL1}(t)$, $T_{SL2}(t)$, $T_{WL1}(t)$ or $T_{WL2}(t)$) is strictly increasing. Also, the representations in Eqs. (4.3), (4.6), (4.9) and (4.12) define the minimum and maximum failure time values in the corresponding focal elements and are not intended to imply that focal elements are intervals.

In the event that the evidence spaces $(\mathcal{T}_{SL1}, \mathbb{T}_{SL1}, m_{SL1})$, $(\mathcal{T}_{SL2}, \mathbb{T}_{SL2}, m_{SL2})$, $(\mathcal{T}_{WL1}, \mathbb{T}_{WL1}, m_{WL1})$ or $(\mathcal{T}_{WL2}, \mathbb{T}_{WL2}, m_{WL2})$ have focal elements that that do not always result in link failure, then the corresponding focal elements for link failure time will include an indicator variable t_∞ as indicated in Sect. 2.4. Conceptually, this inclusion occurs in Eqs. (4.1), (4.4), (4.7) and (4.10) with an assignment of t_∞ to $T_{SL1}^{-1}(T)$, $T_{SL2}^{-1}(T)$, $T_{WL1}^{-1}(T)$ or $T_{WL2}^{-1}(T)$ if the corresponding link does not fail at temperature *T*. The same numeric value is assumed to be used for all occurrences of t_∞. This is important because LOAS occurs only if the SL system fails before WL system. Specifically, LOAS is assumed to not occur if the SL system and the WL system fail at the same time. Thus, if values for t_∞ occur in an analysis for both SL system failure time and WL system failure time, it is important that t_∞ have the same the same value for both systems so that the equality of the assigned values of t_∞ will indicate that LOAS does not occur.

For the present section (i.e., Sect. 4), the evidence spaces $(\mathcal{TM}_{SL1}, \mathbb{TM}_{SL1}, m_{SL1,t})$, $(\mathcal{TM}_{SL2}, \mathbb{TM}_{SL2}, m_{SL2,t})$ and $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1,t})$ for link failure time are combined to produce the product evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ with

$$\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}, \tag{4.13}$$

$$\mathcal{TM}_{ijk} = \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \times \mathcal{TM}_{WL1,k} \in \mathbb{TM}, \tag{4.14}$$

$$\mathbb{TM} = \{\mathcal{TM}_{ijk} : (i, j, k) \in \mathcal{I} = \{1, 2, ..., nSL1\} \times \{1, 2, ..., nSL2\} \times \{1, 2, ..., nWL1\}\} \tag{4.15}$$

and

$$m_{TM}(\mathcal{TM}_{ijk}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{ijk}. \tag{4.16}$$

Example links that will be used for illustration are defined and illustrated in Table 4.1 and Fig. 4.1.

Table 4.1 Definition of 2 WLs and 2 SLs used for illustration.

Link Temperature Function

$$T(t) = \frac{T_0 T_\infty}{T_0 + (T_\infty - T_0)\exp(-rt)}, 0 \le t \le 200 \text{ min}$$

SL 1 Properties

Temperature Function $T_{SL1}(t): T(0) = T_0 = 100\ ^\circ\text{C}, T_\infty = 1100\ ^\circ\text{C}, r = 0.04 \text{ min}^{-1}$

Failure Temperature Focal Elements: $\mathcal{T}_{SL1,1} = [600, 780\ ^\circ\text{C}], \mathcal{T}_{SL1,2} = [625, 875\ ^\circ\text{C}], \mathcal{T}_{SL1,3} = [675, 840\ ^\circ\text{C}],$

$\mathcal{T}_{SL1,4} = [850, 975\ ^\circ\text{C}], \mathcal{T}_{SL1,5} = [925, 1050\ ^\circ\text{C}]$

BPAs: $m_{SL1}(\mathcal{T}_{SL1,1}) = 0.1, m_{SL1}(\mathcal{T}_{SL1,2}) = 0.1, m_{SL1}(\mathcal{T}_{SL1,3}) = 0.2, m_{SL1}(\mathcal{T}_{SL1,4}) = 0.2, m_{SL1}(\mathcal{T}_{SL1,5}) = 0.4$

Failure Time Focal Elements: $\mathcal{TM}_{SL1,1} = [62.12, 79.84 \text{ min}], \mathcal{TM}_{SL1,2} = [64.43, 91.52 \text{ min}],$

$\mathcal{TM}_{SL1,3} = [69.13, 86.88 \text{ min}], \mathcal{TM}_{SL1,4} = [88.16, 108.9 \text{ min}], \mathcal{TM}_{SL1,5} = [99.19, 133.7 \text{ min}]$

SL 2 Properties

Temperature Function $T_{SL2}(t): T(0) = T_0 = 100\ ^\circ\text{C}, T_\infty = 950\ ^\circ\text{C}, r = 0.045 \text{ min}^{-1}$

Failure Temperature Focal Elements: $\mathcal{T}_{SL2,1} = [590, 790\ ^\circ\text{C}], \mathcal{T}_{SL2,2} = [640, 910\ ^\circ\text{C}], \mathcal{T}_{SL2,3} = [800, 1000\ ^\circ\text{C}],$

$\mathcal{T}_{SL2,4} = [870, 940\ ^\circ\text{C}], \mathcal{T}_{SL2,5} = [975, 1175^\circ\text{C}]$

BPAs: $m_{SL2}(\mathcal{T}_{SL2,1}) = 0.1, m_{SL2}(\mathcal{T}_{SL2,2}) = 0.1, m_{SL2}(\mathcal{T}_{SL2,3}) = 0.4, m_{SL2}(\mathcal{T}_{SL2,4}) = 0.2, m_{SL2}(\mathcal{T}_{SL2,5}) = 0.2$

Failure Time Focal Elements: $\mathcal{TM}_{SL2,1} = [58.54, 83.04 \text{ min}], \mathcal{TM}_{SL2,2} = [63.67, 117.0 \text{ min}],$

$\mathcal{TM}_{SL2,3} = [84.76, 200 \text{ min}] \cup t_\infty, \mathcal{TM}_{SL2,4} = [100.6, 148.5 \text{ min}], \mathcal{TM}_{SL2,5} = [t_\infty, t_\infty \text{ min}]$ with $t_\infty = 10^7$

WL 1 Properties

Temperature Function $T_{WL1}(t): T(0) = T_0 = 100\ ^\circ\text{C}, T_\infty = 1000\ ^\circ\text{C}, r = 0.035 \text{ min}^{-1}$

Failure Temperature Focal Elements: $\mathcal{T}_{WL1,1} = [500, 700\ ^\circ\text{C}], \mathcal{T}_{WL1,2} = [550, 750\ ^\circ\text{C}], \mathcal{T}_{WL1,3} = [650, 860\ ^\circ\text{C}],$

$\mathcal{T}_{WL1,4} = [825, 1025\ ^\circ\text{C}], \mathcal{T}_{WL1,5} = [880, 980\ ^\circ\text{C}]$

BPAs: $m_{WL1}(\mathcal{T}_{WL1,1}) = 0.4, m_{WL1}(\mathcal{T}_{WL1,2}) = 0.2, m_{WL1}(\mathcal{T}_{WL1,3}) = 0.2, m_{WL1}(\mathcal{T}_{SL1,4}) = 0.1, m_{WL1}(\mathcal{T}_{WL1,5}) = 0.1$

Failure Time Focal Elements: $\mathcal{TM}_{WL1,1} = [62.78, 86.99 \text{ min}], \mathcal{TM}_{WL1,2} = [68.51, 94.17 \text{ min}],$

$\mathcal{TM}_{WL1,3} = [80.47, 114.6 \text{ min}], \mathcal{TM}_{WL1,4} = [107.1, 200 \text{ min}] \cup t_\infty, \mathcal{TM}_{WL1,5} = [119.7, 174.0 \text{ min}]$ with $t_\infty = 10^7$

WL 2 Properties

Temperature Function $T_{WL2}(t): T(0) = T_0 = 100\ ^\circ\text{C}, T_\infty = 900\ ^\circ\text{C}, r = 0.034 \text{ min}^{-1}$

Failure Temperature Focal Elements: $\mathcal{T}_{WL2,1} = [490, 650\ ^\circ\text{C}], \mathcal{T}_{WL2,2} = [525, 725\ ^\circ\text{C}], \mathcal{T}_{WL2,3} = [575, 680\ ^\circ\text{C}],$

$\mathcal{T}_{WL2,4} = [700, 950\ ^\circ\text{C}], \mathcal{T}_{WL2,5} = [775, 875\ ^\circ\text{C}]$

BPAs: $m_{WL2}(\mathcal{T}_{WL2,1}) = 0.4, m_{WL2}(\mathcal{T}_{WL2,2}) = 0.2, m_{WL2}(\mathcal{T}_{WL2,3}) = 0.2, m_{WL2}(\mathcal{T}_{WL2,4}) = 0.1, m_{WL2}(\mathcal{T}_{WL2,5}) = 0.1$

Failure Time Focal Elements: $\mathcal{TM}_{WL2,1} = [66.40, 89.26 \text{ min}], \mathcal{TM}_{WL2,2} = [71.06, 103.0 \text{ min}],$

$\mathcal{TM}_{WL2,3} = [77.94, 94.35 \text{ min}], \mathcal{TM}_{WL2,4} = [98.01, 200 \text{ min}] \cup t_\infty, \mathcal{TM}_{WL2,5} = [114.8, 165.7 \text{ min}]$ with $t_\infty = 10^7$

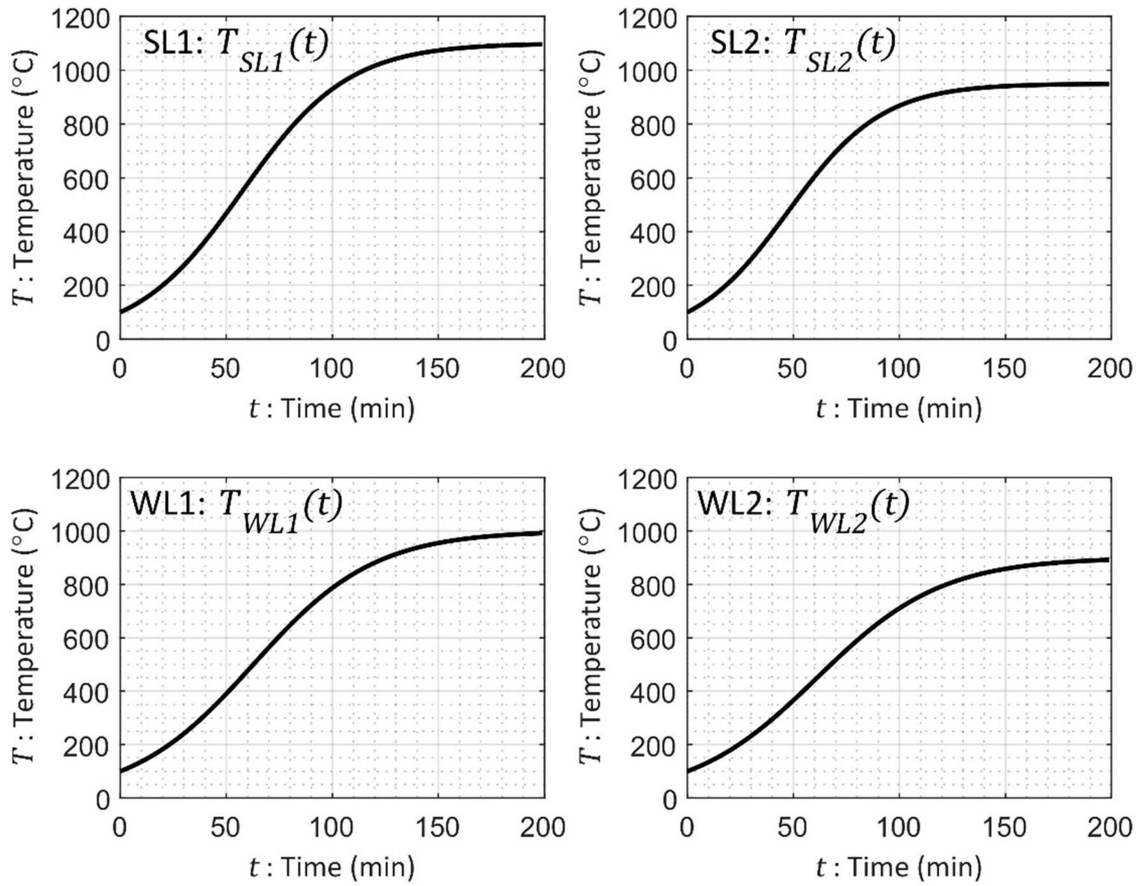


Fig. 4.1 Temperature functions $T_{SL1}(t)$, $T_{SL2}(t)$, $T_{WL1}(t)$ and $T_{WL2}(t)$ defined in Table 4.1.

As an additional illustration, the CPF, CBF and CDF for the link failure temperatures defined in Table 4.1 are shown in Fig. 4.2. The CPFs, CBFs and CDFs in Fig. 4.2 are generated in the same manner as used to generate the CPF, CBF and CDF in Fig. 2.2. Specifically, the CPFs and CBFs are generated with the computational procedure described in conjunction with Eqs. (2.55)-(2.57), and the CDFs are generated by assigning a uniform distribution to each failure temperature focal element as described in conjunction with Eqs. (2.16)-(2.21).

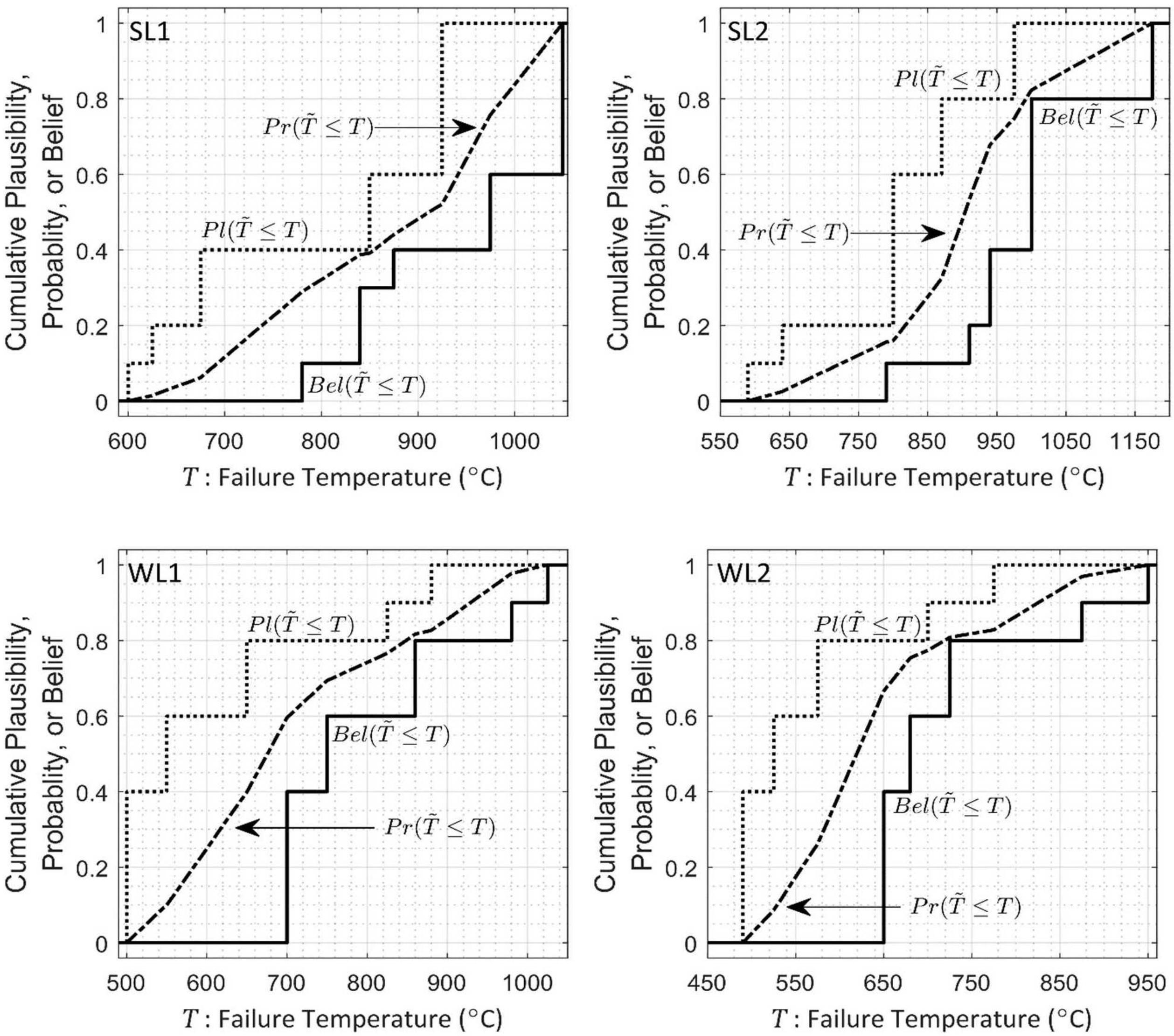


Fig. 4.2 Summary CPFs, CBFs and CDFs for the link failure temperatures defined in Table 4.1: (a) SL 1, (b) SL 2, (c) WL 1, and (d) WL 2.

The CPFs, CBFs and CDFs for the link failure times that result from the link failure temperatures summarized in Table 4.1 and the link temperature functions defined in Table 4.1 and illustrated in Fig. 4.1 are shown in Fig. 4.3. The CPFs and CBFs in Fig. 4.3 are constructed as indicated in conjunction with Eqs. (2.48)-(2.50) with the focal elements for link failure time defined in Table 4.1.

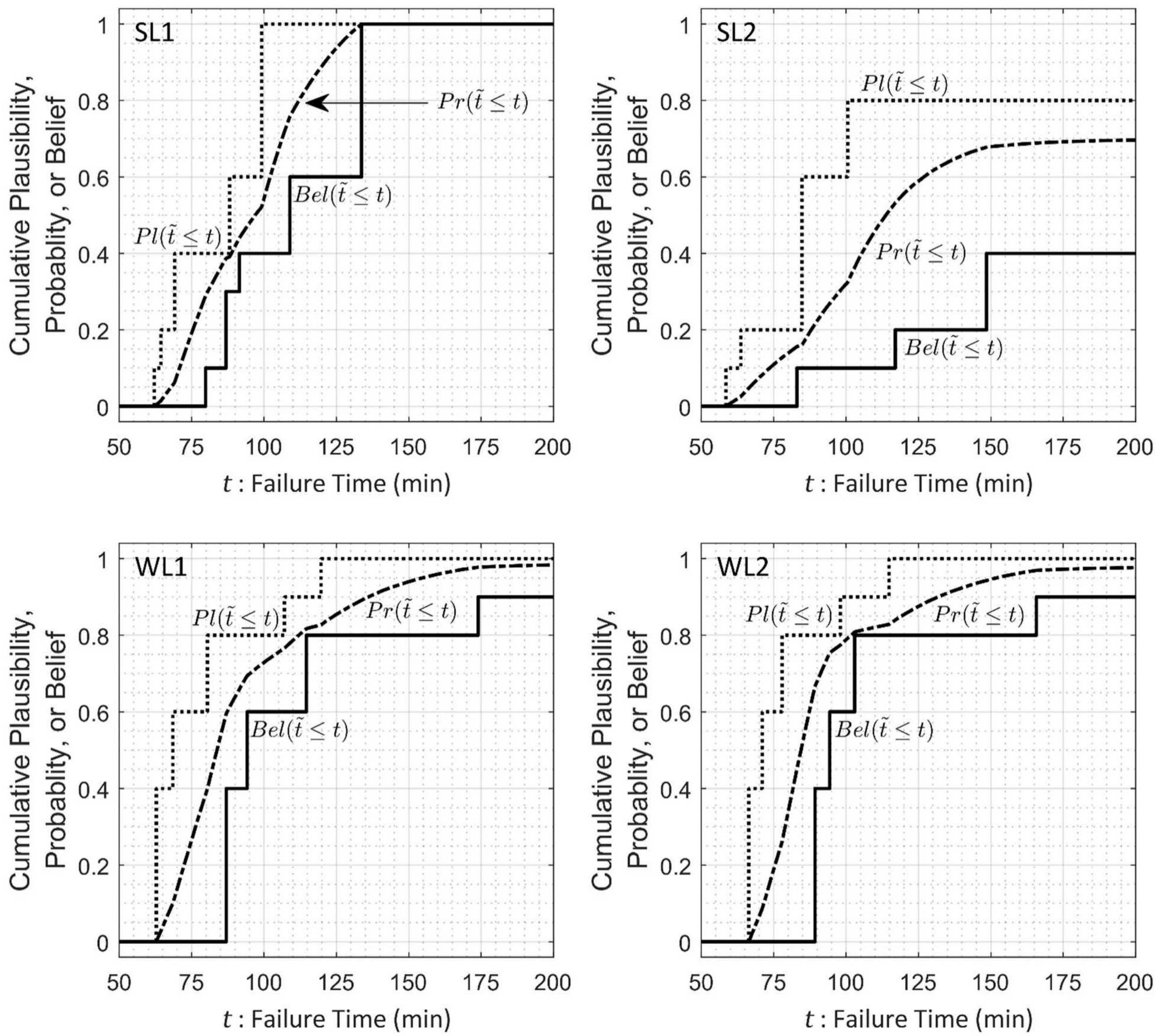


Fig. 4.3 Summary CPFs, CBFs and CDFs for the link failure times that result from the link temperature functions defined in Table 1 and the link failure temperatures summarized in Table 1 and Fig. 4.2: (a) SL 1, (b) SL 2, (c) WL 1, and (d) WL 2.

There are two ways in which the CDFs in Fig. 4.3 can be constructed. One way is to (i) generate a large random sample

$$T_i, i = 1, 2, ..., nR, \tag{4.17}$$

of link failure temperatures from the link failure temperature CDF for the link under consideration, (ii) determine the link failure time $t(T_i)$ for each sampled failure temperature, and (iii) and approximate the CDF by

$$CDF(t) = prob(\tilde{t} \le t) \cong \sum_{i=1}^{nR} \underline{\delta}_t[t(T_i)] / nR \;\; \text{with} \;\; \underline{\delta}_t[t(T_i)] = \begin{cases} 1 \text{ for } t(T_i) \le t \\ 0 \text{ otherwise.} \end{cases} \tag{4.18}$$

For a link temperature curve $T(t)$ defined in Table 4.1 and illustrated in Fig. 4.1,

$$t(T_i) = \begin{cases} (-1/r)\ln[T_0(T_\infty - T_i)/T_i(T_\infty - T_0)] \;\; \text{for} \;\; T(0) \le T_i \le T(200) \\ t_\infty \;\; \text{for} \;\; T(200) < T_i \end{cases} \tag{4.19}$$

as indicated in Eq. (2.73). Another way to generate the CDFs in Fig. 4.3 is to define an appropriate density function over the sample space for link failure time and then integrate this density function to obtain the desired CDF. However, care is required with this approach to appropriately incorporate (i) focal elements for link failure time that may not be closed intervals when the link failure temperature curve is not increasing and (ii) nonzero probabilities associated with the place holder times t_∞ used to indicate nonfailure of a link. The CDFs in Fig. 4.3 were generated with the indicated sampling-based approach in Eqs. (4.17)-(4.19) with a sample of size $nR = 10^5$.

Evidence spaces $(\mathcal{TF}_{SL1}, \mathbb{TF}_{SL1}, m_{SL1,TF})$, $(\mathcal{TF}_{SL2}, \mathbb{TF}_{SL2}, m_{SL2,TF})$, $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{WL1,TF})$ and $(\mathcal{TF}_{WL2}, \mathbb{TF}_{WL2}, m_{WL2,TF})$ for the actual temperatures at which SL 1, SL 2, WL 1 and WL 2 fail can also be defined. These focal elements have a role in the determination evidence spaces for WL/SL failure temperature margins in Sect. 12. As an example, the evidence space $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$ for the temperatures at which at which WL 1 fails is defined by the following transformations of the evidence space $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1,t})$ for WL 1 failure times:

$$\mathcal{TF}_{WL1} = \{T : T = T_{WL1}(t) \;\; \text{for} \;\; t \in \mathcal{TM}_{WL1}\}, \tag{4.20}$$

$$\mathcal{TF}_{WL1,k} = \{T : T = T_{WL1}(t) \;\; \text{for} \;\; t \in \mathcal{TM}_{WL1,k}\}, \tag{4.21}$$

$$\mathbb{TF}_{WL1} = \{\mathcal{TF}_{WL1,k}, k = 1, 2, ..., nWL1\}, \tag{4.22}$$

$$m_{TF,WL1}(\mathcal{TF}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1}(\mathcal{T}_{WL1,k}) \;\; \text{for} \;\; k = 1, 2, ..., nWL1, \tag{4.23}$$

$$(\underline{TF}_{WL1,k}, \overline{TF}_{WL1,k}) = \left(\min(\mathcal{TF}_{WL1,k}), \max(\mathcal{TF}_{WL1,k})\right) = \left(T_{WL1}(\underline{t}_{WL1,k}), T_{WL1}(\overline{t}_{WL1,k})\right) \tag{4.24}$$

with

$$T_{WL1}(t) = \infty \;\; \text{for} \;\; t = \infty \tag{4.25}$$

corresponding to no link failure. The evidence spaces $(\mathcal{TF}_{SL1}, \mathbb{TF}_{SL1}, m_{SL1,TF})$, $(\mathcal{TF}_{SL2}, \mathbb{TF}_{SL2}, m_{SL2,TF})$, and $(\mathcal{TF}_{WL2}, \mathbb{TF}_{WL2}, m_{WL2,TF})$ are defined similarly through transformations of the evidence spaces $(\mathcal{TM}_{SL1}, \mathbb{TM}_{SL1}, m_{SL1,t})$, $(\mathcal{TM}_{SL2}, \mathbb{TM}_{SL2}, m_{SL2,t})$, and $(\mathcal{TM}_{WL2}, \mathbb{TM}_{WL2}, m_{WL2,t})$.

The focal elements associated with the evidence spaces $(\mathcal{TF}_{SL1}, \mathbb{TF}_{SL1}, m_{SL1,TF})$, $(\mathcal{TF}_{SL2}, \mathbb{TF}_{SL2}, m_{SL2,TF})$, $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{WL1,TF})$ and $(\mathcal{TF}_{WL2}, \mathbb{TF}_{WL2}, m_{WL2,TF})$ for the example links defined in Table 4.1 are summarized below:

(i) $\mathcal{TF}_{SL2,1} = [600, 780\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,2} = [625, 875\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,3} = [675, 840\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,4} = [850, 975\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,5} = [925, 1050^\circ\mathrm{C}]$ for $(\mathcal{TF}_{SL1}, \mathbb{TF}_{SL1}, m_{SL1,TF})$.

(ii) $\mathcal{TF}_{SL2,1} = [590, 790\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,2} = [640, 910\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,3} = [800, 950\,^\circ\mathrm{C}] \cup T_\infty$, $\mathcal{TF}_{SL2,4} = [870, 940\,^\circ\mathrm{C}]$, $\mathcal{TF}_{SL2,5} = [T_\infty, T_\infty{}^\circ\mathrm{C}]$ with $T_\infty = \infty$ for $(\mathcal{TF}_{SL2}, \mathbb{TF}_{SL2}, m_{SL2,TF})$.

(iii) $\mathcal{TF}_{WL1,1} = [500, 700\,^\circ\mathrm{C}]$, $\mathcal{TF}_{WL1,2} = [550, 750\,^\circ\mathrm{C}]$, $\mathcal{TF}_{WL1,3} = [650, 860\,^\circ\mathrm{C}]$, $\mathcal{TF}_{WL1,4} = [825, 1000\,^\circ\mathrm{C}] \cup T_\infty$, $\mathcal{TF}_{WL1,5} = [880, 980\,^\circ\mathrm{C}]$ for $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{WL1,TF})$.

(iv) $\mathcal{TF}_{WL2,1} = [490, 650\,^\circ\mathrm{C}]$, $\mathcal{TF}_{WL2,2} = [525, 725\,^\circ\mathrm{C}]$, $\mathcal{TF}_{WL2,3} = [575, 680\,^\circ\mathrm{C}]$, $\mathcal{TF}_{WL2,4} = [700, 900\,^\circ\mathrm{C}] \cup T_\infty$, $\mathcal{TF}_{WL2,5} = [775, 875\,^\circ\mathrm{C}]$ for $(\mathcal{TF}_{WL2}, \mathbb{TF}_{WL2}, m_{WL2,TF})$.

Further, the CPFs, CBFs and CDFs for the actual link failure temperatures that result from the link failure temperatures summarized in Table 4.1and the link temperature functions defined in Table 4.1 and illustrated in Fig. 4.1 are shown in Fig. 4.4. The CPFs, CBFs and CDFs in Fig. 4.4 are constructed in the same manner as the CPFs, CBFs and CDFs in Fig. 4.3.

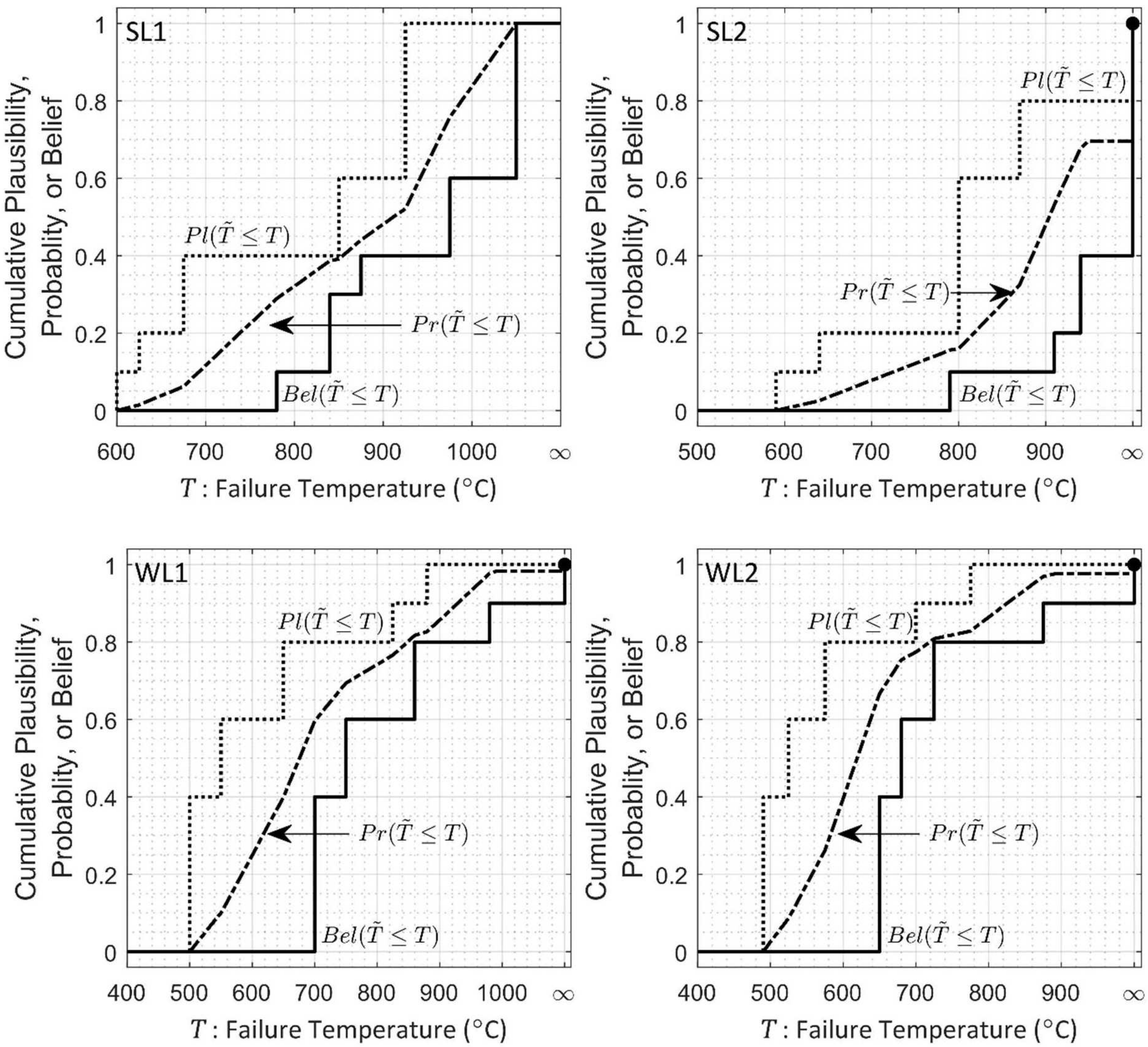


Fig. 4.4 Summary CPFs, CBFs and CDFs for the link failure temperatures that result from the link temperature functions defined in Table 4.1 and the link failure temperatures summarized in Table 4.1 and Fig. 4.2: (a) SL 1, (b) SL 2, (c) WL 1, and (d) WL 2.

5. Representation of LOAS with Evidence Theory for a WL-SL System with 2 SLs and 1 WL

The developments in this section are for a WL-SL system with 2 SLs and 1 WL. Two possibilities for the definition of LOAS are considered: (i) failure of both SLs before failure of the WL, and (ii) failure of either SL before failure of the WL.

5.1 LOAS Defined by Failure of Both SLs before failure of the WL

The occurrence of LOAS defined by the failure of both SLs before the failure of the WL is considered in this section. Specifically, LOAS is assumed to occur for elements of the set

$$\mathcal{L}_1 = \left\{ (t_{SL1}, t_{SL2}, t_{WL}) : (t_{SL1}, t_{SL2}, t_{WL1}) \in \mathcal{TM} \text{ with } \max\{t_{SL1}, t_{SL2}\} < t_{WL1} \right\}. \tag{5.1}$$

In turn, the belief $Bel(\mathcal{L}_1)$ for the occurrence of LOAS is given by

$$Bel(\mathcal{L}_1) = \sum_{\mathcal{TM}_{ijk} \subseteq \mathcal{L}_1} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{B1}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.2}$$

with

$$\delta_{B1}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \mathcal{TM}_{ijk} \subseteq \mathcal{L}_1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \underline{t}_{WL1,k} \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

defined to pick out the elements of $\mathbb{TM}$ that are subsets of $\mathcal{L}_1$. Similarly, the plausibility $Pl(\mathcal{L}_1)$ for the occurrence of LOAS is given by

$$Pl(\mathcal{L}_1) = \sum_{\emptyset \neq \mathcal{TM}_{ijk} \cap \mathcal{L}_1} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{P1}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.4}$$

with

$$\delta_{P1}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \emptyset \neq \mathcal{TM}_{ijk} \cap \mathcal{L}_1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k} \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

defined to pick out the elements of $\mathbb{TM}$ that intersect $\mathcal{L}_1$.

As examples, the calculation of $Bel(\mathcal{L}_1)$ and $Pl(\mathcal{L}_1)$ for the links defined in Table 4.1 and Fig. 4.1 yields the results

$$Bel(\mathcal{L}_1) = \begin{cases} 1.600 \times 10^{-2} & \text{for SL 1, SL 2, WL 1} \\ 1.000 \times 10^{-2} & \text{for SL 1, SL 2, WL 2} \end{cases}$$
$$\cong \begin{cases} 1.602 \times 10^{-2} & \text{for SL 1, SL 2, WL 1} \\ 1.005 \times 10^{-2} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.6}$$

and

$$Pl(\mathcal{L}_1) = \begin{cases} 4.880 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 5.360 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases}$$
$$\cong \begin{cases} 4.881 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 5.361 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.7}$$

with (i) the values for $Bel(\mathcal{L}_1)$ and $Pl(\mathcal{L}_1)$ in the initial equalities determined as indicated in Eqs. (5.2) and (5.4) and (ii) the values for $Bel(\mathcal{L}_1)$ and $Pl(\mathcal{L}_1)$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.2. The agreement of the two computational procedures provides a strong verification result that $Bel(\mathcal{L}_1)$ and $Pl(\mathcal{L}_1)$ are being calculated correctly.

Another analysis outcome of possible interest is an assessment of which SL is the final SL to fail when LOAS occurs. Specifically, belief and plausibility can be determined for the final SL to fail when LOAS occurs. This determination corresponds to determining belief and plausibility for the sets

$$\mathcal{L}_{1;1\le 2} = \{[t_{SL1}, t_{SL2}, t_{WL1}] : [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM} \text{ with } t_{SL1} \le t_{SL2} < t_{WL1}\} \tag{5.8}$$

and

$$\mathcal{L}_{1;2\le 1} = \{[t_{SL1}, t_{SL2}, t_{WL1}] : [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM} \text{ with } t_{SL2} \le t_{SL1} < t_{WL1}\}. \tag{5.9}$$

Specifically, belief and plausibility for the occurrence of LOAS with SL 2 being the last SL to fail are given by

$$Bel(\mathcal{L}_{1;1\le 2}) = \sum_{\mathcal{TM}_{ijk} \subseteq \mathcal{L}_{1;1\le 2}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{B1;1\le 2}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.10}$$

$$Pl(\mathcal{L}_{1;1\le 2}) = \sum_{\varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{1;1\le 2}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{P1;1\le 2}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.11}$$

with

$$\delta_{B1;1\le 2}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \mathcal{TM}_{ijk} \subseteq \mathcal{L}_{1;1\le 2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \overline{t}_{SL1,i} \le \underline{t}_{SL2,j} \text{ and } \overline{t}_{SL2,j} < \underline{t}_{WL1,k} \\ 0 & \text{otherwise,} \end{cases} \tag{5.12}$$

$$\delta_{P1;1\le 2}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{1;1\le 2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \underline{t}_{SL1,i} \le \underline{t}_{SL2,j} < \overline{t}_{WL1,k} \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}$$

Similarly, belief and plausibility for the occurrence of LOAS with SL 1 being the last SL to fail are given by

$$Bel(\mathcal{L}_{1;2\le 1}) = \sum_{\mathcal{TM}_{ijk} \subseteq \mathcal{L}_{1;2\le 1}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{B1;2\le 1}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.14}$$

$$Pl(\mathcal{L}_{1;2\le 1}) = \sum_{\varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{1;2\le 1}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{P1;2\le 1}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.15}$$

with

$$\delta_{B1;2\le 1}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \mathcal{TM}_{ijk} \subseteq \mathcal{L}_{1;2\le 1} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \overline{t}_{SL2,j} \le \underline{t}_{SL1,i} \text{ and } \overline{t}_{SL1,i} < \underline{t}_{WL1,k} \\ 0 & \text{otherwise,} \end{cases} \tag{5.16}$$

$$\delta_{P1;2\le 1}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{1;2\le 1} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \underline{t}_{SL2,j} \le \underline{t}_{SL1,i} < \overline{t}_{WL1,k} \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

As examples, the calculation of $Bel(\mathcal{L}_{1;1\le 2})$, $Pl(\mathcal{L}_{1;1\le 2})$, $Bel(\mathcal{L}_{1;2\le 1})$ and $Pl(\mathcal{L}_{1;2\le 1})$ for the links defined in Table 4.1 and Fig. 4.1 yields the results

$$\begin{aligned} Bel(\mathcal{L}_{1;1\le 2}) &= \begin{cases} 0.000 \times 10^{0} & \text{for SL 1, SL 2, WL 1} \\ 0.000 \times 10^{0} & \text{for SL 1, SL 2, WL 2} \end{cases} \\ &\cong \begin{cases} 0.000 \times 10^{0} & \text{for SL 1, SL 2, WL 1} \\ 0.0000 \times 10^{0} & \text{for SL 1, SL 2, WL 2,} \end{cases} \end{aligned} \tag{5.18}$$

$$\begin{aligned} Pl(\mathcal{L}_{1;1\le 2}) &= \begin{cases} 2.500 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 2.500 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \\ &\cong \begin{cases} 2.501 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 2.502 \times 10^{-1} & \text{for SL 1, SL 2, WL 2,} \end{cases} \end{aligned} \tag{5.19}$$

$$Bel(\mathcal{L}_{1;2\le 1}) = \begin{cases} 2.000\times 10^{-3} & \text{for SL 1, SL 2, WL 1} \\ 2.000\times 10^{-3} & \text{for SL 1, SL 2, WL 2} \end{cases}$$
$$\cong \begin{cases} 1.981\times 10^{-3} & \text{for SL 1, SL 2, WL 1} \\ 2.010\times 10^{-3} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.20}$$

and

$$Pl(\mathcal{L}_{1;2\le 1}) = \begin{cases} 2.380\times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 2.860\times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases}$$
$$\cong \begin{cases} 2.380\times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 2.862\times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.21}$$

with (i) the values for $Bel(\mathcal{L}_{1;1\le 2})$, $Pl(\mathcal{L}_{1;1\le 2})$, $Bel(\mathcal{L}_{1;2\le 1})$ and $Pl(\mathcal{L}_{1;2\le 1})$ in the initial equalities determined as indicated in Eqs. (5.10), (5.11), (5.14) and (5.15), and (ii) the values for $Bel(\mathcal{L}_{1;1\le 2})$, $Pl(\mathcal{L}_{1;1\le 2})$, $Bel(\mathcal{L}_{1;2\le 1})$ and $Pl(\mathcal{L}_{1;2\le 1})$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.3. The agreement of the two computational procedures provides a strong verification result that $Bel(\mathcal{L}_{1;1\le 2})$, $Pl(\mathcal{L}_{1;1\le 2})$, $Bel(\mathcal{L}_{1;2\le 1})$ and $Pl(\mathcal{L}_{1;2\le 1})$ are being calculated correctly.

5.2 LOAS Defined by Failure of Either SL before Failure of the WL

The occurrence of LOAS defined by the failure of a single SL before the failure of the WL is considered in this section. Specifically, LOAS is assumed to occur for elements of the set

$$\mathcal{L}_2 = \left\{(t_{SL1}, t_{SL2}, t_{WL1}) : (t_{SL1}, t_{SL2}, t_{WL1}) \in \mathcal{TM} \text{ with } \min\{t_{SL1}, t_{SL2}\} < t_{WL1}\right\}. \tag{5.22}$$

In turn, the belief $Bel(\mathcal{L}_2)$ for the occurrence of LOAS is given by

$$Bel(\mathcal{L}_2) = \sum_{\mathcal{TM}_{ijk} \subseteq \mathcal{L}_2} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{B2}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.23}$$

with

$$\delta_{B2}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \mathcal{TM}_{ijk} \subseteq \mathcal{L}_2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \underline{t}_{WL1,k} \\ 0 & \text{otherwise} \end{cases} \tag{5.24}$$

defined to pick out the elements of $\mathbb{TM}$ that are subsets of $\mathcal{L}_2$. Similarly, the plausibility $Pl(\mathcal{L}_2)$ for the occurrence of LOAS is given by

$$Pl(\mathcal{L}_2) = \sum_{\varnothing \neq \mathcal{TM}_{ijk} \cap \mathcal{L}_2} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{P2}(\mathcal{TM}_{ijk}) m_{ijk} \tag{5.25}$$

with

$$\delta_{P2}(\mathcal{TM}_{ijk}) = \begin{cases} 1 & \text{for } \varnothing \neq \mathcal{TM}_{ijk} \cap \mathcal{L}_2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k} \\ 0 & \text{otherwise} \end{cases} \tag{5.26}$$

defined to pick out the elements of $\mathbb{TM}$ that intersect $\mathcal{L}_2$.

As examples, the calculation of $Bel(\mathcal{L}_2)$ and $Pl(\mathcal{L}_2)$ for the links defined in Table 4.1 and Fig. 4.1 yields the results

$$\begin{aligned} Bel(\mathcal{L}_2) &= \begin{cases} 1.340 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 1.100 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \\ &\cong \begin{cases} 1.340 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 1.101 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} Pl(\mathcal{L}_2) &= \begin{cases} 8.720 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 9.040 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \\ &\cong \begin{cases} 8.720 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 9.039 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \end{aligned} \tag{5.28}$$

with (i) the values for $Bel(\mathcal{L}_2)$ and $Pl(\mathcal{L}_2)$ in the initial equalities determined as indicated in Eqs. (5.23) and (5.25), and (ii) the values for $Bel(\mathcal{L}_2)$ and $Pl(\mathcal{L}_2)$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.2. The agreement of the two computational procedures provides a strong verification result that $Bel(\mathcal{L}_2)$ and $Pl(\mathcal{L}_2)$ are being calculated correctly.

Similarly to the results presented in Eqs. (5.8)-(5.17), belief and plausibility can be determined for the SL whose failure results in LOAS. This determination corresponds to determining belief and plausibility for the sets

$$\mathcal{L}_{2;1\le 2} = \{[t_{SL1}, t_{SL2}, t_{WL1}] : [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM} \text{ with } t_{SL1} \le t_{SL2} \text{ and } t_{SL1} < t_{WL1}\} \tag{5.29}$$

and

$$\mathcal{L}_{2;2\le 1} = \{[t_{SL1}, t_{SL2}, t_{WL1}] : [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM} \text{ with } t_{SL2} \le t_{SL1} \text{ and } t_{SL2} < t_{WL1}\}. \quad (5.30)$$

Specifically, belief and plausibility for the occurrence of LOAS with SL 1 being the SL whose failure results in LOAS are given by

$$Bel(\mathcal{L}_{2;1\le 2}) = \sum_{\mathcal{TM}_{ijk} \subseteq \mathcal{L}_{2;1\le 2}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{B2;1\le 2}(\mathcal{TM}_{ijk}) m_{ijk}, \quad (5.31)$$

$$Pl(\mathcal{L}_{2;1\le 2}) = \sum_{\varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{2;1\le 2}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{P2;1\le 2}(\mathcal{TM}_{ijk}) m_{ijk} \quad (5.32)$$

with

$$\delta_{B2;1\le 2}(\mathcal{TM}_{ijk}) = \begin{cases} 1 \text{ for } \mathcal{TM}_{ijk} \subseteq \mathcal{L}_{2;1\le 2} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } \overline{t}_{SL1,i} \le \underline{t}_{SL2,j} \text{ and } \overline{t}_{SL1,i} < \underline{t}_{WL1,k} \\ 0 \text{ otherwise,} \end{cases} \quad (5.33)$$

$$\delta_{P2;1\le 2}(\mathcal{TM}_{ijk}) = \begin{cases} 1 \text{ for } \varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{2;1\le 2} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } \underline{t}_{SL1,i} \le \overline{t}_{SL2,j} \text{ and } \underline{t}_{SL1,i} < \overline{t}_{WL1,k} \\ 0 \text{ otherwise.} \end{cases} \quad (5.34)$$

Similarly, belief and plausibility for the occurrence of LOAS with SL 2 being the SL whose failure results in LOAS are given by

$$Bel(\mathcal{L}_{2;2\le 1}) = \sum_{\mathcal{TM}_{ijk} \subseteq \mathcal{L}_{2;2\le 1}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{B2;2\le 1}(\mathcal{TM}_{ijk}) m_{ijk}, \quad (5.35)$$

$$Pl(\mathcal{L}_{2;2\le 1}) = \sum_{\varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{2;2\le 1}} m_{TM}(\mathcal{TM}_{ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_{P2;2\le 1}(\mathcal{TM}_{ijk}) m_{ijk} \quad (5.36)$$

with

$$\delta_{B2;2\le 1}(\mathcal{TM}_{ijk}) = \begin{cases} 1 \text{ for } \mathcal{TM}_{ijk} \subseteq \mathcal{L}_{2;2\le 1} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } \overline{t}_{SL2,j} \le \underline{t}_{SL1,i} \text{ and } \overline{t}_{SL2,j} < \underline{t}_{WL1,k} \\ 0 \text{ otherwise,} \end{cases} \quad (5.37)$$

$$\delta_{P2;2\le 1}(\mathcal{TM}_{ijk}) = \begin{cases} 1 \text{ for } \varnothing \ne \mathcal{TM}_{ijk} \cap \mathcal{L}_{2;2\le 1} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } \underline{t}_{SL2,j} \le \overline{t}_{SL1,i} \text{ and } \underline{t}_{SL2,j} < \overline{t}_{WL1,k} \\ 0 \text{ otherwise.} \end{cases} \quad (5.38)$$

As examples, the calculation of $Bel(\mathcal{L}_{2;1\le 2})$, $Pl(\mathcal{L}_{2;1\le 2})$, $Bel(\mathcal{L}_{2;2\le 1})$ and $Pl(\mathcal{L}_{2;2\le 1})$ for the links defined in Table 4.1 and Fig. 4.1 yields the results

$$Bel(\mathcal{L}_{2;1\le 2}) = \begin{cases} 6.000 \times 10^{-2} & \text{for SL 1, SL 2, WL 1} \\ 4.400 \times 10^{-2} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.39}$$
$$\cong \begin{cases} 6.002 \times 10^{-2} & \text{for SL 1, SL 2, WL 1} \\ 4.401 \times 10^{-2} & \text{for SL 1, SL 2, WL 2,} \end{cases}$$

$$Pl(\mathcal{L}_{2;1\le 2}) = \begin{cases} 6.520 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 7.240 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.40}$$
$$\cong \begin{cases} 6.522 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 7.241 \times 10^{-1} & \text{for SL 1, SL 2, WL 2,} \end{cases}$$

$$Bel(\mathcal{L}_{2;2\le 1}) = \begin{cases} 1.200 \times 10^{-2} & \text{for SL 1, SL 2, WL 1} \\ 1.200 \times 10^{-2} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.41}$$
$$\cong \begin{cases} 1.203 \times 10^{-2} & \text{for SL 1, SL 2, WL 1} \\ 1.199 \times 10^{-2} & \text{for SL 1, SL 2, WL 2} \end{cases}$$

and

$$Pl(\mathcal{L}_{2;2\le 1}) = \begin{cases} 6.080 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 6.080 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases} \tag{5.42}$$
$$\cong \begin{cases} 6.083 \times 10^{-1} & \text{for SL 1, SL 2, WL 1} \\ 6.078 \times 10^{-1} & \text{for SL 1, SL 2, WL 2} \end{cases}$$

with (i) the values for $Bel(\mathcal{L}_{2;1\le 2})$, $Pl(\mathcal{L}_{2;1\le 2})$, $Bel(\mathcal{L}_{2;2\le 1})$ and $Pl(\mathcal{L}_{2;2\le 1})$ in the initial equalities determined as indicated in Eqs. (5.31), (5.32), (5.35) and (5.36), and (ii) the values for $Bel(\mathcal{L}_{2;1\le 2})$, $Pl(\mathcal{L}_{2;1\le 2})$, $Bel(\mathcal{L}_{2;2\le 1})$ and $Pl(\mathcal{L}_{2;2\le 1})$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.3. The agreement of the two computational procedures provides a strong verification result that $Bel(\mathcal{L}_{2;1\le 2})$, $Pl(\mathcal{L}_{2;1\le 2})$, $Bel(\mathcal{L}_{2;2\le 1})$ and $Pl(\mathcal{L}_{2;2\le 1})$ are being calculated correctly.

6. Sampling-Based Verification

6.1 Background

As discussed and illustrated in Refs. [71; 72] for WL/SL systems and in Refs. [73-82] for many additional contexts, model/analysis verification based on the comparison of results obtained in two independent analyses is an important part of the assessment of models and software used in the analysis of high consequence systems. Model verification and model validation are two related, but different and often confused, concepts. Two widely used definitions are (Ref. [82], p. 3):

Verification: The process of determining that a model implementation accurately represents the developers' conceptual description of the model and the solution of the model.

Validation: The process of determining the degree to which a model is an accurate representation of the real world from the perspective of the intended uses of the model.

Thus, verification relates to assessing the correctness of the mathematical development and implementation of a model. It is in this sense that verification is used in this presentation.

With respect to verification, it is possible to define a way to calculate $Bel(\mathcal{L}_1)$, $Pl(\mathcal{L}_1)$, $Bel(\mathcal{L}_2)$ and $Pl(\mathcal{L}_2)$ that is independent of the computational procedures defined in Sects. 5.1 and 5.2. This alternative computational procedure is based on the following previously discussed evidence space properties:

(i) An evidence space is simply an incompletely defined probability space.

(ii) The belief and plausibility of a set as defined for an evidence space correspond to the smallest and largest probabilities that can be assigned to this set for probability spaces that are consistent with the evidence space.

Thus, if a probability space can be defined that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and has the smallest possible probabilities $p(\mathcal{L}_1)$ and $p(\mathcal{L}_2)$ for the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ defined in Eqs. (5.1) and (5.22), then the probabilities $p(\mathcal{L}_1)$ and $p(\mathcal{L}_2)$ obtained in a probabilistically-based calculation should be the same as $Bel(\mathcal{L}_1)$ and $Bel(\mathcal{L}_2)$ obtained as indicated in Eqs. (5.2) and (5.23). Similarly, if a probability space can be defined that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and has the largest possible probabilities $p(\mathcal{L}_1)$ and $p(\mathcal{L}_2)$ for $\mathcal{L}_1$ and $\mathcal{L}_2$, then then the probabilities $p(\mathcal{L}_1)$ and $p(\mathcal{L}_2)$ obtained in a probabilistically-based calculation should be the same as $Pl(\mathcal{L}_1)$ and $Pl(\mathcal{L}_2)$ obtained as indicated in Eqs. (5.4) and (5.25).

For most evidence space problems, defining the indicated probability spaces is too difficult to provide an effective verification procedure. However, such definition is possible for the WL/SL problem under consideration. Specifically, the likelihood of LOAS goes (i) down as the time of

WL failure decreases and the time of SL failure increases and (ii) up as the time of WL failure increases and the time of SL failure decreases.

6.2 Verification for $Bel(\mathcal{L}_1)$ $Bel(\mathcal{L}_2)$, $Pl(\mathcal{L}_1)$ and $Pl(\mathcal{L}_2)$

To obtain the desired probability space for checking the calculation of $Bel(\mathcal{L}_1)$ and $Bel(\mathcal{L}_2)$ with the representations defined in Eqs. (5.1) and (5.22), (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$) is assigned to $\underline{t}_{WL1,k}$, with the result that

$$p(\underline{t}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k} \tag{6.1}$$

(ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$ and $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$, with the result that

$$p(\overline{t}_{SL1,i}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i} \tag{6.2}$$

$$p(\overline{t}_{SL2,j}) = m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2,j}, \tag{6.3}$$

and (iii) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}]$. This produces the probability space that has the smallest possible probabilities for the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$p([\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}]) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{ijk} \tag{6.4}$$

for (i, j, k) belonging to the set $\mathcal{I}$ defined in Eq. (4.15) and (ii) any set that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}]$ has a probability of zero. Then, with a large random sample

$$[\overline{tSL1_r}, \overline{tSL2_r}, \underline{tWL1_r}], r = 1, 2, ..., nR, \tag{6.5}$$

of size nR from the failure time vectors $[\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}]$ generated consistent with the probabilities m_{ijk}, $Bel(\mathcal{L}_1)$ and $Bel(\mathcal{L}_2)$ can be approximated by

$$Bel(\mathcal{L}_i) \cong \sum_{r=1}^{nR} \delta_{Bi}([\overline{tSL1_r}, \overline{tSL2_r}, \underline{tWL1_r}]) / nR \cong p(\mathcal{L}_i) \tag{6.6}$$

for $i = 1, 2$ with

$$\delta_{B1}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r]) = \begin{cases} 1 & \text{for } \max\{\overline{tSL1}_r, \overline{tSL2}_r\} < \underline{tWL1}_r \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

and

$$\delta_{B2}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r]) = \begin{cases} 1 & \text{for } \min\{\overline{tSL1}_r, \overline{tSL2}_r\} < \underline{tWL1}_r \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

The estimates for $Bel(\mathcal{L}_1)$ and $Bel(\mathcal{L}_2)$ indicated in Eq. (6.6) are illustrated in Eqs. (5.6) and (5.27) for the links defined Table 4.1 and Fig. 4.1.

Similarly, to get the desired probability space for checking the calculation of $Pl(\mathcal{L}_1)$ and $Pl(\mathcal{L}_2)$, (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$) is assigned to $\overline{t}_{WL1,k}$, with the result that

$$p(\overline{t}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}, \tag{6.9}$$

(ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$ and $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\underline{t}_{SL1,i}$ and $\underline{t}_{SL2,j}$, with the result that

$$p(\underline{t}_{SL1,i}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i} \tag{6.10}$$

$$p(\underline{t}_{SL2,j}) = m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2,j}, \tag{6.11}$$

and (iii) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]$. This produces the probability space that has the largest possible probabilities for the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$p([\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{ijk} \tag{6.12}$$

for $(i, j, k) \in \mathcal{I}$ and (ii) any set that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]$ has a probability of zero. Then, with a large random sample

$$[\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r], r = 1, 2, ..., nR, \tag{6.13}$$

of size nR from the failure time vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]$ generated consistent with the probabilities m_{ijk}, $Pl(\mathcal{L}_1)$ and $Pl(\mathcal{L}_2)$ can be approximated by

$$Pl(\mathcal{L}_i) \cong \sum_{r=1}^{nR} \delta_{Pi}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r]) / nR \cong p(\mathcal{L}_i) \tag{6.14}$$

for $i = 1, 2$ with

$$\delta_{P1}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r]) = \begin{cases} 1 \text{ for } \max\{\underline{tSL1}_r, \underline{tSL2}_r\} < \overline{tWL1}_r \\ 0 \text{ otherwise} \end{cases} \tag{6.15}$$

and

$$\delta_{P2}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r]) = \begin{cases} 1 \text{ for } \min\{\underline{tSL1}_r, \underline{tSL2}_r\} < \overline{tWL1}_r \\ 0 \text{ otherwise.} \end{cases} \tag{6.16}$$

The estimates for $Pl(\mathcal{L}_1)$ and $Pl(\mathcal{L}_2)$ indicated in Eq. (6.14) are illustrated in Eqs. (5.7) and (5.28) for the links defined Table 4.1 and Fig. 4.1.

The representations for belief and plausibility for $\mathcal{L}_1$ and $\mathcal{L}_2$ in Sects. 4.2 and 4.3 are computationally easier to implement than the sampling-based approximations described in this section. The significance of the sampling-based approximations is that they provide a second independent way to obtain belief and plausibility for verification of results obtained in the manner described in Sects. 4.2 and 4.3.

6.3 Verification for $Bel(\mathcal{L}_{1;1\le 2})$, $Bel(\mathcal{L}_{1;2\le 1})$, $Pl(\mathcal{L}_{1;1\le 2})$ and $Pl(\mathcal{L}_{1;2\le 1})$

Defining probability spaces to verify the calculation of $Bel(\mathcal{L}_{1;1\le 2})$ and $Bel(\mathcal{L}_{1;2\le 1})$ for the sets $\mathcal{L}_{1;1\le 2}$ and $\mathcal{L}_{1;2\le 1}$ defined in Eqs. (5.8) and (5.9) is now considered. The following assignments are made for $Bel(\mathcal{L}_{1;1\le 2})$: (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$) is assigned to $\underline{t}_{WL1,k}$, with the result that

$$p(\underline{t}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}, \tag{6.17}$$

(ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$) is assigned to $\overline{t}_{SL1,i}$, with the result that

$$p(\overline{t}_{SL1,i}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}, \tag{6.18}$$

(iii) the most probability possible (i.e., $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\underline{t}_{SL2,j}$ together with the additional assumption $p(\overline{t}_{SL2,j} \mid \underline{t}_{SL2,j}) = 1.0$ which is made to result in $(\underline{t}_{SL2,j}, \overline{t}_{SL2,j})$ being sampled as a pair in a following sampling-based analysis, with the result that

$$p(\underline{t}_{SL2,j}) = m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2,j} \text{ and } p(\overline{t}_{SL2,j} \mid \underline{t}_{SL2,j}) = 1.0, \tag{6.19}$$

and (iv) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]$. This produces the probability space that has the smallest possible probabilities for the set $\mathcal{L}_{1;1\le 2}$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$p([\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL,k}(\mathcal{TM}_{WL1,k}) = m_{ijk} \tag{6.20}$$

for (i, j, k) belonging to the set $\mathcal{I}$ defined in Eq. (4.15) and (ii) any set that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]$ has a probability of zero. Then, with a large random sample

$$[\overline{tSL1}_r, \underline{tSL2}_r, \overline{tSL2}_r, \underline{tWL1}_r], r = 1, 2, ..., nR, \tag{6.21}$$

of size nR from the failure time vectors $[\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}]$ generated consistent with the probabilities m_{ijk} and the conditional probability $p(\overline{t}_{SL2,j} \mid \underline{t}_{SL2,j}) = 1.0$, $Bel(\mathcal{L}_{1;1\le 2})$ can be approximated by

$$Bel(\mathcal{L}_{1;1\le 2}) \cong \sum_{r=1}^{nR} \delta_B([\overline{tSL1}_r, \underline{tSL2}_r, \overline{tSL2}_r, \underline{tWL1}_r]) / nR \cong p(\mathcal{L}_{1;1\le 2}) \tag{6.22}$$

with

$$\delta_B([\overline{tSL1}_r, \underline{tSL2}_r, \overline{tSL2}_r, \underline{tWL1}_r]) = \begin{cases} 1 \text{ for } \overline{tSL1}_r \le \underline{tSL2}_r \text{ and } \overline{tSL2}_r < \underline{tWL1}_r \\ 0 \text{ otherwise.} \end{cases} \tag{6.23}$$

The approximation for $Bel(\mathcal{L}_{1;2\le 1})$ has the same form as the preceding approximation for $Bel(\mathcal{L}_{1;1\le 2})$ with the roles of SL 1 and SL 2 reversed. The resultant estimates for $Bel(\mathcal{L}_{1;1\le 2})$ indicated in Eq. (6.22) are illustrated in Eq. (5.18) for the links defined Table 4.1 and Fig. 4.1. Similarly obtained estimates for $Bel(\mathcal{L}_{1;2\le 1})$ are illustrated in Eq. (5.20).

Defining probability spaces to verify the calculation of $Pl(\mathcal{L}_{1;1\le 2})$ and $Pl(\mathcal{L}_{1;2\le 1})$ for the sets $\mathcal{L}_{1;1\le 2}$ and $\mathcal{L}_{1;2\le 1}$ defined in Eqs. (5.8) and (5.9) is now considered. The following assignments are made for $Pl(\mathcal{L}_{1;1\le 2})$: (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$) is assigned to $\overline{t}_{WL1,k}$, with the result

$$p(\overline{t}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}, \tag{6.24}$$

(ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$ and $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\underline{t}_{SL1,i}$ and $\underline{t}_{SL2,j}$, with the result that

$$p(\underline{t}_{SL1,i}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i} \tag{6.25}$$

$$p(\underline{t}_{SL2,j}) = m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2,j}, \tag{6.26}$$

and (iii) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]$. This produces the probability space that has the largest possible probability for the set $\mathcal{L}_{1;1\le 2}$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$p([\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]) = m_{SL1,t}(\mathcal{TM}_{SL1,i})m_{SL2,t}(\mathcal{TM}_{SL2,j})m_{WL,k}(\mathcal{TM}_{WL1,k}) = m_{ijk} \tag{6.27}$$

for (i, j, k) belonging to the set $\mathcal{I}$ defined in Eq. (4.15) and (ii) any set that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]$ has a probability of zero. Then, with a large random sample

$$[\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r], r = 1, 2, ..., nR, \tag{6.28}$$

of size nR from the failure time vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}]$ generated consistent with the probabilities m_{ijk}, $Pl(\mathcal{L}_{1;1\le 2})$ can be approximated by

$$Pl(\mathcal{L}_{1;1\le 2}) \cong \sum_{r=1}^{nR} \delta_P([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r]) / nR \cong p(\mathcal{L}_{1;1\le 2}) \tag{6.29}$$

with

$$\delta_P([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r]) = \begin{cases} 1 & \text{for } \underline{tSL1}_r \le \underline{tSL2}_r < \overline{tWL1}_r \\ 0 & \text{otherwise.} \end{cases} \tag{6.30}$$

The approximation for $Pl(\mathcal{L}_{1;2\le 1})$ has the same form as the preceding approximation for $Pl(\mathcal{L}_{1;1\le 2})$ with the roles of SL 1 and SL 2 reversed. The resultant estimates for $Pl(\mathcal{L}_{1;1\le 2})$ indicated in Eq. (6.29) are illustrated in Eq. (5.19) for the links defined Table 4.1 and Fig. 4.1. Similarly obtained estimates for $Pl(\mathcal{L}_{1;2\le 1})$ are illustrated in Eq. (5.21).

6.4 Verification for $Bel(\mathcal{L}_{2;1\le 2})$ $Bel(\mathcal{L}_{2;2\le 1})$, $Pl(\mathcal{L}_{2;1\le 2})$ and $Pl(\mathcal{L}_{2;2\le 1})$

Defining probability spaces to verify the calculation of $Bel(\mathcal{L}_{2;1\le 2})$ and $Bel(\mathcal{L}_{2;2\le 1})$ for the sets $\mathcal{L}_{2;1\le 2}$ and $\mathcal{L}_{2;2\le 1}$ defined in Eqs. (5.29) and (5.30) is now considered. The following assignments are made for $Bel(\mathcal{L}_{2;1\le 2})$: (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$) is assigned to $\underline{t}_{WL1,k}$, with the result that

$$p(\underline{t}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}, \tag{6.31}$$

(ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$) is assigned to $\overline{t}_{SL1,i}$, with the result that

$$p(\overline{t}_{SL1,i}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}, \tag{6.32}$$

(ii) the most probability possible (i.e., $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\underline{t}_{SL2,j}$, with the result that

$$p(\underline{t}_{SL2,j}) = m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}, \tag{6.33}$$

and (iv) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]$. This produces the probability space that has the smallest possible probability for the set $\mathcal{L}_{2;1\le 2}$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$p([\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL,k}(\mathcal{TM}_{WL1,k}) = m_{ijk} \tag{6.34}$$

for (i, j, k) belonging to the set $\mathcal{I}$ defined in Eq. (4.15) and (ii) any set that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]$ has a probability of zero. Then, with a large random sample

$$[\overline{tSL1}_r, \underline{tSL2}_r, \underline{tWL1}_r], r = 1, 2, ..., nR, \tag{6.35}$$

of size nR from the failure time vectors $[\overline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}]$ generated consistent with the probabilities m_{ijk}, $Bel(\mathcal{L}_{2;1\le 2})$ can be approximated by

$$Bel(\mathcal{L}_{2;1\le 2}) \cong \sum_{r=1}^{nR} \delta_B([\overline{tSL1}_r, \underline{tSL2}_r, \underline{tWL1}_r]) / nR \cong p(\mathcal{L}_{2;1\le 2}) \tag{6.36}$$

with

$$\delta_B([\overline{tSL1}_r, \underline{tSL2}_r, \underline{tWL1}_r]) = \begin{cases} 1 \text{ for } \overline{tSL1}_r \le \underline{tSL2}_r \text{ and } \overline{tSL1}_r < \underline{tWL1}_r \\ 0 \text{ otherwise.} \end{cases} \quad (6.37)$$

The approximation for $Bel(\mathcal{L}_{2;2\le1})$ has the same form as the preceding approximation for $Bel(\mathcal{L}_{2;1\le2})$ with the roles of SL 1 and SL 2 reversed. The resultant estimates for $Bel(\mathcal{L}_{2;1\le2})$ indicated in Eq. (6.36) are illustrated in Eq. (5.39) for the links defined Table 4.1 and Fig. 4.1. Similarly obtained estimates for $Bel(\mathcal{L}_{2;2\le1})$ are illustrated in Eq. (5.41).

Defining probability spaces to verify the calculation of $Pl(\mathcal{L}_{2;1\le2})$ and $Pl(\mathcal{L}_{2;2\le1})$ is now considered. The following assignments are made for $Pl(\mathcal{L}_{2;1\le2})$: (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$) is assigned to $\overline{t}_{WL1,k}$, with the result that

$$p(\overline{t}_{WL1,k}) = m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}, \quad (6.38)$$

(ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$) is assigned to $\underline{t}_{SL1,i}$, with the result that

$$p(\underline{t}_{SL1,i}) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}, \quad (6.39)$$

(iii) the most probability possible (i.e., $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2,j}$) is assigned to $\overline{t}_{SL2,j}$, with the result that

$$p(\overline{t}_{SL2,j}) = m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL2,j}, \quad (6.40)$$

and (iv) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \overline{t}_{SL2,j}, \overline{t}_{WL1,k}]$. This produces the probability space that has the largest possible probability for the set $\mathcal{L}_{2;1\le2}$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that

$$p([\underline{t}_{SL1,i}, \overline{t}_{SL2,j}, \overline{t}_{WL1,k}]) = m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{ijk} \quad (6.41)$$

for (i, j, k) belonging to the set $\mathcal{I}$ defined in Eq. (4.15) and (ii) any set that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \overline{t}_{SL2,j}, \overline{t}_{WL1,k}]$ has a probability of zero. Then, with a large random sample

$$[\underline{tSL1}_r, \overline{tSL2}_r, \overline{tWL1}_r], r = 1, 2, ..., nR, \quad (6.42)$$

of size nR from the failure time vectors $[\underline{t}_{SL1,i}, \overline{t}_{SL2,j}, \overline{t}_{WL1,k}]$ generated consistent with the probabilities m_{ijk}, $Pl(\mathcal{L}_{2;1\leq 2})$ can be approximated by

$$Pl(\mathcal{L}_{2;1\leq 2}) \cong \sum_{r=1}^{nR} \delta_P([\underline{tSL1}_r, \overline{tSL2}_r, \overline{tWL1}_r]) / nR \cong p(\mathcal{L}_{2;1\leq 2}) \tag{6.43}$$

with

$$\delta_P([\underline{tSL1}_r, \overline{tSL2}_r, \overline{tWL1}_r]) = \begin{cases} 1 & \text{for } \underline{tSL1}_r \leq \overline{tSL2}_r \text{ and } \underline{tSL1}_r < \overline{tWL1}_r \\ 0 & \text{otherwise.} \end{cases} \tag{6.44}$$

The approximation for $Pl(\mathcal{L}_{2;2\leq 1})$ has the same form as the preceding approximation for $Pl(\mathcal{L}_{2;1\leq 2})$ with the roles of SL 1 and SL 2 reversed. The resultant estimates for $Pl(\mathcal{L}_{2;1\leq 2})$ indicated in Eq. (6.43) are illustrated in Eq. (5.40) for the links defined Table 4.1 and Fig. 4.1. Similarly obtained estimates for $Pl(\mathcal{L}_{2;2\leq 1})$ are illustrated in Eq. (5.42).

7. Cumulative and Complementary Cumulative Belief and Plausibility for Time at which LOAS Occurs

For simplicity, this section considers a system with 2 SLs and 1 WL and two definitions of system failure: (i) LOAS occurs when both SLs fail before the WL fails and (ii) LOAS occurs when either SL fails before the WL fails. As discussed below, a useful visual summary of an analysis for LOAS is provided by a plot of time-dependent (i.e., cumulative and complementary) values of belief and plausibility for the occurrence time of LOAS.

7.1 Cumulative and Complementary Cumulative Belief and Plausibility for Time at which LOAS Occurs when Both SLs Fail before the WL fails

For the first definition (i.e., LOAS occurs when both SLs fail before the WL fails), the function

$$TML_1(t_{SL1}, t_{SL2}, t_{WL1}) = \begin{cases} \max\{t_{SL1}, t_{SI2}\} \text{ for } \max\{t_{SL1}, t_{SI2}\} < t_{WL1} \\ \infty \text{ for } t_{WL1} \le \max\{t_{SIL}, t_{SL2}\} \end{cases} \tag{7.1}$$

maps the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13)-(4.16) into an evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ for the time at which LOAS occurs. Specifically,

$$\mathcal{TML}_1 = \{t : t = TML_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}\}, \tag{7.2}$$

$$\mathcal{TML}_{1,ijk} = \{t : t = TML_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}_{ijk}\} \tag{7.3}$$

$$\mathbb{TML}_1 = \{\mathcal{TML}_{1,ijk} : (i,j,k) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\} \times \{1,2,...,nWL1\}\} \tag{7.4}$$

and

$$m_{TML1}(\mathcal{TML}_{1,ijk}) = m_{TM}(\mathcal{TM}_{ijk}) = m_{t,ijk}. \tag{7.5}$$

In addition, the bounds

$$(\underline{t}_{1,ijk}, \overline{t}_{1,ijk}) = \left(\min(\mathcal{TML}_{1,ijk}), \max(\mathcal{TML}_{1,ijk})\right) \tag{7.6}$$

are introduced for use in the determination of the cumulative values of belief and plausibility for the occurrence of LOAS as indicated in in conjunction with Eqs. (2.48)-(2.50). Use of the min and max functions in Eq. (7.6) is correct because of the assumptions that (i) focal elements for link failure time are closed intervals and (ii) link temperature functions are continuous. Without these assumptions, the min and max in Eq. (7.6) would have to be replaced by the greatest lower bound (glb) and least upper bound (lub) functions. Specifically, $\underline{t}_{1,ijk}$ is defined by

$$\underline{t}_{1,ijk} = \begin{cases} \infty \ \text{ for } \ \bar{t}_{WL1,k} \le \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \\ \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \ \text{ for } \ \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \bar{t}_{WL1,k}, \end{cases} \tag{7.7}$$

with (i) the first definition resulting because LOAS cannot occur for elements of $\mathcal{TML}_{1,ijk}$ when the indicated inequality holds and (ii) the second definition corresponding to the earliest time at which LOAS could occur for elements of $\mathcal{TML}_{1,ijk}$ when the indicated inequality holds. Similarly,

$$\bar{t}_{1,ijk} = \begin{cases} \infty \ \text{ for } \ \underline{t}_{WL1,k} \le \max\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \\ \max\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \ \text{ for } \ \max\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} < \underline{t}_{WL1,k}, \end{cases} \tag{7.8}$$

with (i) the first definition resulting because LOAS cannot occur for at least some elements of $\mathcal{TML}_{1,ijk}$ when the indicated inequality holds and (ii) the second definition corresponding to the last time at which LOAS could occur for elements of $\mathcal{TML}_{1,ijk}$ when the indicated inequality holds. In the event that the first inequality in Eq. (7.7) holds, then the first inequality in Eq. (7.8) also holds, with the result that $(\underline{t}_{1,ijk}, \bar{t}_{1,ijk}) = (\infty, \infty)$.

Once the evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for the time at which LOAS occurs can be obtained from the pairs $(\underline{t}_{1,ijk}, \bar{t}_{1,ijk})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 7.1.

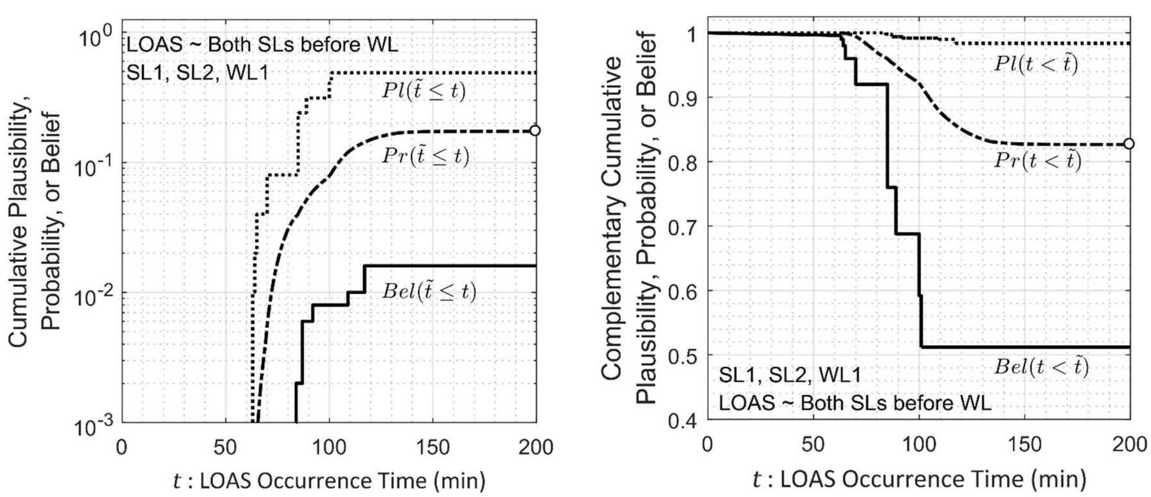


Fig. 7.1 Graphical summary of evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ for time t at which LOAS occurs for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of both SLs before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

In addition, Fig. 7.1 also contains the CDF and the CCDF for the time at which LOAS occurs obtained by assigning uniform distributions to the individual focal elements for link failure temperature as indicated for the construction of the link failure time CDFs in Fig. 4.4. For SL 1, SL 2 and WL 1, the indicated CDF and CCDF are constructed by: (i) generating a large random sample

$$(T_{SL1,r}, T_{SL2,r}, T_{WL1,r}), r = 1, 2, ..., nR, \tag{7.9}$$

from the constructed distributions for link failure temperatures, (ii) determining the corresponding link failure times

$$\begin{aligned} \mathbf{t}_r &= \left(t_{SL1,r} = T_{SL1}^{-1}(T_{SL1,r}), t_{SL2,r} = T_{SL2}^{-1}(T_{SL2,r}), t_{WL1,r} = T_{WL1}^{-1}(T_{WL1,r})\right) \\ &= (t_{SL1,r}, t_{SL2,r}, t_{WL1,r}), \end{aligned} \tag{7.10}$$

for $r = 1, 2, , , nR$ with the inverse functions indicating the earliest possible failure times for the sampled link failure temperatures, (iii) determining the LOAS occurrence times

$$TML_1(\mathbf{t}_r) = TML_1(t_{SL1,r}, t_{SL2,r}, t_{WL1,r}), r = 1, 2, ..., nR, \tag{7.11}$$

as indicated in Eq. (7.1), and (iv) defining the desired CDF and CCDF by

$$CDF(t) = prob(\tilde{t} \le t) = \sum_{r=1}^{nR} \underline{\delta}_t[TML_1(\mathbf{t}_r)] / nR \text{ with } \underline{\delta}_t[TML_1(\mathbf{t}_r)] = \begin{cases} 1 \text{ for } TML_1(\mathbf{t}_r) \le t \\ 0 \text{ otherwise} \end{cases} \tag{7.12}$$

and

$$CCDF(t) = prob(t < \tilde{t}) = \sum_{r=1}^{nR} \bar{\delta}_t[TML_1(\mathbf{t}_r)] / nR \text{ with } \bar{\delta}[TML_1(\mathbf{t}_r)] = \begin{cases} 1 \text{ for } t < TML_1(\mathbf{t}_r) \\ 0 \text{ otherwise.} \end{cases} \tag{7.13}$$

The cumulative and complementary cumulative plausibility and belief results in Fig. 7.1 provide related, but not identical, information about the potential occurrence of LOAS. In most analyses, the results of most interest pertain to whether or not LOAS occurs. Specifically, the cumulative $t = 200$ min results in Fig. 7.1a provide the analysis outcomes

$$Pl(\tilde{t} \le 200) = \text{plausibility that LOAS occurs before or at 200 min} = 0.488, \tag{7.14}$$

$$Bel(\tilde{t} \le 200) = \text{belief that LOAS occurs before or at 200 min} = 0.016, \tag{7.15}$$

and the complementary cumulative $t = 200$ min results in Fig. 7.1b provide the analysis outcomes

$$
\begin{aligned}
Pl(200 < \tilde{t}) &= \text{plausibility that LOAS did not occur prior to 200 min} \\
&= Pl(t = t_\infty) \\
&= 0.984,
\end{aligned}
\tag{7.16}
$$

$$
\begin{aligned}
Bel(200 < \tilde{t}) &= \text{belief that LOAS did not occur prior to 200 min} \\
&= Bel(t = t_\infty) \\
&= 0.512.
\end{aligned}
\tag{7.17}
$$

Thus, as illustrated, the cumulative outcomes in Fig. 7.1a provide results on the timing and occurrence of LOAS, and the complementary cumulative outcomes in Fig. 7.1b provide results on the timing and nonoccurrence of LOAS.

Initially, some results may seem counterintuitive (e.g., $Pl(200 < \tilde{t}) = 0.984$). The values of plausibility and belief depend on (i) the number of focal elements that are consistent with the plausibility or belief under consideration and (ii) the BPAs associated with these focal elements. A large number of consistent focal elements may, but not necessarily, be associated with a large plausibility or belief. The evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ under consideration has

$$nSL1 \times nSL2 \times nWL1 = 5 \times 5 \times 5 = 125 \tag{7.18}$$

focal elements. In turn,

$$
\begin{aligned}
81 = \;& \text{number focal elements consistent with } Pl(\tilde{t} \le 200) \\
& \text{i.e., focal elements that contain times in [0, 200 min],}
\end{aligned}
\tag{7.19}
$$

$$
\begin{aligned}
11 = \;& \text{number focal elements consistent with } Bel(\tilde{t} \le 200) \\
& \text{i.e., focal elements that are subsets of [0, 200 min],}
\end{aligned}
\tag{7.20}
$$

$$
\begin{aligned}
114 = \;& \text{number focal elements consistent with } Pl(200 < \tilde{t}) \\
& \text{i.e., focal elements that contain } t_\infty,
\end{aligned}
\tag{7.21}
$$

$$
\begin{aligned}
44 = \;& \text{number focal elements consistent with } Bel(200 < \tilde{t}) \\
& \text{i.e., focal elements that contain only } t_\infty.
\end{aligned}
\tag{7.22}
$$

Given the large number of focal elements consistent with $Pl(200 < \tilde{t})$ and $Bel(200 < \tilde{t})$, the resultant large values of $Pl(200 < \tilde{t}) = 0.984$ and $Bel(200 < \tilde{t}) = 0.512$ are not surprising.

The cumulative and complementary cumulative plausibility and belief results in Fig. 7.1 are related through the relationship

$$Bel(\mathcal{S}) + Pl(\mathcal{S}^c) = 1 \tag{7.23}$$

previously stated in Eq. (2.24). As examples,

$$Pl(200 < \tilde{t}) = Pl(t = t_\infty) = 1 - Bel(\tilde{t} \le 200) = 1 - 0.016 = 0.984 \tag{7.24}$$

$$Bel(200 < \tilde{t}) = Bel(t = t_\infty) = 1 - Pl(\tilde{t} \le 200) = 1 - 0.488 = 0.512. \tag{7.25}$$

The equality of the results in Eqs. (7.16)-(7.17) with the corresponding results in Eqs. (7.24)-(7.25) provides a verification result that indicates that the plausibility and belief results in Fig. 7.1 have been correctly constructed. An additional verification is provided by the match of $Pl(\tilde{t} \le 200)$ and $Bel(\tilde{t} \le 200)$ with the independently determined values for the occurrence of LOAS in Eqs. (5.6)-(5.7).

As should be the case, the CDF in Fig. 7.1a for LOAS occurrence time, denoted by $Pr(\tilde{t} \le t)$, falls between the corresponding CBF and CPF, denoted by $Bel(\tilde{t} \le t)$ and $Pl(\tilde{t} \le t)$. Similarly, the CCDF in Fig. 7.1b for LOAS occurrence time, denoted by $Pr(t < \tilde{t})$, falls between the corresponding CCBF and CCPF, denoted by $Bel(t < \tilde{t})$ and $Pl(t < \tilde{t})$. For perspective, it is recommended that CDFs and CCDFs as described for Fig. 7.1 be included in presentations of CPFs, CBFs, CCPFs and CCBFs.

It is informative to know the smallest time at which LOAS could occur (i.e., $\underline{t}_{1L}$) and the largest time at which LOAS could occur (i.e., $\overline{t}_{1L}$). As described below, these two times can be determined from the focal elements associated with the evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$. However, this closed form determination can be rather tedious and error prone, especially the determination of the largest time at which LOAS could occur. In practice, it is easier to employ sampling-based results of the form used to obtain the CDF and CCDF illustrated in Fig. 7.1 to estimate the indicated times than it is to carry out a closed form analysis. As an example, the sampling procedure indicated in Eqs. (7.9)-(7.13) to obtain the CDF and CCDF in Fig. 7.1 also yields values of

$$\underline{t}_{1L} \cong 62.12 \text{ min and } \overline{t}_{1L} \cong 200 \text{ min.} \tag{7.26}$$

In this example, $\overline{t}_{1L}$ is the same as the end time (i.e., 200 min) for the analysis, which may not be the case in other analyses.

The closed form determination of $\underline{t}_{1L}$ and $\overline{t}_{1L}$ is now considered. A focal element $\mathcal{TML}_{1,ijk}$ associated with the evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ contains times corresponding to the actual occurrence of LOAS (i.e., times $< \infty$) only if $(i, j, k) \in \mathcal{I}_{1L}$ with

$$\mathcal{I}_{1L} = \{(i, j, k) : \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL,k}\}. \tag{7.27}$$

In turn, the earliest time $\underline{t}_{1L}$ at which LOAS can occur is defined by

$$\underline{t}_{1L} = \min\{\underline{t}_{1,ijk} : (i,j,k) \in \mathcal{I}_{1L}\}$$
$$= \min\left\{\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} : (i,j,k) \in \mathcal{I}_{1L}\right\} \tag{7.28}$$

with $\underline{t}_{1,ijk}$ initially defined in Eq. (7.7). As an example, the preceding equality produces $\underline{t}_{1L} = 62.12$ min for the results illustrated in Fig. 7.1, which is consistent with the sampling-based value of 62.12 min in Eq. (7.26).

When some of the pairs $(\underline{t}_{1,ijk}, \overline{t}_{1,ijk})$ are of the form $(\underline{t}_{1,ijk}, \infty)$ with $\underline{t}_{1,ijk} < \infty$, plots for plausibility and belief constructed as indicated in conjunction with Eqs. (2.48)-(2.50) and illustrated in Fig. 7.1 will not include a step that corresponds to the least upper bound (lub) $\overline{t}_{1L}$ of the times at which LOAS could occur. Closed form representation for $\overline{t}_{1L}$ are now determined for the case in which (i) all link temperature curves are continuous functions and (ii) all focal elements for link failure temperature are closed intervals.

The following two possibilities for $\overline{t}_{WL,k}$ require consideration:

$$\overline{t}_{WL,k} = \infty \text{ and } \overline{t}_{WL,k} \le t_{mx}. \tag{7.29}$$

Given the two preceding possibilities, the lub $\overline{t}_{1L,ijk}$ of the times at which LOAS could occur for $\mathcal{TML}_{1,ijk}$ with $(i,j,k) \in \mathcal{I}_{1L}$ is defined by

$$\overline{t}_{WL,k} = \infty \text{ and } (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \overline{t}_{1L,ijk} = \begin{cases} \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \le t_{mx} \\ \max\{t : t \in \mathcal{TML}_{1,ijk} \text{ and } t \ne \infty\} & \text{for } t_{mx} < \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \end{cases} \tag{7.30}$$

and

$$\overline{t}_{WL,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \overline{t}_{1L,ijk} = \begin{cases} \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \overline{t}_{WL,k} \\ \text{lub}\{t : t \in \mathcal{TML}_{1,ijk} \text{ and } t \ne \infty\} & \text{for } \overline{t}_{WL,k} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}. \end{cases} \tag{7.31}$$

The lub is needed in Eq. (7.31) because, under the stated conditions, it is possible the that LOAS could occur at times with an lub of $\overline{t}_{WL,k}$ (e.g., if $\overline{t}_{SL1,i} < \overline{t}_{SL2,j} = \overline{t}_{WL,k}$).

In turn, the resultant lub time $\overline{t}_{1L}$ for LOAS occurrence is defined by

$$\overline{t}_{1L} = \max\{\overline{t}_{1L,ijk} : (i,j,k) \in \mathcal{I}_{1L}\}. \tag{7.32}$$

Only a max rather than an lub is needed in the preceding definition of $\bar{t}_{1L}$ because the number of focal elements indexed by the set $\mathcal{I}_{1L}$ is finite. In contrast, it is possible that some of the times $\bar{t}_{1L,ijk}$ involved in the definition of $\bar{t}_{1L}$ could be defined as lub's.

As an example, the indicated procedure produces $\bar{t}_{1L} = 200$ min for the results illustrated in Fig. 7.1, which matches the sampling-based value in Eq. (7.26).

The LOAS occurrence time evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ and its associated CPF, CBF, CCPF and CCBF for SL 1 and SL 2 both failing before WL 1 fails can also be defined with use of the evidence spaces $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ and $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$. Specifically, (i) $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ is defined in Sect. 8.1 for the times at which a system consisting of SL 1 and SL 2 fails with system failure time corresponding to the time at which the second SL fails and (ii) $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$ is defined in Sect. 4 for the time at which WL 1 fails.

7.2 Cumulative and Complementary Cumulative Belief and Plausibility for Time at which LOAS Occurs when Either SL Fails before the WL Fails

For the second definition (i.e., LOAS occurs when either SL fails before the WL fails), the function

$$TML_2(t_{SL1}, t_{SL2}, t_{WL1}) = \begin{cases} \min\{t_{SL1}, t_{SL2}\} & \text{for } \min\{t_{SL1}, t_{Sl2}\} < t_{WL1} \\ \infty & \text{for } t_{WL1} \le \min\{t_{SL1}, t_{SL2}\} \end{cases} \tag{7.33}$$

maps the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13)-(4.16) into an evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ with

$$\mathbb{TML}_2 = \{\mathcal{TML}_{2,ijk} : (i,j,k) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\} \times \{1,2,...,nWL1\}\} \tag{7.34}$$

for the time at which LOAS occurs as shown in Eqs. (7.2)-(7.5) to obtain the evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$.

Similarly to the definitions of $\underline{t}_{1,ijk}$ and $\bar{t}_{1,ijk}$ in Eqs. (7.6)-(7.8), the bounds

$$(\underline{t}_{2,ijk}, \bar{t}_{2,ijk}) = \left(\min(\mathcal{TML}_{2,ijk}), \max(\mathcal{TML}_{2,ijk})\right) \tag{7.35}$$

are defined by

$$\underline{t}_{2,ijk} = \begin{cases} \infty & \text{for } \bar{t}_{WL1,k} \le \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \\ \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} & \text{for } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \bar{t}_{WL1,k} \end{cases} \tag{7.36}$$

and

$$\overline{t}_{2,ijk} = \begin{cases} \infty \text{ for } \underline{t}_{WL1,k} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \\ \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \text{ for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \underline{t}_{WL1,k}. \end{cases} \tag{7.37}$$

Once the evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for the time at which LOAS occurs can be obtained from the pairs $(\underline{t}_{2,ijk}, \overline{t}_{2,ijk})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 7.2. In addition, Fig. 7.2 also contains CDFs and CCDFs for the time at which LOAS occurs constructed as described in Eqs. (7.9)-(7.13) with $TML_2(\mathbf{t}_r)$ replacing $TML_1(\mathbf{t}_r)$ in Eqs. (7.11)-(7.13).

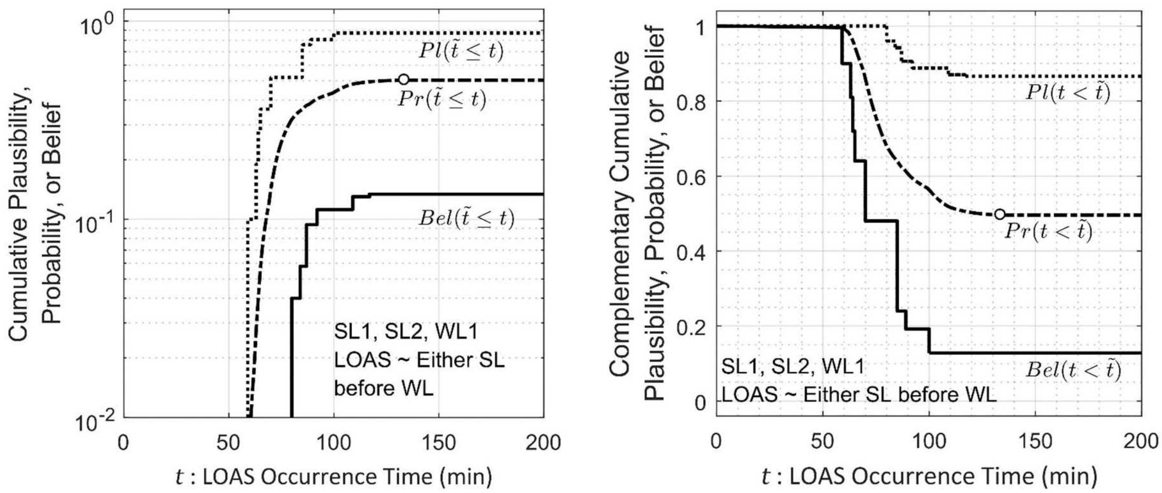


Fig. 7.2 Graphical summary of evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ for time t at which LOAS occurs for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of either SL before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

The sampling-based analysis used to construct the CDF and CCDF in Fig. 7.2 also establishes that the smallest time at which LOAS can occur (i.e., $\underline{t}_{2L}$) and the largest time at which LOAS actually occurs (i.e., $\overline{t}_{2L}$) are approximated by

$$[\underline{t}_{2L}, \overline{t}_{2L}] \cong [58.5 \text{ min}, 133.7 \text{ min}]. \tag{7.38}$$

Combination of $\overline{t}_{2L}$ with the cumulative plausibility and belief results at t = 200 min in Fig. 7.2a provides the analysis outcomes

$$0.872 = Pl(\tilde{t} \le 200) = Pl(\tilde{t} \le \overline{t}_{2L}) = Pl(\tilde{t} \le 133.7) \tag{7.39}$$

$$0.134 = Bel(\tilde{t} \le 200) = Bel(\tilde{t} \le \overline{t}_{2L}) = Bel(\tilde{t} \le 133.7) \tag{7.40}$$

and combination of $\overline{t}_{2L}$ with the complementary cumulative plausibility and belief results at t = 200 min in Fig. 7.2b provides the analysis outcomes

$$0.866 = Pl(200 < \tilde{t}) = Pl(\overline{t}_{2L} < \tilde{t}) = Pl(133.7 < \tilde{t}) = Pl(t = t_\infty) \tag{7.41}$$

$$0.128 = Bel(200 < \tilde{t}) = Bel(\overline{t}_{2L} < \tilde{t}) = Bel(133.7 < \tilde{t}) = Bel(t = t_\infty). \tag{7.42}$$

In addition, the results in Fig. 7.2 provide information on the potential timing of LOAS, which could be important in some analyses.

An important point here is that the construction and subsequent display of the evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ or a similar evidence space for failure time may not include the time of last failure due to the need to include the marker time t_∞ for nonfailure in the definition of focal elements for failure time and their associated BPAs. As a consequence, it is important to have a method to determine maximum failure time as well as the evidence space for failure time. The sampling-based procedure just illustrated is one way to determine $\underline{t}_{2L}$ and $\overline{t}_{2L}$ and similar results. Another way is to use a closed form representation for $\underline{t}_{2L}$ and $\overline{t}_{2L}$.

The closed form determination of $\underline{t}_{2L}$ and $\overline{t}_{2L}$ is now considered. A focal element $\mathcal{TML}_{2,ijk}$ associated with the evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ contains times corresponding to the occurrence of LOAS (i.e., times < ∞) only if $(i, j, k) \in \mathcal{I}_{2L}$ with

$$\mathcal{I}_{2L} = \{(i, j, k) : \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k}\}. \tag{7.43}$$

In turn, the earliest time $\underline{t}_{2L}$ at which LOAS could occur is defined by

$$\underline{t}_{2L} = \min\{\underline{t}_{2,ijk} : (i, j, k) \in \mathcal{I}_{2L}\}. \tag{7.44}$$

As an example,

$$\underline{t}_{2L} = \begin{cases} 58.536 \text{ min for the sampling-based approach} \\ 58.537 \text{ min for the closed form representation} \end{cases} \tag{7.45}$$

for the results illustrated in Fig. 7.2.

The lub $\overline{t}_{2L}$ for the times at which LOAS occurs for LOAS corresponding to failure of either SL before failure of the WL is obtained in a manner similar to that used in Eqs. (7.30)-(7.32) to obtain $\overline{t}_{1L}$ for LOAS corresponding to failure of both SLs before failure of the WL. As for $\overline{t}_{1L}$, closed form representation for $\overline{t}_{2L}$ are now determined for the case in which (i) all link temperature curves are continuous functions and (ii) all focal elements for link failure temperature are closed intervals.

A focal element $\mathcal{TML}_{2,ijk}$ associated with the evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ contains times corresponding to the occurrence of LOAS (i.e., times < ∞) only if $(i,j,k) \in \mathcal{I}_{2L}$. The following two possibilities for $\overline{t}_{WL,k}$ require consideration:

$$\overline{t}_{WL,k} = \infty \text{ and } \overline{t}_{WL,k} \le t_{mx}. \tag{7.46}$$

Given the two preceding possibilities, the lub $\overline{t}_{2L,ijk}$ of the times at which LOAS could occur for $\mathcal{TML}_{2,ijk}$ with $(i,j,k) \in \mathcal{I}_{2L}$ is defined by

$$\begin{aligned}&\overline{t}_{WL,k} = \infty \text{ and } (i,j,k) \in \mathcal{I}_{2L}\\ &\quad \Rightarrow \overline{t}_{2L,ijk} = \begin{cases}\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \le t_{mx}\\ \max\{t : t \in \mathcal{TML}_{2,ijk} \text{ and } t \ne \infty\} & \text{for } t_{mx} < \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}\end{cases}\end{aligned} \tag{7.47}$$

and

$$\begin{aligned}&\overline{t}_{WL,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{2L}\\ &\quad \Rightarrow \overline{t}_{2L,ijk} = \begin{cases}\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \overline{t}_{WL,k}\\ \text{lub}\{t : t \in \mathcal{TML}_{2,ijk} \text{ and } t \ne \infty\} & \text{for } \overline{t}_{WL,k} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}.\end{cases}\end{aligned} \tag{7.48}$$

The lub is needed in Eq. (7.31) because, under the stated conditions, it is possible the that LOAS could occur at times with an lub of $\overline{t}_{WL,k}$ (e.g., if $\overline{t}_{WL,k} = \overline{t}_{SL1,i} < \overline{t}_{SL2,j}$). In turn, the resultant lub time $\overline{t}_{2L}$ for LOAS occurrence is defined by

$$\overline{t}_{2L} = \max\{\overline{t}_{2L,ijk} : (i,j,k) \in \mathcal{I}_{2L}\}. \tag{7.49}$$

For the first case (i.e., all link temperature curves are continuous functions), the lub $\overline{t}_{2L,ijk}$ for the times at which LOAS could occur for $\mathcal{TML}_{2,ijk}$ with $(i,j,k) \in \mathcal{I}_{2L}$ is defined by

$$\overline{t}_{2L,ijk} = \text{lub}\{t : t \in \mathcal{TML}_{2,ijk} \text{ and } t \ne \infty\}, \tag{7.50}$$

and the resultant lub $\overline{t}_{2L}$ for LOAS occurrence time is defined by

$$\bar{t}_{2L} = \max\{\bar{t}_{2L,ijk} : (i,j,k) \in \mathcal{I}_{2L}\}. \quad (7.51)$$

As an example,

$$\bar{t}_{2L} = \begin{cases} 133.675 & \text{for the sampling-based approach} \\ 133.682 & \text{for the closed form representation} \end{cases} \quad (7.52)$$

for the results illustrated in Fig. 7.2.

The LOAS occurrence time evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ and its associated CPF, CBF, CCPF and CCBF for either SL 1 or SL 2 failing before WL 1 fails can also be defined with use of the evidence spaces $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ and $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$. Specifically, (i) $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ is defined in Sect. 8.2 for the times at which a system consisting of SL 1 and SL 2 fails with system failure time corresponding to the time at which the first SL fails and (ii) $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$ is defined in Sect. 4 for the time at which WL 1 fails.

8. Cumulative and Complementary Cumulative Belief and Plausibility for Time at which a System of Two Links Fails

For simplicity, this section considers a system of 2 SLs and two definitions of system failure: (i) system failure occurs when both links have failed and (ii) system failure occurs when either link has failed. The development is identical for a system of 2 WLs.

8.1 Cumulative and Complementary Cumulative Belief and Plausibility for Time at which a System of Two Links Fails Due to Failure of Both Links

The development for both definitions of link system failure starts with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for the times at which the links could fail. Specifically, the evidence spaces $(\mathcal{TM}_{SL1}, \mathbb{TM}_{SL1}, m_{SL1,t})$ and $(\mathcal{TM}_{SL2}, \mathbb{TM}_{SL2}, m_{SL2,t})$ for link failure time defined in Sect. 4 are combined to produce the product evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ with

$$\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \tag{8.1}$$

$$\mathcal{TM}_{ij} = \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \in \mathbb{TM}, \tag{8.2}$$

$$\mathbb{TM} = \{\mathcal{TM}_{ij} : (i,j) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\}\} \tag{8.3}$$

and

$$m_{TM}(\mathcal{TM}_{ij}) = m_{SL1,i}(\mathcal{TM}_{SL1,i}) m_{SL2,j}(\mathcal{TM}_{SL2,j}) = m_{t,ij}. \tag{8.4}$$

Example SL links that will be used for illustration are defined and illustrated in Table 4.1 and Fig. 4.1.

For the first definition (i.e., system failure occurs when both links have failed), the function

$$TMF_1(\mathbf{t}) = \max\{t_{SL1}, t_{SL2}\} \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM} \tag{8.5}$$

is used to map the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ into the evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for link system failure time with

$$\mathcal{TMF}_1 = \{t : t = TMF_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM}\}, \tag{8.6}$$

$$\mathcal{TMF}_{1,ij} = \{t : t = TMF_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM}_{ij}\} \tag{8.7}$$

$$\mathbb{TMF}_1 = \{\mathcal{TMF}_{1,ij} : (i,j) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\}\} \tag{8.8}$$

and

$$m_{TMF1}(\mathcal{TMF}_{1,ij}) = m_{TM}(\mathcal{TM}_{ij}) = m_{t,ij}. \tag{8.9}$$

Next, the bounds

$$(\underline{t}_{1,ij}, \overline{t}_{1,ij}) = \left(\min(\mathcal{TMF}_{1,ij}), \max(\mathcal{TMF}_{1,ij})\right) \tag{8.10}$$

are introduced for use in the determination of the cumulative values of belief and plausibility for the time at which the link system fails as indicated in in conjunction with Eqs. (2.48)-(2.50). Specifically,

$$\underline{t}_{1,ij} = \begin{cases} \infty \text{ for } t_{mx} < \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \\ \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \text{ for } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx} \end{cases} \tag{8.11}$$

and

$$\overline{t}_{1,ij} = \begin{cases} \infty \text{ for } t_{mx} < \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \\ \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \text{ for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \le t_{mx}. \end{cases} \tag{8.12}$$

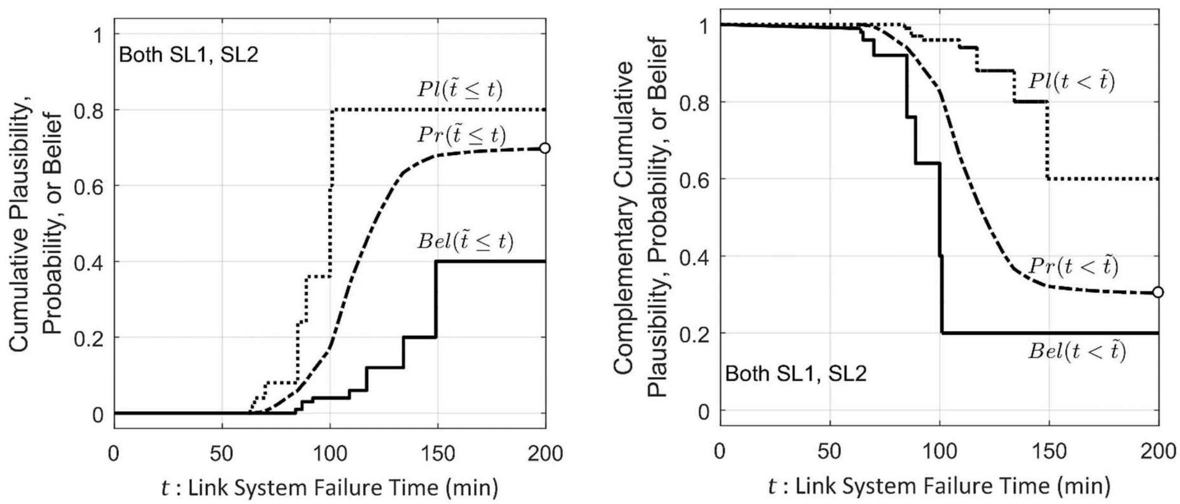


Fig. 8.1 Graphical summary of evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for time t at which link system failure occurs for (i) a two link system composed of SLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of both links: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

Once the evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for link system failure time can be obtained from the pairs $(\underline{t}_{1,ij}, \overline{t}_{1,ij})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 8.1. In addition, Fig. 8.1 also contains CDFs and CCDFs for the time at which link system failure occurs obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 8.1 are constructed as indicated in Eqs. (7.9)-(7.13) with $TMF_1(t_{SL1}, t_{SL2})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

The cumulative t = 200 min results in Fig. 8.1a provide the analysis outcomes

$$\begin{aligned} Pl(\tilde{t} \le 200) &= \text{plausibility that link system failure occurs before or at 200 min} \\ &= 0.800, \end{aligned} \tag{8.13}$$

$$\begin{aligned} Bel(\tilde{t} \le 200) &= \text{belief that link system failure occurs before or at 200 min} \\ &= 0.400, \end{aligned} \tag{8.14}$$

and the complementary cumulative t = 200 min results in Fig. 8.1b provide the analysis outcomes

$$\begin{aligned} Pl(200 < \tilde{t}) &= \text{plausibility that link system failure did not occur before} \\ &\qquad \text{or at 200 min} \\ &= Pl(t = t_\infty) \\ &= 0.600, \end{aligned} \tag{8.15}$$

$$\begin{aligned} Bel(200 < \tilde{t}) &= \text{belief that link system failure did not occur before} \\ &\qquad \text{or at 200 min} \\ &= Bel(t = t_\infty) \\ &= 0.200. \end{aligned} \tag{8.16}$$

In addition, the results in Fig. 8.1 provide information on the potential timing of link system failure, which could be important in some analyses.

As an additional example, cumulative and complementary cumulative plausibility, probability and belief functions for link system failure time are presented in Fig. 8.2 for a link system consisting of WLs 1 and 2 defined in Sect. 4. The construction of the results in Fig. 8.2 is the same as the construction of the results in Fig. 8.1

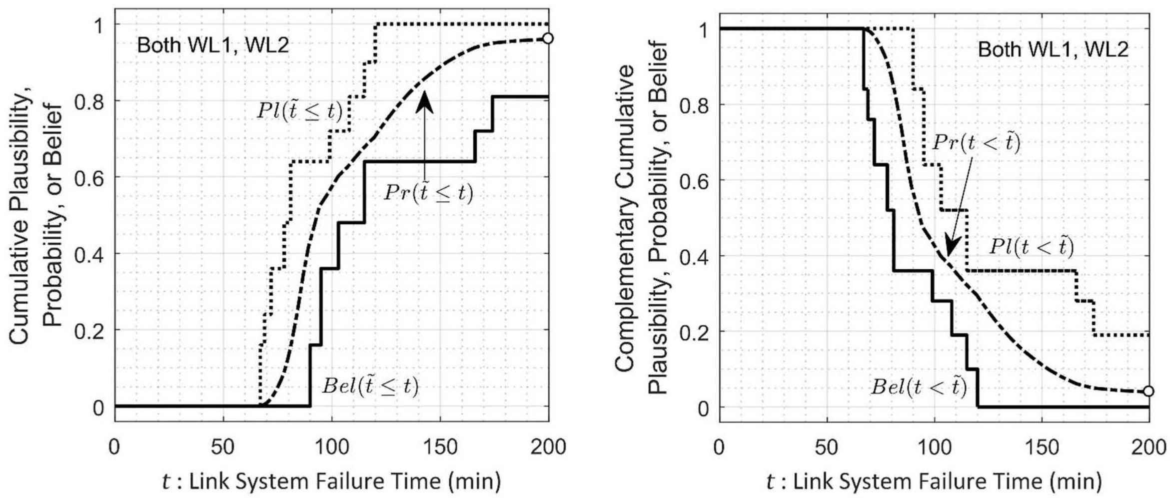


Fig. 8.2 Graphical summary of evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for time t at which link system failure occurs for (i) a two link system composed of WLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of both links: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

A focal element $\mathcal{TMF}_{1,ij}$ associated with the evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ contains times corresponding to link system failure (i.e., times $< \infty$) only if $(i, j) \in \mathcal{I}_{1F}$ with

$$\mathcal{I}_{1F} = \{(i, j) : \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx}\}. \tag{8.17}$$

In turn, the earliest time $\underline{t}_{1F}$ at which link system failure can occur is defined by

$$\underline{t}_{1F} = \min\{\underline{t}_{1,ij} : (i, j) \in \mathcal{I}_{1F}\}. \tag{8.18}$$

As an example,

$$\begin{aligned} \underline{t}_{1F} &= \begin{cases} 62.123 \text{ min for SL 1, SL 2} \\ 66.403 \text{ min for WL 1, WL 2} \end{cases} \\ &\cong \begin{cases} 62.295 \text{ min for SL 1, SL 2} \\ 66.406 \text{ min for WL 1, WL 2} \end{cases} \end{aligned} \tag{8.19}$$

for the results illustrated in Fig. 8.1 and Fig. 8.2, with (i) the first results obtained from Eq. (8.18) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 8.1 and Fig. 8.2.

Determination of the maximum value $\overline{t}_{1F}$ for the times at which link system failure could occur is now considered for system failure corresponding to failure of both SLs. As in Sect. 7.1, two cases for the definition of $\overline{t}_{1F}$ are considered: (i) All link temperature curves are continuous functions, and (ii) All link temperature curves are continuous increasing functions.

For the first case (i.e., all link temperature curves are continuous functions), the link system maximum failure time $\overline{t}_{1F,ij}$ for $\mathcal{TMF}_{1,ij}$ with $(i,j) \in \mathcal{I}_{1F}$ is defined by

$$\overline{t}_{1F,ij} = \max\{t : t \in \mathcal{TMF}_{1,ij} \text{ and } t \neq \infty\}, \tag{8.20}$$

and the resultant maximum value $\overline{t}_{1F}$ for link system failure time is defined by

$$\overline{t}_{1F} = \max\{\overline{t}_{1F,ij} : (i,j) \in \mathcal{I}_{1F}\}. \tag{8.21}$$

For the second case (i.e., all link temperature curves are continuous increasing functions), the maximum value $\overline{t}_{1F,ij}$ for link failure time for $\mathcal{TMF}_{1,ij}$ with $(i,j) \in \mathcal{I}_{1F}$ is defined by

$$\overline{t}_{1F,ij} = \begin{cases} t_{mx} & \text{for } t_{mx} < \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \\ \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \leq t_{mx} \end{cases} \tag{8.22}$$

and the resultant time $\overline{t}_{1F}$ of the last link system failure is defined as indicated in Eq. (8.21).

As an example,

$$\begin{aligned} \overline{t}_{1F} &= \begin{cases} 200.000 \text{ min for SL 1, SL 2} \\ 200.000 \text{ min for WL 1, WL 2} \end{cases} \\ &\cong \begin{cases} 199.964 \text{ min for SL 1, SL 2} \\ 199.977 \text{ min for WL 1, WL 2} \end{cases} \end{aligned} \tag{8.23}$$

for the results illustrated in Fig. 8.1 and Fig. 8.2, with (i) the first results obtained from Eq. (8.22) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 8.1 and Fig. 8.2.

8.2 Cumulative and Complementary Cumulative Belief and Plausibility for Time at which a System of Two links Fails Due to Failure of Either Link

The determination of cumulative belief and plausibility for the second definition (i.e., system failure occurs when either link has failed) is similar to the determination for the first definition. Specifically, the function

$$TMF_2(\mathbf{t}) = \min\{t_{SL1}, t_{SL2}\} \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM} \tag{8.24}$$

is used to map the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ into an evidence space $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ for link system failure time as indicated in Eqs. (8.6)-(8.9). Further, the focal element bounds $\underline{t}_{2,ij}$ and $\overline{t}_{2,ij}$ are now defined by

$$(\underline{t}_{2,ij}, \overline{t}_{2,ij}) = \left(\min(\mathcal{TMF}_{2,ij}), \max(\mathcal{TMF}_{2,ij})\right), \tag{8.25}$$

with

$$\underline{t}_{2,ij} = \begin{cases} \infty \text{ for } t_{mx} < \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \\ \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \text{ for } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx} \end{cases} \tag{8.26}$$

and

$$\overline{t}_{2,ij} = \begin{cases} \infty \text{ for } t_{mx} < \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \\ \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \text{ for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \le t_{mx}. \end{cases} \tag{8.27}$$

In turn, cumulative and complementary plausibility and belief functions for the second definition of link system failure can be obtained from the pairs $(\underline{t}_{2,ij}, \overline{t}_{2,ij})$ as indicated in conjunction with Eqs. (2.48)-(2.50) and illustrated in Fig. 8.3. In addition, Fig. 8.3also contains the CDF and CCDF for the time at which link system failure occurs obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDF and CCDF in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 8.3 are constructed as indicated in Eqs. (7.9)-(7.13) with $TMF_2(t_{SL1}, t_{SL2})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

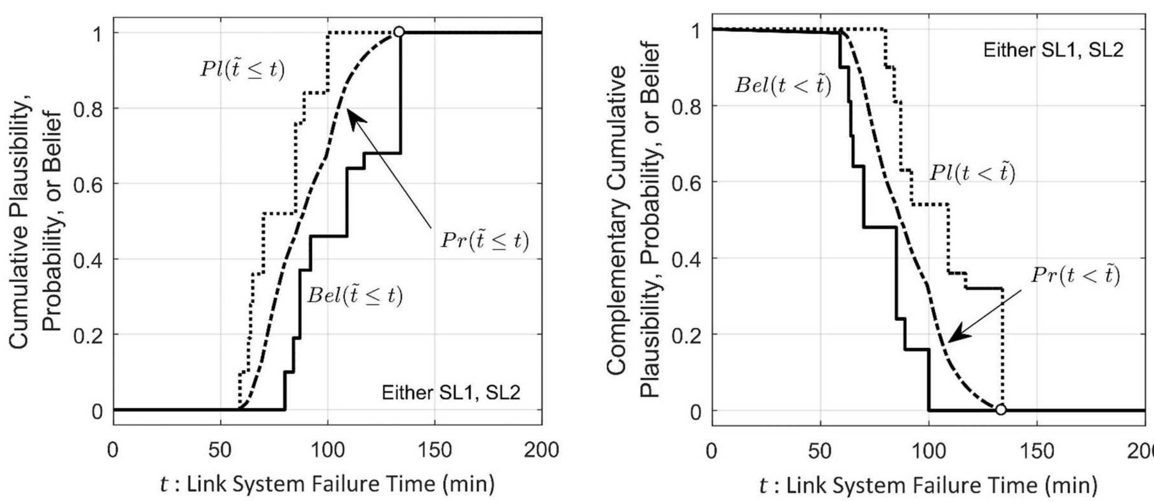


Fig. 8.3 Graphical summary of evidence space $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ for time t at which link system failure occurs for (i) a two link system composed of SLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of either link: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$,

probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

As an additional example, cumulative and complementary cumulative plausibility, probability and belief functions for link system failure time are presented in Fig. 8.4 for a link system consisting of WLs 1 and 2 defined in Sect. 4. The construction of the results in Fig. 8.4 is the same as the construction of the results in Fig. 8.1.

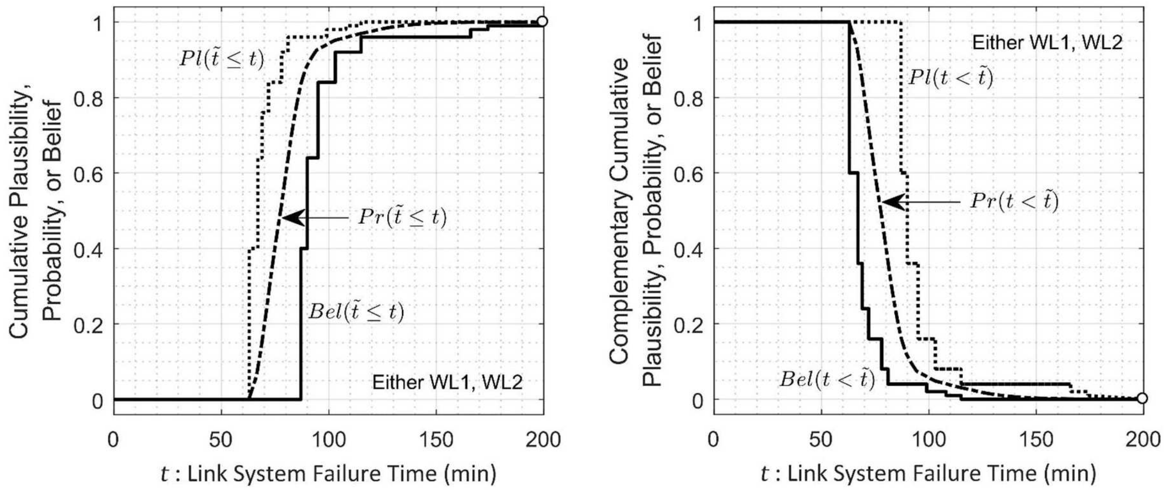


Fig. 8.4 Graphical summary of evidence space $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ for time t at which link system failure occurs for (i) a two link system composed of WLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of either link: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

A focal element $\mathcal{TMF}_{2,ij}$ associated with the evidence space $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ contains times corresponding to link system failure (i.e., times $< \infty$) only if $(i, j) \in \mathcal{I}_{2F}$ with

$$\mathcal{I}_{2F} = \{(i, j) : \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx}\}. \tag{8.28}$$

In turn, the earliest time $\underline{t}_{2L}$ at which LOAS can occur is defined by

$$\underline{t}_{2L} = \min\{\underline{t}_{1,ijk} : (i, j, k) \in \mathcal{I}_{2L}\}. \tag{8.29}$$

As an example,

$$\underline{t}_{2F} = \begin{cases} 58.537 \text{ min for SL 1, SL 2} \\ 62.778 \text{ min for WL 1, WL 2} \end{cases}$$
$$\cong \begin{cases} 58.535 \text{ min or SL 1, SL 2} \\ 62.778 \text{ min for WL 1, WL 2} \end{cases} \tag{8.30}$$

for the results illustrated in Fig. 8.3 and Fig. 8.4, with (i) the first results obtained from Eq. (8.29) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 8.3 and Fig. 8.4.

Determination of the maximum value $\bar{t}_{2F}$ for the times at which link system failure could occur is now considered for system failure corresponding to failure of either SL. As in Sect. 7.1, two cases for the definition of $\bar{t}_{1F}$ are considered: (i) All link temperature curves are continuous functions, and (ii) All link temperature curves are continuous increasing functions.

For the first case (i.e., all link temperature curves are continuous functions), the maximum value $\bar{t}_{2F,ij}$ for link system failure time for $\mathcal{TMF}_{2,ij}$ with $(i, j) \in \mathcal{I}_{2F}$ is defined by

$$\bar{t}_{2F,ij} = \max\{t : t \in \mathcal{TMF}_{2,ij} \text{ and } t \neq \infty\}, \tag{8.31}$$

and the resultant maximum value $\bar{t}_{2F}$ for link system failure time is defined by

$$\bar{t}_{2F} = \max\{\bar{t}_{2F,ij} : (i, j) \in \mathcal{I}_{2F}\}. \tag{8.32}$$

For the second case (i.e., link temperature curves are continuous increasing functions), the maximum value $\bar{t}_{2F,ij}$ for link system failure time for $\mathcal{TMF}_{2,ij}$ with $(i, j) \in \mathcal{I}_{2F}$ is defined by

$$\bar{t}_{2F,ij} = \begin{cases} t_{mx} \text{ for } t_{mx} < \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \\ \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \text{ for } \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \leq t_{mx} \end{cases} \tag{8.33}$$

and the resultant maximum value $\bar{t}_{2F}$ for link system failure time is defined as indicated in Eq. (8.32).

As an example,

$$\bar{t}_{2F} = \begin{cases} 133.682 \text{ min for SL 1, SL 2} \\ 200.000 \text{ min for WL 1, WL 2} \end{cases}$$
$$\cong \begin{cases} 133.678 \text{ min for SL 1, SL 2} \\ 199.749 \text{ min for WL 1, WL 2} \end{cases} \tag{8.34}$$

for the results illustrated in Fig. 8.3 and Fig. 8.4, with (i) the first results obtained from Eq. (8.29) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 8.3 and Fig. 8.4.

9. Cumulative and Complementary Cumulative Belief and Plausibility for Temperature at which a System of Two Links Fails

For simplicity, this section considers a system of 2 SLs and two definitions of system failure: (i) system failure occurs when both links have failed and (ii) system failure occurs when either link has failed. The development is identical for a system of 2 WLs

9.1 Cumulative and Complementary Cumulative Belief and Plausibility for Temperature at which a System of Two Links Fails Due to Failure of Both Links

The development for both definitions of link system failure starts with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for the times at which the links could fail as developed in Eqs. (8.1)-(8.4). For the first definition (i.e., system failure occurs when both links have failed), the function

$$TF_1(\mathbf{t}) = \begin{cases} \infty \text{ for } \max\{t_{SL1}, t_{SL2}\} = \infty \\ T_{SL1}(t_{SL1}) \text{ for } t_{SL2} < t_{SL1} < \infty \\ T_{SL2}(t_{SL2}) \text{ for } t_{SL1} < t_{SL2} < \infty \\ \max\{T_{SL1}(t_{12}), T_{SL2}(t_{12})\} \text{ for } t_{12} = t_{SL1} = t_{SL2} < \infty \end{cases} \quad (9.1)$$

with $\mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM}$ is used to map the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ into an evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ for link system failure temperature. As indicated in the definition of $TF_1(\mathbf{t})$, link system failure is assumed to be the maximum of the individual link failure temperatures when the individual links fail at the same time. Further, the notational assumptions

$$T_{SL1}(t_{SL1}) = \infty \text{ for } t_{SL1} = \infty \text{ and } T_{SL2}(t_{SL2}) = \infty \text{ for } t_{SL2} = \infty \quad (9.2)$$

are used to indicate that link failure temperature was not reached and hence that link failure did not occur.

The evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ for link system failure temperature is defined by

$$\mathcal{TF}_1 = \{T : T = TF_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM}\}, \quad (9.3)$$

$$\mathcal{TF}_{1,ij} = \{T : T = TF_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM}_{ij}\}, \quad (9.4)$$

$$\mathbb{TF}_1 = \{\mathcal{TF}_{1,ij} : (i,j) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\}\} \quad (9.5)$$

and

$$m_{TF1}(\mathcal{TF}_{1,ij}) = m_{TM}(\mathcal{TM}_{ij}) = m_{t,ij}. \quad (9.6)$$

Similarly to the results in Eqs. (8.10)-(8.12), the bounds

$$(\underline{TF}_{1,ij}, \overline{TF}_{1,ij}) = \left(\text{glb}(\mathcal{TF}_{1,ij}), \max(\mathcal{TF}_{1,ij})\right) \tag{9.7}$$

are introduced for use in the determination of the cumulative values of belief and plausibility for the temperature at which the link system fails as indicated in conjunction with Eqs. (2.48)-(2.50).

Definition of the focal element bound $\underline{TF}_{1,ij}$ is considered first. Specifically, $\underline{TF}_{1,ij}$ has a definition that (i) involves greatest lower bounds (glb's) for sets of link failure temperatures and (ii) is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$. The following possibilities exist for the definition of $\underline{TF}_{1,ij}$:

Possibility (1): If $\underline{t}_{SL1,i} = \underline{t}_{SL2,j} = \underline{t}_{ij}$, then either (1.1) $\underline{t}_{ij} = \infty$ and

$$\underline{TF}_{1,ij} = \infty, \tag{9.8}$$

or (1.2) $\underline{t}_{ij} = t_{mx}$ and

$$\underline{TF}_{1,ij} = \max\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\} = \max\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\}, \tag{9.9}$$

or (1.3) $\underline{t}_{ij} < t_{mx}$, $T_{SL1}(\underline{t}_{ij}) = T_{SL2}(\underline{t}_{ij})$ and

$$\underline{TF}_{1,ij} = T_{SL1}(\underline{t}_{ij}) = T_{SL2}(\underline{t}_{ij}), \tag{9.10}$$

or (1.4) $\underline{t}_{ij} < t_{mx}$, $T_{SL1}(\underline{t}_{ij}) < T_{SL2}(\underline{t}_{ij})$ and

$$\underline{TF}_{1,ij} = \begin{cases} T_{SL2}(\underline{t}_{ij}) \text{ if } (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL1,i} = \varnothing \\ T_{SL1}(\underline{t}_{ij}) = \text{glb}\{T : T = T_{SL1}(t) \text{ for } t \in (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL1,i} \neq \varnothing\}, \end{cases} \tag{9.11}$$

or (1.5) $\underline{t}_{ij} < t_{mx}$, $T_{SL2}(\underline{t}_{ij}) < T_{SL1}(\underline{t}_{ij})$ and

$$\underline{TF}_{1,ij} = \begin{cases} T_{SL1}(\underline{t}_{ij}) \text{ if } (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL2,i} = \varnothing \\ T_{SL2}(\underline{t}_{ij}) = \text{glb}\{T : T = T_{SL2}(t) \text{ for } t \in (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL2,i} \neq \varnothing\}. \end{cases} \tag{9.12}$$

Possibility (2): If $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, then

$$\underline{TF}_{1,ij} = \begin{cases} \text{glb}_{T1}(\underline{t}_{SL2,j}) \text{ if } (\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{SL1,i} \neq \varnothing \text{ and } \text{glb}_{T1}(\underline{t}_{SL2,j}) < T_{SL2}(\underline{t}_{SL2,j}) \\ T_{SL2}(\underline{t}_{SL2,j}) \text{ otherwise} \end{cases} \tag{9.13}$$

with

$$\text{glb}_{T1}(\underline{t}_{SL2,j}) = \begin{cases} \text{glb}\{T : T = T_{SL1}(t) \text{ for } t \in (\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{SL1,i} \neq \varnothing\} \\ \text{undefined if } (\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{SL1,i} = \varnothing. \end{cases} \tag{9.14}$$

Possibility (3): If $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$, then

$$\underline{TF}_{1,ij} = \begin{cases} \text{glb}_{T2}(\underline{t}_{SL1,i}) \text{ if } (\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{SL2,j} \neq \varnothing \text{ and } \text{glb}_{T2}(\underline{t}_{SL1,i}) < T_{SL1}(\underline{t}_{SL1,i}) \\ T_{SL1}(\underline{t}_{SL1,i}) \text{ otherwise} \end{cases} \tag{9.15}$$

with

$$\text{glb}_{T2}(\underline{t}_{SL1,i}) = \begin{cases} \text{glb}\{T : T = T_{SL2}(t) \text{ for } t \in (\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{SL2,j} \neq \varnothing\} \\ \text{undefined if } (\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{SL2,j} = \varnothing. \end{cases} \tag{9.16}$$

The need for the use of glb's in Eqs. (9.11)-(9.16) is illustrated by the use of the glb in Eq. (9.11). For this case (i.e., $\underline{t}_{ij} < t_{mx}$ and $T_{SL1}(\underline{t}_{ij}) < T_{SL2}(\underline{t}_{ij})$), the earliest time at which the link system will fail is $\underline{t}_{ij}$ and the corresponding link failure temperature at time $\underline{t}_{ij}$ is

$$\max\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\} = T_{SL2}(\underline{t}_{ij}), \tag{9.17}$$

which initially suggests that $T_{SL2}(\underline{t}_{ij})$ should be the minimum failure temperature. However, this is not correct in general because it is possible that SL 1 will fail after $\underline{t}_{ij}$ at a temperature less than $T_{SL2}(\underline{t}_{ij})$. In this case, the set

$$\mathcal{S} = \{T : T = T_{SL1}(t) \text{ for } t \in (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL1,i} \neq \varnothing\} \tag{9.18}$$

will (i) exist if $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL1,i} \neq \varnothing$ and (ii) contain failure times $t > t_{ij}$ at which $T_{SL1}(t) < T_{SL2}(\underline{t}_{ij})$. Thus, the smallest of these times and the associated temperature for SL 1 rather than $T_{SL2}(\underline{t}_{ij})$ will define $\underline{T}_{1,ij}$.However, there is a complication because the indicated temperatures do not have a smallest value. Rather, they have a glb, which is equal to $T_{SL1}(\underline{t}_{ij})$. Thus, although $T_{SL1}(\underline{t}_{ij})$ is not formally equal to $\underline{T}_{1,ij}$, it in effect defines $\underline{T}_{1,ij}$ by being the glb of SL 1 failure temperatures that occur after t_{ij} and are less than $T_{SL2}(\underline{t}_{ij})$.

The bound $\overline{TF}_{1,ij}$ also has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ as stated for the following possibilities:

Possibility (1): If $\overline{t}_{SL1,i} = \overline{t}_{SL2,j} = \overline{t}_{ij}$, then

$$\overline{TF}_{1,ij} = \max\{T_{SL1}(\bar{t}_{ij}), T_{SL2}(\bar{t}_{ij})\}. \tag{9.19}$$

Possibility (2): If $\bar{t}_{SL1,i} < \bar{t}_{SL2,j}$, then either (2.1) the inequality $\underline{t}_{SL2,j} \le \bar{t}_{SL1,i} < \bar{t}_{SL2,j}$ holds and

$$\overline{TF}_{1,ij} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\} \tag{9.20}$$

or (2.2) the inequality $\bar{t}_{SL1,i} < \underline{t}_{SL2,j} \le \bar{t}_{SL2,j}$ holds and

$$\bar{T}_{1,ij} = T_{SL2}(\bar{t}_{SL2,j}). \tag{9.21}$$

Possibility (3): If $\bar{t}_{SL2,j} < \bar{t}_{SL1,i}$, then either (3.1) the inequality $\underline{t}_{SL1,i} \le \bar{t}_{SL2,j} < \bar{t}_{SL1,i}$ holds and

$$\overline{TF}_{1,ij} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\} \tag{9.22}$$

or (3.2) the inequality $\bar{t}_{SL2,j} < \underline{t}_{SL1,i} \le \bar{t}_{SL1,i}$ holds and

$$\overline{TF}_{1,ij} = T_{SL1}(\bar{t}_{SL1,i}). \tag{9.23}$$

Once the evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for system failure temperature can be obtained from the pairs $(\underline{TF}_{1,ij}, \overline{TF}_{1,ij})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 9.1 and Fig. 9.2. In addition, Fig. 9.1 and Fig. 9.2 also contain CDFs and CCDFs for link system failure temperature obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the indicated CDFs and CCDFs are constructed as indicated in Eqs. (7.9) -(7.13) with $TF_1(t_{SL1}, t_{SL2})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

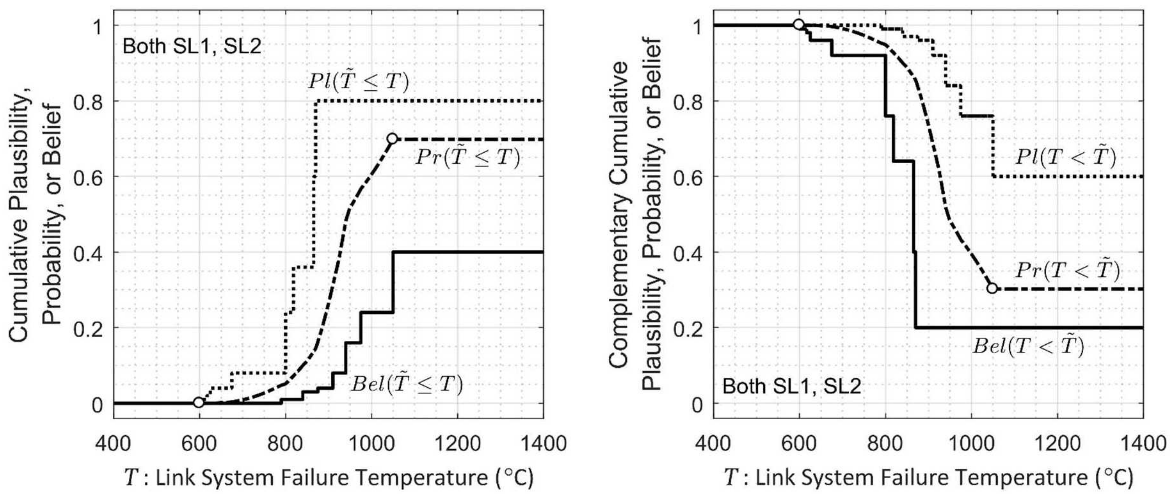


Fig. 9.1 Graphical summary of evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for temperature T at which link system failure occurs for (i) a two link system composed of SLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of both links: (a) Cumulative plausibility $Pl(\tilde{T} \le T)$, probability $Pr(\tilde{T} \le T)$ and belief $Bel(\tilde{T} \le T)$, and (b) Complementary cumulative plausibility $Pl(T < \tilde{T})$, probability $Pr(T < \tilde{T})$ and belief $Bel(T < \tilde{T})$.

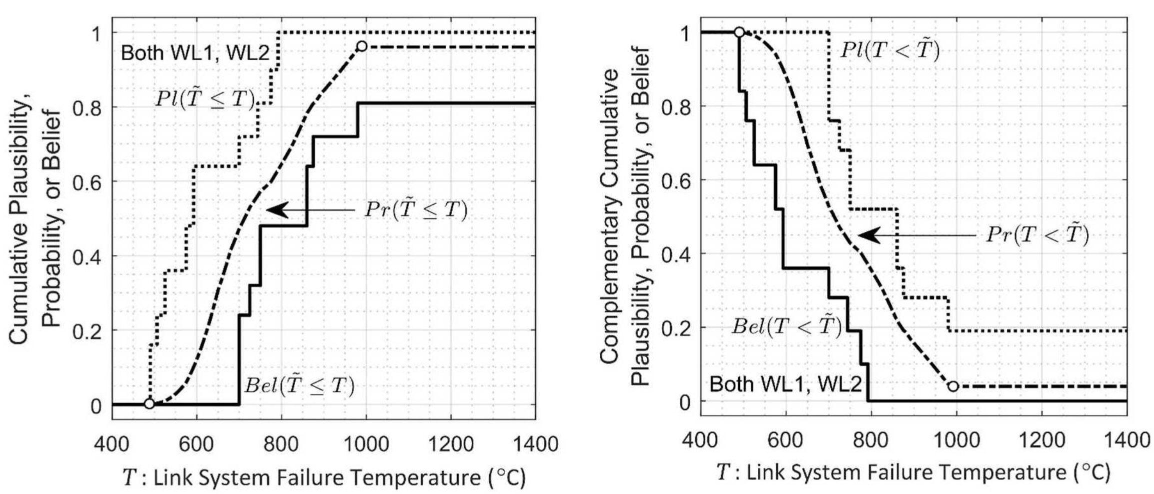


Fig. 9.2 Graphical summary of evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for temperature T at which link system failure occurs for (i) a two link system composed of WLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of both links: (a) Cumulative plausibility $Pl(\tilde{T} \le T)$, probability $Pr(\tilde{T} \le T)$ and belief $Bel(\tilde{T} \le T)$, and (b) Complementary cumulative plausibility $Pl(T < \tilde{T})$, probability $Pr(T < \tilde{T})$ and belief $Bel(T < \tilde{T})$.

A focal element $\mathcal{TF}_{1,ij}$ associated with the evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ contains temperatures corresponding to link system failure (i.e., temperatures $< \infty$) only if $(i, j) \in \mathcal{I}_{1F}$ with

$$\mathcal{I}_{1F} = \{(i,j) : \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx}\}. \tag{9.24}$$

In turn, the glb $\underline{TF}_1$ for the temperatures at which link system failure can occur is defined by

$$\underline{TF}_1 = \min\{\underline{TF}_{1,ij} : (i,j) \in \mathcal{I}_{1F}\}. \tag{9.25}$$

As an example,

$$\begin{aligned} \underline{TF}_1 &= \begin{cases} 600.000 \text{ °C for SL 1, SL 2} \\ 490.000 \text{ °C for WL 1, WL 2} \end{cases} \\ &\cong \begin{cases} 600.011 \text{ °C for SL 1, SL 2} \\ 490.004 \text{ °C for WL 1, WL 2} \end{cases} \end{aligned} \tag{9.26}$$

for the results illustrated in Fig. 9.1 and Fig. 9.2, with (i) the first results obtained from Eq. (9.25) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 9.1 and Fig. 9.2.

Determination of the maximum temperature $\overline{TF}_1$ at which link system failure could occur is now considered for system failure corresponding to failure of both SLs. As in Sect. 7.1, two cases for the definition of $\overline{TF}_1$ are considered: (i) All link temperature curves are continuous functions, and (ii) All link temperature curves are continuous increasing functions. For the first case (i.e., all link temperature curves are continuous functions), the maximum realized link system failure temperature $\overline{T}_{1F,ij}$ for $\mathcal{TF}_{1,ij}$ with $(i,j) \in \mathcal{I}_{1F}$ is defined by

$$\overline{T}_{1F,ij} = \max\{T : T \in \mathcal{TF}_{1,ij} \text{ and } T \ne \infty\}, \tag{9.27}$$

and the resultant maximum link system failure temperature $\overline{TF}_1$ is defined by

$$\overline{TF}_1 = \max\{\overline{T}_{1F,ij} : (i,j) \in \mathcal{I}_{1F}\}. \tag{9.28}$$

For the second case (i.e., all link temperature curves are continuous increasing functions), the determination of a closed-form representation for the maximum temperature $\overline{TF}_1$ at which link system failure could occur requires consideration of a number of special relationships involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$. The following development considers focal elements $\mathcal{TF}_{1,ij}$ associated with the evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ with $(i,j) \in \mathcal{I}_{1F}$.

To start, the following two possibilities

$$t_{mx} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \text{ and } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < t_{mx} \tag{9.29}$$

for $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ are identified and then used to identify more possibilities involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ as indicated below:

$$t_{mx} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \Rightarrow \begin{cases} t_{mx} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \text{ or} \\ \overline{t}_{SL1,i} < t_{mx} \le \overline{t}_{SL2,j} \text{ or} \\ \overline{t}_{SL2,i} < t_{mx} \le \overline{t}_{SL1,j} \end{cases} \tag{9.30}$$

with

$$\begin{gathered} t_{mx} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \text{ and } (i,j) \in \mathcal{I}_{1F} \\ \Rightarrow \overline{T}_{1F,ij} = \max\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\}, \end{gathered} \tag{9.31}$$

$$\begin{aligned} &\overline{t}_{SL1,i} < t_{mx} \le \overline{t}_{SL2,j} \text{ and } (i,j) \in \mathcal{I}_{1F} \\ &\Rightarrow \begin{cases} \overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx} \le \overline{t}_{SL2,j} \Rightarrow \overline{T}_{1F,ij} = T_{SL2}(t_{mx}) \\ \text{or} \\ \underline{t}_{SL2,j} \le \overline{t}_{SL1,i} < t_{mx} \le \overline{t}_{SL2,j} \\ \qquad \Rightarrow \overline{T}_{1F,ij} = \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(t_{mx})\}, \end{cases} \end{aligned} \tag{9.32}$$

$$\begin{aligned} &\overline{t}_{SL2,i} < t_{mx} \le \overline{t}_{SL1,j} \text{ and } (i,j) \in \mathcal{I}_{1F} \\ &\Rightarrow \begin{cases} \overline{t}_{SL2,i} < \underline{t}_{SL1,i} \le t_{mx} \le \overline{t}_{SL1,j} \Rightarrow \overline{T}_{1F,ij} = T_{SL1}(t_{mx}) \\ \text{or} \\ \underline{t}_{SL1,i} \le \overline{t}_{SL2,i} < t_{mx} \le \overline{t}_{SL1,j} \\ \qquad \Rightarrow \overline{T}_{1F,ij} = \max\{T_{SL1}(t_{mx}), T_{SL2}(\overline{t}_{SL2,i})\}. \end{cases} \end{aligned} \tag{9.33}$$

Similarly,

$$\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < t_{mx} \Rightarrow \begin{cases} \overline{t}_{SL1,i} = \overline{t}_{SL2,j} = \overline{t}_{ij} < t_{mx} \text{ or} \\ \overline{t}_{SL1,i} < \overline{t}_{SL2,j} < t_{mx} \text{ or} \\ \overline{t}_{SL2,j} < \overline{t}_{SL1,i} < t_{mx} \end{cases} \tag{9.34}$$

with

$$\begin{gathered} \overline{t}_{SL1,i} = \overline{t}_{SL2,j} = \overline{t}_{ij} < t_{mx} \text{ and } (i,j) \in \mathcal{I}_{1F} \\ \Rightarrow \overline{T}_{1F,ij} = \max\{T_{SL1}(\overline{t}_{ij}), T_{SL2}(\overline{t}_{ij})\}, \end{gathered} \tag{9.35}$$

$$\bar{t}_{SL1,i} < \bar{t}_{SL2,j} < t_{mx} \text{ and } (i,j) \in \mathcal{I}_{1F}$$

$$\Rightarrow \begin{cases} \underline{t}_{SL2,j} \le \bar{t}_{SL1,i} < \bar{t}_{SL2,j} < t_{mx} \\ \qquad \Rightarrow \bar{T}_{1F,ij} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\} \\ \text{or} \\ \bar{t}_{SL1,i} < \underline{t}_{SL2,j} < \bar{t}_{SL2,j} < t_{mx} \Rightarrow \bar{T}_{1F,ij} = T_{SL2}(\bar{t}_{SL2,j}), \end{cases} \tag{9.36}$$

$$\bar{t}_{SL2,j} < \bar{t}_{SL1,i} < t_{mx} \text{ and } (i,j) \in \mathcal{I}_{1F}$$

$$\Rightarrow \begin{cases} \underline{t}_{SL1,i} \le \bar{t}_{SL2,j} < \bar{t}_{SL1,i} < t_{mx} \\ \qquad \Rightarrow \bar{T}_{1F,ij} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\} \\ \text{or} \\ \bar{t}_{SL2,j} < \underline{t}_{SL1,i} < \bar{t}_{SL1,i} < t_{mx} \Rightarrow \bar{T}_{1F,ij} = T_{SL1}(\bar{t}_{SL1,i}). \end{cases} \tag{9.37}$$

Given the possible definitions for $\bar{T}_{1F,ij}$ in Eqs. (9.31)-(9.33) and (9.35)-(9.37) obtained with the assumption that the link temperature curves are increasing, the resultant value for $\bar{T}_{1F}$ is obtained as indicated in Eq. (9.28). As an example,

$$\begin{aligned} \overline{TF}_1 &= \begin{cases} 1050.000 \text{ °C for SL 1, SL 2} \\ 991.860 \text{ °C for WL 1, WL 2} \end{cases} \\ &\cong \begin{cases} 1050.000 \text{ °C for SL 1, SL 2} \\ 991.859 \text{ °C for WL 1, WL 2} \end{cases} \end{aligned} \tag{9.38}$$

for the results illustrated in Fig. 9.1 and Fig. 9.2, with (i) the first results obtained from Eqs. (9.29) -(9.37) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 9.1 and Fig. 9.2.

9.2 Cumulative and Complementary Cumulative Belief and Plausibility for Temperature at which a System of Two Links Fails Due to Failure of Either Link

The determination of cumulative belief and plausibility for the second definition (i.e., system failure occurs when either link has failed) is similar to the determination for the first definition. Specifically, the function

$$TF_2(\mathbf{t}) = \begin{cases} \infty \text{ for } t_{mx} < \min\{t_{SL1}, t_{SL2}\} = \infty \\ T_{SL2}(t_{SL2}) \text{ for } t_{SL2} \le t_{mx} \text{ and } t_{SL2} < t_{SL1} \\ T_{SL1}(t_{SL1}) \text{ for } t_{SL1} \le t_{mx} \text{ and } t_{SL1} < t_{SL2} \\ \min\{T_{SL1}(t_{12}), T_{SL2}(t_{12})\} \text{ for } t_{12} = t_{SL1} = t_{SL2} \le t_{mx} \end{cases} \tag{9.39}$$

with $\mathbf{t} = [t_{SL1}, t_{SL2}] \in \mathcal{TM}$ is used to map the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ into the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ for link system failure temperature as indicated in Eqs. (9.3)-(9.6) to obtain the evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$. Further, bounds

$$(\underline{TF}_{2,ij}, \overline{TF}_{2,ij}) = \left(\min(\mathcal{TF}_{2,ij}), \text{lub}(\mathcal{TF}_{2,ij})\right) \tag{9.40}$$

for focal elements $\mathcal{TF}_{2,ij}$ associated with the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ are introduced for use in the determination of the cumulative values of belief and plausibility for the temperature at which the link system fails as indicated in conjunction with Eqs. (2.48)-(2.50).

Definition of the focal element bounds $\underline{TF}_{2,ij}$ and $\overline{TF}_{2,ij}$ has an organizational structure that is similar to the structure used in the definition of the bounds $\underline{T}_{1,ij}$ and $\overline{T}_{1,ij}$ in Eqs. (9.8)-(9.23). Specifically, $\underline{TF}_{2,ij}$ is defined conditional on the following possibilities:

Possibility (1): If $\underline{t}_{ij} = \underline{t}_{SL1,i} = \underline{t}_{SL2,j}$, then

$$\underline{TF}_{2,ij} = \min\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\}, \tag{9.41}$$

Possibility (2): If $\underline{t}_{SL1,i} < \underline{t}_{SL2,j}$, then either: (2.1) the inequality $\underline{t}_{SL1,i} \le \overline{t}_{SL1,i} < \underline{t}_{SL2,j}$ holds and

$$\underline{TF}_{2,ij} = T_{SL1}(\underline{t}_{SL1,i}) \tag{9.42}$$

or (2.2) the inequality $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$ holds and

$$\underline{TF}_{2,ij} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\}, \tag{9.43}$$

Possibility: (3) If $\underline{t}_{SL2,j} < \underline{t}_{SL1,i}$, then either: (3.1) the inequality $\underline{t}_{SL2,j} \le \overline{t}_{SL2,j} < \underline{t}_{SL1,i}$ holds and

$$\underline{TF}_{2,ij} = T_{SL2}(\underline{t}_{SL2,j}) \tag{9.44}$$

or (3.2) the inequality $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$ holds and

$$\underline{TF}_{2,ij} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\}. \tag{9.45}$$

Similarly, $\overline{TF}_{2,ij}$ is defined by

Possibility (1): If $\overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j}$, then either: (1.1) the equality $\overline{t}_{ij} = \infty$ holds and

$$\overline{TF}_{2,ij} = \infty \tag{9.46}$$

or (1.2) $\overline{t}_{ij} \le t_{mx}$, $T_{SL1}(\overline{t}_{ij}) = T_{SL2}(\overline{t}_{ij})$ and

$$\overline{TF}_{2,ij} = T_{SL1}(\overline{t}_{ij}) = T_{SL2}(\overline{t}_{ij}), \tag{9.47}$$

or (1.3) $\overline{t}_{ij} \le t_{mx}$, $T_{SL1}(\overline{t}_{ij}) < T_{SL2}(\overline{t}_{ij})$ and

$$\overline{TF}_{2,ij} = T_{SL2}(\overline{t}_{ij}) = \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL2,j}\}, \tag{9.48}$$

or (1.4) $\overline{t}_{ij} \le t_{mx}$, $T_{SL2}(\overline{t}_{ij}) < T_{SL1}(\overline{t}_{ij})$ and

$$\overline{TF}_{2,ij} = T_{SL1}(\overline{t}_{ij}) = \text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL1,i}\}. \tag{9.49}$$

With respect to Eqs. (9.48) and (9.49), relationships

$$[\underline{t}_{SL2,j}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL2,j} \ne \varnothing \text{ and } [\underline{t}_{SL1,i}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL1,i} \ne \varnothing \tag{9.50}$$

hold as a consequence of the assumed inequality $\overline{t}_{ij} \le t_{mx}$.

Possibility (2): If $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$, then either: (2.1) the inequality $\underline{t}_{SL2,j} < \overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ holds and

$$\begin{aligned} \overline{TF}_{2,ij} &= \max\left\{T_{SL1}(\overline{t}_{SL1,i}), \text{lub}\{T : T = T_{SL2}(t) \text{ for } [\underline{t}_{SL2,j}, \overline{t}_{SL1,i}) \cap \mathcal{TM}_{SL2,j}\}\right\} \\ &= \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL1,i})\} \text{ if } \overline{t}_{SL1,i} \in \mathcal{TM}_{SL2,j}, \end{aligned} \tag{9.51}$$

or (2.2) the inequality $\underline{t}_{SL2,j} = \overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ holds and

$$\overline{TF}_{2,ij} = \min\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\}, \tag{9.52}$$

or (2.3) the inequality $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le \overline{t}_{SL2,j}$ holds and

$$\overline{TF}_{2,ij} = T_{SL1}(\overline{t}_{SL1,i}). \tag{9.53}$$

Possibility (3): If $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$, then either: (3.1) the inequality $\underline{t}_{SL1,i} < \overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ holds and

$$\begin{aligned}\overline{TF}_{2,ij} &= \max\left\{T_{SL2}(\overline{t}_{SL2,j}), \text{lub}\{T : T = T_{SL1}(t) \text{ for } [\underline{t}_{SL1,i}, \overline{t}_{SL2,j}) \cap \mathcal{TM}_{SL1,i}\}\right\} \\ &= \max\{T_{SL2}(\overline{t}_{SL2,j}), T_{SL1}(\overline{t}_{SL2,j})\} \text{ if } \overline{t}_{SL2,j} \in \mathcal{TM}_{SL1,i},\end{aligned} \quad (9.54)$$

or (3.2) the inequality $\underline{t}_{SL1,i} = \overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ holds and

$$\overline{TF}_{2,ij} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\}, \quad (9.55)$$

or (3.3) the inequality $\overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le \overline{t}_{SL1,i}$ holds and

$$\overline{TF}_{2,ij} = T_{SL2}(\overline{t}_{SL2,j}). \quad (9.56)$$

Once the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for system failure temperature can be obtained from the pairs $(\underline{TF}_{2,ij}, \overline{TF}_{2,ij})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 9.3 and Fig. 9.4. In addition, Fig. 9.3 and Fig. 9.4 also contain CDFs and CCDFs for link system failure temperature obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the indicated CDFs and CCDFs are constructed as indicated in Eqs. (7.9) -(7.13) with $TF_2(t_{SL1}, t_{SL2})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

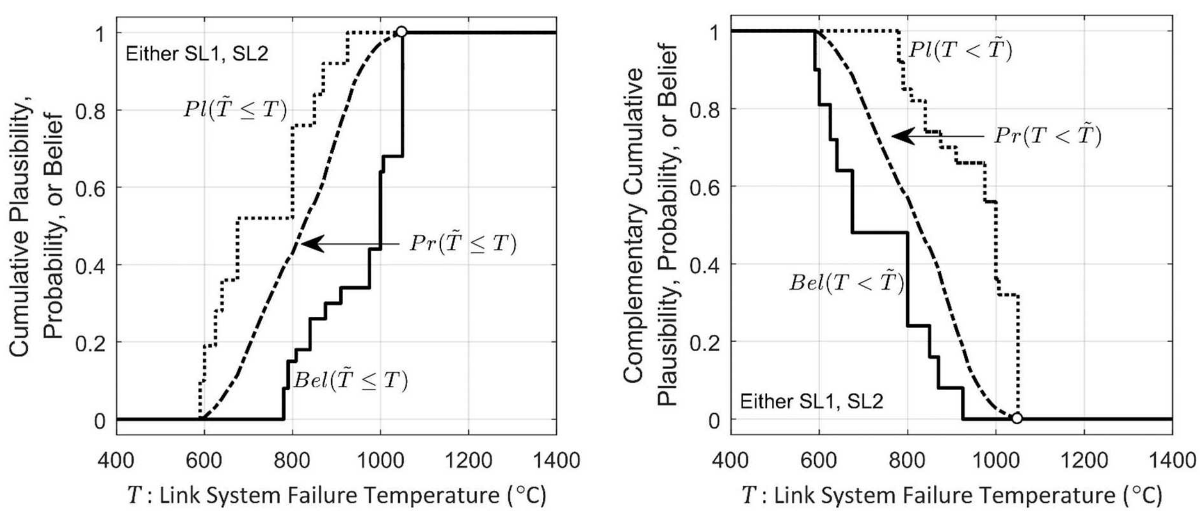


Fig. 9.3 Graphical summary of evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for temperature T at which link system failure occurs for (i) a two link system composed of SLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of either link: (a) Cumulative plausibility $Pl(\tilde{T} \le T)$, probability $Pr(\tilde{T} \le T)$ and belief $Bel(\tilde{T} \le T)$, and (b) Complementary cumulative plausibility $Pl(T < \tilde{T})$, probability $Pr(T < \tilde{T})$ and belief $Bel(T < \tilde{T})$.

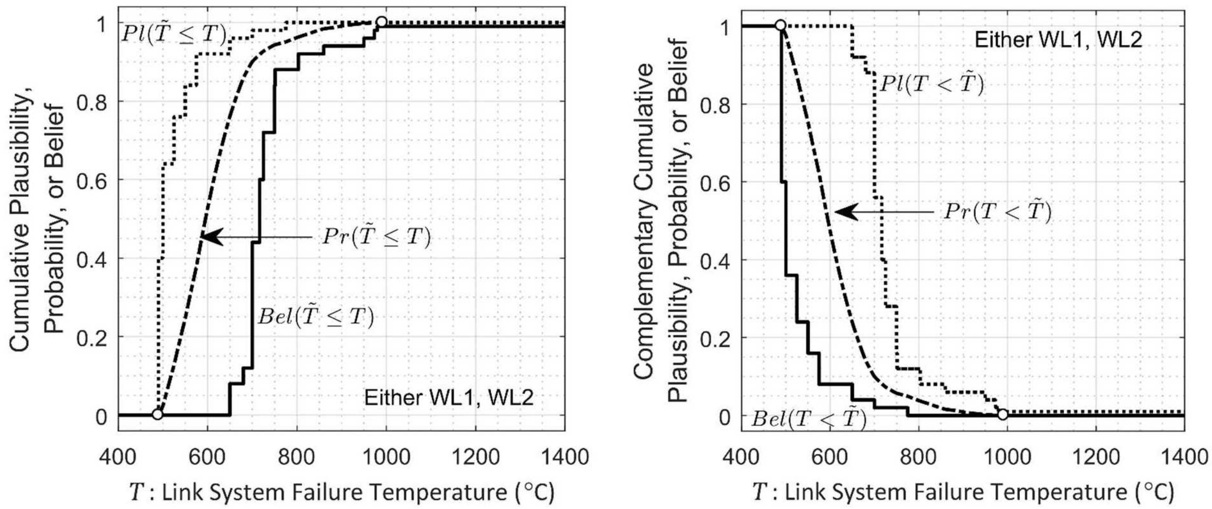


Fig. 9.4 Graphical summary of evidence space $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ for temperature T at which link system failure occurs for (i) a two link system composed of WLs 1 and 2 defined in Sect. 4.1 and (ii) system failure corresponding to failure of either link: (a) Cumulative plausibility $Pl(\tilde{T} \le T)$, probability $Pr(\tilde{T} \le T)$ and belief $Bel(\tilde{T} \le T)$, and (b) Complementary cumulative plausibility $Pl(T < \tilde{T})$, probability $Pr(T < \tilde{T})$ and belief $Bel(T < \tilde{T})$.

A focal element $\mathcal{TF}_{2,ij}$ associated with the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ contains temperatures corresponding to link system failure (i.e., temperatures $< \infty$) only if $(i, j) \in \mathcal{I}_{2F}$ with

$$\mathcal{I}_{2F} = \{(i, j) : \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx}\} \ne \varnothing. \tag{9.57}$$

In turn, the glb $\underline{TF}_2$ for the temperatures at which link system failure can occur is defined by

$$\underline{TF}_2 = \min\{\underline{TF}_{2,ij} : (i, j) \in \mathcal{I}_{2F}\}. \tag{9.58}$$

As an example,

$$\begin{aligned} \underline{TF}_2 &= \begin{cases} 590.000\ ^\circ\text{C for SL 1, SL 2} \\ 490.000\ ^\circ\text{C for WL 1, WL 2} \end{cases} \\ &\cong \begin{cases} 590.008\ ^\circ\text{C for SL 1, SL 2} \\ 490.000\ ^\circ\text{C for WL 1, WL 2} \end{cases} \end{aligned} \tag{9.59}$$

for the results illustrated in Fig. 9.3 and Fig. 9.4, with (i) the first results obtained from Eq. (9.58) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 9.3 and Fig. 9.4.

Determination of the lub $\overline{TF}_2$ of the link temperatures at which link system failure could occur is now considered for system failure corresponding to failure of either SL. As in Sect. 7.1, two cases for the definition of $\overline{TF}_2$ are considered: (i) All link temperature curves are continuous functions, and (ii) All link temperature curves are continuous increasing functions. For the first case (i.e., all link temperature curves are continuous functions), the lub $\overline{T}_{2F,ij}$ for system failure temperature for $\mathcal{TF}_{2,ij}$ with $(i, j) \in \mathcal{I}_{2F}$ is defined by

$$\overline{T}_{2F,ij} = \text{lub}\{T : T \in \mathcal{TF}_{2,ij} \text{ and } T \neq \infty\}, \tag{9.60}$$

and the resultant lub $\overline{TF}_2$ for realized link system failure temperatures is defined by

$$\overline{TF}_2 = \max\{\overline{T}_{2F,ij} : (i, j) \in \mathcal{I}_{2F}\}. \tag{9.61}$$

For the second case (i.e., all link temperature curves are continuous increasing functions), the determination of a closed-form representation for the lub $\overline{TF}_2$ of the temperatures at which link system failure could occur requires consideration of a number of special relationships involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$. The following development considers focal elements $\mathcal{TF}_{2,ij}$ associated with the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ with $(i, j) \in \mathcal{I}_{2F}$ and has a structure that is similar to structure in Eqs. (9.29)-(9.37) used in the determination of $\overline{TF}_1$ for continuous link temperature curves.

An important property that contributes to the following results derives from the assumptions that (i) the link temperature curves are continuous increasing functions and (ii) the focal elements for link failure temperatures are closed intervals. As a consequence, both links have the property indicated below for SL 1:

$$\begin{aligned}&\{T : T = T_{SL1}(t) \text{ for } t \in \mathcal{TM}_{SL1,i} \text{ and } t < \infty\}\\ &\quad = \begin{cases}[T_{SL1}(\underline{t}_{SL1,i}), T_{SL1}(\overline{t}_{SL1,i})] & \text{for } \overline{t}_{SL1,i} \leq t_{mx}\\ [T_{SL1}(\underline{t}_{SL1,i}), T_{SL1}(t_{mx})] & \text{for } t_{mx} < \overline{t}_{SL1,i}.\end{cases}\end{aligned} \tag{9.62}$$

Further, for a continuous, increasing link temperature curve $T(t)$ defined on $[t_{mn}, t_{mx}]$, a focal element $[\underline{T}, \overline{T}]$ for link failure temperature, and $T(t_{mn}) < \underline{T}$, the minimum time $\underline{t}$ and maximum time $\overline{t}$ for link failure (e.g., $\underline{t}_{SL1,i}$ and $\overline{t}_{SL1,i}$ in Eq. (9.62)) are equal if, and only if, $T(t_{mx}) = \underline{T}$. Specifically, existence of the indicated conditions means that the following statements are equivalent (i.e., that each statement implies the other): (i) $\underline{t} = \overline{t}$ and (ii) $T(t_{mx}) = \underline{T}$. Further, the condition $\underline{t} = \overline{t} = t_{mx}$ results for both statements.

To start, the following two disjoint possibilities

$$\min\{\overline{t}_{SL1,i},\overline{t}_{SL2,j}\} \le t_{mx} \text{ and } t_{mx} < \min\{\overline{t}_{SL1,i},\overline{t}_{SL2,j}\} \tag{9.63}$$

for $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ are used to identify more possibilities involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ for $(i, j) \in \mathcal{I}_{2F}$ as indicated below:

$$\min\{\overline{t}_{SL1,i},\overline{t}_{SL2,j}\} \le t_{mx} \Rightarrow \begin{cases} \overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} \le t_{mx} \text{ or} \\ \overline{t}_{SL1,i} < \overline{t}_{SL2,j} \text{ and } \overline{t}_{SL1,i} \le t_{mx} \text{ or} \\ \overline{t}_{SL2,i} < \overline{t}_{SL1,i} \text{ and } \overline{t}_{SL2,i} \le t_{mx} \end{cases} \tag{9.64}$$

and

$$t_{mx} < \min\{\overline{t}_{SL1,i},\overline{t}_{SL2,j}\} \Rightarrow \begin{cases} \underline{t}_{SL1,i} \le t_{mx} < \min\{\underline{t}_{SL2,j},\overline{t}_{SL1,i},\overline{t}_{SL2,j}\} \text{ or} \\ \underline{t}_{SL2,j} \le t_{mx} < \min\{\underline{t}_{SL1,i},\overline{t}_{SL1,i},\overline{t}_{SL2,j}\} \text{ or} \\ \max\{\underline{t}_{SL1,i},\underline{t}_{SL2,j}\} \le t_{mx} < \min\{\overline{t}_{SL1,i},\overline{t}_{SL2,j}\}. \end{cases} \tag{9.65}$$

In turn, the six possibilities indicated in Eqs. (9.64) and (9.65) result in the following values for $\overline{T}_{2F,ij}$ with $(i, j) \in \mathcal{I}_{2F}$:

Possibility (1): If $\overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} \le t_{mx}$ and $(i, j) \in \mathcal{I}_{2F}$, then either (1.1) $\underline{t}_{SL1,i} = \overline{t}_{SL1,i} = \overline{t}_{ij}$ (which implies $\underline{t}_{SL1,i} = \overline{t}_{SL1,i} = t_{mx}$), $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} = \overline{t}_{ij} = t_{mx}$ and

$$\begin{aligned} \overline{T}_{2F,ij} &= \max\left\{T_{SL1}(t_{mx}), \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{ij} = t_{mx}\}\right\} \\ &= \max\left\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\right\}, \end{aligned} \tag{9.66}$$

or (1.2) $\underline{t}_{SL2,j} = \overline{t}_{SL2,j} = \overline{t}_{ij}$ (which implies $\underline{t}_{SL2,j} = \overline{t}_{SL2,j} = t_{mx}$), $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} = \overline{t}_{ij} = t_{mx}$ and

$$\begin{aligned} \overline{T}_{2F,ij} &= \max\left\{T_{SL1}(t_{mx}), \text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{ij} = t_{mx}\}\right\} \\ &= \max\left\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\right\}, \end{aligned} \tag{9.67}$$

or (1.3) $\underline{t}_{SL1,i} = \overline{t}_{SL1,i} = \overline{t}_{ij}$ (which implies $\underline{t}_{SL1,i} = \overline{t}_{SL1,i} = t_{mx}$), $\underline{t}_{SL2,j} = \overline{t}_{SL2,j} = \overline{t}_{ij} = t_{mx}$ and

$$\overline{T}_{2F,ij} = \min\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\}, \tag{9.68}$$

or (1.4) $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} = \overline{t}_{ij}$, $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} = \overline{t}_{ij}$ and

$$\begin{aligned}\overline{T}_{2F,ij} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{ij}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{ij}\}\Big\} \quad . \\ &= \max\{T_{SL1}(\overline{t}_{ij}), T_{SL2}(\overline{t}_{ij})\}.\end{aligned} \tag{9.69}$$

Possibility (2): If $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$, $\overline{t}_{SL1,i} \le t_{mx}$ and $(i,j) \in \mathcal{I}_{2F}$, then either (2.1) the inequalities $\underline{t}_{SL2,j} < \overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and $\overline{t}_{SL1,i} \le t_{mx}$ hold and

$$\begin{aligned}\overline{T}_{2F,ij} &= \max\Big\{T_{SL1}(\overline{t}_{SL1,i}), \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{SL1,i}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL1,i})\}.\end{aligned} \tag{9.70}$$

or (2.2) the relationships $\tilde{t}_{ij} = \underline{t}_{SL2,j} = \overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and $\overline{t}_{SL1,i} \le t_{mx}$ hold and

$$\begin{aligned}\overline{T}_{2F,ij} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \tilde{t}_{ij}\}, T_{SL2}(\tilde{t}_{ij})\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL1,i})\},\end{aligned} \tag{9.71}$$

or (2.3) the inequalities $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} < \overline{t}_{SL2,j}$ and $\overline{t}_{SL1,i} \le t_{mx}$ hold and

$$\overline{T}_{2F,ij} = T_{SL1}(\overline{t}_{SL1,i}). \tag{9.72}$$

Possibility (3): If $\overline{t}_{SL2,i} < \overline{t}_{SL1,i}$, $\overline{t}_{SL2,i} \le t_{mx}$ and $(i,j) \in \mathcal{I}_{2F}$, then either: (3.1) the inequalities $\underline{t}_{SL1,i} < \overline{t}_{SL2,i} < \overline{t}_{SL1,j}$ and $\overline{t}_{SL2,i} \le t_{mx}$ hold and

$$\begin{aligned}\overline{T}_{2F,ij} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{SL2,j}\}, T_{SL2}(\overline{t}_{SL2,j})\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL2,j}), T_{SL2}(\overline{t}_{SL2,j})\},\end{aligned} \tag{9.73}$$

or (3.2) the relationships $\tilde{t}_{ij} = \underline{t}_{SL1,i} = \overline{t}_{SL2,i} < \overline{t}_{SL1,j}$ and $\overline{t}_{SL2,i} \le t_{mx}$ hold and

$$\begin{aligned}\overline{T}_{2F,ij} &= \max\Big\{T_{SL1}(\tilde{t}_{ij}), \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \tilde{t}_{ij}\}, T_{SL1}(\tilde{t}_{ij})\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL2,i}), T_{SL2}(\overline{t}_{SL2,i})\},\end{aligned} \tag{9.74}$$

or (3.3) the inequalities $\overline{t}_{SL2,i} < \underline{t}_{SL1,i} < \overline{t}_{SL1,j}$ and $\overline{t}_{SL2,i} \le t_{mx}$ hold and

$$\overline{T}_{2F,ij} = T_{SL2}(\overline{t}_{SL2,j}). \tag{9.75}$$

Possibility (4): If $\underline{t}_{SL1,i} \le t_{mx} < \min\{\underline{t}_{SL2,j}, \overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}$ and $(i,j) \in \mathcal{I}_{2F}$, then

$$\overline{T}_{2F,ij} = T_{SL1}(t_{mx}). \tag{9.76}$$

Possibility (5): If $\underline{t}_{SL2,j} \leq t_{mx} < \min\{\underline{t}_{SL1,i}, \overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}$ and $(i,j) \in \mathcal{I}_{2F}$, then

$$\overline{T}_{2F,ij} = T_{SL2}(t_{mx}) \tag{9.77}$$

Possibility (6): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \leq t_{mx} < \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}$ and $(i,j) \in \mathcal{I}_{2F}$, then

$$\overline{T}_{2F,ij} = \max\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\}. \tag{9.78}$$

Given the possible definitions for $\overline{T}_{2F,ij}$ in Eqs. (9.66)-(9.78) obtained with the assumption that the link temperature curves are continuous increasing functions, the resultant value for $\overline{T}_{2F}$ is obtained as indicated in Eq. (9.61). As an example,

$$\begin{aligned}\overline{TF}_2 &= \begin{cases} 1050.000\ °\text{C for SL 1, SL 2} \\ 991.860\ °\text{C for WL 1, WL 2} \end{cases} \\ &\cong \begin{cases} 1049.999\ °\text{C for SL 1, SL 2} \\ 991.862\ °\text{C for WL 1, WL 2} \end{cases}\end{aligned} \tag{9.79}$$

for the results illustrated in Fig. 9.3 and Fig. 9.4, with (i) the first results obtained from Eqs. (9.63) -(9.78) and (ii) the following approximate results obtained from the sampling-based analysis used to construct the CDFs and CCDFs in Fig. 9.3 and Fig. 9.4.

10. Cumulative and Complementary Cumulative Belief and Plausibility for SL Temperature at Which LOAS Occurs

For simplicity, this section considers a system with 2 SLs, 1 WL and two definitions of system failure: (i) LOAS occurs when both SLs fail before the WL fails and (ii) LOAS occurs when either SL fails before the WL fails.

10.1 Cumulative and Complementary Cumulative Belief and Plausibility for SL Temperature at which LOAs occurs when Both SLs Fail before the WL Fails

The development for both definitions of LOAS starts with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13)-(4.16). For the first definition (i.e., LOAS occurs when both SLs fail before the WL fails), the function

$$TL_1(\mathbf{t}) = \begin{cases} \infty \text{ for } t_{WL1} \le \max\{t_{SL1}, t_{SL2}\} \\ T_{SL1}(t_{SL1}) \text{ for } t_{SL2} < t_{SL1} < t_{WL1} \\ T_{SL2}(t_{SL2}) \text{ for } t_{SL1} < t_{SL2} < t_{WL1} \\ \max\{T_{SL1}(t_{SL1}), T_{SL2}(t_{SL2})\} \text{ for } t_{SL1} = t_{SL2} < t_{WL1} \end{cases} \tag{10.1}$$

with $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}$ is used to map the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ into the evidence space $(\mathcal{TL}_1, \mathbb{TL}_1, m_{TL1})$ for the SL temperature at which LOAS occurs. As indicated in the definition of $TL_1(\mathbf{t})$, SL link system failure is assumed to be the maximum of the individual link failure temperatures when the individual links fail at the same time. Further, the notational assumption

$$T_{SL1}(t_{SL1}) = \infty \text{ for } t_{SL1} = \infty \text{ and } T_{SL2}(t_{SL2}) = \infty \text{ for } t_{SL2} = \infty \tag{10.2}$$

is used to indicate that link failure temperature was not reached and hence that link failure did not occur.

The evidence space $(\mathcal{TL}_1, \mathbb{TL}_1, m_{TL1})$ for the SL temperature at which LOAS occurs (i.e., the temperature of the second SL to fail at the time that its failure results in LOAS) is defined by

$$\mathcal{TL}_1 = \{T : T = TL_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}\}, \tag{10.3}$$

$$\mathcal{TL}_{1,ijk} = \{T : T = TL_1(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}_{ijk}\}, \tag{10.4}$$

$$\mathbb{TL}_1 = \{\mathcal{TL}_{1,ijk} : (i,j,k) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\} \times \{1,2,...,nWL1\}\} \tag{10.5}$$

and

$$m_{TL1}(\mathcal{TL}_{1,ijk}) = m_{TML}(\mathcal{TM}_{ijk}) = m_{t,ijk}. \tag{10.6}$$

Further, the bounds

$$(\underline{T}_{1,ijk}, \overline{T}_{1,ijk}) = \left(\min(\mathcal{TL}_{1,ijk}), \max(\mathcal{TL}_{1,ijk})\right) \tag{10.7}$$

are introduced for use in the determination of the cumulative values of belief and plausibility for the SL temperature at which LOAS occurs as indicated in conjunction with Eqs. (2.48)-(2.50).

Definition of the focal element bounds $\underline{T}_{1,ijk}$ and $\overline{T}_{1,ijk}$ is now considered. Specifically, $\underline{T}_{1,ijk}$ has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$ as defined for the following disjoint conditions:

$$\overline{t}_{WL1,k} \le \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \tag{10.8}$$

and

$$\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k} \Rightarrow \begin{cases} \underline{t}_{ij} = \underline{t}_{SL1,i} = \underline{t}_{SL2,j} < \overline{t}_{WL1,k} & \text{or} \\ \underline{t}_{SL1,i} < \underline{t}_{SL2,j} < \overline{t}_{WL1,k} & \text{or} \\ \underline{t}_{SL2,j} < \underline{t}_{SL1,i} < \overline{t}_{WL1,k}. \end{cases} \tag{10.9}$$

In turn, the preceding four inequalities result in the following possibilities for the definition of $\underline{T}_{1,ijk}$:

Possibility (1): If $\overline{t}_{WL1,k} \le \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\}$, then

$$\underline{T}_{1,ijk} = \infty. \tag{10.10}$$

Possibility (2): If $\underline{t}_{ij} = \underline{t}_{SL1,i} = \underline{t}_{SL2,j} < \overline{t}_{WL1,k}$, then either (2.1) $T_{SL1}(\underline{t}_{ij}) = T_{SL2}(\underline{t}_{ij})$ and

$$\underline{T}_{1,ijk} = T_{SL1}(\underline{t}_{ij}) = T_{SL2}(\underline{t}_{ij}), \tag{10.11}$$

or (2.2) $T_{SL1}(\underline{t}_{ij}) < T_{SL2}(\underline{t}_{ij})$ and

$$\underline{T}_{1,ijk} = \begin{cases} \text{glb}\{\mathcal{S}_{21}\} = T_{SL1}(\underline{t}_{ij}) & \text{if } \mathcal{S}_{21} \ne \varnothing \\ T_{SL2}(\underline{t}_{ij}) & \text{if } \mathcal{S}_{21} = \varnothing \end{cases} \tag{10.12}$$

with

$$\mathcal{S}_{21} = \left\{T : T = T_{SL1}(t) \text{ for } T_{SL1}(\underline{t}_{ij}) < T_{SL1}(t) < T_{SL2}(\underline{t}_{ij}) \text{ and } t \in (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL1,i}\right\}, \tag{10.13}$$

or (2.3) $T_{SL2}(\underline{t}_{ij}) < T_{SL1}(\underline{t}_{ij})$ and

$$\underline{T}_{1,ijk} = \begin{cases} \text{glb}\{\mathcal{S}_{22}\} = T_{SL2}(\underline{t}_{ij}) & \text{if } \mathcal{S}_{22} \neq \varnothing \\ T_{SL1}(\underline{t}_{ij}) \text{ if } \mathcal{S}_{22} = \varnothing \end{cases} \tag{10.14}$$

with

$$\mathcal{S}_{22} = \left\{T : T = T_{SL2}(t) \text{ for } T_{SL2}(\underline{t}_{ij}) \leq T_{SL2}(t) < T_{SL1}(\underline{t}_{ij}) \text{ and } t \in (\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{SL2,j}\right\}. \tag{10.15}$$

Possibility (3): If $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} < \overline{t}_{WL1,k}$, then either: (3.1) the inequalities $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} \leq \underline{t}_{SL2,j} < \overline{t}_{WL1,k}$ hold and

$$\underline{T}_{1,ijk} = \begin{cases} T_{SL2}(\underline{t}_{SL2,j}) \text{ for } \overline{t}_{SL1,i} < \underline{t}_{SL2,j} \\ T_{SL2}(\underline{t}_{SL2,j}) \text{ for } \overline{t}_{SL1,i} = \underline{t}_{SL2,j} \text{ and } T_{SL1}(\overline{t}_{SL1,i}) \leq T_{SL2}(\underline{t}_{SL2,j}) \\ \text{glb}(\mathcal{S}_{31}) = T_{SL2}(\underline{t}_{SL2,j}) \text{ for } \overline{t}_{SL1,i} = \underline{t}_{SL2,j}, T_{SL2}(\underline{t}_{SL2,j}) < T_{SL1}(\overline{t}_{SL1,i}) \\ \qquad \text{and } \mathcal{S}_{31} \neq \varnothing \\ T_{SL1}(\overline{t}_{SL1,i}) \text{ for } \overline{t}_{SL1,i} = \underline{t}_{SL2,j}, T_{SL2}(\underline{t}_{SL2,j}) < T_{SL1}(\overline{t}_{SL1,i}) \text{ and } \mathcal{S}_{31} = \varnothing \end{cases} \tag{10.16}$$

with

$$\mathcal{S}_{31} = \left\{T : T = T_{SL2,j}(t) \text{ for } T_{SL2,j}(t) < T_{SL1}(\overline{t}_{SL1,i}) \text{ and } t \in (\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{SL2,j}\right\} \tag{10.17}$$

or (3.2) the inequalities $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} < \overline{t}_{SL1,i}$ and $\underline{t}_{SL2,j} < \overline{t}_{WL1,k}$ hold and

$$\underline{T}_{1,ijk} = \begin{cases} \text{glb}\{\mathcal{S}_{32}\} \text{ if } \mathcal{S}_{32} \neq \varnothing \\ T_{SL1,i}(\underline{t}_{SL2,j}) = \text{glb}\{\mathcal{S}_{32}\} \text{ if } \mathcal{S}_{32} \neq \varnothing \text{ and } \underline{t}_{SL2,j} \in \mathcal{TM}_{SL1,i} \\ T_{SL2,j}(\underline{t}_{SL2,j}) \text{ if } \mathcal{S}_{32} = \varnothing \end{cases} \tag{10.18}$$

with

$$\mathcal{S}_{32} = \left\{T : T = T_{SL1,i}(t) \text{ for } T_{SL1,i}(t) < T_{SL2,j}(\underline{t}_{SL2,j}) \text{ and } t \in (\underline{t}_{SL2,j}, \overline{t}_{SL1,i}] \cap \mathcal{TM}_{SL1,i}\right\}. \tag{10.19}$$

Possibility (4): If $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} < \overline{t}_{WL1,k}$, then either: (4.1) the inequalities $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} \leq \underline{t}_{SL1,i} < \overline{t}_{WL1,k}$ hold and

$$\underline{T}_{1,ijk} = \begin{cases} T_{SL1}(\underline{t}_{SL1,i}) \text{ for } \overline{t}_{SL2,j} < \underline{t}_{SL1,i} \\ T_{SL1}(\underline{t}_{SL1,i}) \text{ for } \overline{t}_{SL2,j} = \underline{t}_{SL1,i} \text{ and } T_{SL2}(\overline{t}_{SL2,j}) \le T_{SL1}(\underline{t}_{SL1,i}) \\ \text{glb}(\mathcal{S}_{41}) = T_{SL1}(\underline{t}_{SL1,i}) \text{ for } \overline{t}_{SL2,j} = \underline{t}_{SL1,i}, T_{SL1}(\underline{t}_{SL1,i}) < T_{SL2}(\overline{t}_{SL2,j}) \\ \qquad \text{and } \mathcal{S}_{41} \ne \varnothing \\ T_{SL2}(\overline{t}_{SL2,j}) \text{ for } \overline{t}_{SL2,j} = \underline{t}_{SL1,i}, T_{SL1}(\underline{t}_{SL1,i}) < T_{SL2}(\overline{t}_{SL2,j}) \text{ and } \mathcal{S}_{41} = \varnothing \end{cases} \quad (10.20)$$

with

$$\mathcal{S}_{41} = \left\{ T : T = T_{SL1,i}(t) \text{ for } T_{SL1,i}(t) < T_{SL2}(\overline{t}_{SL2,j}) \text{ and } t \in (\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{SL1,i} \right\} \quad (10.21)$$

or (4.2) the inequalities $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and $\underline{t}_{SL1,i} < \overline{t}_{WL1,k}$ hold and

$$\underline{T}_{1,ijk} = \begin{cases} \text{glb}\{\mathcal{S}_{42}\} \text{ if } \mathcal{S}_{42} \ne \varnothing \\ T_{SL2,j}(\underline{t}_{SL1,i}) = \text{glb}\{\mathcal{S}_{42}\} \text{ if } \mathcal{S}_{42} \ne \varnothing \text{ and } \underline{t}_{SL1,i} \in \mathcal{TM}_{SL2,j} \\ T_{SL1,i}(\underline{t}_{SL1,i}) \text{ if } \mathcal{S}_{42} = \varnothing \end{cases} \quad (10.22)$$

with

$$\mathcal{S}_{42} = \left\{ T : T = T_{SL2,j}(t) \text{ for } T_{SL2,j}(t) < T_{SL1,i}(\underline{t}_{SL1,i}) \text{ and } t \in (\underline{t}_{SL1,i}, \overline{t}_{SL2,j}] \cap \mathcal{TM}_{SL2,j} \right\}. \quad (10.23)$$

The bound $\overline{T}_{1,ij}$ also has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$ as defined for the following disjoint conditions:

$$\underline{t}_{WL1,k} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \quad (10.24)$$

and

$$\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \underline{t}_{WL1,k} \Rightarrow \begin{cases} \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < \underline{t}_{WL1,k} \text{ or} \\ \overline{t}_{SL1,i} < \overline{t}_{SL2,j} < \underline{t}_{WL1,k} \text{ or} \\ \overline{t}_{SL2,j} < \overline{t}_{SL1,i} < \underline{t}_{WL1,k}. \end{cases} \quad (10.25)$$

Specifically, the preceding four inequalities result in the following possibilities for the definition of $\overline{T}_{1,ijk}$:

Possibility (1): If $\underline{t}_{WL1,k} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}$, then

$$\overline{T}_{1,ijk} = \infty. \quad (10.26)$$

Possibility (2): If $\overline{t}_{SL1,i} = \overline{t}_{SL2,j} < \underline{t}_{WL1,k}$, then

$$\overline{T}_{1,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\}. \tag{10.27}$$

Possibility (3): If $\overline{t}_{SL1,i} < \overline{t}_{SL2,j} < \underline{t}_{WL1,k}$, then either: (3.1) the inequalities $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i} < \overline{t}_{SL2,j} < \underline{t}_{WL1,k}$ hold and

$$\overline{T}_{1,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\} \tag{10.28}$$

or (3.2) the inequalities $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le \overline{t}_{SL2,j} < \underline{t}_{WL1,k}$ hold and

$$\overline{T}_{1,ijk} = T_{SL2}(\overline{t}_{SL2,j}). \tag{10.29}$$

Possibility (4): If $\overline{t}_{SL2,j} < \overline{t}_{SL1,i} < \underline{t}_{WL1,k}$, then either: (4.1) the inequalities $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j} < \overline{t}_{SL1,i} < \underline{t}_{WL1,k}$ hold and

$$\overline{T}_{1,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\} \tag{10.30}$$

or (4.2) the inequalities $\overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le \overline{t}_{SL1,i} < \underline{t}_{WL1,k}$ hold and

$$\overline{T}_{1,ijk} = T_{SL1}(\overline{t}_{SL1,i}). \tag{10.31}$$

Once the evidence space $(\mathcal{TL}_1, \mathbb{TL}_1, m_{TL1})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for SL temperature at which LOAS occurs can be obtained from the pairs $(\underline{T}_{1,ijk}, \overline{T}_{1,ijk})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 10.1. In addition, Fig. 10.1 also contains the CDF and CCDF for the SL temperature at which LOAS occurs obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the indicated CDF and CCDF are constructed as described in Eqs. (7.9)-(7.13) with $TL_2(t_{SL1}, t_{SL2}, t_{WL1})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

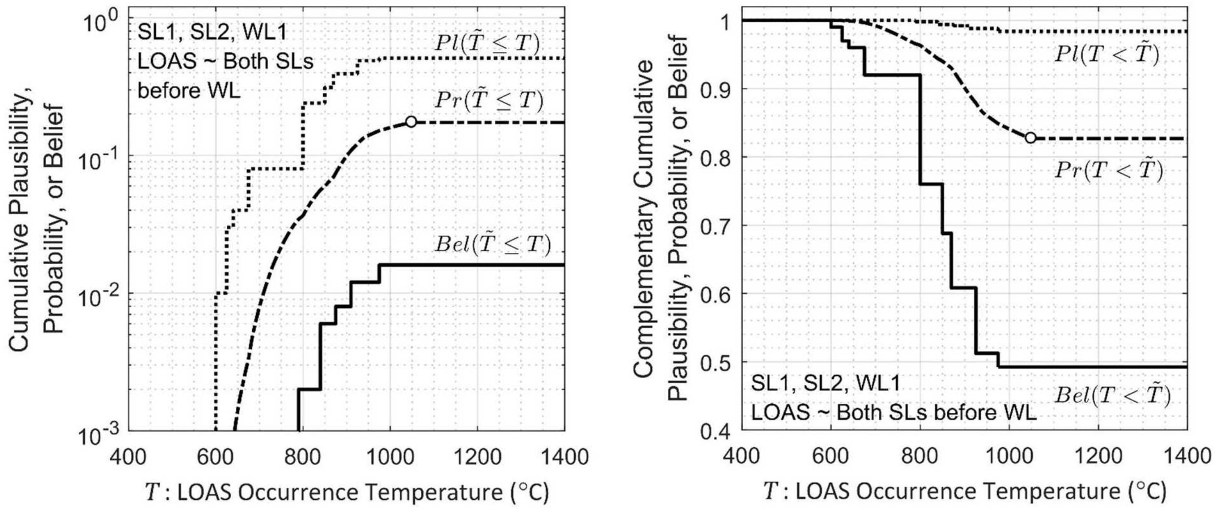


Fig. 10.1 Graphical summary of evidence space $(\mathcal{TL}_1, \mathbb{TL}_1, m_{TL1})$ for temperature T at which LOAS occurs for (i) a three link system composed of SL 1, SL 2 and WL 1 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of both SL links before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{T} \le T)$, probability $Pr(\tilde{T} \le T)$ and belief $Bel(\tilde{T} \le T)$, and (b) Complementary cumulative plausibility $Pl(T < \tilde{T})$, probability $Pr(T < \tilde{T})$ and belief $Bel(T < \tilde{T})$.

A focal element $\mathcal{TL}_{1,ijk}$ associated with the evidence space $(\mathcal{TL}_1, \mathbb{TL}_1, m_{TL1})$ contains temperatures corresponding to SL temperatures at which LOAS could actually occur only if $(i,j,k) \in \mathcal{I}_{1L}$ with

$$\mathcal{I}_{1L} = \left\{(i,j,k) : \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k}\right\} \ne \varnothing \tag{10.32}$$

as previously indicated in Eq. (6.14). In turn, the glb $\underline{T}_{1L}$ for SL temperatures at which LOAS could occur is defined by

$$\underline{T}_{1L} = \min\{\underline{T}_{1,ijk} : (i,j,k) \in \mathcal{I}_{1L}\}. \tag{10.33}$$

As an example,

$$\underline{T}_{1L} = 600.000 \text{ °C} \cong 600.383 \text{ °C} \tag{10.34}$$

for the results illustrated in Fig. 10.1, with (i) the first result obtained from Eq. (10.33) and (ii) the following approximate result obtained from the sampling-based analysis used to construct the CDF and CCDF in Fig. 10.1.

Determination of the lub $\overline{T}_{1L}$ for the SL temperatures at which LOAS could occur is now considered for system failure corresponding to failure of both SLs before failure of the WL. Two cases for the definition of $\overline{T}_{1L}$ are considered: (i) All link temperature curves are continuous functions, and (ii) All link temperature curves are continuous increasing functions.

For the first case (i.e., all link temperature curves are continuous functions), the lub $\overline{T}_{1L,ijk}$ of the temperatures corresponding to the occurrence of LOAS for $\mathcal{TL}_{1,ijk}$ with $(i,j,k) \in \mathcal{I}_{1L}$ is defined by

$$\overline{T}_{1L,ijk} = \text{lub}\{T : T \in \mathcal{TL}_{1,ijk} \text{ and } T \neq \infty\}, \tag{10.35}$$

and the resultant maximum temperature $\overline{T}_{1L}$ corresponding to the occurrence of LOAS is defined by

$$\overline{T}_{1L} = \max\{\overline{T}_{1L,ijk} : (i,j,k) \in \mathcal{I}_{1L}\}. \tag{10.36}$$

For the second case (i.e., all link temperature curves are continuous increasing functions), the determination of a closed-form representation for the lub $\overline{T}_{1F}$ of the SL temperatures at which LOAS could occur requires consideration of a number of special relationships involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$.

To start, the following two disjoint possibilities

$$\overline{t}_{WL1,k} = \infty \text{ and } \overline{t}_{WL1,k} \le t_{mx} \tag{10.37}$$

for $\overline{t}_{WL1,k}$ are identified and then used to define more possibilities involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$ as indicated below:

$$\overline{t}_{WL1,k} = \infty \Rightarrow \begin{cases} t_{mx} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \le \overline{t}_{WL1,k} & \text{or} \\ \overline{t}_{SL1,i} < t_{mx} \le \overline{t}_{SL2,j} \le \overline{t}_{WL1,k} & \text{or} \\ \overline{t}_{SL2,j} < t_{mx} \le \overline{t}_{SL1,i} \le \overline{t}_{WL1,k} & \text{or} \\ \overline{t}_{SL1,i} < \overline{t}_{SL2,j} < t_{mx} < \overline{t}_{WL1,k} & \text{or} \\ \overline{t}_{SL2,j} < \overline{t}_{SL1,i} < t_{mx} < \overline{t}_{WL1,k} & \text{or} \\ \overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < t_{mx} < \overline{t}_{WL1,k} \end{cases} \tag{10.38}$$

with

$$t_{mx} \le \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \le \bar{t}_{WL1,k} \quad \text{and} \quad (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\}, \tag{10.39}$$

$$\bar{t}_{SL1,i} < t_{mx} \le \bar{t}_{SL2,j} \le \bar{t}_{WL1,k} \quad \text{and} \quad (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \begin{cases} \bar{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx} \le \bar{t}_{SL2,j} \le \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = T_{SL2}(t_{mx}) \\ \text{or} \\ \underline{t}_{SL2,j} \le \bar{t}_{SL1,i} < t_{mx} \le \bar{t}_{SL2,j} \le \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(t_{mx})\}, \end{cases} \tag{10.40}$$

$$\bar{t}_{SL2,j} < t_{mx} \le \bar{t}_{SL1,i} \le \bar{t}_{WL1,k} \quad \text{and} \quad (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \begin{cases} \bar{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx} \le \bar{t}_{SL1,i} \le \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = T_{SL1}(t_{mx}) \\ \text{or} \\ \underline{t}_{SL1,i} \le \bar{t}_{SL2,j} < t_{mx} \le \bar{t}_{SL1,i} \le \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(t_{mx}), T_{SL2}(\bar{t}_{SL2,j})\}, \end{cases} \tag{10.41}$$

$$\bar{t}_{SL1,i} < \bar{t}_{SL2,j} < t_{mx} < \bar{t}_{WL1,k} \quad \text{and} \quad (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \begin{cases} \bar{t}_{SL1,i} < \underline{t}_{SL2,j} \le \bar{t}_{SL2,j} < t_{mx} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = T_{SL2}(\bar{t}_{SL2,j}) \\ \text{or} \\ \underline{t}_{SL2,j} \le \bar{t}_{SL1,i} < \bar{t}_{SL2,j} < t_{mx} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\}, \end{cases} \tag{10.42}$$

$$\bar{t}_{SL2,j} < \bar{t}_{SL1,i} < t_{mx} < \bar{t}_{WL1,k} \quad \text{and} \quad (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \begin{cases} \bar{t}_{SL2,j} < \underline{t}_{SL1,i} \le \bar{t}_{SL1,i} < t_{mx} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = T_{SL1}(\bar{t}_{SL1,i}) \\ \text{or} \\ \underline{t}_{SL1,i} \le \bar{t}_{SL2,j} < \bar{t}_{SL1,i} < t_{mx} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\}, \end{cases} \tag{10.43}$$

$$\bar{t}_{ij} = \bar{t}_{SL1,i} = \bar{t}_{SL2,j} < t_{mx} < \bar{t}_{WL1,k} \quad \text{and} \quad (i,j,k) \in \mathcal{I}_{1L}$$
$$\Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{ij}), T_{SL2}(\bar{t}_{ij})\}. \tag{10.44}$$

Similarly,

$$\bar{t}_{WL1,k} \le t_{mx} \Rightarrow \begin{cases} \bar{t}_{WL1,k} \le \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \quad \text{or} \\ \bar{t}_{SL1,i} < \bar{t}_{WL1,k} \le \bar{t}_{SL2,j} \quad \text{or} \\ \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \le \bar{t}_{SL1,i} \quad \text{or} \\ \bar{t}_{SL1,i} < \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \quad \text{or} \\ \bar{t}_{SL2,j} < \bar{t}_{SL1,i} < \bar{t}_{WL1,k} \quad \text{or} \\ \bar{t}_{ij} = \bar{t}_{SL1,i} = \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \end{cases} \tag{10.45}$$

with

$$\begin{aligned}&\bar{t}_{WL1,k} \le \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\},\ \bar{t}_{WL1,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L}\\ &\quad \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{WL1,k}), T_{SL2}(\bar{t}_{WL1,k})\},\end{aligned} \tag{10.46}$$

$$\begin{aligned}&\bar{t}_{SL1,i} < \bar{t}_{WL1,k} \le \bar{t}_{SL2,j},\ \bar{t}_{WL1,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L}\\ &\quad \Rightarrow \begin{cases} \bar{t}_{SL1,i} < \underline{t}_{SL2,j} < \bar{t}_{WL1,k} \le \bar{t}_{SL2,j} \Rightarrow \bar{T}_{1L,ijk} = T_{SL2}(\bar{t}_{WL1,k}) \\ \text{or} \\ \underline{t}_{SL2,j} \le \bar{t}_{SL1,i} < \bar{t}_{WL1,k} \le \bar{t}_{SL2,j} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{WL1,k})\},\end{cases}\end{aligned} \tag{10.47}$$

$$\begin{aligned}&\bar{t}_{SL2,j} < \bar{t}_{WL1,k} \le \bar{t}_{SL1,i},\ \bar{t}_{WL1,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L}\\ &\quad \Rightarrow \begin{cases} \bar{t}_{SL2,j} < \underline{t}_{SL1,i} \le \bar{t}_{WL1,k} \le \bar{t}_{SL1,i} \Rightarrow \bar{T}_{1L,ijk} = T_{SL1}(\bar{t}_{WL1,k}) \\ \text{or} \\ \underline{t}_{SL1,i} \le \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \le \bar{t}_{SL1,i} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{WL1,k}), T_{SL2}(\bar{t}_{SL2,j})\},\end{cases}\end{aligned} \tag{10.48}$$

$$\begin{aligned}&\bar{t}_{SL1,i} < \bar{t}_{SL2,j} < \bar{t}_{WL1,k},\ \bar{t}_{WL1,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L}\\ &\quad \Rightarrow \begin{cases} \bar{t}_{SL1,i} < \underline{t}_{SL2,j} \le \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = T_{SL2}(\bar{t}_{SL2,j}) \\ \text{or} \\ \underline{t}_{SL2,j} \le \bar{t}_{SL1,i} < \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\},\end{cases}\end{aligned} \tag{10.49}$$

$$\begin{aligned}&\bar{t}_{SL2,j} < \bar{t}_{SL1,i} < \bar{t}_{WL1,k},\ \bar{t}_{WL1,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L}\\ &\quad \Rightarrow \begin{cases} \bar{t}_{SL2,j} < \underline{t}_{SL1,i} \le \bar{t}_{SL1,i} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = T_{SL1}(\bar{t}_{SL1,i}) \\ \text{or} \\ \underline{t}_{SL1,i} \le \bar{t}_{SL2,j} < \bar{t}_{SL1,i} < \bar{t}_{WL1,k} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL2,j})\},\end{cases}\end{aligned} \tag{10.50}$$

$$\bar{t}_{ij} = \bar{t}_{SL1,i} = \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \le t_{mx} \text{ and } (i,j,k) \in \mathcal{I}_{1L} \Rightarrow \bar{T}_{1L,ijk} = \max\{T_{SL1}(\bar{t}_{ij}), T_{SL2}(\bar{t}_{ij})\}. \tag{10.51}$$

Technically, the quantity $T_{SL1}(\bar{t}_{WL1,k})$ in Eqs. (10.46) and (10.48) corresponds to the lub of the set

$$\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \bar{t}_{WL1,k} \text{ and } t \in \mathcal{TM}_{SL1,i}\}, \tag{10.52}$$

which is $T_{SL1}(\bar{t}_{WL1,k})$. The use of the indicated lub is appropriate in Eqs. (10.46) and (10.48) because (i) LOAS can occur for SL 1 temperatures approaching $T_{SL1}(\bar{t}_{WL1,k})$ at time $\bar{t}_{WL1,k}$ but (ii) LOAS cannot occur due to the failure of SL 1 at time $\bar{t}_{WL1,k}$ as LOAS is assumed to not occur for the simultaneous occurrence of the second SL failure and the WL failure. For a similar reason, the quantity $T_{SL2}(\overline{t}_{WL1,k})$ in Eqs. (10.46) and (10.47) corresponds to the lub of the set

$$\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{WL1,k} \text{ and } t \in \mathcal{TM}_{SL2,j}\}, \tag{10.53}$$

which is $T_{SL2}(\overline{t}_{WL1,k})$.

Given the possible definitions for $\overline{T}_{1L,ijk}$ in Eqs. (10.39)-(10.44) and (10.46)-(10.51) obtained with the assumption that the link temperature curves are continuous increasing functions, the resultant value for $\overline{T}_{1L}$ is obtained as indicated in Eq. (10.36). As an example,

$$\overline{T}_{1L} = 1050.000\ °\text{C} \cong 1049.999\ °\text{C} \tag{10.54}$$

for the results illustrated in Fig. 10.1, with (i) the first result obtained from Eqs. (10.39)-(10.44) and (10.46)-(10.51), and (ii) the following approximate result obtained from the sampling-based analysis used to construct the CDF and CCDF in Fig. 10.1.

10.2 Cumulative and Complementary Cumulative Belief and Plausibility for SL Temperature at which LOAS Occurs when Either SL Fails before the WL Fails

For the second definition (i.e., LOAS occurs when either SL fails before the WL fails), the function

$$TL_2(\mathbf{t}) = \begin{cases} \infty \text{ for } t_{WL1} \le \min\{t_{SL1}, t_{SL2}\} \\ T_{SL2}(t_{SL2}) \text{ for } t_{SL2} < t_{SL1} \text{ and } t_{SL2} < t_{WL1} \\ T_{SL1}(t_{SL1}) \text{ for } t_{SL1} < t_{SL2} \text{ and } t_{SL1} < t_{WL1} \\ \min\{T_{SL1}(t_{SL1}), T_{SL2}(t_{SL2})\} \text{ for } t_{SL1} = t_{SL2} < t_{WL1} \end{cases} \tag{10.55}$$

with $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}$ is used to map the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ into the evidence space $(\mathcal{TL}_2, \mathbb{TL}_2, m_{TL2})$ for the SL temperature at which LOAS occurs as shown in Eqs. (10.3)-(10.6) to obtain the evidence space $(\mathcal{TL}_1, \mathbb{TL}_1, m_{TL1})$. Specifically, $(\mathcal{TL}_2, \mathbb{TL}_2, m_{TL2})$ is defined by

$$\mathcal{TL}_2 = \{T : T = TL_2(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}\}, \tag{10.56}$$

$$\mathcal{TL}_{2,ijk} = \{T : T = TL_2(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}_{ijk}\}, \tag{10.57}$$

$$\mathbb{TL}_2 = \{\mathcal{TL}_{2,ijk} : (i,j,k) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\} \times \{1,2,...,nWL1\}\} \tag{10.58}$$

and

$$m_{TL2}(\mathcal{TL}_{2,ijk}) = m_{TML}(\mathcal{TM}_{ijk}) = m_{t,ijk}. \tag{10.59}$$

Further, the bounds

$$(\underline{T}_{2,ijk}, \overline{T}_{2,ijk}) = \left(\min(\mathcal{TL}_{2,ijk}), \max(\mathcal{TL}_{2,ijk})\right) \tag{10.60}$$

are introduced for use in the determination of the cumulative values of belief and plausibility for the SL temperature at which LOAS occurs as indicated in conjunction with Eqs. (2.48)-(2.50).

As indicated in the definition of $TF_2(\mathbf{t})$, link system failure temperature is assumed to be the minimum of the individual SL failure temperatures when the individual SLs fail at the same time. Further, the notational assumption

$$T_{SL1}(t_{SL1}) = \infty \text{ for } t_{SL1} = \infty \text{ and } T_{SL2}(t_{SL2}) = \infty \text{ for } t_{SL2} = \infty \tag{10.61}$$

is used to indicate that link failure temperature was not reached and hence that link failure did not occur.

The bound $\underline{T}_{2,ijk}$ has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$ as defined for the following disjoint conditions:

$$\overline{t}_{WL1,k} \le \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \tag{10.62}$$

and

$$\min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k} \Rightarrow \begin{cases} \underline{t}_{ij} = \underline{t}_{SL1,i} = \underline{t}_{SL2,j} < \overline{t}_{WL1,k} \text{ or} \\ \underline{t}_{SL1,i} < \min\{\overline{t}_{WL1,k}, \underline{t}_{SL2,j}\} \text{ or} \\ \underline{t}_{SL2,j} < \min\{\overline{t}_{WL1,k}, \underline{t}_{SL1,i}\}. \end{cases} \tag{10.63}$$

In turn, the preceding four inequalities result in the following possibilities for the definition of $\underline{T}_{2,ijk}$:

Possibility (1): If $\overline{t}_{WL1,k} \le \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\}$, then

$$\underline{T}_{2,ijk} = \infty. \tag{10.64}$$

Possibility (2): If $\underline{t}_{ij} = \underline{t}_{SL1,i} = \underline{t}_{SL2,j} < \overline{t}_{WL1,k}$, then

$$\underline{T}_{2,ijk} = \min\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\}. \tag{10.65}$$

Possibility (3): If $\underline{t}_{SL1,i} < \min\{\overline{t}_{WL1,k}, \underline{t}_{SL2,j}\}$, then either (3.1) the inequality $\underline{t}_{SL1,i} < \overline{t}_{WL1,k} \le \underline{t}_{SL2,j}$ holds and

$$\underline{T}_{2,ijk} = T_{SL1}(\underline{t}_{SL1,i}) \tag{10.66}$$

or (3.2) the inequalities $\underline{t}_{SL2,j} < \overline{t}_{WL1,k}$ and $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$ hold and

$$\underline{T}_{2,ijk} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\}, \tag{10.67}$$

or (3.3) the inequalities $\underline{t}_{SL2,j} < \overline{t}_{WL1,k}$ and $\underline{t}_{SL1,i} \le \overline{t}_{SL1,i} < \underline{t}_{SL2,j}$ hold and

$$\underline{T}_{2,ijk} = T_{SL1}(\underline{t}_{SL1,i}). \tag{10.68}$$

Possibility (4) If $\underline{t}_{SL2,j} < \min\{\overline{t}_{WL1,k}, \underline{t}_{SL1,i}\}$, then either (4.1) the inequality $\underline{t}_{SL2,j} < \overline{t}_{WL1,k} \le \underline{t}_{SL1,i}$ holds and

$$\underline{T}_{2,ijk} = T_{SL2}(\underline{t}_{SL2,j}) \tag{10.69}$$

or (4.2) the inequalities $\underline{t}_{SL1,i} < \overline{t}_{WL1,k}$ and $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$ hold and

$$\underline{T}_{2,ijk} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\}, \tag{10.70}$$

or (4.3) the inequalities $\underline{t}_{SL1,i} < \overline{t}_{WL1,k}$ and $\underline{t}_{SL2,j} \le \overline{t}_{SL2,j} < \underline{t}_{SL1,i}$ hold and

$$\underline{T}_{2,ijk} = T_{SL2}(\underline{t}_{SL2,j}). \tag{10.71}$$

The definition of $\overline{T}_{2,ijk}$ is now considered. Specifically, $\overline{T}_{2,ijk}$ has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}, \overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$ as defined for the following disjoint conditions:

$$\underline{t}_{WL1,k} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \tag{10.72}$$

and

$$\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \underline{t}_{WL1,k} \Rightarrow \begin{cases} \overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < \underline{t}_{WL1,k} \text{ or} \\ \overline{t}_{SL1,i} < \min\{\overline{t}_{SL2,j}, \underline{t}_{WL1,k}\} \text{ or} \\ \overline{t}_{SL2,j} < \min\{\overline{t}_{SL1,i}, \underline{t}_{WL1,k}\}. \end{cases} \tag{10.73}$$

In turn, the preceding four inequalities result in the following possibilities for the definition of $\overline{T}_{2,ijk}$:

Possibility (1): If $\underline{t}_{WL1,k} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}$, then

$$\overline{T}_{2,ijk} = \infty. \quad (10.74)$$

Possibility (2): If $\overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < \underline{t}_{WL1,k}$, then

$$\overline{T}_{2,ijk} = \begin{cases} T_{SL1}(\overline{t}_{ij}) = \text{lub}(\mathcal{S}_1) \text{ if } \underline{t}_{SL1,i} < \underline{t}_{SL2,j} = \overline{t}_{ij} \\ T_{SL2}(\overline{t}_{ij}) = \text{lub}(\mathcal{S}_2) \text{ if } \underline{t}_{SL2,j} < \underline{t}_{SL1,i} = \overline{t}_{ij} \\ \max\{T_{SL1}(\overline{t}_{ij}) = \text{lub}(\mathcal{S}_1), T_{SL2}(\overline{t}_{ij}) = \text{lub}(\mathcal{S}_2)\} \text{ if } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{ij} \\ \min\{T_{SL1}(\overline{t}_{ij}), T_{SL2}(\overline{t}_{ij})\} \text{ if } \underline{t}_{SL1,i} = \underline{t}_{SL2,j} = \overline{t}_{ij} \end{cases} \quad (10.75)$$

with

$$\begin{aligned} \mathcal{S}_1 &= \{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL1,i}\} \\ \mathcal{S}_2 &= \{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL2,j}\}. \end{aligned} \quad (10.76)$$

Possibility (3): If $\overline{t}_{SL1,i} < \min\{\overline{t}_{SL2,j}, \underline{t}_{WL1,k}\}$, then either (3.1) the inequality $\underline{t}_{SL2,j} < \overline{t}_{SL1,i} < \min\{\overline{t}_{SL2,j}, \underline{t}_{WL1,k}\}$ holds and

$$\begin{aligned} \overline{T}_{2,ijk} &= \max\left\{T_{SL1}(\overline{t}_{SL1,i}), \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{SL1,i}) \cap \mathcal{TM}_{SL2,j}\}\right\} \\ &= \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL1,i})\} \text{ if } \overline{t}_{SL1,i} \in \mathcal{TM}_{SL2,j}, \end{aligned} \quad (10.77)$$

or (3.2) the inequality $\underline{t}_{SL2,j} = \overline{t}_{SL1,i} < \min\{\overline{t}_{SL2,j}, \underline{t}_{WL1,k}\}$ holds and

$$\overline{T}_{2,ijk} = \begin{cases} T_{SL1}(\overline{t}_{SL1,i}) = \text{lub}\{\mathcal{S}_3\} \text{ if } \mathcal{S}_3 \neq \varnothing \\ \min\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\} \text{ if } \mathcal{S}_3 = \varnothing, \end{cases} \quad (10.78)$$

with

$$\mathcal{S}_3 = \{T : T = T_{SL1}(t) \text{ for } t \in \mathcal{TM}_{SL1,i} \text{ and } t < \overline{t}_{SL1,i}\} \quad (10.79)$$

or (3.3) the inequalities $\overline{t}_{SL1,i} < \underline{t}_{SL2,j}$ and $\overline{t}_{SL1,i} < \min\{\overline{t}_{SL2,j}, \underline{t}_{WL1,k}\}$ hold and

$$\overline{T}_{2,ijk} = T_{SL1}(\overline{t}_{SL1,i}). \quad (10.80)$$

Possibility (4): If $\overline{t}_{SL2,j} < \min\{\overline{t}_{SL1,i}, \underline{t}_{WL1,k}\}$, then either (4.1) the inequality $\underline{t}_{SL1,i} < \overline{t}_{SL2,j} < \min\{\overline{t}_{SL1,i}, \underline{t}_{WL1,k}\}$ holds and

$$\begin{aligned}\overline{T}_{2,ijk} &= \max\left\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{SL2,j}) \cap \mathcal{TM}_{SL1,i}\}, T_{SL2}(\overline{t}_{SL2,j})\right\} \\ &= \max\{T_{SL1}(\overline{t}_{SL2,j}), T_{SL2}(\overline{t}_{SL2,j})\} \text{ if } \overline{t}_{SL2,j} \in \mathcal{TM}_{SL1,i},\end{aligned} \tag{10.81}$$

or (4.2) the inequality $\underline{t}_{SL1,i} = \overline{t}_{SL2,j} < \min\{\overline{t}_{SL1,i}, \underline{t}_{WL1,k}\}$ holds and

$$\overline{T}_{2,ijk} = \begin{cases} T_{SL2}(\overline{t}_{SL2,j}) = \text{lub}\{\mathcal{S}_4\} \text{ if } \mathcal{S}_4 \neq \varnothing \\ \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\} \text{ if } \mathcal{S}_4 = \varnothing \end{cases} \tag{10.82}$$

with

$$\mathcal{S}_4 = \{T : T = T_{SL2}(t) \text{ for } t \in \mathcal{TM}_{SL2,j} \text{ and } t < \overline{t}_{SL2,j}\}, \tag{10.83}$$

or (4.3) the inequalities $\overline{t}_{SL2,j} < \underline{t}_{SL1,i}$ and $\overline{t}_{SL2,j} < \min\{\overline{t}_{SL1,i}, \underline{t}_{WL1,k}\}$ hold and

$$\overline{T}_{2,ijk} = T_{SL2}(\overline{t}_{SL2,j}). \tag{10.84}$$

Once the evidence space $(\mathcal{TL}_2, \mathbb{TL}_2, m_{TL2})$ is constructed, cumulative plausibility and belief functions for SL temperature at which LOAS occurs can be obtained from the pairs $(\underline{T}_{1,ijk}, \overline{T}_{1,ijk})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 10.2. In addition, Fig. 10.2 also contains the CDF and CCDF for the SL temperature at which LOAS occurs obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the indicated CDF and CCDF are constructed as described in Eqs. (7.9)-(7.13) with $TL_2(t_{SL1}, t_{SL2}, t_{WL1})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

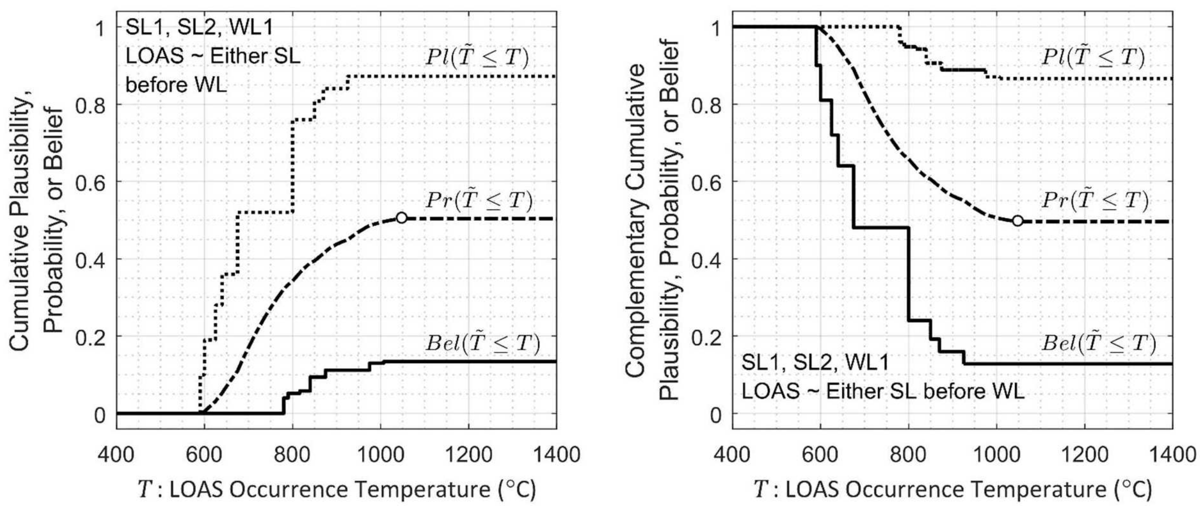


Fig. 10.2 Graphical summary of evidence space $(\mathcal{TL}_2, \mathbb{TL}_2, m_{TL2})$ for temperature T at which LOAS occurs for (i) a three link system composed of SL 1, SL 2 and WL 1 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of both SL links before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{T} \le T)$, probability $Pr(\tilde{T} \le T)$ and belief $Bel(\tilde{T} \le T)$, and (b) Complementary cumulative plausibility $Pl(T < \tilde{T})$, probability $Pr(T < \tilde{T})$ and belief $Bel(T < \tilde{T})$.

A focal element $\mathcal{TL}_{2,ijk}$ associated with the evidence space $(\mathcal{TL}_2, \mathbb{TL}_2, m_{TL2})$ contains temperatures corresponding to SL temperatures at which LOAS could occur only if $(i, j, k) \in \mathcal{I}_{2L}$ with

$$\mathcal{I}_{2L} = \left\{(i, j, k) : \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k}\right\} \ne \varnothing \tag{10.85}$$

as previously indicated in Eq. (7.43). In turn, the glb $\underline{T}_{2L}$ for the SL temperatures at which LOAS could occur is defined by

$$\underline{T}_{2L} = \min\{\underline{T}_{2,ijk} : (i, j, k) \in \mathcal{I}_{2L}\}. \tag{10.86}$$

As an example,

$$\underline{T}_{2L} = 590 \text{ °C} \cong 590.008 \text{ °C} \tag{10.87}$$

for the results illustrated in Fig. 10.2, with (i) the first result obtained from Eq. (10.86) and (ii) the following approximate result obtained from the sampling-based analysis used to construct the CDF and CCDF in Fig. 10.2.

Determination of the lub $\overline{T}_{2L}$ of the SL temperatures at which LOAS could occur is now considered for LOAS corresponding to failure of either SL before failure of the WL. Two cases for the definition of $\overline{T}_{2L}$ are considered: (i) All link temperature curves are continuous functions, and (ii) All link temperature curves are continuous increasing functions.

For the first case (i.e., all link temperature curves are continuous functions), the lub $\overline{T}_{2L,ijk}$ of SL temperatures corresponding to the occurrence of LOAS for $\mathcal{TL}_{2,ijk}$ with $(i,j,k) \in \mathcal{I}_{2L}$ is defined by

$$\overline{T}_{2L,ijk} = \text{lub}\{T : T \in \mathcal{TL}_{2,ijk} \text{ and } T \neq \infty\}, \tag{10.88}$$

and the resultant lub $\overline{T}_{2L}$ of SL temperatures corresponding to the occurrence of LOAS is defined by

$$\overline{T}_{2L} = \max\{\overline{T}_{2L,ijk} : (i,j,k) \in \mathcal{I}_{2L}\}. \tag{10.89}$$

For the second case (i.e., all link temperature curves are continuous increasing functions), the determination of a closed-form representation for the lub $\overline{T}_{2L}$ of the SL temperatures at which LOAS could occur requires consideration of a number of special relationships involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$. The following development considers focal elements $\mathcal{TL}_{2,ijk}$ associated with the evidence space $(\mathcal{TL}_2, \mathbb{TL}_2, m_{TL2})$ with $(i,j,k) \in \mathcal{I}_{2L}$.

To start, the following two disjoint possibilities

$$\overline{t}_{WL1,k} = \infty \text{ and } \overline{t}_{WL1,k} \le t_{mx} \tag{10.90}$$

for $\overline{t}_{WL1,k}$ are identified and then used to identify more possibilities involving $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$ as indicated below:

$$\overline{t}_{WL1,k} = \infty \Rightarrow \begin{cases} t_{mx} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \le \overline{t}_{WL1,k} = \infty \text{ or} \\ \overline{t}_{SL1,i} < \min\{t_{mx}, \overline{t}_{SL2,j}\} \text{ and } \overline{t}_{SL2,j} \le \overline{t}_{WL1,k} = \infty \text{ or} \\ \overline{t}_{SL2,j} < \min\{t_{mx}, \overline{t}_{SL1,i}\} \text{ and } \overline{t}_{SL1,i} \le \overline{t}_{WL1,k} = \infty \text{ or} \\ \overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < t_{mx} < \overline{t}_{WL1,k} = \infty \end{cases} \tag{10.91}$$

and

$$\bar{t}_{WL1,k} \le t_{mx} \Rightarrow \begin{cases} \bar{t}_{WL1,k} \le \min\{t_{mx}, \bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \ \text{ or} \\ \bar{t}_{SL1,i} < \min\{\bar{t}_{WL1,k}, \bar{t}_{SL2,j}\} \ \text{ and } \ \bar{t}_{WL1,k} \le t_{mx} \ \text{ or} \\ \bar{t}_{SL2,j} < \min\{\bar{t}_{WL1,k}, \bar{t}_{SL1,i}\} \ \text{ and } \ \bar{t}_{WL1,k} \le t_{mx} \ \text{ or} \\ \bar{t}_{ij} = \bar{t}_{SL1,i} = \bar{t}_{SL2,j} < \bar{t}_{WL1,k} \le t_{mx}. \end{cases} \tag{10.92}$$

In turn, the four possibilities in Eq. (10.91) for $\bar{t}_{WL1,k} = \infty$ result in the following definitions for $\bar{T}_{2L,ijk}$:

Possibility (1): If $t_{mx} \le \min\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}\} \le \bar{t}_{WL1,k} = \infty$ and $(i,j,k) \in \mathcal{I}_{2L}$, then either (1.1) the inequality $\underline{t}_{SL1,i} \le t_{mx} < \underline{t}_{SL2,j}$ holds and

$$\bar{T}_{2L,ijk} = T_{SL1}(t_{mx}), \tag{10.93}$$

or (1.2) the inequality $\underline{t}_{SL2,j} \le t_{mx} < \underline{t}_{SL1,i}$ holds and

$$\bar{T}_{2L,ijk} = T_{SL2}(t_{mx}), \tag{10.94}$$

or (1.3) the inequality $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} \le t_{mx}$ holds and

$$\bar{T}_{2L,ijk} = \begin{cases} \max\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\} \ \text{ if } \ \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < t_{mx} \\ \min\{T_{SL1}(t_{mx}), T_{SL2}(t_{mx})\} \ \text{ if } \ \underline{t}_{SL1,i} = \underline{t}_{SL2,j} = t_{mx} \end{cases} \tag{10.95}$$

with

$$T_{SL1}(t_{mx}) = \text{lub}\{T : T = T_{SL1}(t) \ \text{ for } \ t \in [\underline{t}_{SL1,i}, t_{mx})\} \ \text{ if } \ \underline{t}_{SL1,i} < t_{mx}$$
$$T_{SL2}(t_{mx}) = \text{lub}\{T : T = T_{SL2}(t) \ \text{ for } \ t \in [\underline{t}_{SL2,j}, t_{mx})\} \ \text{ if } \ \underline{t}_{SL2,j} < t_{mx}.$$

Possibility (2): If $\bar{t}_{SL1,i} < \min\{t_{mx}, \bar{t}_{SL2,j}\}$, $\bar{t}_{SL2,j} \le \bar{t}_{WL1,k}$ and $(i,j,k) \in \mathcal{I}_{2L}$, then either (2.1) the inequality $\underline{t}_{SL2,j} < \bar{t}_{SL1,i}$ holds and

$$\begin{aligned} \bar{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \ \text{ for } \ \underline{t}_{SL1,i} \le t < \bar{t}_{SL1,i}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \ \text{ for } \ \underline{t}_{SL2,j} \le t < \bar{t}_{SL1,i}\}\Big\} \\ &= \max\{T_{SL1}(\bar{t}_{SL1,i}), T_{SL2}(\bar{t}_{SL1,i})\}, \end{aligned} \tag{10.96}$$

or (2.2) $\underline{t}_{SL2,j} = \bar{t}_{SL1,i}$ and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{SL1,i}\} \\ &= T_{SL1}(\overline{t}_{SL1,i}),\end{aligned} \tag{10.97}$$

or (2.3) the inequality $\overline{t}_{SL1,i} < \underline{t}_{SL2,j}$ holds and

$$\overline{T}_{2L,ijk} = T_{SL1}(\overline{t}_{SL1,i}). \tag{10.98}$$

Possibility (3): If $\overline{t}_{SL2,j} < \min\{t_{mx}, \overline{t}_{SL1,i}\}$, $\overline{t}_{SL1,i} \le \overline{t}_{WL1,k}$ and $(i,j,k) \in \mathcal{I}_{2L}$, then either (3.1) the inequality $\underline{t}_{SL1,i} < \overline{t}_{SL2,j}$ holds and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{SL2,j}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{SL2,j}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL2,j}), T_{SL2}(\overline{t}_{SL2,j})\},\end{aligned} \tag{10.99}$$

or (3.2) $\underline{t}_{SL1,i} = \overline{t}_{SL2,j}$ and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{SL2,j}\} \\ &= T_{SL1}(\overline{t}_{SL2,j}),\end{aligned} \tag{10.100}$$

or (3.3) the inequality $\overline{t}_{SL2,j} < \underline{t}_{SL1,i}$ holds and

$$\overline{T}_{2L,ijk} = T_{SL2}(\overline{t}_{SL2,j}). \tag{10.101}$$

Possibility (4): If $\overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < t_{mx} < \overline{t}_{WL1,k}$ and $(i,j,k) \in \mathcal{I}_{2L}$ holds, then

$$\begin{aligned}\overline{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{ij}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{ij}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{ij}), T_{SL2}(\overline{t}_{ij})\}.\end{aligned} \tag{10.102}$$

Similarly, the four possibilities in Eq. (10.92) for $\overline{t}_{WL1,k} \le t_{mx}$ result in the following definitions for $\overline{T}_{2F,ijk}$:

Possibility (1) If $\overline{t}_{WL1,k} \le \min\{t_{mx}, \overline{t}_{SL1,i}, \overline{t}_{SL2,j}\}$ and $(i,j,k) \in \mathcal{I}_{2L}$, then either (1.1) the inequalities $\underline{t}_{SL1,i} < \overline{t}_{WL1,k} \le \underline{t}_{SL2,j}$ hold and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{WL1,k}\} \\ &= T_{SL1}(\overline{t}_{WL1,k}),\end{aligned} \tag{10.103}$$

or (1.2) the inequalities $\underline{t}_{SL2,j} < \overline{t}_{WL1,k} \le \underline{t}_{SL1,i}$ hold and

$$\begin{aligned}\overline{T}_{2F,ijk} &= \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{WL1,k}\} \\ &= T_{SL2}(\overline{t}_{WL1,k}),\end{aligned} \tag{10.104}$$

or (1.3) the inequality $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{WL1,k}$ holds and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{WL1,k}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{WL1,k}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{WL1,k}), T_{SL2}(\overline{t}_{WL1,k})\}.\end{aligned} \tag{10.105}$$

Possibility (2) If $\overline{t}_{SL1,i} < \min\{\overline{t}_{WL1,k}, \overline{t}_{SL2,j}\}$, $\overline{t}_{WL1,k} \le t_{mx}$ and $(i, j, k) \in \mathcal{I}_{2L}$, then either: (2.1) the inequality $\underline{t}_{SL2,j} < \overline{t}_{SL1,i}$ holds and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{SL1,i}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{SL1,i}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL1,i})\},\end{aligned} \tag{10.106}$$

or (2.2) $\underline{t}_{SL2,j} = \overline{t}_{SL1,i}$ and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{SL1,i}\} \\ &= T_{SL1}(\overline{t}_{SL1,i}),\end{aligned} \tag{10.107}$$

or (2.3) the inequality $\overline{t}_{SL1,i} < \underline{t}_{SL2,j}$ holds and

$$\overline{T}_{2L,ijk} = T_{SL1}(\overline{t}_{SL1,i}). \tag{10.108}$$

Possibility (3) If $\overline{t}_{SL2,j} < \min\{\overline{t}_{WL1,k}, \overline{t}_{SL1,i}\}$, $\overline{t}_{WL1,k} \le t_{mx}$ and $(i, j, k) \in \mathcal{I}_{2L}$, then either: (3.1) the inequality $\underline{t}_{SL1,i} < \overline{t}_{SL2,j}$ holds and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{SL2,j}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{SL2,j}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{SL2,j}), T_{SL2}(\overline{t}_{SL2,j})\},\end{aligned} \tag{10.109}$$

or (3.2) $\underline{t}_{SL2,j} = \overline{t}_{SL1,i}$ and

$$\begin{aligned}\overline{T}_{2L,ijk} &= \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{SL2,j}\} \\ &= T_{SL2}(\overline{t}_{SL2,j}),\end{aligned} \tag{10.110}$$

or (3.3) the inequality $\overline{t}_{SL2,j} < \underline{t}_{SL1,i}$ holds and

$$\overline{T}_{2L,ijk} = T_{SL2}(\overline{t}_{SL2,j}). \tag{10.111}$$

Possibility (4) If $\overline{t}_{ij} = \overline{t}_{SL1,i} = \overline{t}_{SL2,j} < \overline{t}_{WL1,k} \le t_{mx}$ and $(i,j,k) \in \mathcal{I}_{2L}$ holds, then

$$\begin{aligned}\overline{T}_{2L,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL1,i} \le t < \overline{t}_{ij}\}, \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL2,j} \le t < \overline{t}_{ij}\}\Big\} \\ &= \max\{T_{SL1}(\overline{t}_{ij}), T_{SL2}(\overline{t}_{ij})\}.\end{aligned} \tag{10.112}$$

Given the possible definitions for $\overline{T}_{2L,ijk}$ in Eqs. (10.93)-(10.102) and (10.103)-(10.112) obtained with the assumption that the link temperature curves are increasing, the resultant value for $\overline{T}_{2L}$ is obtained as indicated in Eq. (10.89). As an example,

$$\overline{T}_{2L} = 1050.000 \text{ °C} \cong 1049.995 \text{ °C} \tag{10.113}$$

for the results illustrated in Fig. 10.2, with (i) the first result obtained from Eqs. (10.93)-(10.102) and (10.103)-(10.112), and (ii) the following approximate result obtained from the sampling-based analysis used to construct the CDF and CCDF in Fig. 10.2.

11. Cumulative and Complementary Cumulative Belief and Plausibility for Failure Time Margins

For simplicity, this section considers a system with 2 SLs and 1 WL and two definitions of system failure: (i) LOAS occurs when both SLs fail before the WL fails and (ii) LOAS occurs when either SL fails before the WL fails.

11.1 Cumulative and Complementary Cumulative Belief and Plausibility for Failure Time Margins with LOAS Defined by Failure of Both SLs before Failure of the WL

Failure time margins defined by

$$\begin{aligned} M_t = \; & \text{(time at which SL failure potentially causes LOAS)} \\ & - \text{(time at which WL failure potentially prevents LOAS)} \end{aligned} \tag{11.1}$$

are an important summary result in the analysis of WL/SL systems. The descriptor "potentially" is used in the definition of M_t because the occurrence of LOAS depends on the relative timing of SL failure and WL failure. Specifically, M_t is positive if SL failure occurs after WL failure (i.e., the desired occurrence) and negative if SL failure occurs before WL failure (i.e., the undesired occurrence).

This section presents failure time margin results for a 2 SL, 1 WL system for which LOAS occurs if both SLs fail before the WL fails. Nonfailure of either of the SLs or the WL is a possibility for this system that must be addressed as part of the analysis of margins. To handle this situation, a generalized margin defined by

$$\begin{aligned} M_{1t}(\mathbf{t}) &= M_{1t}([t_{SL1}, t_{SL2}, t_{WL1}]) \\ &= \begin{cases} -\infty \;\text{ for } t_{WL1} = \infty,\; \max\{t_{SL1}, t_{SL2}\} < \infty \\ \max\{t_{SL1}, t_{SL2}\} - t_{WL1} \;\text{ for }\; t_{WL1} < \infty,\; \max\{t_{SL1}, t_{SL2}\} < \infty \\ \infty \;\text{ for }\; \max\{t_{SL1}, t_{SL2}\} = \infty \end{cases} \end{aligned} \tag{11.2}$$

for $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ belonging to the set $\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}$ defined in Eq. (4.13) is considered for analysis.

Application of the function $M_{1t}(\mathbf{t})$ defined in Eq. (11.2) to the elements $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ of the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13) -(4.16) results in the evidence space $(\mathcal{MTM}_1, \mathbb{MTM}_1, m_{MTM1})$ for failure time margins with

$$\mathcal{MTM}_1 = \{m_t : m_t = M_{1t}(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}\}, \tag{11.3}$$

$$\mathcal{MTM}_{1,ijk} = \{m_t : m_t = M_t(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}_{ijk}\} \tag{11.4}$$

$$(\underline{m}_{1t,ijk}, \overline{m}_{1t,ijk}) = \left(\min(\mathcal{MTM}_{1,ijk}), \max(\mathcal{MTM}_{1,ijk})\right) \tag{11.5}$$

$$\mathbb{MTM}_1 = \{\mathcal{MTM}_{1,ijk} : (i,j,k) \in \mathcal{I} = \{1,2,...,nSL1\} \times \{1,2,...,nSL2\} \times \{1,2,...,nWL1\}\} \tag{11.6}$$

and

$$m_{MTM1}(\mathcal{MTM}_{1,ijk}) = m_{TM}(\mathcal{TM}_{ijk}) = m_{ijk}. \tag{11.7}$$

Further, the focal element bounds $\underline{m}_{1t,ijk}$ and $\overline{m}_{1t,ijk}$ are defined by

$$\underline{m}_{1t,ijk} = \begin{cases} -\infty \text{ for } \overline{t}_{WL1,k} = \infty \text{ and } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty \\ \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} - \overline{t}_{WL1,k} \text{ for } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty \\ \infty \text{ for } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} = \infty \end{cases} \tag{11.8}$$

and

$$\overline{m}_{1t,ijk} = \begin{cases} -\infty \text{ for } \underline{t}_{WL1,k} = \infty \text{ and } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty \\ \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} - \underline{t}_{WL1,k} \text{ for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}\} < \infty \\ \infty \text{ for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} = \infty. \end{cases} \tag{11.9}$$

Once the evidence space $(\mathcal{MTM}_1, \mathbb{MTM}_1, m_{MTM1})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for SL/WL failure time margins can be obtained from the pairs $(\underline{m}_{1t,ijk}, \overline{m}_{1t,ijk})$ as indicated in conjunction with Eqs. (2.48)-(2.50). As examples, cumulative and complementary cumulative plausibility and belief functions for failure time margins are presented in Fig. 11.1 for a system with 2 SLs and 1 WL. In addition, Fig. 11.1 also contains the CDF and CCDF for SL/WL failure time margins obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 11.1 are constructed as indicated in Eqs. (7.9)-(7.13) with $M_{1t}(t_{SL1}, t_{SL2}, t_{WL1})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

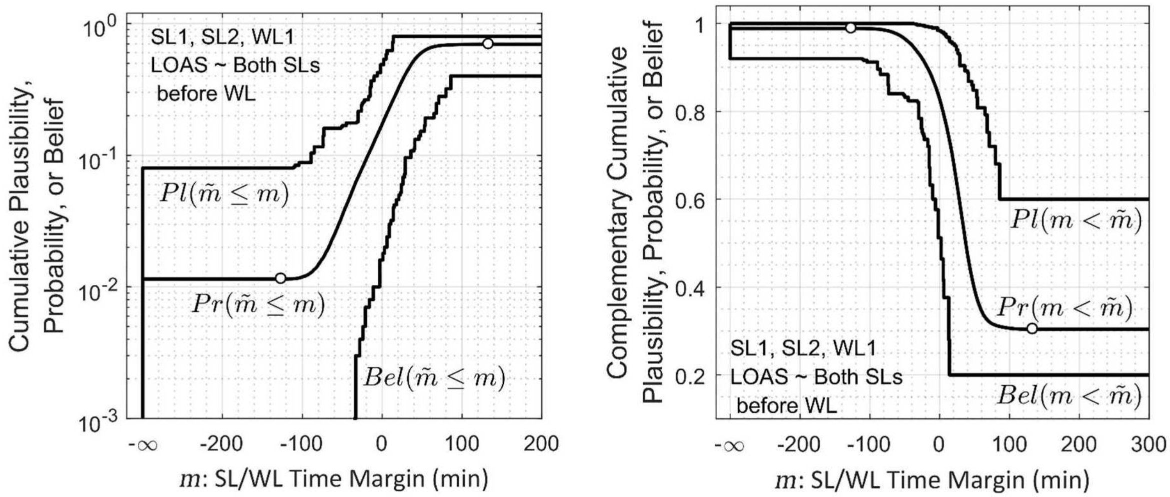


Fig. 11.1 Graphical summary of evidence space $(\mathcal{MTM}_1, \mathbb{MTM}_1, m_{MTM1})$ for SL/WL failure time margins for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4 and (ii) LOAS corresponding to failure of both SLs before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{m} \leq m)$, probability $Pr(\tilde{m} \leq m)$ and belief $Bel(\tilde{m} \leq m)$, and (b) Complementary cumulative plausibility $Pl(m < \tilde{m})$, probability $Pr(m < \tilde{m})$ and belief $Bel(m < \tilde{m})$.

Margin results of the form shown in Fig. 11.1 are valuable because they show and quantify the uncertainty in the time between when (i) failure of the SL system potentially results in LOAS and (ii) failure of the WL potentially averts LOAS.

Belief $Bel(\mathcal{S})$ and plausibility $Pl(\mathcal{S})$ for specific subsets $\mathcal{S}$ of $\mathcal{MTM}_1$ can be calculated from the relationships

$$Bel(\mathcal{S}) = \sum_{\mathcal{MTM}_{1,ijk} \subseteq \mathcal{S}} m_{MTM1}(\mathcal{MTM}_{1,ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_B(\mathcal{MTM}_{1,ijk}) m_{ijk}, \tag{11.10}$$

$$Pl(\mathcal{S}) = \sum_{\varnothing \neq \mathcal{MTM}_{1,ijk} \cap \mathcal{S}} m_{MTM1}(\mathcal{MTM}_{1,ijk}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \delta_P(\mathcal{MTM}_{1,ijk}) m_{ijk} \tag{11.11}$$

with

$$\delta_B(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 \text{ for } \mathcal{MTM}_{1,ijk} \subseteq \mathcal{S} \\ 0 \text{ otherwise} \end{cases} \tag{11.12}$$

$$\delta_P(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 \text{ for } \varnothing \neq \mathcal{MTM}_{1,ijk} \cap \mathcal{S} \\ 0 \text{ otherwise.} \end{cases} \tag{11.13}$$

Many possibilities exist for the definition of the set $\mathcal{S}$. For example, suitable definitions of $\mathcal{S}$ can be used to define CBFs, CPFs, CCBFs and CCPFs for failure time margins. Specifically, with

$$\mathcal{S}(m_t) = \{\tilde{m}_t : \tilde{m}_t \in \mathcal{MTM}_1 \text{ and } \tilde{m}_t \le m_t\} \tag{11.14}$$

and

$$\mathcal{S}^c(m_t) = \{\tilde{m}_t : \tilde{m}_t \in \mathcal{MTM}_1 \text{ and } m_t < \tilde{m}_t\}, \tag{11.15}$$

the indicator functions

$$\delta_B(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 & \text{for } \mathcal{MTM}_{1,ijk} \subseteq \mathcal{S}(m_t) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \overline{m}_{1t,ijk} \le m_t \\ 0 & \text{otherwise,} \end{cases} \tag{11.16}$$

and

$$\delta_B^c(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 & \text{for } \mathcal{MTM}_{1,ijk} \subseteq \mathcal{S}^c(m_t) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } m_t < \underline{m}_{1t,ijk} \\ 0 & \text{otherwise} \end{cases} \tag{11.17}$$

can be used in Eq. (11.10) to define $Bel[\mathcal{S}(m_t)]$ and $Bel[\mathcal{S}^c(m_t)]$ for use in the construction of CBFs and CCBFs. Similarly, the indicator functions

$$\delta_P(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 & \text{for } \varnothing \ne \mathcal{MTM}_{1,ijk} \cap \mathcal{S}(m_t) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } \underline{m}_{1t,ijk} \le m_t \\ 0 & \text{otherwise,} \end{cases} \tag{11.18}$$

and

$$\delta_P^c(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 & \text{for } \varnothing \ne \mathcal{MTM}_{1,ijk} \cap \mathcal{S}^c(m_t) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{for } m_t < \underline{m}_{1t,ijk} \\ 0 & \text{otherwise} \end{cases} \tag{11.19}$$

can be used in Eq. (11.11) to define $Pl[\mathcal{S}(m_t)]$ and $Pl[\mathcal{S}^c(m_t)]$ for use in the construction of CPFs and CCPFs. However, this approach to the construction of CPFs and CCPFs is not as efficient as the procedure described in conjunction with Eqs. (2.48)-(2.50).

The approach described in Eqs. (11.14)-(11.19) as a possible way to construct CBFs, CCBFs, CBFs and CCBFs is not as computationally efficient as the procedure described in conjunction with Eqs. (2.48)-(2.50). However, it is useful for determining belief and plausibility for specific sets of margins on the form defined in Eqs. (11.14) and (11.15).

As examples, the calculation of $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ for the WL/SL system in Fig. 11.1 produces the results

$$Bel[\mathcal{S}(0)] = 1.600 \times 10^{-2} \cong 1.595 \times 10^{-2} \tag{11.20}$$

and

$$Pl[\mathcal{S}(0)] = 4.880 \times 10^{-1} \cong 4.879 \times 10^{-1} \tag{11.21}$$

with (i) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the initial equalities determined as indicated in Eqs. (11.10) and (11.11), and (ii) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 5.2. The agreement of the plotted results in Fig. 11.1a (i.e., $Bel[\mathcal{S}(0)] = 0.016$ and $Pl[\mathcal{S}(0)] = 0.488$) and the two numerical results in Eqs. (11.20) and (11.21) provides a strong verification result that $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ are being calculated correctly. Additional verification is provided by the agreement of (i) the preceding values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$, and (ii) the corresponding values for LOAS in Eqs. (5.6) and (5.7).

As additional examples, two special cases of potential interest are now considered: (i) belief $Bel(\{-\infty\})$ and plausibility $Pl(\{-\infty\})$ for $-\infty$ (i.e., for nonfailure of the WL and failure of both SLs) and (ii) belief $Bel(\{\infty\})$ and plausibility $Pl(\{\infty\})$ for ∞ (i.e., for nonfailure of one of the SLs). Specifically, $Bel(\{-\infty\})$ and $Bel(\{\infty\})$ are defined as in Eq. (11.10) with

$$\delta_{B,-\infty}(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 \text{ for } \mathcal{MTM}_{1,ijk} \subseteq \{-\infty\} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } -\infty = \underline{m}_{1t,ijk} = \overline{m}_{1t,ijk} \\ 0 \text{ otherwise} \end{cases} \tag{11.22}$$

and

$$\delta_{B,\infty}(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 \text{ for } \mathcal{MTM}_{1,ijk} \subseteq \{\infty\} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } \infty = \underline{m}_{1t,ijk} = \overline{m}_{1t,ijk} \\ 0 \text{ otherwise.} \end{cases} \tag{11.23}$$

Similarly, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ are defined as in Eq. (11.11) with

$$\delta_{P,-\infty}(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 \text{ for } \varnothing \neq \mathcal{MTM}_{1,ijk} \cap \{-\infty\} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } -\infty = \underline{m}_{1t,ijk} \\ 0 \text{ otherwise} \end{cases} \tag{11.24}$$

and

$$\delta_{P,\infty}(\mathcal{MTM}_{1,ijk}) = \begin{cases} 1 \text{ for } \varnothing \neq \mathcal{MTM}_{1,ijk} \cap \{\infty\} \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ for } \infty = \overline{m}_{1t,ijk} \\ 0 \text{ otherwise.} \end{cases} \tag{11.25}$$

As examples, the calculation of $Bel(\{-\infty\})$, $Bel(\{\infty\})$, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ for the WL/SL system in Fig. 11.1 produces the results

$$Bel(\{-\infty\}) = 0.000 \times 10^{0} \cong 0.000 \times 10^{0}, \tag{11.26}$$

$$Bel(\{\infty\}) = 0.2000 \cong 0.2002, \tag{11.27}$$

$$Pl(\{-\infty\}) = 8.000 \times 10^{-2} \cong 7.987 \times 10^{-2} \tag{11.28}$$

and

$$Pl(\{\infty\}) = 0.6000 \cong 0.5998 \tag{11.29}$$

with (i) the values for $Bel(\{-\infty\})$, $Bel(\{\infty\})$, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ in the initial equalities determined as indicated in Eqs. (11.10) and (11.11) and (ii) the values for $Bel(\{-\infty\})$, $Bel(\{\infty\})$, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.2. The agreement of the two computational procedures provides a strong verification result that $Bel(\{-\infty\})$, $Bel(\{\infty\})$, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ are being calculated correctly.

The sampling-based procedure used to obtain the CDF and CCDF in Fig. 11.1 yields values of

$$\underline{m}_{1Ft} \cong -137.212 \text{ min} \text{ and } \overline{m}_{1Ft} \cong 137.131 \text{ min} \tag{11.30}$$

for the smallest failure time margin $\underline{m}_{1Ft} > -\infty$ and the largest failure time margin $\overline{m}_{1Ft} < \infty$. As a verification test, a closed form determination of $\underline{m}_{1Ft}$ and $\overline{m}_{1Ft}$ can be performed as described below.

Focal elements $\mathcal{TM}_{ijk}$ for link failure times result in one or more failure time margins m_t in $\mathcal{MTM}_{1,ijk}$ satisfying $-\infty < m_t < \infty$ only if $(i, j, k) \in \mathcal{I}_{1M}$ with

$$\mathcal{I}_{1M} = \left\{(i, j, k) : \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \underline{t}_{WL1,k}\} < \infty\right\}. \tag{11.31}$$

The maximum of the link failure times for $\mathcal{TM}_{ijk}$ with $(i, j, k) \in \mathcal{I}_{1M}$ that result in time margins satisfying $-\infty < m_t < \infty$ can be represented by

$$\overline{t}_{1F,SL1,i} = \max\{t : t \in \mathcal{TM}_{SL1,i} \text{ and } t < \infty\}, \tag{11.32}$$

$$\overline{t}_{1F,SL2,j} = \max\{t : t \in \mathcal{TM}_{SL2,j} \text{ and } t < \infty\}, \tag{11.33}$$

$$\overline{t}_{1F,WL1,k} = \max\{t : t \in \mathcal{TM}_{WL1,k} \text{ and } t < \infty\}. \tag{11.34}$$

The minimum $\underline{m}_{1Ft,ijk}$ and maximum $\overline{m}_{1Ft,ijk}$ of the time margins contained in $\mathcal{MTM}_{1,ijk}$ with $(i, j, k) \in \mathcal{I}_{1M}$ that satisfy $-\infty < m_t < \infty$ are defined by

$$\underline{m}_{1Ft,ijk} = \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} - \overline{t}_{1F,WL1,k} \tag{11.35}$$

and

$$\overline{m}_{1Ft,ijk} = \max\{\overline{t}_{1F,SL1,i}, \overline{t}_{1F,SL2,j}\} - \underline{t}_{WL1,k}. \tag{11.36}$$

In turn, the minimum $\underline{m}_{1Ft}$ and maximum $\overline{m}_{1Ft}$ of the time margins contained in $\mathcal{MTM}_1$ that satisfy $-\infty < m_t < \infty$ are defined by

$$\underline{m}_{1Ft} = \min\{\underline{m}_{1Ft,ijk} : (i,j,k) \in \mathcal{I}_{1F}\} \tag{11.37}$$

and

$$\overline{m}_{1Ft} = \max\{\overline{m}_{1Ft,ijk} : (i,j,k) \in \mathcal{I}_{1F}\}, \tag{11.38}$$

respectively.

As an example,

$$\underline{m}_{1Ft} = -137.877 \text{ min} \cong -137.212 \text{ min and } \overline{m}_{1Ft} = 137.222 \text{ min} \cong 137.131 \text{ min} \tag{11.39}$$

for the results illustrated in Fig. 11.1, with (i) the first values for $\underline{m}_{1Ft}$ and $\overline{m}_{1Ft}$ obtained as indicated in Eqs. (11.37) and (11.38) and (ii) the following approximate values obtained as indicated in Eq. (11.30).

The failure time margin evidence space $(\mathcal{MTM}_1, \mathbb{MTM}_1, m_{MTM1})$ and its associated CPF, CBF, CCPF and CCBF for SL 1 and SL 2 both failing before WL 1 fails can also be defined with use of the evidence spaces $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ and $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$. Specifically, (i) $(\mathcal{TMF}_1, \mathbb{TMF}_1, m_{TMF1})$ is defined in Sect. 8.1 for the times at which a system consisting of SL 1 and SL 2 fails with system failure time corresponding to the time at which the second SL fails and (ii) $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$ is defined in Sect. 4 for the time at which WL 1 fails.

11.2 Cumulative and Complementary Cumulative Belief and Plausibility for Failure Time Margins with LOAS Defined by Failure of Either SL before Failure of the WL

This section presents failure time margin results for a 2 SL, 1 WL system with LOAS occurring if either SL fails before the WL fails. Similarly to Eq. (11.2), the failure time margin under consideration is defined by

$$\begin{aligned} M_{2t}(\mathbf{t}) &= M_{2t}([t_{SL1}, t_{SL2}, t_{WL1}]) \\ &= \begin{cases} -\infty & \text{for } t_{WL1} = \infty,\ \min\{t_{SL1}, t_{SL2}\} < \infty \\ \min\{t_{SL1}, t_{SL2}\} - t_{WL1} & \text{for } t_{WL1} < \infty,\ \min\{t_{SL1}, t_{SL2}\} < \infty \\ \infty & \text{for } \min\{t_{SL1}, t_{SL2}\} = \infty \end{cases} \end{aligned} \tag{11.40}$$

for $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ belonging to the set $\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}$ defined in Eq. (4.13) is considered for analysis.

Application as indicated in Eqs. (11.3)-(11.7) of the function $M_{2t}(\mathbf{t})$ defined in Eq. (11.40) to the elements $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ of the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13) -(4.16) results in the evidence space $(\mathcal{MTM}_2, \mathbb{MTM}_2, m_{MTM2})$ for failure time margins. The difference in the functions $M_{1t}(\mathbf{t})$ and $M_{2t}(\mathbf{t})$ defined in Eqs. (11.2) and (11.40) results in the focal element bounds $\underline{m}_{1t,ijk}$ and $\overline{m}_{1t,ijk}$ defined in Eqs. (11.8) and (11.9) now being defined by

$$\underline{m}_{2t,ijk} = \begin{cases} -\infty \text{ for } \overline{t}_{WL1,k} = \infty \text{ and } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty \\ \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} - \overline{t}_{WL1,k} \text{ for } \overline{t}_{WL1,k} < \infty \text{ and } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty \\ \infty \text{ for } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} = \infty \end{cases} \quad (11.41)$$

and

$$\overline{m}_{2t,ijk} = \begin{cases} -\infty \text{ for } \underline{t}_{WL1,k} = \infty \text{ and } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty \\ \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} - \underline{t}_{WL1,k} \text{ for } \underline{t}_{WL1,k} < \infty \text{ and } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty \\ \infty \text{ for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} = \infty. \end{cases} \quad (11.42)$$

Once the evidence space $(\mathcal{MTM}_2, \mathbb{MTM}_2, m_{MTM2})$ is constructed, cumulative plausibility and belief functions for failure time margins can be obtained from the pairs $(\underline{m}_{2t,ijk}, \overline{m}_{2t,ijk})$ as indicated in conjunction with Eqs. (2.48)-(2.50). As examples, cumulative and complementary cumulative plausibility and belief functions for failure time margins are presented in Fig. 11.2 for systems with 2 SLs and 1 WL. In addition, Fig. 11.2 also contains the CDF and CCDF for SL/WL failure time margins obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 11.2 are constructed as indicated in Eqs. (7.9)-(7.13)with $M_{2t}(t_{SL1}, t_{SL2}, t_{WL1})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

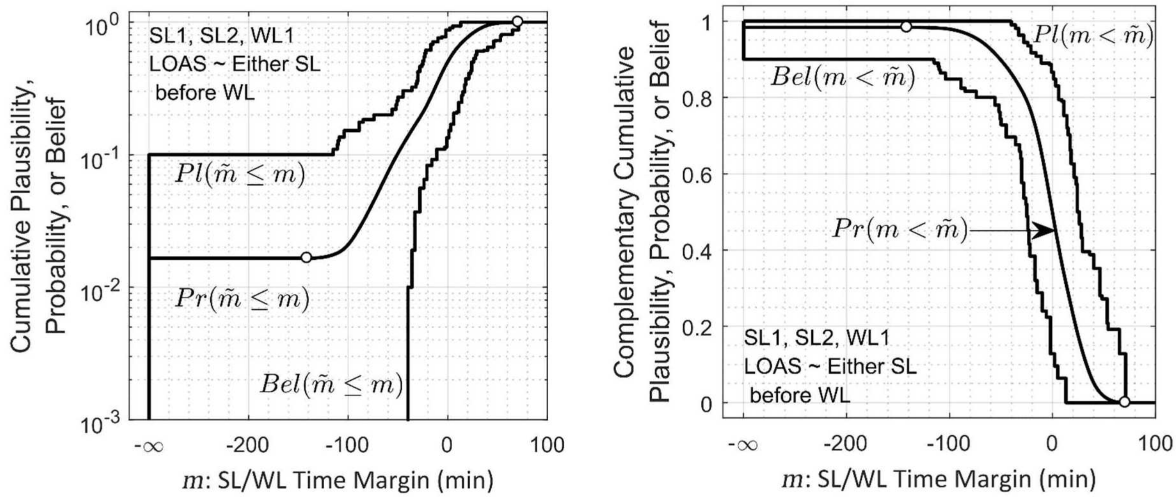


Fig. 11.2 Graphical summary of evidence space $(\mathcal{MTM}_2, \mathbb{MTM}_2, m_{MTM2})$ for SL/WL failure time margins for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4 and (ii) LOAS corresponding to failure of either SL before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{m} \le m)$, probability $Pr(\tilde{m} \le m)$ and belief $Bel(\tilde{m} \le m)$, and (b) Complementary cumulative plausibility $Pl(m < \tilde{m})$, probability $Pr(m < \tilde{m})$ and belief $Bel(m < \tilde{m})$.

Belief $Bel(\mathcal{S})$ and plausibility $Pl(\mathcal{S})$ for subsets $\mathcal{S}$ of $\mathcal{MTM}_2$ can be calculated from the relationships defined in Eqs. (11.10)-(11.19). As a reminder, this requires that the evidence space $(\mathcal{MTM}_2, \mathbb{MTM}_2, m_{MTM2})$ for failure time margins be defined to be consistent with the function $M_{2t}(\mathbf{t})$ defined in Eq. (11.40) and the corresponding focal element bounds $\underline{m}_{2t,ijk}$ and $\overline{m}_{2t,ijk}$ defined in Eqs. (11.41) and (11.42).

As examples, calculation of $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ for the WL/SL system in Fig. 11.2 yields the results

$$Bel[\mathcal{S}(0)] = 1.340 \times 10^{-1} \cong 1.341 \times 10^{-1} \tag{11.43}$$

and

$$Pl[\mathcal{S}(0)] = 8.720 \times 10^{-1} \cong 8.719 \times 10^{-1} \tag{11.44}$$

with (i) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the initial equalities determined as indicated in Eqs. (11.10) and (11.11) and (ii) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.2. The agreement of the plotted results in Fig. 11.2 (i.e., $Bel[\mathcal{S}(0)] = 1.340 \times 10^{-1}$ and $Pl[\mathcal{S}(0)] = 8.720 \times 10^{-1}$) and the numerical results in Eqs. (11.43) and (11.44) provides a strong verification result that $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ are being calculated correctly. Additional verification is provided by the agreement of (i) the preceding values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ and (ii) the corresponding values for LOAS in Eqs. (5.27) and (5.28).

If desired, $Bel(\{-\infty\})$, $Bel(\{\infty\})$, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ can be calculated as indicated in Eqs. (11.22)-(11.29).

The sampling-based procedure used to obtain the CDF and CCDF in Fig. 11.2 yields values of

$$\underline{m}_{2Ft} \cong -135.228 \text{ min} \quad \text{and} \quad \overline{m}_{2Ft} \cong 70.681 \text{ min} \tag{11.45}$$

for the smallest failure time margin $\underline{m}_{2Ft} > -\infty$ and the largest failure time margin $\overline{m}_{2Ft} < \infty$. As a verification test, a closed form determination of $\underline{m}_{2Ft}$ and $\overline{m}_{2Ft}$ can be performed as described below.

Focal elements $\mathcal{TM}_{ijk}$ for link failure times result in one or more failure time margins m_t in $\mathcal{MTM}_{2,ijk}$ satisfying $-\infty < m_t < \infty$ only if $(i,j,k) \in \mathcal{I}_{2M}$ with

$$\mathcal{I}_{2M} = \left\{(i,j,k) : \underline{t}_{WL1,k} < \infty \text{ and } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty\right\}. \tag{11.46}$$

The maximum of the link failure times for $\mathcal{TM}_{ijk}$ with $(i,j,k) \in \mathcal{I}_{2M}$ that result in time margins satisfying $-\infty < m_{1Tt,ijk} < \infty$ are

$$\overline{t}_{2F,SL1,i} = \begin{cases} \max\{t : t \in \mathcal{TM}_{SL1,i} \text{ and } t < \infty\} \text{ for } \underline{t}_{SL1,i} < \infty \\ \text{undefined for } \underline{t}_{SL1,i} = \infty, \end{cases} \tag{11.47}$$

$$\overline{t}_{2F,SL2,j} = \begin{cases} \max\{t : t \in \mathcal{TM}_{SL2,j} \text{ and } t < \infty\} \text{ for } \underline{t}_{SL2,j} < \infty \\ \text{undefined for } \underline{t}_{SL2,j} = \infty, \end{cases} \tag{11.48}$$

and

$$\overline{t}_{2F,WL1,k} = \max\{t : t \in \mathcal{TM}_{WL1,k} \text{ and } t < \infty\}. \tag{11.49}$$

The minimum $\underline{m}_{2Ft,ijk}$ and maximum $\overline{m}_{2Ft,ijk}$ of the time margins contained in $\mathcal{MTM}_{2,ijk}$ with $(i,j,k) \in \mathcal{I}_{2M}$ that satisfy $-\infty < m_t < \infty$ are defined by

$$\underline{m}_{2Ft,ijk} = \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} - \overline{t}_{2F,WL1,k} \tag{11.50}$$

and

$$\bar{m}_{2Ft,ijk} = \begin{cases} \min\{\bar{t}_{2F,SL1,i}, \bar{t}_{2F,SL2,j}\} - \underline{t}_{WL1,k} & \text{for } \bar{t}_{2F,SL1,i}, \bar{t}_{2F,SL2,j} \text{ both defined} \\ \bar{t}_{2F,SL1,i} - \underline{t}_{WL1,k} & \text{for only } \bar{t}_{2F,SL1,i} \text{ defined} \\ \bar{t}_{2F,SL2,j} - \underline{t}_{WL1,k} & \text{for only } \bar{t}_{2F,SL2,j} \text{ defined.} \end{cases} \quad (11.51)$$

In turn, the minimum $\underline{m}_{2Ft}$ and maximum $\bar{m}_{2Ft}$ of the time margins contained in $\mathcal{MTM}_2$ that satisfy $-\infty < m_t < \infty$ are defined in the same manner as shown in Eqs. (11.37) and (11.38).

As an example,

$$\underline{m}_{2Ft} = -141.463 \cong -141.182 \text{ and } \bar{m}_{2Ft} = 70.904 \cong 70.681 \quad (11.52)$$

for the results illustrated in Fig. 11.2, with (i) the first value for $\underline{m}_{2Ft}$ and $\bar{m}_{2Ft}$ obtained as indicated in conjunction with Eqs. (11.37) and (11.38) and (ii) the following approximate value obtained as indicated in Eq. (11.45).

The failure time margin evidence space $(\mathcal{MTM}_2, \mathbb{MTM}_2, m_{MTM2})$ and its associated CPF, CBF, CCPF and CCBF for either SL 1 or SL 2 failing before WL 1 fails can also be defined with use of the evidence spaces $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ and $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$. Specifically, (i) $(\mathcal{TMF}_2, \mathbb{TMF}_2, m_{TMF2})$ is defined in Sect. 8.2 for the times at which a system consisting of SL 1 and SL 2 fails with system failure time corresponding to the time at which the first SL fails and (ii) $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1})$ is defined in Sect. 4 for the time at which WL 1 fails.

12. Cumulative and Complementary Cumulative Belief and Plausibility for WL/SL Temperature Margins for a System with 2 SLs and 1 WL

For simplicity, this section considers a system with 2 SLs and 1 WL and two definitions of system failure: (i) LOAS occurs when both SLs fail before the WL fails and (ii) LOAS occurs when either SL fails before the WL fails.

12.1 Cumulative and Complementary Cumulative Belief and Plausibility for WL/SL Temperature Margins with LOAS defined by Failure of Both SLs before Failure of the WL

Failure temperature margins defined by

$$\begin{aligned} M_T = &\ \text{(temperature at which SL failure potentially causes LOAS)} \\ &- \text{(temperature at which WL failure potentially prevents LOAS)} \end{aligned} \tag{12.1}$$

are another possible summary result in the analysis of WL/SL systems. Again, the descriptor "potentially" is used in the definition of M_T because the occurrence of LOAS depends on the relative timing of SL failure and WL failure. The margin M_T is positive if SL failure occurs at a higher temperature than WL failure (i.e., the desired occurrence) and negative if SL failure occurs at a lower temperature than WL failure (i.e., the undesired occurrence). However, a negative failure temperature margin is not necessarily associated with the occurrence of LOAS.

This section presents failure temperature margin results for a 2 SL, 1 WL system for which LOAS occurs if both SLs fail before the WL fails. Nonfailure of either of the SLs or the WL is a possibility for this system that must be addressed as part of the analysis of margins. To handle this situation, a generalized margin defined by

$$\begin{aligned} M_{1T}(\mathbf{t}) &= M_{1T}([t_{SL1}, t_{SL2}, t_{WL1}]) \\ &= \begin{cases} -\infty \ \text{ for } t_{WL1} = \infty, \max\{t_{SL1}, t_{SL2}\} < \infty \\ T_{SL1}(t_{SL1}) - T_{WL1}(t_{WL1}) \ \text{ for } t_{WL1} < \infty, t_{SL2} < t_{SL1} < \infty \\ T_{SL2}(t_{SL2}) - T_{WL1}(t_{WL1}) \ \text{ for } t_{WL1} < \infty, t_{SL1} < t_{SL2} < \infty \\ \max\{T_{SL1}(t_{ij}), T_{SL2}(t_{ij})\} - T_{WL1}(t_{WL1}) \ \text{ for } t_{WL1} < \infty, t_{ij} = t_{SL1} = t_{SL2} < \infty \\ \infty \ \text{ for } \ \max\{t_{SL1}, t_{SL2}\} = \infty \end{cases} \end{aligned} \tag{12.2}$$

for $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ belonging to the set $\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}$ defined in Eq. (4.13) is considered for analysis.

Application of the function $M_{1T}(\mathbf{t})$ defined in Eq. (12.2) to the elements $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ of the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13) -(4.16) results in the evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$ for failure temperature margins with

$$\mathcal{MT}_1 = \{m_{1T} : m_{1T} = M_{1T}(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}\}, \tag{12.3}$$

$$\mathcal{MT}_{1,ijk} = \{m_{1T} : m_{1T} = M_{1T}(\mathbf{t}) \text{ for } \mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}] \in \mathcal{TM}_{ijk}\} \tag{12.4}$$

$$\mathbb{MT}_1 = \{\mathcal{MT}_{1,ijk} : (i, j, k) \in \mathcal{I} = \{1, 2, ..., nSL1\} \times \{1, 2, ..., nSL2\} \times \{1, 2, ..., nWL1\}\} \tag{12.5}$$

and

$$m_{MT1}(\mathcal{MT}_{1,ijk}) = m_{TM}(\mathcal{TM}_{ijk}) = m_{ijk}. \tag{12.6}$$

In addition, the bounds

$$(\underline{m}_{1T,ijk}, \overline{m}_{1T,ijk}) = \left(\text{glb}(\mathcal{MT}_{1,ijk}), \max(\mathcal{MT}_{1,ijk})\right) \tag{12.7}$$

are introduced for use in the determination of the cumulative values of belief and plausibility for the WL/WL failure temperature margins defined in Eq. (12.2).

Definition of the focal element bound $\underline{m}_{1T,ijk}$ is considered first. Specifically, $\underline{m}_{1T,ijk}$ has a definition that (i) involves greatest lower bounds (glb's) for sets of link failure temperatures and (ii) is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$. The following possibilities exist for the definition of $\underline{m}_{1T,ijk}$:

Possibility (1): If $\overline{t}_{WL1,k} = \infty$ and $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty$, then

$$\underline{m}_{1T,ijk} = -\infty. \tag{12.8}$$

Possibility (2): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} = \infty$, then

$$\underline{m}_{1T,ijk} = \infty. \tag{12.9}$$

Possibility (3): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty$ and $\underline{t}_{SL1,i} < \underline{t}_{SL2,j}$, then either (3.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$ and

$$\underline{m}_{1T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{WL1}(\overline{t}_{WL1,k}) \tag{12.10}$$

or (3.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$, $[\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{1i} \ne \varnothing$ and

$$\underline{m}_{1T,ijk} = \max\left\{\text{glb}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL2,j} \le t \text{ and } t \in \mathcal{TM}_{SL1,i}\}, T_{SL2}(\underline{t}_{SL2,j})\right\} - T_{WL1}(\overline{t}_{WL1,k}) = \max\{T_{SL1}(\underline{t}_{SL2,j}), T_{SL2}(\underline{t}_{SL2,j})\} - T_{WL1}(\overline{t}_{WL1,k}) \text{ if } \underline{t}_{SL2,j} \in \mathcal{TM}_{1i}, \tag{12.11}$$

or (3.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$, $[\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{1i} = \varnothing$ and

$$\underline{m}_{1T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{WL1}(\overline{t}_{WL1,k}). \tag{12.12}$$

Possibility (4): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty$ and $\underline{t}_{SL2,j} < \underline{t}_{SL1,i}$, then either (4.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$ and

$$\underline{m}_{1T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{WL1}(\overline{t}_{WL1,k}), \tag{12.13}$$

or (4.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$, $[\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{2j} \ne \varnothing$ and

$$\underline{m}_{1T,ijk} = \max\left\{T_{SL1}(\underline{t}_{SL1,i}), \text{glb}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL1,i} \le t \text{ and } t \in \mathcal{TM}_{SL2,j}\}\right\} - T_{SL1}(\overline{t}_{WL1,k})\} = \max\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL1,i})\} \text{ if } \underline{t}_{SL1,i} \in \mathcal{TM}_{2j}, \tag{12.14}$$

or (4.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$, $[\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{2j} = \varnothing$ and

$$\underline{m}_{1T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{WL1}(\overline{t}_{WL1,k}). \tag{12.15}$$

Possibility (5): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty$ and $\underline{t}_{SL1,i} = \underline{t}_{SL2,j} = \underline{t}_{ij} \le t_{mx}$, then either (5.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{ij} = t_{mx}$ and

$$\underline{m}_{1T,ijk} = \max\left\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\right\} - T_{WL1}(\overline{t}_{WL1,k}) \tag{12.16}$$

or (5.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{ij} = t_{mx}$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{1i} \ne \varnothing$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{2j} \ne \varnothing$ and

$$\underline{m}_{1T,ijk} = \min\left\{\text{glb}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{1i}\}, \text{glb}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{2j}\}\right\} - T_{WL1}(\overline{t}_{WL1,k}) = \min\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\} - T_{WL1}(\overline{t}_{WL1,k}), \tag{12.17}$$

or (5.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{ij} = t_{mx}$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{1i} \ne \varnothing$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{2j} = \varnothing$ and

$$\underline{m}_{1T,ijk} = \text{glb}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{1i}\} - T_{WL1}(\overline{t}_{WL1,k}) = T_{SL1}(\underline{t}_{ij}) - T_{WL1}(\overline{t}_{WL1,k}), \quad (12.18)$$

or (5.4) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{ij} = t_{mx}$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{1i} = \varnothing$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{2j} \ne \varnothing$ and

$$\underline{m}_{1T,ijk} = \text{glb}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{2j}\} - T_{WL1}(\overline{t}_{WL1,k}) = T_{SL2}(\underline{t}_{ij}) - T_{WL1}(\overline{t}_{WL1,k}), \quad (12.19)$$

or (5.5) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{ij} = t_{mx}$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{1i} = \varnothing$, $(\underline{t}_{ij}, t_{mx}] \cap \mathcal{TM}_{2j} = \varnothing$ and

$$\underline{m}_{1T,ijk} = \max\left\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\right\} - T_{WL1}(\overline{t}_{WL1,k}). \quad (12.20)$$

The bound $\overline{m}_{1T,ijk}$ also has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\underline{t}_{WL1,k}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ as stated for the following possibilities:

Possibility (1): If $\underline{t}_{WL1,k} = \infty$ and $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty$, then

$$\overline{m}_{1T,ijk} = -\infty. \quad (12.21)$$

Possibility (2): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} = \infty$, then

$$\overline{m}_{1T,ijk} = \infty. \quad (12.22)$$

Possibility (3): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}\} < \infty$ and $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$, then either (3.1) $\underline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i} < \overline{t}_{SL2,j} \le t_{mx}$ and

$$\overline{m}_{1T,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\} - T_{WL1}(\underline{t}_{WL1,k}), \quad (12.23)$$

or (3.2) $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} < \overline{t}_{SL2,j} \le t_{mx}$ and

$$\overline{m}_{1T,ijk} = T_{SL2}(\overline{t}_{SL2,j}) - T_{SL1}(\underline{t}_{WL1,k}). \quad (12.24)$$

Possibility (4): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}\} < \infty$ and $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$, then either (4.1) $\underline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j} < \overline{t}_{SL1,i} \le t_{mx}$ and

$$\overline{m}_{1T,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL2,j})\} - T_{WL1}(\underline{t}_{WL1,k}), \quad (12.25)$$

or (4.2) $\underline{t}_{WL1} \le t_{mx}$, $\bar{t}_{SL2,j} < \underline{t}_{SL1,i} < \bar{t}_{SL1,i} \le t_{mx}$ and

$$\bar{m}_{1T,ijk} = T_{SL1}(\bar{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}). \tag{12.26}$$

Possibility (5): If $\max\{\bar{t}_{SL1,i}, \bar{t}_{SL2,j}, \underline{t}_{WL1}\} < \infty$ and $\bar{t}_{SL1,i} = \bar{t}_{SL2,j} = \bar{t}_{ij}$, then

$$\bar{m}_{1T,ijk} = \max\{T_{SL1}(\bar{t}_{ij}), T_{SL2}(\bar{t}_{ij})\} - T_{WL1}(\underline{t}_{WL1,k}). \tag{12.27}$$

Once the evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for SL/WL failure temperature margins can be obtained from the pairs $(\underline{m}_{1T,ijk}, \bar{m}_{1T,ijk})$ as indicated in conjunction with Eqs. (2.48)-(2.50). As examples, cumulative and complementary cumulative plausibility and belief functions for failure time margins are presented in Fig. 12.1 for a system with 2 SLs and 1 WL. In addition, also contains the CDF and CCDF for SL/WL failure temperature margins obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 12.1 are constructed as indicated in Eqs. (7.9)-(7.13) with $M_{1T}([t_{SL1}, t_{SL2}, t_{WL1}])$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

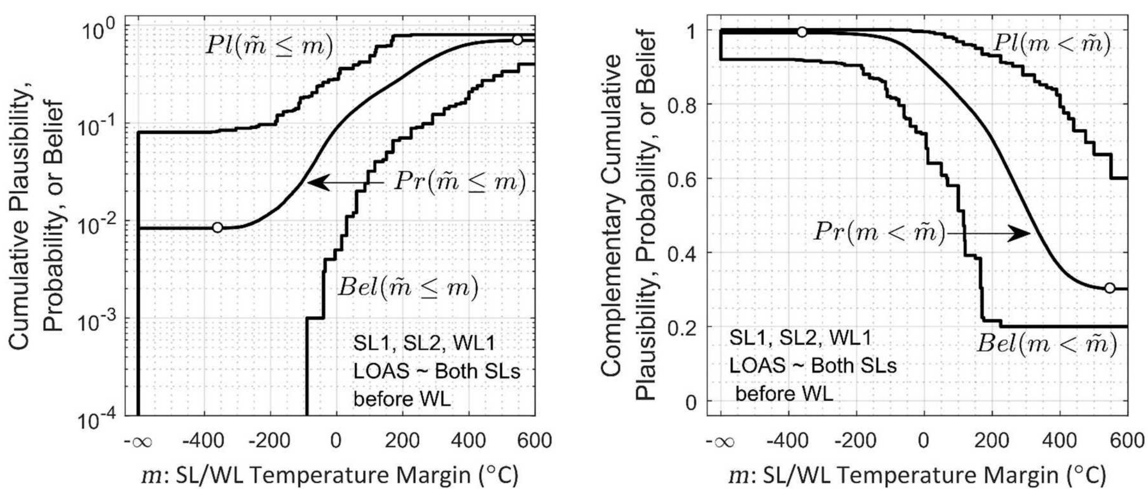


Fig. 12.1 Graphical summary of evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$ for SL/WL failure temperature margins for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4 and (ii) LOAS corresponding to failure of both SLs before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{m} \le m)$, probability $Pr(\tilde{m} \le m)$ and belief $Bel(\tilde{m} \le m)$, and (b) Complementary cumulative plausibility $Pl(m < \tilde{m})$, probability $Pr(m < \tilde{m})$ and belief $Bel(m < \tilde{m})$.

Margin results of the form shown in Fig. 12.1 are valuable because they show and quantify the uncertainty in the temperature difference between when (i) failure of the SL system potentially results in LOAS and (ii) failure of the WL potentially averts LOAS. In addition, the sampling-based procedure used to obtain the CDF and CCDF in Fig. 12.1 produced values of

$$\underline{m}_{1FT} \cong -358.059\ °\text{C} \text{ and } \overline{m}_{1FT} \cong 549.204\ °\text{C} \tag{12.28}$$

for the smallest failure temperature margin $\underline{m}_{1FT} > -\infty$ and the largest failure temperature. margin $\overline{m}_{1FT} < \infty$.

Belief $Bel(\mathcal{S})$ and plausibility $Pl(\mathcal{S})$ for subsets $\mathcal{S}$ of $\mathcal{MT}_1$ can be calculated from the relationships defined in Eqs. (11.10)-(11.19). As a reminder, this requires that the evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$ for failure temperature margins be defined to be consistent with the function $M_{1T}(\mathbf{t})$ defined in Eq. (12.2) and the corresponding focal element bounds $\underline{m}_{T,ijk}$ and $\overline{m}_{T,ijk}$ defined in Eqs. (12.8)-(12.27).

As examples, the calculation of $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ for temperature margin values of 0 for the WL/SL system in Fig. 12.1 produces the results

$$Bel[\mathcal{S}(0)] = 5.000 \times 10^{-3} \cong 5.002 \times 10^{-3} \tag{12.29}$$

and

$$Pl[\mathcal{S}(0)] = 2.800 \times 10^{-1} \cong 2.800 \times 10^{-1} \tag{12.30}$$

with (i) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the initial equalities determined as indicated in Eqs. (11.10) and (11.11), and (ii) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 6.2. The agreement of the plotted results in Fig. 12.1 (i.e., $Bel[\mathcal{S}(0)] = 0.005$, $Pl[\mathcal{S}(0)] = 0.280$) and the numerical results in Eqs. (12.29) and (12.30) provides a strong verification result that $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ are being calculated correctly.

If desired, $Bel(\{-\infty\})$, $Bel(\{\infty\})$, $Pl(\{-\infty\})$ and $Pl(\{\infty\})$ can be calculated as indicated in Eqs. (11.22)-(11.29).

The failure temperature margin evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$ and its associated CPF, CBF, CCPF and CCBF for SL 1 and SL 2 both failing before WL 1 fails can also be defined with use of the evidence spaces $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ and $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$. Specifically, (i) $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ is defined in Sect. 9.1 for the temperatures at which a system consisting of SL 1 and SL 2 fails with system failure temperature corresponding to the temperature at which the second SL fails and (ii) $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$ is defined in Sect. 4 for the temperature at which WL 1 fails.

In turn, the focal element bounds $\underline{m}_{1T,ijk}$ and $\overline{m}_{1T,ijk}$ can be defined on the basis of (i) the bounds $(\underline{TF}_{1,ij}, \overline{TF}_{1,ij})$ for focal elements $\mathcal{TF}_{1,ij}$ associated with the evidence space $(\mathcal{TF}_1, \mathbb{TF}_1, m_{TF1})$ defined in Eqs. (9.8)-(9.23) for the failure temperatures for a system consisting of SL 1 and SL 2 with system failure corresponding to failure of the second SL and (ii) the bounds $(\underline{TF}_{WL1,k}, \overline{TF}_{WL1,k})$ for focal elements $\mathcal{TF}_{WL1,k}$ associated with the evidence space $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{WL1,TF})$ defined in Eqs. (4.20)- (4.25) for the actual temperatures at which WL 1 fails. Given known values for the bounds $(\underline{TF}_{1,ij}, \overline{TF}_{1,ij})$ and $(\underline{TF}_{WL1,k}, \overline{TF}_{WL1,k})$, the bounds $(\underline{m}_{1T,ijk}, \overline{m}_{1T,ijk})$ for $\mathcal{MT}_{1,ijk}$ are defined by

$$\underline{m}_{1T,ijk} = \underline{TF}_{1,ij} - \overline{TF}_{WL1,k} \text{ and } \overline{m}_{1T,ijk} = \overline{TF}_{1,ij} - \underline{TF}_{WL1,k} \tag{12.31}$$

with the assumption that

$$a-b = \begin{cases} \infty & \text{if } a = \infty \\ -\infty & \text{if } a < \infty \text{ and } b = \infty. \end{cases} \tag{12.32}$$

The evidence space $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$ is for the actual temperatures at which WL 1 fails with $T_\infty = \infty$ included to indicate that that link system failure did not occur. A slightly different failure temperature margin evidence space is obtained if the WL 1 failure temperature evidence space $(\mathcal{T}_{WL1}, \mathbb{T}_{WL1}, m_{WL1})$ defined in Sect. 4 is used instead of $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$. The sample space $\mathcal{T}_{WL1}$ contains all originally specified possible failure times for WL 1. In contrast, $\mathcal{TF}_{WL1}$ contains only (i) the failure temperatures that actually occurred and (ii) the indicator T_∞ assigned to failure temperatures in $\mathcal{T}_{WL1}$ that were never reached by the WL 1 temperature curve $T_{WL1}(t)$ for $t_{mn} \le t \le t_{mx}$.

12.2 Cumulative and Complementary Cumulative Belief and Plausibility for WL/SL Temperature Margin with LOAS Defined by Failure of Either SL before failure of the WL

This section presents failure temperature margin results for a 2 SL, 1 WL system for which LOAS occurs if either SL fails before the WL fails. Nonfailure of either of the SLs or the WL is a possibility for this system that must be addressed as part of the analysis of margins. To handle this situation, a generalized margin defined by

$$M_{2T}(\mathbf{t}) = M_{2T}([t_{SL1}, t_{SL2}, t_{WL1}])$$
$$= \begin{cases} -\infty \text{ for } t_{WL1} = \infty, \min\{t_{SL1}, t_{SL2}\} < \infty \\ T_{SL2}(t_{SL2}) - T_{WL1}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{SL2} < t_{SL1} \\ T_{SL1}(t_{SL1}) - T_{WL1}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{SL1} < t_{SL2} \\ \min\{T_{SL1}(t_{ij}), T_{SL2}(t_{ij})\} - T_{WL1}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{ij} = t_{SL1} = t_{SL2} < \infty \\ \infty \text{ for } \min\{t_{SL1}, t_{SL2}\} = \infty \end{cases} \tag{12.33}$$

for $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ belonging to the set $\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}$ defined in Eq. (4.13) is considered for analysis.

Application of the function $M_{2T}(\mathbf{t})$ defined in Eq. (12.33) to the elements $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ of the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13) -(4.16) results in the evidence space $(\mathcal{MT}_2, \mathbb{MT}_2, m_{MT2})$ for failure temperature margins defined in the same manner as used in Eqs. (12.3)-(12.7) to define the evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$.

Definition of the focal element bound $\underline{m}_{2T,ijk}$ is considered first. Specifically, $\underline{m}_{2T,ijk}$ has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\underline{t}_{WL1,k}$ $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$. The following possibilities exist for the definition of $\underline{m}_{2T,ijk}$:

Possibility (1): If $\overline{t}_{WL1,k} = \infty$ and $\min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty$, then

$$\underline{m}_{2T,ijk} = -\infty. \tag{12.34}$$

Possibility (2): If $\min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} = \infty$, then

$$\underline{m}_{2T,ijk} = \infty. \tag{12.35}$$

Possibility (3): If $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} \le t_{mx}$ and $\underline{t}_{SL1,i} < \underline{t}_{SL2,j}$, then either: (3.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} = \infty$ and

$$\underline{m}_{2T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{WL1}(\overline{t}_{WL1,k}), \tag{12.36}$$

or (3.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$ and

$$\underline{m}_{2T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{WL1}(\overline{t}_{WL1,k}), \tag{12.37}$$

or (3.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$ and

$$\underline{m}_{2T,ijk} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\} - T_{WL1}(\overline{t}_{WL1,k}). \tag{12.38}$$

Possibility (4): If $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} \le t_{mx}$ and $\underline{t}_{SL2,j} < \underline{t}_{SL1,i}$, then either: (4.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} = \infty$ and

$$\underline{m}_{2T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{WL1}(\overline{t}_{WL1,k}), \tag{12.39}$$

or (4.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$ and

$$\underline{m}_{2T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{WL1}(\overline{t}_{WL1,k}), \tag{12.40}$$

or (4.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$ and

$$\underline{m}_{2T,ijk} = \min\{T_{SL1}(\underline{t}_{SL1,i}), T_{SL2}(\underline{t}_{SL2,j})\} - T_{WL1}(\overline{t}_{WL1,k}). \tag{12.41}$$

Possibility (5): If $\overline{t}_{WL1,k} \le t_{mx}$ and $\underline{t}_{SL2,j} = \underline{t}_{SL1,i} = \underline{t}_{ij} \le t_{mx}$, then

$$\underline{m}_{2T,ijk} = \min\{T_{SL1}(\underline{t}_{ij}), T_{SL2}(\underline{t}_{ij})\} - T_{WL1}(\overline{t}_{WL1,k}). \tag{12.42}$$

Similarly, $\overline{m}_{2T,ijk}$ is defined by

Possibility (1): If $\underline{t}_{WL1,k} = \infty$ and $\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty$, then

$$\overline{m}_{2T,ijk} = -\infty. \tag{12.43}$$

Possibility (2): If $\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} = \infty$, then

$$\overline{m}_{2T,ijk} = \infty. \tag{12.44}$$

Possibility (3): If $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL1,i} \le t_{mx}$ and $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$, then either: (3.1) $\underline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \overline{t}_{SL1,i} \le t_{mx}$, $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and

$$\begin{aligned}\overline{m}_{2T,ijk} &= \max\left\{T_{SL1}(\overline{t}_{SL1,i}), \mathrm{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{SL1,i}) \cap \mathcal{TM}_{SL2,j}\}\right\} \\ &\quad - T_{WL1}(\underline{t}_{WL1,k}) \\ &= \max\left\{T_{SL1}(\overline{t}_{SL1,i}), T_{SL2}(\overline{t}_{SL1,i})\right\} - T_{WL1}(\underline{t}_{WL1,k}) \text{ if } \overline{t}_{SL1,i} \in \mathcal{TM}_{SL2,j},\end{aligned} \tag{12.45}$$

or (3.2) $\underline{t}_{WL1,k} \le t_{mx}$, $t_{ij} = \underline{t}_{SL2,j} = \overline{t}_{SL1,i} \le t_{mx}$, $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and

$$\begin{aligned} \overline{m}_{2T,ijk} &= \text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{SL1,i}) \cap \mathcal{TM}_{SL1,i}\} - T_{WL1}(\underline{t}_{WL1,k}) \\ &= T_{SL1}(\overline{t}_{SL1,i}) - T_{WL1}(\underline{t}_{WL1,k}), \end{aligned} \tag{12.46}$$

or (3.3) $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le \overline{t}_{SL2,j}$, $\overline{t}_{SL1,i} \le t_{mx}$ and

$$\overline{m}_{2T,ijk} = T_{SL1}(\overline{t}_{SL1,i}) - T_{WL1}(\underline{t}_{WL1,k}). \tag{12.47}$$

Possibility (4): If $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL2,j} \le t_{mx}$ and $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$, then either: (4.1) $\underline{t}_{WL1,k} \le t_{mx}$ $\underline{t}_{SL1,i} < \overline{t}_{SL2,j} \le t_{mx}$, $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ and

$$\begin{aligned} \overline{m}_{2T,ijk} &= \max\left\{T_{SL2}(\overline{t}_{SL2,j}), \text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{SL2,j}) \cap \mathcal{TM}_{SL1,i}\}\right\} \\ &\quad - T_{WL1}(\underline{t}_{WL1,k}) \\ &= \max\left\{T_{SL2}(\overline{t}_{SL2,j}), T_{SL1}(\overline{t}_{SL2,j})\right\} - T_{WL1}(\underline{t}_{WL1,k}) \text{ if } \overline{t}_{SL2,j} \in \mathcal{TM}_{SL1,i}, \end{aligned} \tag{12.48}$$

or (4.2) $\underline{t}_{WL1,k} \le t_{mx}$, $t_{ij} = \underline{t}_{SL1,i} = \overline{t}_{SL2,j} \le t_{mx}$, $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ and

$$\begin{aligned} \overline{m}_{2T,ijk} &= \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{SL2,j}) \cap \mathcal{TM}_{SL2,j}\} - T_{WL1}(\underline{t}_{WL1,k}) \\ &= T_{SL2}(\overline{t}_{SL2,j}) - T_{WL1}(\underline{t}_{WL1,k}), \end{aligned} \tag{12.49}$$

or (4.3) $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le \overline{t}_{SL1,i}$, $\overline{t}_{SL2,j} \le t_{mx}$ and

$$\overline{m}_{2T,ijk} = T_{SL2}(\overline{t}_{SL2,j}) - T_{WL1}(\underline{t}_{WL1,k}). \tag{12.50}$$

Possibility (5): If $\underline{t}_{WL1,k} \le t_{mx}$ and $\overline{t}_{ij} = \overline{t}_{SL2,j} = \overline{t}_{SL1,i} \le t_{mx}$, then $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{ij}$ and

$$\begin{aligned} \overline{m}_{2T,ijk} &= \max\left\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL1,i}\},\right. \\ &\qquad \left.\text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL2,j}\}\right\} \\ &\qquad - T_{WL1}(\underline{t}_{WL1,k}) \\ &= \max\left\{T_{SL1}(\overline{t}_{ij}), T_{SL2}(\overline{t}_{ij})\right\} - T_{WL1}(\underline{t}_{WL1,k}). \end{aligned} \tag{12.51}$$

Once the evidence space $(\mathcal{MT}_2, \mathbb{MT}_2, m_{2MT})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for SL/SL failure temperature margins can be obtained from the pairs $(\underline{m}_{2T,ijk}, \overline{m}_{2T,ijk})$ as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 12.2. In addition, Fig. 12.2 also contains the CDF and CCDF for WL/SL failure temperature margins obtained by assigning uniform distributions to the individual focal elements for link failure temperature as indicated for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the indicated CDF and CCDF are constructed as indicated in Eqs. (7.9)-(7.13) with $M_{2T}(t_{SL1}, t_{SL2}, t_{WL1})$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

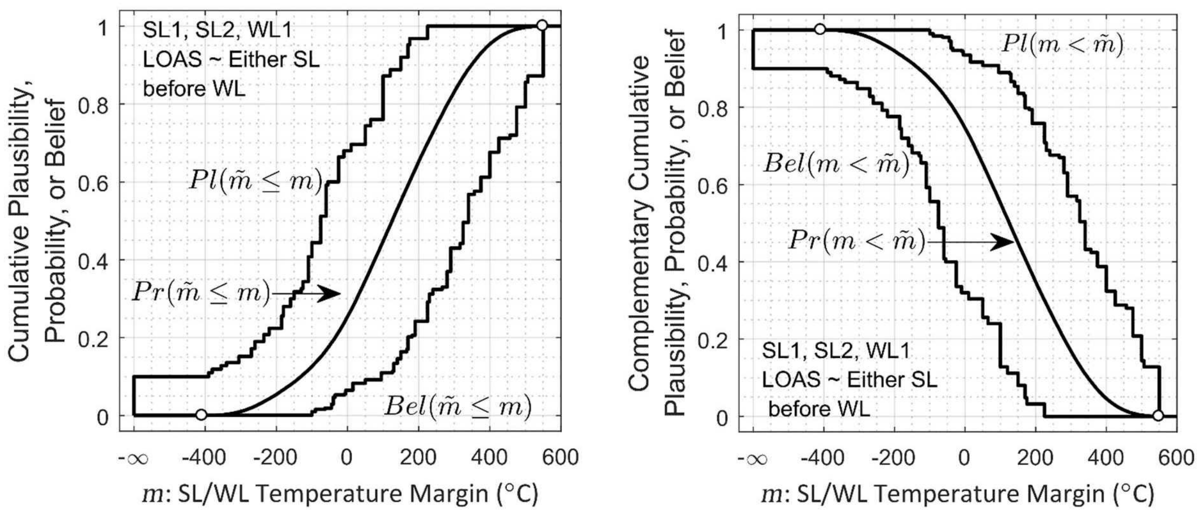


Fig. 12.2 Graphical summary of evidence space $(\mathcal{MT}_1, \mathbb{MT}_1, m_{MT1})$ for SL/WL failure temperature margins for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4 and (ii) LOAS corresponding to failure of either SL before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{m} \le m)$, probability $Pr(\tilde{m} \le m)$ and belief $Bel(\tilde{m} \le m)$, and (b) Complementary cumulative plausibility $Pl(m < \tilde{m})$, probability $Pr(m < \tilde{m})$ and belief $Bel(m < \tilde{m})$.

As for Fig. 12.1, margin results of the form shown in Fig. 12.2 are valuable because they show and quantify the uncertainty in the temperature difference between when (i) failure of the SL system potentially results in LOAS and (ii) failure of the WL potentially averts LOAS. In addition, the sampling-based procedure used to obtain the CDF and CCDF in Fig. 12.2 produced values of

$$\underline{m}_{2FT} \cong -408.174 \text{ °C} \text{ and } \bar{m}_{2FT} \cong 549.552 \text{ °C} \tag{12.52}$$

for the smallest failure temperature margin $\underline{m}_{2FT} > -\infty$ and the largest failure temperature. margin $\bar{m}_{2FT} < \infty$.

Failure temperature margin evidence spaces for either SL failing before the WL fails can also be defined with use of (i) the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ defined in Sect. 9.2 for the temperatures at which a system of 2 SLs fails with system failure temperature corresponding to the temperature at which the first SL fails and (ii) and the evidence space for WL failure temperature (e.g., $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$ for WL 1 failure time defined in Sect. 4).

The failure temperature margin evidence space $(\mathcal{MT}_2, \mathbb{MT}_2, m_{MT2})$ and its associated CPF, CBF, CCPF and CCBF for either SL 1 or SL 2 failing before WL 1 fails can also be defined with use of the evidence spaces $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ and $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$. Specifically, (i) $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ is defined in Sect. 9.2 for the temperatures at which a system consisting of SL 1 and SL 2 fails with system failure temperature corresponding to the temperature at which the first SL fails and (ii) $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$ is defined in Sect. 4 for the temperature at which WL 1 fails.

The bounds $\underline{m}_{2T,ijk}$ and $\overline{m}_{2T,ijk}$ for focal elements $\mathcal{MT}_{2,ijk}$ associated with the evidence space $(\mathcal{MT}_2, \mathbb{MT}_2, m_{MT2})$ can be defined on the basis of (i) the bounds $(\underline{TF}_{2,ij}, \overline{TF}_{2,ij})$ for focal elements $\mathcal{TF}_{2,ij}$ associated with the evidence space $(\mathcal{TF}_2, \mathbb{TF}_2, m_{TF2})$ defined in Sect. 9.2 for the failure temperatures for a system consisting of SL 1 and SL 2 with system failure corresponding to failure of either SL and (ii) the bounds $(\underline{TF}_{WL1,k}, \overline{TF}_{WL1,k})$ for focal elements $\mathcal{TF}_{WL1,k}$ associated with the evidence space $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{WL1,TF})$ defined in Eqs. (4.20)- (4.25) for the actual temperatures at which WL 1 fails.

Given known values for the bounds $(\underline{TF}_{2,ij}, \overline{TF}_{2,ij})$ and $(\underline{TF}_{WL1,k}, \overline{TF}_{WL1,k})$, the bounds $(\underline{m}_{2T,ijk}, \overline{m}_{2T,ijk})$ for $\mathcal{MT}_{2,ijk}$ are defined by

$$\underline{m}_{2T,ijk} = \underline{TF}_{2,ij} - \overline{TF}_{WL1,k} \text{ and } \overline{m}_{2T,ijk} = \overline{TF}_{2,ij} - \underline{TF}_{WL1,k} \tag{12.53}$$

with the assumption that

$$a - b = \begin{cases} \infty & \text{if } a = \infty \\ -\infty & \text{if } a < \infty \text{ and } b = \infty \end{cases} \tag{12.54}$$

as previously indicated in Eq. (12.32). Once the focal element BPAs $m_{MT1}(\mathcal{MT}_{1,ijk})$ and the focal element bounds $(\underline{m}_{1T,ijk}, \overline{m}_{1T,ijk})$ for $\mathcal{MT}_{1,ijk}$ are obtained, cumulative and complementary cumulative plausibility and belief functions for temperature margin can be obtained as (i) indicated in conjunction with Eqs. (2.48)-(2.50) and (ii) illustrated in Fig. 12.2.

The evidence space $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$ is for the actual temperatures at which WL 1 fails with $T_\infty = \infty$ included to indicate that that link system failure did not occur. A slightly different failure temperature margin evidence space is obtained if the WL 1 failure temperature evidence space $(\mathcal{T}_{WL1}, \mathbb{T}_{WL1}, m_{WL1})$ defined in Sect. 4 is used instead of $(\mathcal{TF}_{WL1}, \mathbb{TF}_{WL1}, m_{TF,WL1})$. The sample space $\mathcal{T}_{WL1}$ contains all originally specified possible failure times for WL 1. In contrast, $\mathcal{TF}_{WL1}$ contains only (i) the failure temperatures that actually occurred and (ii) the indicator T_∞ assigned to failure temperatures in $\mathcal{T}_{WL1}$ that were never reached by the WL 1 temperature curve $T_{WL1}(t)$ for $t_{mn} \leq t \leq t_{mx}$.

13. Cumulative and Complementary Cumulative Belief and Plausibility for SL/SL Temperature Margin for a System with 2 SLs and 1 WL

For simplicity, this section considers a system with 2 SLs and 1 WL and two definitions of system failure: (i) LOAS occurs when both SLs fail before the WL fails and (ii) LOAS occurs when either SL fails before the WL fails.

13.1 Cumulative and Complementary Cumulative Belief and Plausibility for SL/SL Temperature Margin with LOAS defined by Failure of Both SLs before failure of the WL

Another possibility is a SL/SL failure temperature margin M_{3T} defined by

$$\begin{aligned} M_T = {} & \text{(SL temperature at which second SL failure potentially causes LOAS)} \\ & - \text{(temperature of second SL to fail when WL failure potentially prevents LOAS).} \end{aligned} \tag{13.1}$$

for (i) systems with two SLs and one WL and (ii) LOAS corresponding to failure of both SLs before failure of the WL. If the SL temperature curves are increasing as illustrated in Fig. 4.1, then the margin M_T is (i) positive if the second SL failure occurs after failure of the WL (i.e., the desired occurrence) and (ii) negative if the second SL failure occurs before failure of the WL (i.e., the undesired occurrence).

To incorporate the possibility of nonfailure of individual links, a generalized margin $M_T(\mathbf{t})$ is considered for analysis with $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ belonging to the set $\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}$ defined in Eq. (4.13). Specifically, $M_T(\mathbf{t})$ is defined by (i)

$$\begin{aligned} M_{3T}(\mathbf{t}) &= M_{3T}([t_{SL1}, t_{SL2}, t_{WL1}]) \\ &= \begin{cases} -\infty \text{ for } t_{WL1} = \infty, \max\{t_{SL1}, t_{SL2}\} < \infty \\ T_{SL1}(t_{SL1}) - T_{SL1}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{SL2} < t_{SL1} < \infty \\ T_{SL2}(t_{SL2}) - T_{SL2}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{SL1} < t_{SL2} < \infty \\ \min\{T_{SL1}(t_{ij}) - T_{SL1}(t_{WL1}), T_{SL2}(t_{ij}) - T_{SL2}(t_{WL1})\} \text{ for } t_{WL1} < \infty, t_{ij} = t_{SL1} = t_{SL2} < \infty \\ \infty \text{ for } \max\{t_{SL1}, t_{SL2}\} = \infty. \end{cases} \end{aligned} \tag{13.2}$$

Application of the function $M_{3T}(\mathbf{t})$ to the elements $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ of the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time as indicated in Eqs. (12.3)-(12.6) results in the evidence space $(\mathcal{MT}_3, \mathbb{MT}_3, m_{3MT})$ for SL/SL failure temperature margins. Definition of the focal element bounds $\underline{m}_{3T,ijk}$ and $\overline{m}_{3T,ijk}$ for focal elements $\mathcal{MT}_{3,ijk}$ associated with the evidence space $(\mathcal{MT}_3, \mathbb{MT}_3, m_{3MT})$ is now considered.

Definition of the focal element bound $\underline{m}_{3T,ijk}$ is considered first. Specifically, $\underline{m}_{3T,ijk}$ has a definition that (i) involves greatest lower bounds (glb's) for sets of link failure temperatures and

(ii) is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\underline{t}_{WL1,k}$ $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$. The following possibilities exist for the definition of $\underline{m}_{3T,ijk}$:

Possibility (1): If $\overline{t}_{WL1,k} = \infty$ and $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty$, then

$$\underline{m}_{3T,ijk} = -\infty. \tag{13.3}$$

Possibility (2): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} = \infty$, then

$$\underline{m}_{3T,ijk} = \infty. \tag{13.4}$$

Possibility (3): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty$ and $\underline{t}_{SL1,i} < \underline{t}_{SL2,j}$, then either (3.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$ and

$$\underline{m}_{3T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k}), \tag{13.5}$$

or (3.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$, $[\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{1i} \ne \varnothing$ and

$$\begin{aligned}\underline{m}_{3T,ijk} &= \min\Big\{\text{glb}\{T : T = T_{SL1}(t) \text{ for } \underline{t}_{SL2,j} \le t \text{ and } t \in \mathcal{TM}_{SL1,i}\} - T_{SL1}(\overline{t}_{WL1,k}),\\ &\qquad T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k})\Big\}\\ &= \min\{T_{SL1}(\underline{t}_{SL2,j}) - T_{SL1}(\overline{t}_{WL1,k}), T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k})\} \text{ if } \underline{t}_{SL2,j} \in \mathcal{TM}_{1i},\end{aligned} \tag{13.6}$$

or (3.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$, $[\underline{t}_{SL2,j}, t_{mx}] \cap \mathcal{TM}_{1i} = \varnothing$ and

$$\underline{m}_{3T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k}). \tag{13.7}$$

Possibility (4): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty$ and $\underline{t}_{SL2,j} < \underline{t}_{SL1,i}$, then either (4.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$ and

$$\underline{m}_{3T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k}), \tag{13.8}$$

or (4.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$, $[\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{2j} \ne \varnothing$ and

$$\begin{aligned}\underline{m}_{3T,ijk} &= \min\Big\{\text{glb}\{T : T = T_{SL2}(t) \text{ for } \underline{t}_{SL1,i} \le t \text{ and } t \in \mathcal{TM}_{SL2,j}\} - T_{SL2}(\overline{t}_{WL1,k}),\\ &\qquad T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k})\Big\}\\ &= \min\{T_{SL2}(\underline{t}_{SL1,i}) - T_{SL2}(\overline{t}_{WL1,k}), T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k})\} \text{ if } \underline{t}_{SL1,i} \in \mathcal{TM}_{2j},\end{aligned} \tag{13.9}$$

or (4.3) $\overline{t}_{WL1,k} \leq t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \leq t_{mx}$, $\underline{t}_{SL1,i} \leq \overline{t}_{SL2,j}$, $[\underline{t}_{SL1,i}, t_{mx}] \cap \mathcal{TM}_{2j} = \varnothing$ and

$$\underline{m}_{3T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k}). \tag{13.10}$$

Possibility (5): If $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}\} < \infty$ and $\underline{t}_{SL1,i} = \underline{t}_{SL2,j} = \underline{t}_{ij} \leq t_{mx}$, then

$$\overline{m}_{3T,ijk} = \min\left\{T_{SL1}(\underline{t}_{ij}) - T_{SL1}(\overline{t}_{WL1,k}), T_{SL2}(\underline{t}_{ij}) - T_{SL2}(\overline{t}_{WL1,k})\right\}. \tag{13.11}$$

The bound $\overline{m}_{3T,ijk}$ also has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\underline{t}_{WL1,k}$, $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$ as stated for the following possibilities:

Possibility (1): If $\underline{t}_{WL1,k} = \infty$ and $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty$, then

$$\overline{m}_{3T,ijk} = -\infty. \tag{13.12}$$

Possibility (2): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} = \infty$, then

$$\overline{m}_{3T,ijk} = \infty. \tag{13.13}$$

Possibility (3): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}\} < \infty$ and $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$, then either (3.1) $\underline{t}_{WL1,k} \leq t_{mx}$, $\underline{t}_{SL2,j} \leq \overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and

$$\overline{m}_{3T,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}), T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k})\} \tag{13.14}$$

or (2.2) $\underline{t}_{WL1,k} \leq t_{mx}$, $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} < \overline{t}_{SL2,j}$ and

$$\overline{m}_{3T,ijk} = T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k}). \tag{13.15}$$

Possibility (4): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}\} < \infty$ and $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$, then either (4.1) $\underline{t}_{WL1,k} \leq t_{mx}$, $\underline{t}_{SL1,i} \leq \overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ and

$$\overline{m}_{3T,ijk} = \max\{T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}), T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k})\} \tag{13.16}$$

or (4.2) $\underline{t}_{WL1} \leq t_{mx}$, $\overline{t}_{SL2,j} < \underline{t}_{SL1,i} < \overline{t}_{SL1,i}$ and

$$\overline{m}_{3T,ijk} = T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}). \tag{13.17}$$

Possibility (5): If $\max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1}\} < \infty$ and $\overline{t}_{SL1,i} = \overline{t}_{SL2,j} = \overline{t}_{ij}$, then

$$
\begin{aligned}
\overline{m}_{3T,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL1,i}\} - T_{SL1}(\underline{t}_{WL1,k}), \\
&\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{ij}) \cap \mathcal{TM}_{SL2,j}\} - T_{SL2}(\underline{t}_{WL1,k})\Big\} \qquad (13.18) \\
&= \max\Big\{T_{SL1}(\overline{t}_{ij}) - T_{SL1}(\underline{t}_{WL1,k}), T_{SL2}(\overline{t}_{ij}) - T_{SL2}(\underline{t}_{WL1,k})\Big\}.
\end{aligned}
$$

Once the evidence space $(\mathcal{MT}_3, \mathbb{MT}_3, m_{3MT})$ is constructed, cumulative and complementary cumulative plausibility and belief functions for SL/SL failure temperature margins can be obtained from the pairs $(\underline{m}_{3T,ijk}, \overline{m}_{3T,ijk})$ as indicated in conjunction with Eqs. (2.48)-(2.50). As examples, cumulative and complementary cumulative plausibility and belief functions for SL/SL failure temperature margins are presented in Fig. 13.1 for a system with 2 SLs and 1 WL. In addition, Fig. 13.1 also contains the CDF and CCDF for SL/SL failure temperature margins obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 13.1 are constructed as indicated in Eqs. (7.9-(7.13) with $M_{3T}([t_{SL1}, t_{SL2}, t_{WL1}])$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

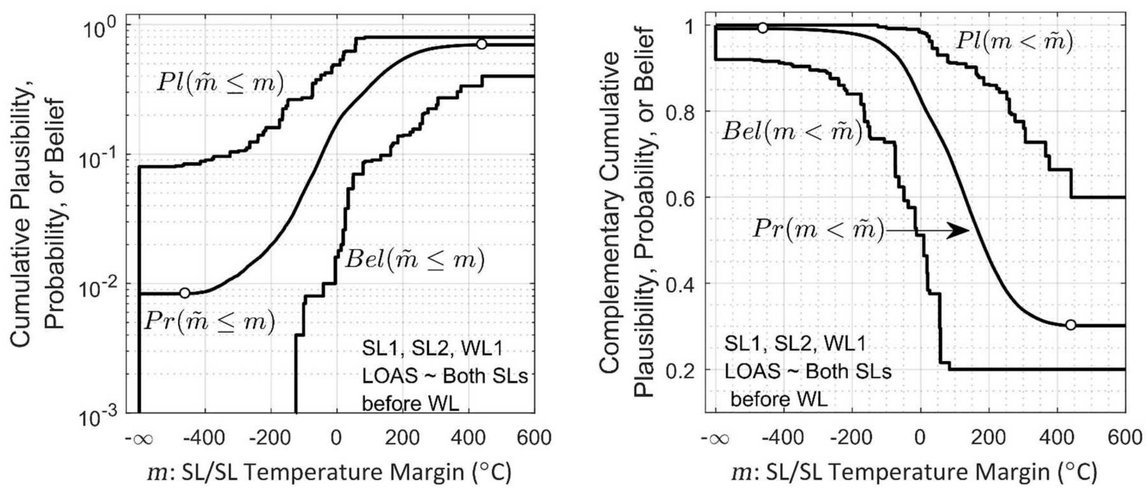


Fig. 13.1 Graphical summary of evidence space $(\mathcal{MT}_3, \mathbb{MT}_3, m_{MT3})$ for SL/SL failure temperature margins for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4 and (ii) LOAS corresponding to failure of both SLs before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{m} \leq m)$, probability $Pr(\tilde{m} \leq m)$ and belief $Bel(\tilde{m} \leq m)$, and (b) Complementary cumulative plausibility $Pl(m < \tilde{m})$, probability $Pr(m < \tilde{m})$ and belief $Bel(m < \tilde{m})$.

Margin results of the form shown in Fig. 13.1 are valuable because they show and quantify the uncertainty in the temperature difference between (i) temperature of the second SL to fail when this failure corresponds to failure of the SL link system and thus potentially results in LOAS and (ii) the temperature of the second SL to fail at the time that the WL fails and potentially averts LOAS. This provides perspective on how far apart failure of the SL system and failure of the WL are in temperature space relative to the potential occurrence of LOAS.

In addition, the sampling-based procedure used to obtain the CDF and CCDF in Fig. 13.1 produced values of

$$\underline{m}_{3FT} \cong -459.674\ ^\circ\mathrm{C} \text{ and } \overline{m}_{3FT} \cong 441.949\ ^\circ\mathrm{C} \tag{13.19}$$

for the smallest failure temperature margin $\underline{m}_{3FT} > -\infty$ and the largest failure temperature. margin $\overline{m}_{3FT} < \infty$.

Belief $Bel(\mathcal{S})$ and plausibility $Pl(\mathcal{S})$ for subsets $\mathcal{S}$ of $\mathcal{MT}_3$ can be calculated from the relationships defined in Eqs. (11.10)-(11.19). As a reminder, this requires that the evidence space $(\mathcal{MT}_3, \mathbb{MT}_3, m_{3MT})$ for failure temperature margins be defined to be consistent with the function $M_{3T}(\mathbf{t})$ defined in Eq. (13.2) and the corresponding focal element bounds $\underline{m}_{3T,ijk}$ and $\overline{m}_{3T,ijk}$ defined in Eqs. (13.3)-(13.18).

As examples, the calculation of $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ for a SL/SL temperature margin value of 0 for the WL/SL system in Fig. 13.1 produces the results

$$Bel[\mathcal{S}(0)] = 1.600 \times 10^{-2} \cong 1.602 \times 10^{-2} \tag{13.20}$$

and

$$Pl[\mathcal{S}(0)] = 4.880 \times 10^{-1} \cong 4.880 \times 10^{-1} \tag{13.21}$$

with (i) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the initial equalities determined as indicated in Eqs. (11.10)-(11.19) and (ii) the values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as described in Sect. 5.2. The agreement of the plotted results in Fig. 13.1 (i.e., $Bel[\mathcal{S}(0)] = 0.016$, $Pl[\mathcal{S}(0)] = 0.488$) and the numerical results in Eqs. (13.20) and (13.21) provides a strong verification result that $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ are being calculated correctly. Additional verification is provided by the agreement of (i) the preceding values for $Bel[\mathcal{S}(0)]$ and $Pl[\mathcal{S}(0)]$ and (ii) the corresponding values for LOAS in Eqs. (4.31) and (4.32).

13.2 Cumulative and Complementary Cumulative Belief and Plausibility for SL/SL Temperature Margin with LOAS defined by Failure of Either SL before Failure of the WL

This section presents SL/SL failure temperature margin results defined by

$$\begin{aligned} M_T = {} & \text{(SL temperature at which first SL failure potentially causes LOAS)} \\ & - \text{(temperature of first SL to fail when WL failure potentially prevents LOAS).} \end{aligned} \tag{13.22}$$

for a 2 SL, 1 WL system for which LOAS occurs if either SL fails before the WL fails. To incorporate the possibility of nonfailure of individual links, a generalized margin $M_T(\mathbf{t})$ is considered for analysis with $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ belonging to the set $\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1}$ defined in Eq. (4.13). Specifically, $M_T(\mathbf{t})$ is defined by (i)

$$M_{4T}(\mathbf{t}) = M_{4T}([t_{SL1}, t_{SL2}, t_{WL1}])$$
$$= \begin{cases} -\infty \text{ for } t_{WL1} = \infty, \min\{t_{SL1}, t_{SL2}\} < \infty \\ T_{SL2}(t_{SL2}) - T_{SL2}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{SL2} < t_{SL1} \\ T_{SL1}(t_{SL1}) - T_{SL1}(t_{WL1}) \text{ for } t_{WL1} < \infty, t_{SL1} < t_{SL2} \\ \min\{T_{SL1}(t_{ij}) - T_{SL1}(t_{WL1}), T_{SL2}(t_{ij}) - T_{SL2}(t_{WL1})\} \text{ for } t_{WL1} < \infty, t_{ij} = t_{SL1} = t_{SL2} < \infty \\ \infty \text{ for } \min\{t_{SL1}, t_{SL2}\} = \infty. \end{cases} \tag{13.23}$$

Application of the function $M_{4T}(\mathbf{t})$ to the elements $\mathbf{t} = [t_{SL1}, t_{SL2}, t_{WL1}]$ of the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time defined in conjunction with Eqs. (4.13)-(4.16) results in the evidence space $(\mathcal{MT}_4, \mathbb{MT}_4, m_{4MT})$ for SL/SL failure temperature margins defined as indicated in Eqs. (12.3)-(12.6). Definition of the focal element bounds $\underline{m}_{4T,ijk}$ and $\overline{m}_{4T,ijk}$ for focal elements $\mathcal{MT}_{4,ijk}$ associated with the evidence space $(\mathcal{MT}_4, \mathbb{MT}_4, m_{4MT})$ is now considered.

Definition of the focal element bound $\underline{m}_{4T,ijk}$ is considered first. Specifically, $\underline{m}_{4T,ijk}$ has a definition that is conditional on various equalities and inequalities involving the times $\underline{t}_{SL1,i}$, $\underline{t}_{SL2,j}$, $\underline{t}_{WL1,k}$ $\overline{t}_{SL1,i}$, $\overline{t}_{SL2,j}$ and $\overline{t}_{WL1,k}$. The following possibilities exist for the definition of $\underline{m}_{4T,ijk}$:

Possibility (1): If $\overline{t}_{WL1,k} = \infty$ and $\min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \infty$, then

$$\underline{m}_{4T,ijk} = -\infty. \tag{13.24}$$

Possibility (2): If $\min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} = \infty$, then

$$\underline{m}_{4T,ijk} = \infty. \tag{13.25}$$

Possibility (3): If $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} \le t_{mx}$ and $\underline{t}_{SL1,i} < \underline{t}_{SL2,j}$, then either: (3.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} = \infty$ and

$$\underline{m}_{4T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k}), \tag{13.26}$$

or (3.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$ and

$$\underline{m}_{4T,ijk} = T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k}), \tag{13.27}$$

or (3.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL1,i} < \underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} \le \overline{t}_{SL1,i}$ and

$$\underline{m}_{4T,ijk} = \min\{T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k}), T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k})\}. \tag{13.28}$$

Possibility (4): If $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} \le t_{mx}$ and $\underline{t}_{SL2,j} < \underline{t}_{SL1,i}$, then either: (4.1) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} = \infty$ and

$$\underline{m}_{4T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k}), \tag{13.29}$$

or (4.2) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$ and

$$\underline{m}_{4T,ijk} = T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k}) \tag{13.30}$$

or (4.3) $\overline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \underline{t}_{SL1,i} \le t_{mx}$, $\underline{t}_{SL1,i} \le \overline{t}_{SL2,j}$ and

$$\underline{m}_{4T,ijk} = \min\{T_{SL1}(\underline{t}_{SL1,i}) - T_{SL1}(\overline{t}_{WL1,k}), T_{SL2}(\underline{t}_{SL2,j}) - T_{SL2}(\overline{t}_{WL1,k})\}. \tag{13.31}$$

Possibility (5): If $\overline{t}_{WL1,k} \le t_{mx}$ and $\underline{t}_{SL2,j} = \underline{t}_{SL1,i} = \underline{t}_{ij} \le t_{mx}$, then

$$\underline{m}_{4T,ijk} = \min\{T_{SL1}(\underline{t}_{ij}) - T_{SL1}(\overline{t}_{WL1,k}), T_{SL2}(\underline{t}_{ij}) - T_{SL2}(\overline{t}_{WL1,k})\}. \tag{13.32}$$

Similarly, $\overline{m}_{4T,ijk}$ is defined by

Possibility (1): If $\underline{t}_{WL1,k} = \infty$ and $\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \infty$, then

$$\overline{m}_{4T,ijk} = -\infty. \tag{13.33}$$

Possibility (2): If $\min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} = \infty$, then

$$\overline{m}_{4T,ijk} = \infty. \tag{13.34}$$

Possibility (3): If $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL1,i} \le t_{mx}$ and $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ then either: (3.1) $\underline{t}_{WL1,k} \le t_{mx}$, $\underline{t}_{SL2,j} < \overline{t}_{SL1,i} \le t_{mx}$, $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and

$$
\begin{aligned}
\overline{m}_{4T,ijk} &= \max\Big\{T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}), \\
&\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{SL1,i}) \cap \mathcal{TM}_{SL2,j}\} - T_{SL2}(\underline{t}_{WL1,k})\Big\} \\
&= \max\Big\{T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}), T_{SL2}(\overline{t}_{SL1,i}) - T_{SL2}(\underline{t}_{WL1,k})\Big\} \\
&\qquad \text{if } \overline{t}_{SL1,i} \in \mathcal{TM}_{SL2,j},
\end{aligned}
\tag{13.35}
$$

or (3.2) $\underline{t}_{WL1,k} \le t_{mx}$, $t_{ij} = \underline{t}_{SL2,j} = \overline{t}_{SL1,i} \le t_{mx}$, $\overline{t}_{SL1,i} < \overline{t}_{SL2,j}$ and

$$
\begin{aligned}
\overline{m}_{4T,ijk} &= \text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{SL1,i}) \cap \mathcal{TM}_{SL1,i}\} - T_{SL1}(\underline{t}_{WL1,k}) \\
&= T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}),
\end{aligned}
\tag{13.36}
$$

or (3.3) $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL1,i} < \underline{t}_{SL2,j} \le \overline{t}_{SL2,j}$, $\overline{t}_{SL1,i} \le t_{mx}$ and

$$
\overline{m}_{4T,ijk} = T_{SL1}(\overline{t}_{SL1,i}) - T_{SL1}(\underline{t}_{WL1,k}). \tag{13.37}
$$

Possibility (4): If $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL2,j} \le t_{mx}$ and $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$, then either: (4.1) $\underline{t}_{WL1,k} \le t_{mx}$ $\underline{t}_{SL1,i} < \overline{t}_{SL2,j} \le t_{mx}$, $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ and

$$
\begin{aligned}
\overline{m}_{4T,ijk} &= \max\Big\{T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k}), \\
&\qquad \text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \overline{t}_{SL2,j}) \cap \mathcal{TM}_{SL1,i}\} - T_{SL1}(\underline{t}_{WL1,k})\Big\} \\
&= \max\Big\{T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k}), T_{SL1}(\overline{t}_{SL2,j}) - T_{SL1}(\underline{t}_{WL1,k})\Big\} \\
&\qquad \text{if } \overline{t}_{SL2,j} \in \mathcal{TM}_{SL1,i},
\end{aligned}
\tag{13.38}
$$

or (4.2) $\underline{t}_{WL1,k} \le t_{mx}$, $t_{ij} = \underline{t}_{SL1,i} = \overline{t}_{SL2,j} \le t_{mx}$, $\overline{t}_{SL2,j} < \overline{t}_{SL1,i}$ and

$$
\begin{aligned}
\overline{m}_{4T,ijk} &= \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \overline{t}_{SL2,j}) \cap \mathcal{TM}_{SL2,j}\} - T_{SL2}(\underline{t}_{WL1,k}) \\
&= T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k}),
\end{aligned}
\tag{13.39}
$$

or (4.3) $\underline{t}_{WL1,k} \le t_{mx}$, $\overline{t}_{SL2,j} < \underline{t}_{SL1,i} \le \overline{t}_{SL1,i}$, $\overline{t}_{SL2,j} \le t_{mx}$ and

$$
\overline{m}_{4T,ijk} = T_{SL2}(\overline{t}_{SL2,j}) - T_{SL2}(\underline{t}_{WL1,k}). \tag{13.40}
$$

Possibility (5): If $\underline{t}_{WL1,k} \le t_{mx}$ and $\overline{t}_{ij} = \overline{t}_{SL2,j} = \overline{t}_{SL1,i} \le t_{mx}$, then $\max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \overline{t}_{ij}$ and

$$\begin{aligned}\bar{m}_{4T,ijk} &= \max\Big\{\text{lub}\{T : T = T_{SL1}(t) \text{ for } t \in [\underline{t}_{SL1,i}, \bar{t}_{ij}) \cap \mathcal{TM}_{SL1,i}\} - T_{SL1}(\underline{t}_{WL1,k}), \\ &\qquad \text{lub}\{T : T = T_{SL2}(t) \text{ for } t \in [\underline{t}_{SL2,j}, \bar{t}_{ij}) \cap \mathcal{TM}_{SL2,j}\} - T_{SL2}(\underline{t}_{WL1,k})\Big\} \\ &= \max\Big\{T_{SL1}(\bar{t}_{ij}) - T_{SL1}(\underline{t}_{WL1,k}), T_{SL2}(\bar{t}_{ij}) - T_{SL2}(\underline{t}_{WL1,k})\Big\}.\end{aligned} \quad (13.41)$$

As examples, cumulative and complementary cumulative plausibility and belief functions for SL/SL failure temperature margins are presented in Fig. 13.2 for a system with 2 SLs and 1 WL. In addition, Fig. 13.2 also contains the CDF and CCDF for SL/SL failure temperature margins obtained by assigning uniform distributions to the individual focal elements for link failure temperature as described for the construction of the link failure time CDFs in Fig. 4.4. Specifically, the CDF and CCDF in Fig. 13.2 are constructed as indicated in Eqs. (7.9)-(7.13) with $M_{4T}([t_{SL1}, t_{SL2}, t_{WL1}])$ replacing $TML_1(t_{SL1}, t_{SL2}, t_{WL1})$.

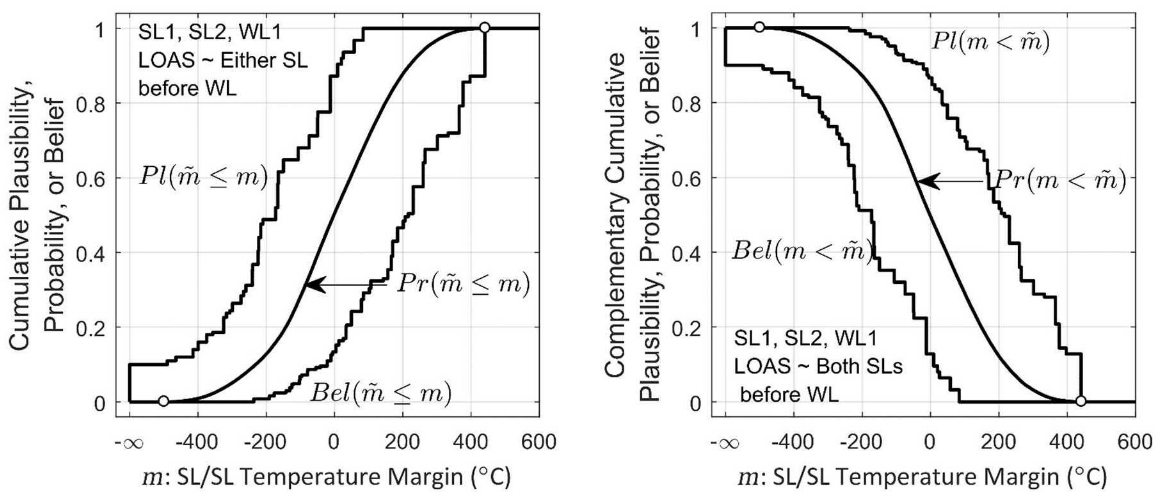


Fig. 13.2 Graphical summary of evidence space $(\mathcal{MT}_3, \mathbb{MT}_3, m_{MT3})$ for SL/SL failure temperature margins for (i) a system composed of SL 1, SL 2 and WL 1 defined in Sect. 4 and (ii) LOAS corresponding to failure of either SL before failure of the WL: (a) Cumulative plausibility $Pl(\tilde{m} \le m)$, probability $Pr(\tilde{m} \le m)$ and belief $Bel(\tilde{m} \le m)$, and (b) Complementary cumulative plausibility $Pl(m < \tilde{m})$, probability $Pr(m < \tilde{m})$ and belief $Bel(m < \tilde{m})$.

The sampling-based procedure used to obtain the CDF and CCDF in Fig. 13.2 produced values of

$$\underline{m}_{4FT} \cong -498.310 \text{ °C} \quad \text{and} \quad \bar{m}_{4FT} \cong 442.404 \text{ °C} \qquad (13.42)$$

for the smallest failure temperature margin $\underline{m}_{4FT} > -\infty$ and the largest failure temperature. margin $\bar{m}_{4FT} < \infty$.

14. Plausibility and Belief for LOAS with Two WLs and Two SLs

To this point, plausibility and belief for LOAS have been considered for 2 SLs and 1 WL with LOAS corresponding to (i) both SLs failing before the WL fails or (ii) either SL failing before the WL fails. Past analyses have considered the following four definitions of LOAS for systems with *nSL* SLs and *nWL* WLs (e.g., see Table 1, Ref. [83]): (i) Failure of all SLs before failure of any WL, (ii) Failure of any SL before failure of any WL, (iii) Failure of all SLs before failure of all WLs, and (iv) Failure of any SL before failure of all WLs. For notational simplicity, plausibility and belief representations for LOAS will be developed for the preceding four definitions of LOAS for a system with 2 SLs and 2 WLs.

14.1 Plausibility and Belief for Occurrence of LOAS

The two SLs (i.e., SL 1 and SL 2) and two WLs (i.e., WL 1 and WL 2) are assumed to have the evidence theory representations and properties defined in Sect. 4.1. In turn, combining the evidence spaces $(\mathcal{TM}_{SL1}, \mathbb{TM}_{SL1}, m_{SL1,t})$, $(\mathcal{TM}_{SL2}, \mathbb{TM}_{SL2}, m_{SL2,t})$, $(\mathcal{TM}_{WL1}, \mathbb{TM}_{WL1}, m_{WL1,t})$ and $(\mathcal{TM}_{WL2}, \mathbb{TM}_{WL2}, m_{WL2,t})$ for link failure time produces the product evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ with

$$\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{WL1} \times \mathcal{TM}_{WL2}, \tag{14.1}$$

$$\mathcal{TM}_{ijkl} = \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \times \mathcal{TM}_{WL1,k} \times \mathcal{TM}_{WL2,l} \in \mathbb{TM}, \tag{14.2}$$

$$\mathcal{I}_{SL1} = \{1,2,...,nSL1\}, \mathcal{I}_{SL2} = \{1,2,...,nSL2\}, \mathcal{I}_{WL1} = \{1,2,...,nWL1\}, \mathcal{I}_{WL2} = \{1,2,...,nWL2\}, \tag{14.3}$$

$$\mathbb{TM} = \{\mathcal{TM}_{ijkl} : (i,j,k,l) \in \mathcal{I} = \mathcal{I}_{SL1} \times \mathcal{I}_{SL2} \times \mathcal{I}_{WL1} \times \mathcal{I}_{WL2}\} \tag{14.4}$$

and

$$\begin{aligned} m_{TM}(\mathcal{TM}_{ijkl}) &= m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL1,t}(\mathcal{TM}_{WL1,k}) m_{WL2,t}(\mathcal{TM}_{WL2,l}) \\ &= m_{ijkl}. \end{aligned} \tag{14.5}$$

Indicator functions analogous those defined in Eqs. (4.28) and (4.30) for use in Eqs. (4.27) and (4.29) in the determination of plausibility and belief are now defined for 2 SLs and 2 WLs. Specifically, $\delta_{Ps}(\mathcal{TM}_{ijkl})$ and $\delta_{Bs}(\mathcal{TM}_{ijkl})$ as defined below for $s = 1,2,3,4$ are the indicator functions used in the determination plausibility and belief for the subsets $\mathcal{L}_s$ of $\mathcal{TM}$ that satisfy the definitions of the four failure patterns:

(i) Pattern 1, Failure of all SLs before failure of any WL:

$$\mathcal{L}_1 = \{(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) \in \mathcal{TM} \text{ with } \max\{t_{SL1}, t_{SL2}\} < \min\{t_{WL1}, t_{WL2}\}\} \tag{14.6}$$

$$\delta_{P1}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \min\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ 0 \text{ otherwise,} \end{cases} \tag{14.7}$$

$$\delta_{B1}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \min\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ 0 \text{ otherwise.} \end{cases} \tag{14.8}$$

(ii) Pattern 2, Failure of any SL before failure of any WL:

$$\mathcal{L}_2 = \{(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) \in \mathcal{TM} \text{ with } \min\{t_{SL1}, t_{SL2}\} < \min\{t_{WL1}, t_{WL2}\}\} \tag{14.9}$$

$$\delta_{P2}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \min\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ 0 \text{ otherwise,} \end{cases} \tag{14.10}$$

$$\delta_{B2}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \min\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ 0 \text{ otherwise.} \end{cases} \tag{14.11}$$

(iii) Pattern 3, Failure of all SLs before failure of all WLs:

$$\mathcal{L}_3 = \{(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) \in \mathcal{TM} \text{ with } \max\{t_{SL1}, t_{SL2}\} < \max\{t_{WL1}, t_{WL2}\}\} \tag{14.12}$$

$$\delta_{P3}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \max\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ 0 \text{ otherwise,} \end{cases} \tag{14.13}$$

$$\delta_{B3}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \max\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ 0 \text{ otherwise.} \end{cases} \tag{14.14}$$

(iv) Pattern 4, Failure of any SL before failure of all WLs:

$$\mathcal{L}_4 = \{(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) \in \mathcal{TM} \text{ with } \min\{t_{SL1}, t_{SL2}\} < \max\{t_{WL1}, t_{WL2}\}\} \tag{14.15}$$

$$\delta_{P4}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \max\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ 0 \text{ otherwise,} \end{cases} \tag{14.16}$$

$$\delta_{B4}(\mathcal{TM}_{ijkl}) = \begin{cases} 1 \text{ if } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \max\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ 0 \text{ otherwise.} \end{cases} \tag{14.17}$$

In turn,

$$\begin{aligned} Pl(\mathcal{L}_s) &= \text{ plausibility that LOAS occurs for failure pattern } s, s = 1,2,3,4 \\ &= \sum_{\varnothing \neq \mathcal{TM}_{ijkl} \cap \mathcal{L}_s} m_{TM}(\mathcal{TM}_{ijkl}) \\ &= \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \sum_{l=1}^{nWL2} \delta_{Ps}(\mathcal{TM}_{ijkl}) m_{ijkl} \end{aligned} \quad (14.18)$$

and

$$\begin{aligned} Bel(\mathcal{L}_s) &= \text{ belief that LOAS occurs for failure pattern } s, s = 1,2,3,4 \\ &= \sum_{\mathcal{TM}_{ijkl} \subset \mathcal{L}_s} m_{TM}(\mathcal{TM}_{ijkl}) \\ &= \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nWL1} \sum_{l=1}^{nWL2} \delta_{Bs}(\mathcal{TM}_{ijkl}) m_{ijkl}. \end{aligned} \quad (14.19)$$

As examples, the calculation of $Bel(\mathcal{L}_s)$ and $Pl(\mathcal{L}_s)$ for the links defined in Table 4.1 and Fig. 4.1 yields the results

$$\begin{aligned} Bel(\mathcal{L}_s) &= \begin{cases} 1.800 \times 10^{-3} & \text{for } s = 1 \text{ (i.e., failure of all SLs before failure of any WL)} \\ 2.420 \times 10^{-2} & \text{for } s = 2 \text{ (i.e., failure of any SL before failure of any WL)} \\ 2.420 \times 10^{-2} & \text{for } s = 3 \text{ (i.e., failure of all SLs before failure of all WLs)} \\ 2.198 \times 10^{-1} & \text{for } s = 4 \text{ (i.e., failure of any SL before failure of all WLs)} \end{cases} \\ &\cong \begin{cases} 1.791 \times 10^{-3} & \text{for } s = 1 \\ 2.428 \times 10^{-2} & \text{for } s = 2 \\ 2.427 \times 10^{-2} & \text{for } s = 3 \\ 2.196 \times 10^{-1} & \text{for } s = 4 \end{cases} \end{aligned} \quad (14.20)$$

and

$$Pl(\mathcal{L}_s) = \begin{cases} 3.824\times 10^{-1} & \text{for } s=1 \text{ (i.e., failure of all SLs before failure of any WL)} \\ 8.336\times 10^{-1} & \text{for } s=2 \text{ (i.e., failure of any SL before failure of any WL)} \\ 6.416\times 10^{-1} & \text{for } s=3 \text{ (i.e., failure of all SLs before failure of all WLs)} \\ 9.424\times 10^{-1} & \text{for } s=4 \text{ (i.e., failure of any SL before failure of all WLs)} \end{cases}$$

$$\cong \begin{cases} 3.826\times 10^{-1} & \text{for } s=1 \\ 8.334\times 10^{-1} & \text{for } s=2 \\ 6.415\times 10^{-1} & \text{for } s=3 \\ 9.423\times 10^{-1} & \text{for } s=4 \end{cases} \qquad (14.21)$$

with (i) the values for $Bel(\mathcal{L}_s)$ and $Pl(\mathcal{L}_s)$ in the initial equalities determined as indicated above in Eqs. (14.6)-(14.19) and (ii) the values for $Bel(\mathcal{L}_s)$ and $Pl(\mathcal{L}_s)$ in the following approximate equalities determined in a sampling-based verification procedure with a sample of size 10^7 as indicated below in Eqs. (14.22)-(14.35). The agreement of the two computational procedures provides a strong verification result that $Bel(\mathcal{L}_1)$ and $Pl(\mathcal{L}_1)$ are being calculated correctly.

The results in Eqs. (14.6)-(14.19) also generalize in a straight forward manner to representations for plausibility and belief for the 4 link failure patterns for *nSL* SLs and *nWL* WLs. In this generalized form, each min and max in Eqs. (14.7)-(14.17) will contain the analogous times for *nSL* SLs rather than for 2 SLs or *nWL* WLs rather than for 2 WLs.

To obtain the desired probability space for checking the calculation of $Bel(\mathcal{L}_s)$ with the representations defined in Eqs. (14.6)--(14.19), (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$ and $m_{WL2,t}(\mathcal{TM}_{WL2,l}) = m_{WL2,l}$) is assigned to $\underline{t}_{WL1,k}$ and $\underline{t}_{WL2,l}$, (ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$ and $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\overline{t}_{SL1,i}$ and $\overline{t}_{SL2,j}$, and (iii) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}, \underline{t}_{WL2,l}]$. This produces the probability space that has the smallest possible probabilities for the sets $\mathcal{L}_s$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$\begin{aligned} p([\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}, \underline{t}_{WL2,l}]) \\ &= m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) m_{WL1,k}(\mathcal{TM}_{WL1,k}) m_{WL2,l}(\mathcal{TM}_{WL2,l}) \\ &= m_{ijkl} \end{aligned} \qquad (14.22)$$

for (i, j, k, l) belonging to the set $\mathcal{I}$ defined in Eq. (14.4) and (ii) any set that does not contain one or more of the vectors $[\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}, \underline{t}_{WL2,l}]$ has a probability of zero. Then, with a large random sample

$$[\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r, \underline{tWL2}_r], r = 1, 2, ..., nR, \quad (14.23)$$

of size nR from the failure time vectors $[\overline{t}_{SL1,i}, \overline{t}_{SL2,j}, \underline{t}_{WL1,k}, \underline{t}_{WL2,l}]$ generated consistent with the probabilities m_{ijkl}, $Bel(\mathcal{L}_s)$ can be approximated by

$$Bel(\mathcal{L}_s) \cong \sum_{r=1}^{nR} \delta_{Bs}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r, \underline{tWL2}_r]) / nR \cong p(\mathcal{L}_s) \quad (14.24)$$

for $s = 1, 2, 3, 4$ with

$$\delta_{B1}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r, \underline{tWL2}_r]) = \begin{cases} 1 & \text{for } \max\{\overline{tSL1}_r, \overline{tSL2}_r\} < \min\{\underline{tWL1}_r, \underline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \quad (14.25)$$

$$\delta_{B2}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r, \underline{tWL2}_r]) = \begin{cases} 1 & \text{for } \min\{\overline{tSL1}_r, \overline{tSL2}_r\} < \min\{\underline{tWL1}_r, \underline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \quad (14.26)$$

$$\delta_{B3}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r, \underline{tWL2}_r]) = \begin{cases} 1 & \text{for } \max\{\overline{tSL1}_r, \overline{tSL2}_r\} < \max\{\underline{tWL1}_r, \underline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \quad (14.27)$$

and

$$\delta_{B4}([\overline{tSL1}_r, \overline{tSL2}_r, \underline{tWL1}_r, \underline{tWL2}_r]) = \begin{cases} 1 & \text{for } \min\{\overline{tSL1}_r, \overline{tSL2}_r\} < \max\{\underline{tWL1}_r, \underline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \quad (14.28)$$

The estimates for $Bel(\mathcal{L}_s)$ indicated in Eq. (14.24) are illustrated in Eq. (14.20) for the links defined in Table 4.1 and Fig. 4.1.

Similarly, to obtain the desired probability space for checking the calculation of $Pl(\mathcal{L}_s)$ with the representations defined in Eqs. (14.6)--(14.19), (i) the most probability possible (i.e., $m_{WL1,t}(\mathcal{TM}_{WL1,k}) = m_{WL1,k}$ and $m_{WL2,t}(\mathcal{TM}_{WL2,l}) = m_{WL2,l}$) is assigned to $\overline{t}_{WL1,k}$ and $\overline{t}_{WL2,l}$, (ii) the most probability possible (i.e., $m_{SL1,t}(\mathcal{TM}_{SL1,i}) = m_{SL1,i}$ and $m_{SL2,t}(\mathcal{TM}_{SL2,j}) = m_{SL1,j}$) is assigned to $\underline{t}_{SL1,i}$ and $\underline{t}_{SL2,i}$, and (iii) a probability of zero is assigned to every subset of $\mathcal{TM}$ that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}, \overline{t}_{WL2,l}]$. This produces the probability space that has the largest possible probabilities for the sets $\mathcal{L}_s$ for a probability space that is consistent with the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the properties that (i)

$$
\begin{aligned}
&p([\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}, \overline{t}_{WL2,l}]) \\
&= m_{_{SL1,t}}(\mathcal{TM}_{SL1,i}) m_{_{SL2,t}}(\mathcal{TM}_{SL2,j}) m_{_{WL1,k}}(\mathcal{TM}_{WL1,k}) m_{_{WL2,l}}(\mathcal{TM}_{WL2,l}) \\
&= m_{ijkl}
\end{aligned}
\qquad (14.29)
$$

for (i, j, k, l) belonging to the set $\mathcal{I}$ defined in Eq. (14.4) and (ii) any set that does not contain one or more of the vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}, \overline{t}_{WL2,l}]$ has a probability of zero. Then, with a large random sample

$$
[\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r, \overline{tWL2}_r], r = 1, 2, ..., nR, \qquad (14.30)
$$

of size nR from the failure time vectors $[\underline{t}_{SL1,i}, \underline{t}_{SL2,j}, \overline{t}_{WL1,k}, \overline{t}_{WL2,l}]$ generated consistent with the probabilities m_{ijkl}, $Pl(\mathcal{L}_s)$ can be approximated by

$$
Pl(\mathcal{L}_s) \cong \sum_{r=1}^{nR} \delta_{Ps}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r, \overline{tWL2}_r]) / nR \cong p(\mathcal{L}_s) \qquad (14.31)
$$

for $s = 1, 2, 3, 4$ with

$$
\delta_{P1}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r, \overline{tWL2}_r]) = \begin{cases} 1 & \text{for } \max\{\underline{tSL1}_r, \underline{tSL2}_{rr}\} < \min\{\overline{tWL1}_r, \overline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \qquad (14.32)
$$

$$
\delta_{P2}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r, \overline{tWL2}_r]) = \begin{cases} 1 & \text{for } \min\{\underline{tSL1}_r, \underline{tSL2}_r\} < \min\{\overline{tWL1}_r, \overline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \qquad (14.33)
$$

$$
\delta_{P3}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r, \overline{tWL2}_r]) = \begin{cases} 1 & \text{for } \max\{\underline{tSL1}_r, \underline{tSL2}_r\} < \max\{\overline{tWL1}_r, \overline{tWL2}_r\} \\ 0 & \text{otherwise,} \end{cases} \qquad (14.34)
$$

and

$$
\delta_{P4}([\underline{tSL1}_r, \underline{tSL2}_r, \overline{tWL1}_r, \overline{tWL2}_r]) = \begin{cases} 1 & \text{for } \min\{\underline{tSL1}_r, \underline{tSL2}_r\} < \max\{\overline{tWL1}_r, \overline{tWL2}_r\} \\ 0 & \text{otherwise.} \end{cases} \qquad (14.35)
$$

The estimates for $Pl(\mathcal{L}_s)$ indicated in Eq. (14.31) are illustrated in Eq. (14.21) for the links defined in Table 4.1 and Fig. 4.

14.2 Cumulative and Complementary Cumulative Belief and Plausibility for LOAS Occurrence Time

Evidence spaces $(\mathcal{TML}_s, \mathbb{TML}_s, m_{TML,s})$, $s = 1,2,3,4$, associated with the four failure patterns for the time at which LOAS occurs can also be defined and used to obtain CPFs, CBFs, CCPFs and CCBFs for LOAS occurrence time. Construction of the indicated evidence spaces is based on the following definitions for the function TML_s for $s = 1,2,3,4$ that maps the sample space $\mathcal{TM}$ for the evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ defined in Eqs. (14.1)-(14.5) into the sample space $\mathcal{TML}_s$ for the evidence space $(\mathcal{TML}_s, \mathbb{TML}_s, m_{TML,s})$:

$$TML_1(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) = \begin{cases} \max\{t_{SL1}, t_{SL2}\} & \text{for } \max\{t_{SL1}, t_{SL2}\} < \min\{t_{WL1}, t_{WL2}\} \\ \infty & \text{for} \min\{t_{WL1}, t_{WL2}\} \le \max\{t_{SL1}, t_{SL2}\} \end{cases} \quad (14.36)$$

for failure pattern 1 (i.e., failure of all SLs before failure of any WL),

$$TML_2(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) = \begin{cases} \min\{t_{SL1}, t_{SL2}\} & \text{for } \min\{t_{SL1}, t_{SL2}\} < \min\{t_{WL1}, t_{WL2}\} \\ \infty & \text{for} \min\{t_{WL1}, t_{WL2}\} \le \min\{t_{SL1}, t_{SL2}\} \end{cases} \quad (14.37)$$

for failure pattern 2 (i.e., failure of any SL before failure of any WL),

$$TML_3(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) = \begin{cases} \min\{t_{SL1}, t_{SL2}\} & \text{for} \max\{t_{SL1}, t_{SL2}\} < \max\{t_{WL1}, t_{WL2}\} \\ \infty & \text{for} \max\{t_{WL1}, t_{WL2}\} \le \max\{t_{SL1}, t_{SL2}\} \end{cases} \quad (14.38)$$

for failure pattern 3 (i.e., failure of all SLs before failure of all WLs), and

$$TML_4(t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) = \begin{cases} \min\{t_{SL1}, t_{SL2}\} & \text{for} \min\{t_{SL1}, t_{SL2}\} < \max\{t_{WL1}, t_{WL2}\} \\ \infty & \text{for} \max\{t_{WL1}, t_{WL2}\} \le \min\{t_{SL1}, t_{SL2}\} \end{cases} \quad (14.39)$$

for failure pattern 4 (i.e., failure of any SL before failure of all WLs).

In turn, the evidence space $(\mathcal{TML}_s, \mathbb{TML}_s, m_{TML,s})$ is defined by

$$\mathcal{TML}_s = \{t : t = TML_s(\mathbf{t}) \text{ for } \mathbf{t} = (t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) \in \mathcal{TM}\}, \quad (14.40)$$

$$\mathcal{TML}_{s,ijkl} = \{t : t = TML_s(\mathbf{t}) \text{ for } \mathbf{t} = (t_{SL1}, t_{SL2}, t_{WL1}, t_{WL2}) \in \mathcal{TM}_{ijkl}\}, \quad (14.41)$$

$$\mathbb{TML}_s = \{\mathcal{TML}_{s,ijkl} : (i,j,k,l) \in \mathcal{I} = \mathcal{I}_{SL1} \times \mathcal{I}_{SL2} \times \mathcal{I}_{WL1} \times \mathcal{I}_{WL2};\ \text{see Eq. (13.3)}\} \quad (14.42)$$

and

$$m_{TML,s}(\mathcal{TML}_{s,ijkl}) = m_{TM}(\mathcal{TM}_{ijkl}) = m_{ijkl} \quad \text{(see Eq. (13.5))}.$$

Further, the minimum $\underline{t}_{s,ijkl}$ and maximum $\overline{t}_{s,ijkl}$ for $\mathcal{TML}_{s,ijkl}$ are defined by

$$\underline{t}_{1,ijkl} = \begin{cases} \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} & \text{for } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \min\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ \infty & \text{for } \min\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \le \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\}, \end{cases} \tag{14.43}$$

$$\overline{t}_{1,ijkl} = \begin{cases} \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \min\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ \infty & \text{for } \min\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \end{cases} \tag{14.44}$$

for failure pattern 1 (i.e., failure of all SLs before failure of any WL),

$$\underline{t}_{2,ijkl} = \begin{cases} \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} & \text{for } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \min\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ \infty & \text{for } \min\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \le \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\}, \end{cases} \tag{14.45}$$

$$\overline{t}_{2,ijkl} = \begin{cases} \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \min\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ \infty & \text{for } \min\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \end{cases} \tag{14.46}$$

for failure pattern 2 (i.e., failure of any SL before failure of any WL),

$$\underline{t}_{3,ijkl} = \begin{cases} \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} & \text{for } \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \max\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ \infty & \text{for } \max\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \le \max\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\}, \end{cases} \tag{14.47}$$

$$\overline{t}_{3,ijkl} = \begin{cases} \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \max\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ \infty & \text{for } \max\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \le \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \end{cases} \tag{14.48}$$

for failure pattern 3 (i.e., failure of all SLs before failure of all WLs), and

$$\underline{t}_{4,ijkl} = \begin{cases} \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} & \text{for } \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\} < \max\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \\ \infty & \text{for } \max\{\overline{t}_{WL1,k}, \overline{t}_{WL2,l}\} \le \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\}, \end{cases} \tag{14.49}$$

$$\overline{t}_{4,ijkl} = \begin{cases} \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} & \text{for } \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} < \max\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \\ \infty & \text{for } \max\{\underline{t}_{WL1,k}, \underline{t}_{WL2,l}\} \le \min\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\} \end{cases} \tag{14.50}$$

for failure pattern 4 (i.e., failure of any SL before failure of all WLs).

Once the focal element bounds $(\underline{t}_{s,ijkl}, \overline{t}_{s,ijkl})$ are available, CPFs, CBFs, CCPFs and CCBFs for LOAS occurrence time can be obtained as (i) described in Eqs. (2.48) - (2.50) and (ii) illustrated in Fig. 14.1. The indicated figures also contain CDFs and CCDFs for LOAS occurrence time obtained by assigning uniform distributions to the focal elements for link failure temperature as described in conjunction with Eqs. (7.9)-(7.13).

Example results for failure pattern 1 (i.e., failure of all SLs before failure of any WL) are presented in Fig. 14.1.

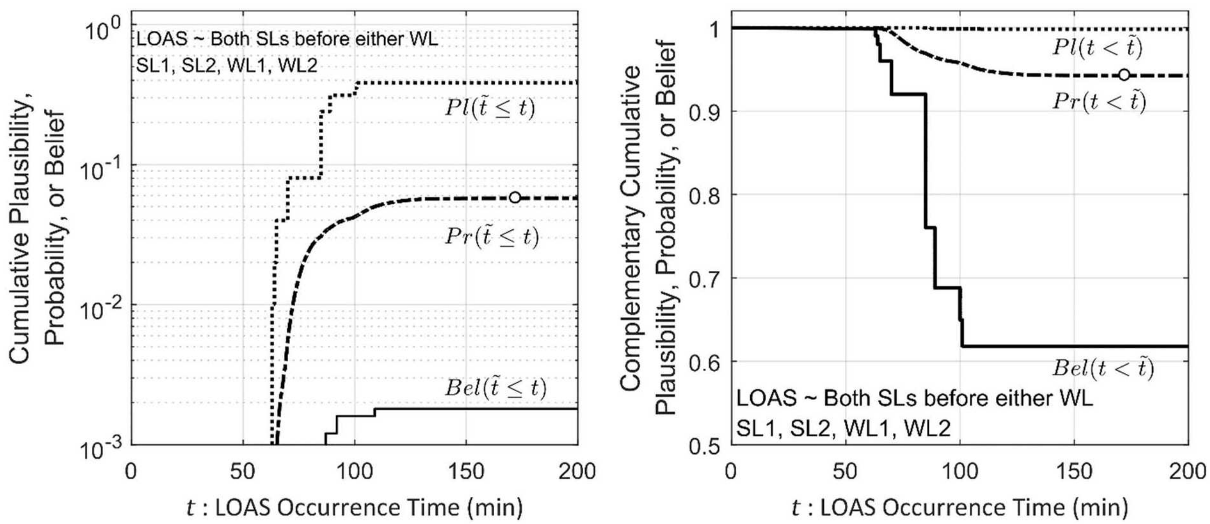


Fig. 14.1 Graphical summary of evidence space $(\mathcal{TML}_1, \mathbb{TML}_1, m_{TML1})$ for time t at which LOAS occurs for (i) a system composed of SL 1, SL 2, WL 1 and WL 2 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of both SLs before failure of either WL: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

The sampling-based procedure used to obtain the CDF and CCDF in Fig. 14.1 also yields values of

$$\underline{t}_{1L} = 62.295 \text{ min and } \overline{t}_{1L} = 172.279 \text{ min} \tag{14.51}$$

for the first time $\underline{t}_{1L}$ and last time $\overline{t}_{2L}$ that LOAS occurs. In turn, combination of $\overline{t}_{2L}$ with the cumulative plausibility and belief results at t = 200 min in Fig. 14.1a provides the analysis outcomes

$$0.382 = Pl(\tilde{t} \le 200) = Pl(\tilde{t} \le \overline{t}_{2L}) = Pl(\tilde{t} \le 172.279) \tag{14.52}$$

$$0.002 = Bel(\tilde{t} \le 200) = Bel(\tilde{t} \le \overline{t}_{2L}) = Bel(\tilde{t} \le 172.279), \tag{14.53}$$

and combination of $\overline{t}_{2L}$ with the complementary cumulative plausibility and belief results at t = 200 min in Fig. 14.1b provides the analysis outcomes

$$0.998 = Pl(200 < \tilde{t}) = Pl(\overline{t}_{2L} < \tilde{t}) = Pl(172.279 < \tilde{t}) = Pl(t = t_\infty) \qquad (14.54)$$

$$0.618 = Bel(200 < \tilde{t}) = Bel(\overline{t}_{2L} < \tilde{t}) = Bel(172.279 < \tilde{t}) = Bel(t = t_\infty). \qquad (14.55)$$

Example results for failure patterns 2, 3 and 4 are presented in Fig. 14.2, Fig. 14.3 and Fig. 14.4. If desired, summaries of the form shown in Eqs. (14.51) -(14.55) for Fig. 14.1 can be defined for Fig. 14.2, Fig. 14.3 and Fig. 14.4.

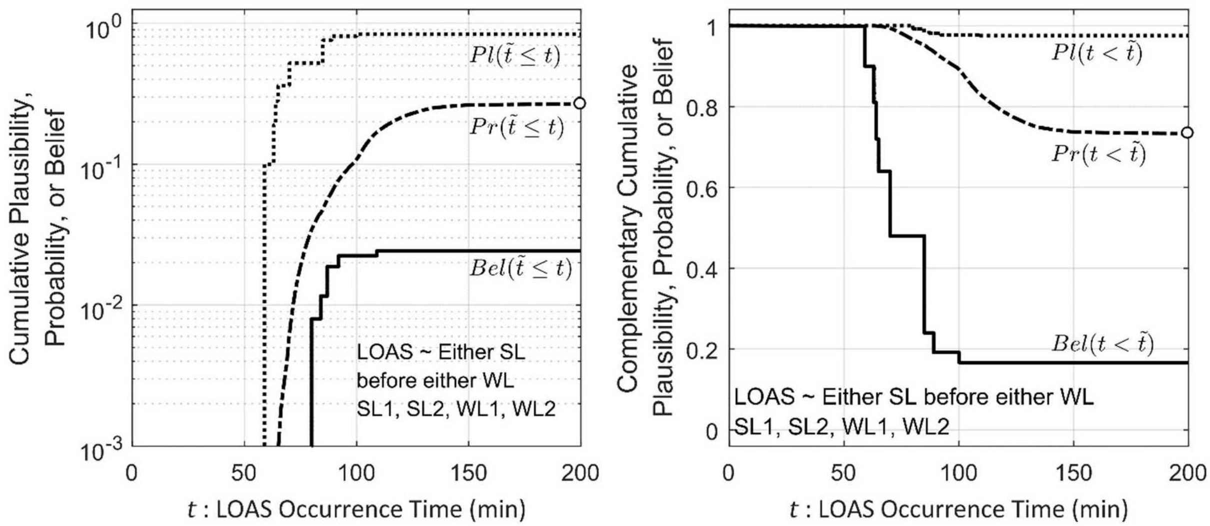


Fig. 14.2 Graphical summary of evidence space $(\mathcal{TML}_2, \mathbb{TML}_2, m_{TML2})$ for time t at which LOAS occurs for (i) a system composed of SL 1, SL 2, WL 1 and WL 2 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of either SL before failure of either WL: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

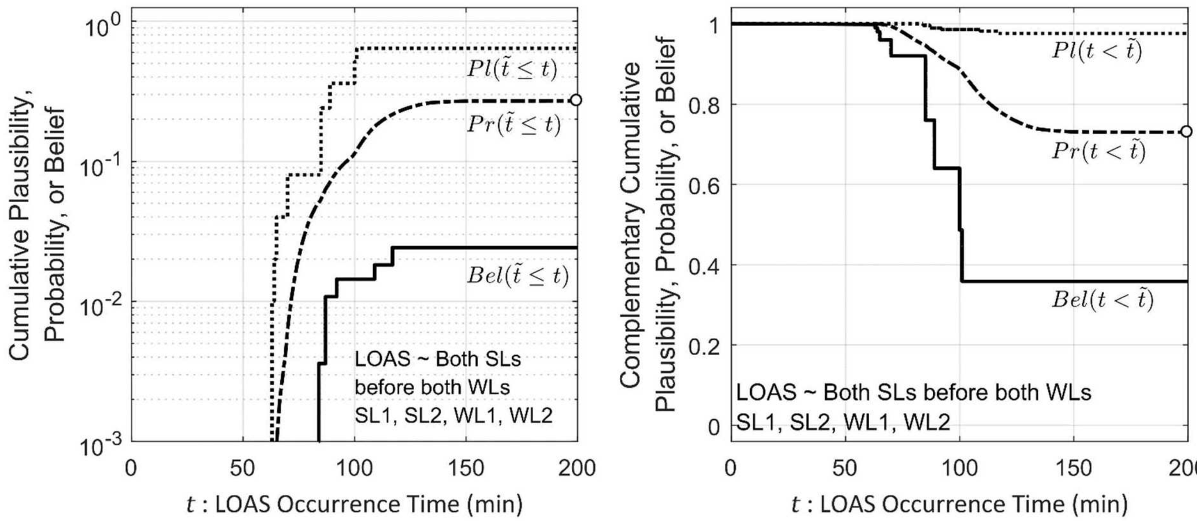


Fig. 14.3 Graphical summary of evidence space $(\mathcal{TML}_3, \mathbb{TML}_3, m_{TML3})$ for time t at which LOAS occurs for (i) a system composed of SL 1, SL 2, WL 1 and WL 2 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of both SLs before failure of both WLs: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

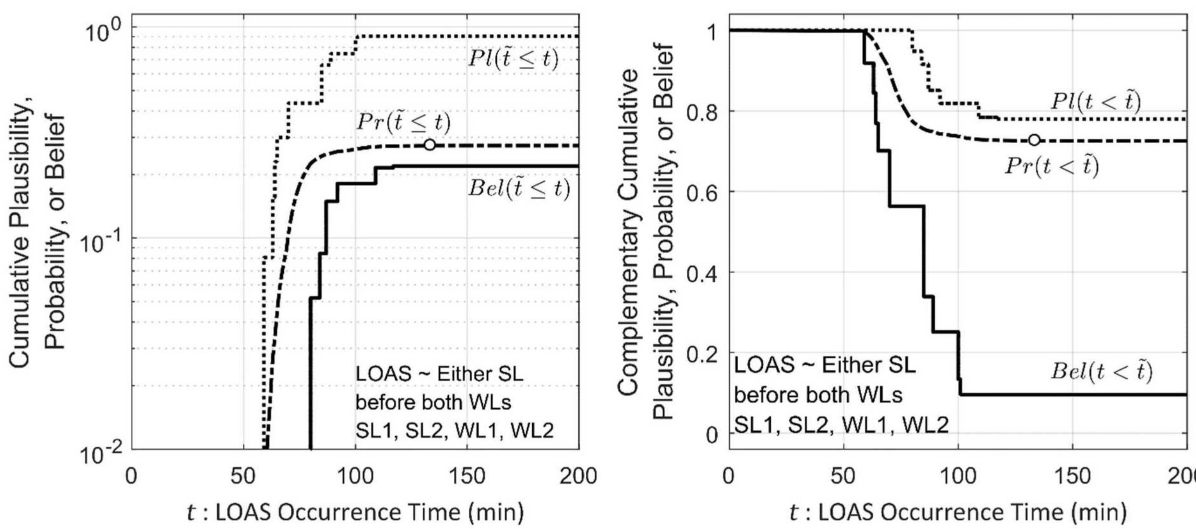


Fig. 14.4 Graphical summary of evidence space $(\mathcal{TML}_4, \mathbb{TML}_4, m_{TML4})$ for time t at which LOAS occurs for (i) a system composed of SL 1, SL 2, WL 1 and WL 2 defined in Sect. 4.1 and (ii) LOAS corresponding to failure of either SL before failure of both WLs: (a) Cumulative plausibility $Pl(\tilde{t} \le t)$, probability $Pr(\tilde{t} \le t)$ and belief $Bel(\tilde{t} \le t)$, and (b) Complementary cumulative plausibility $Pl(t < \tilde{t})$, probability $Pr(t < \tilde{t})$ and belief $Bel(t < \tilde{t})$.

If desired, the evidence spaces developed in Sect. 8 for the time at which a system of two links fails can be used define evidence spaces for failure time margins for the four failure patterns for a 2 SL, 2 WL system. Similarly, the evidence spaces developed in Sect. 9 for the temperature at which a system of two links fails can be used define evidence spaces for failure temperature margins for the four failure patterns for a 2 SL, 2 WL system.

The results in this section generalize to systems with *nSL* SLs and *nWL* WLs in a reasonably straight forward manner.

15. Incorporation of Evidence Spaces for Link Temperature Curves

Up to this point, there has been no consideration of the uncertainty in the link temperature curves that underlie the determination of LOAS. However, this is likely to be a major source of uncertainty in the determination of LOAS in most analyses. Fortunately, if evidence spaces for the link temperature curves for individual links can be obtained, then these evidence spaces can be combined with the failure temperature evidence spaces for the individual links and used to determine the evidence space for the time at which LOAS occurs.

This combination process described below for a link system with

$$nL = nSL + nWL \tag{15.1}$$

links, where nSL is the number of SLs and nWL is the number of WLs.

The following evidence spaces are involved in the incorporation of an evidence space for the link i, i=1, 2, …, nL, temperature curve into the determination of plausibility and belief for the occurrence of LOAS:

(i) An evidence space $(\mathcal{T}_i, \mathbb{T}_i, m_{Ti})$ for possible failure temperatures for link i with $\mathbb{T}_i = \{\mathcal{T}_{i1}, \mathcal{T}_{i2}, ..., \mathcal{T}_{i,nT(i)}\}$ and $m_{Ti}(\mathcal{T}_{ij}) = m_{Tij}$ for $j \in \mathcal{I}_{Ti} = \{1, 2, ..., nT(i)\}$.

(ii) An evidence space $(\mathcal{C}_i, \mathbb{C}_i, m_{Ci})$ for possible link temperature curves for link i with $\mathbb{C}_i = \{\mathcal{C}_{i1}, \mathcal{C}_{i2}, ..., \mathcal{C}_{i,nC(i)}\}$ and $m_{Ci}(\mathcal{C}_{ik}) = m_{Cik}$ for $k \in \mathcal{I}_{Ci} = \{1, 2, ...nC(i)\}$.

(iii) A resultant product evidence space $(\mathcal{P}_i, \mathbb{P}_i, m_{Pi})$ obtained by combining the evidence spaces $(\mathcal{T}_i, \mathbb{T}_i, m_{Ti})$ and $(\mathcal{C}_i, \mathbb{C}_i, m_{Ci})$ with

$$\mathcal{P}_i = \mathcal{T}_i \times \mathcal{C}_i, \tag{15.2}$$

$$\begin{aligned} \mathbb{P}_i = \{\mathcal{P}_{il} : \mathcal{P}_{il} = \mathcal{T}_{ij} \times \mathcal{C}_{ik} \text{ for } (j,k) \in \mathcal{I}_{Ti} \times \mathcal{I}_{Ci} \text{ and} \\ l = (j-1)nC(i) + k \in \{1, 2, ..., nT(i) \times nC(i)\} = \mathcal{I}_{Pi}\}, \end{aligned} \tag{15.3}$$

and

$$\begin{aligned} m_{Pi}(\mathcal{P}_{il}) = m_{Ti}(\mathcal{T}_{ij}) m_{Ci}(\mathcal{C}_{ik}) = m_{Tij} m_{Cik} = m_{Pil} \\ \text{for } (j,k) \in \mathcal{I}_{Ti} \times \mathcal{I}_{Ci} \text{ and } l = (j-1)nC(i) + k. \end{aligned} \tag{15.4}$$

(iv) A resultant evidence space $(\mathcal{TM}_i, \mathbb{TM}_i, m_{TMi})$ for possible failure time for link i constructed from the evidence space $(\mathcal{P}_i, \mathbb{P}_i, m_{Pi})$ with

$$f_i(T,C) = \text{ link failure time for } (T,C) \in \mathcal{P}_i, \tag{15.5}$$

$$\mathcal{TM}_i = \{t : t = f_i(T, C) \text{ for } (T, C) \in \mathcal{P}_i\} \tag{15.6}$$

$$\mathcal{TM}_{il} = \{t : t = f(T, C) \text{ for } (T, C) \in \mathcal{P}_{il}\} \text{ for } l \in \mathcal{I}_{Pi}, \tag{15.7}$$

$$\mathbb{TM}_i = \{\mathcal{TM}_{il} \text{ for } l \in \mathcal{I}_{Pi}\} \tag{15.8}$$

$$m_{TMi}(\mathcal{TM}_{il}) = m_{Pi}(\mathcal{P}_{il}) \text{ for } l \in \mathcal{I}_{Pi} \tag{15.9}$$

and

$$(\underline{t}_{il}, \overline{t}_{il}) = \left(\min(\mathcal{TM}_{il}), \max(\mathcal{TM}_{il})\right). \tag{15.10}$$

(v) A resultant evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for possible link failure times constructed from the failure time evidence spaces $(\mathcal{TM}_i, \mathbb{TM}_i, m_{TMi})$ for the *nL* links that constitute the system under consideration with

$$\mathcal{TM} = \mathcal{TM}_1 \times \mathcal{TM}_2 \times \cdots \times \mathcal{TM}_{nL}, \tag{15.11}$$

$$\begin{aligned} \mathcal{TM}_{\mathbf{l}} &= \mathcal{TM}_{1,l(1)} \times \mathcal{TM}_{2,l(2)} \times \cdots \times \mathcal{TM}_{nL,l(nL)} \\ &\text{for } \mathbf{l} = [l(1), l(2), ..., l(nL)] \in \mathcal{I}_{TM} = \mathcal{I}_{P1} \times \mathcal{I}_{P2} \times \cdots \mathcal{I}_{P,nL}, \end{aligned} \tag{15.12}$$

$$\mathbb{TM} = \{\mathcal{TM}_{\mathbf{l}} \text{ for } \mathbf{l} = [l(1), l(2), ..., l(nL)] \in \mathcal{I}_{TM}\} \tag{15.13}$$

and

$$m_{TM}(\mathcal{TM}_{\mathbf{l}}) = \prod_{i=1}^{nL} m_{TMi}(\mathcal{TM}_{i,l(i)}) \text{ for } \mathbf{l} = [l(1), l(2), ..., l(nL)] \in \mathcal{I}_{TM}. \tag{15.14}$$

The evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ and the associated time intervals $(\underline{t}_{il}, \overline{t}_{il})$ can be used in the determination of plausibility and belief for the occurrence of LOAS in the same manner as the evidence spaces for link failure time are used in Sects. 4, 6 and 13 in the determination of plausibility and belief for the occurrence of LOAS.

Caveat: Development of evidence spaces for link temperature curves may be easy or very difficult depending on the specifics of a particular analysis.

For the links defined in Sect. 4.1, example evidence spaces for link temperature curves can be obtained by defining evidences spaces for the quantities T_∞ and *r* defined in Table 4.1 and then constructing the evidence spaces for the resultant temperature curves.

16. Illustration of Plausibility and Belief for LOAS for WL/SL Systems with SL Subsystems

16.1 WL/SL System with 2 SL Subsystems

The example WL/SL system illustrated in this section has two SL subsystems, with (i) one SL subsystem comprised of SL 1 and SL 2 with subsystem failure corresponding to failure of SL 1 or SL 2, (ii) the other SL subsystem comprised of SL 3 and SL 4 with subsystem failure corresponding to failure of SL 3 or SL 4, (iii) SL system failure corresponding to failure of both SL subsystems, and (iv) LOAS corresponding to SL system failure before failure of WL1. Each link is assumed to be characterized by (i) a continuous time-dependent temperature curve and (ii) an evidence space characterizing the uncertainty in link failure temperature. Each SL subsystem could correspond to different failure locations on the same SL.

The following notation is needed in the development of plausibility and belief for LOAS. The same properties are defined for each of the 5 links (i.e., for SL 1, SL 2, SL 3, SL 4, WL 1). To eliminate unnecessary repetition, these properties will be defined for an arbitrary link L with the understanding that the properties for the 5 links are defined by replacing L in the following definitions by SL1, SL2, SL3, SL4 and WL1.

The following entities are assumed to be known for the notional link L: (i) an evidence space $(\mathcal{T}_L, \mathbb{T}_L, m_L)$ for link failure temperature with *nL* focal elements $\mathcal{T}_{L,1}, \mathcal{T}_{L,2}, ..., \mathcal{T}_{L,nL}$, (ii) a function $T_L(t)$ that defines link temperature as a function of time, and (iii) a corresponding evidence space $(\mathcal{TM}_L, \mathbb{TM}_L, m_{L,t})$ for link failure time constructed from $(\mathcal{T}_L, \mathbb{T}_L, m_L)$ and $T_L(t)$ as discussed in Sect. 3 with

$$\mathcal{TM}_{L,i} = T_L^{-1}(\mathcal{T}_{L,i}) = \left\{t : t = T_T^{-1}(T) = \min\{t : T = T_L(t)\} \text{ for } T \in \mathcal{T}_{L,i}\right\} \tag{16.1}$$

$$m_{L,t}(\mathcal{TM}_{L,i}) = m_L(\mathcal{T}_{L,i}) = m_{L,i} \tag{16.2}$$

$$(\underline{t}_{L,i}, \overline{t}_{L,i}) = \left(\min(\mathcal{TM}_{L,i}), \max(\mathcal{TM}_{L,i})\right) \tag{16.3}$$

for $\mathcal{T}_{L,i} \in \mathbb{T}_L$, $\mathcal{TM}_{L,i} \in \mathbb{TM}_L$ and $i \in \mathcal{I}_L = \{1, 2, ..., nL\}$.

The link failure time evidence spaces for the 5 links can be combined to produce a product evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time as indicated below:

$$\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \mathcal{TM}_{SL3} \times \mathcal{TM}_{SL4} \times \mathcal{TM}_{WL1}, \tag{16.4}$$

$$\mathcal{TM}_{ijklr} = \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \times \mathcal{TM}_{SL3,k} \times \mathcal{TM}_{SL4,l} \times \mathcal{TM}_{WL1,r} \text{ for } (i,j,k,l,r) \in \mathcal{I}, \tag{16.5}$$

$$\mathbb{TM} = \{\mathcal{TM}_{ijklr} : (i,j,k,l,r) \in \mathcal{I}\}, \tag{16.6}$$

and

$$
\begin{aligned}
m_{TM}(\mathcal{TM}_{ijklr}) &= m_{SL1,t}(\mathcal{TM}_{SL1,i})m_{SL2,t}(\mathcal{TM}_{SL2,j})m_{SL3,t}(\mathcal{TM}_{SL3,k}) \\
&\quad \times m_{SL4,t}(\mathcal{TM}_{SL4,l})m_{WL1,t}(\mathcal{TM}_{WL1,r}) \text{ for } (i,j,k,l,r)\in\mathcal{I} \\
&= m_{SL1}(\mathcal{T}_{SL1,i})m_{SL2}(\mathcal{T}_{SL2,j})m_{SL3}(\mathcal{T}_{SL3,k})m_{SL4}(\mathcal{T}_{SL4,l})m_{WL1}(\mathcal{T}_{WL1,r}) \\
&= m_{SL1,i}m_{SL2,j}m_{SL3,k}m_{SL4,l}m_{WL1,r}
\end{aligned} \tag{16.7}
$$

with $\mathcal{I} = \mathcal{I}_{SL1} \times \mathcal{I}_{SL2} \times \mathcal{I}_{SL3} \times \mathcal{I}_{SL4} \times \mathcal{I}_{WL1}$.

Given the link system failure definitions for the WL/SL system under consideration, LOAS occurs for elements of the set

$$
\begin{aligned}
\mathcal{L}_1 = \Big\{ &(t_{SL1,i}t_{SL2,j}t_{SL3,k}t_{SL4,l}t_{WL1,r}) \in \mathcal{TM} \text{ with } \min\{t_{SL1,i}t_{SL2,j}\} < t_{WL1,r} \\
&\text{and } \min\{t_{SL3,k}t_{SL4,l}\} < t_{WL1,r} \Big\}.
\end{aligned} \tag{16.8}
$$

The following additional time definitions are now needed to define plausibility and belief for the set $\mathcal{L}_1$:

$$
\begin{aligned}
\underline{t}_{S1,ij} &= \text{earliest SL subsytem 1 failure time for } \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \\
&= \min\{\underline{t}_{SL1,i}, \underline{t}_{SL2,j}\},
\end{aligned} \tag{16.9}
$$

$$
\begin{aligned}
\underline{t}_{S2,kl} &= \text{earliest SL subsytem 2 failure time for } \mathcal{TM}_{SL3,k} \times \mathcal{TM}_{SL4,l} \\
&= \min\{\underline{t}_{SL3,k}, \underline{t}_{SL4,l}\},
\end{aligned} \tag{16.10}
$$

$$
\begin{aligned}
\overline{t}_{S1,ij} &= \text{last SL subsytem 1 failure time for } \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \\
&= \max\{\overline{t}_{SL1,i}, \overline{t}_{SL2,j}\},
\end{aligned} \tag{16.11}
$$

$$
\begin{aligned}
\overline{t}_{S2,kl} &= \text{last SL subsytem 2 failure time for } \mathcal{TM}_{SL3,k} \times \mathcal{TM}_{SL4,l} \\
&= \max\{\overline{t}_{SL3,k}, \overline{t}_{SL4,l}\}.
\end{aligned} \tag{16.12}
$$

With use of the preceding time definitions, the definition of plausibility and belief for the occurrence of LOAS for the system under consideration is based on the following two indicator functions:

$$\delta_{P1}(\mathcal{TM}_{ijkls}) = \begin{cases} 1 \text{ if } \mathcal{TM}_{ijklr} \cap \mathcal{L}_1 \neq \varnothing \\ 0 \text{ if } \mathcal{TM}_{ijklr} \cap \mathcal{L}_1 = \varnothing \end{cases}$$

$$= \begin{cases} 1 \text{ if } \max\{\underline{t}_{S1,ij}, \underline{t}_{S2,kl}\} < \overline{t}_{WL1,r} \\ 0 \text{ if } \overline{t}_{WL1,r} \leq \max\{\underline{t}_{S1,ij}, \underline{t}_{S2,kl}\} \end{cases} \tag{16.13}$$

and

$$\delta_{B1}(\mathcal{TM}_{ijklr}) = \begin{cases} 1 \text{ if } \mathcal{TM}_{ijklr} \subset \mathcal{L}_1 \\ 0 \text{ if } \mathcal{TM}_{ijklr} \not\subset \mathcal{L}_1 \end{cases}$$

$$= \begin{cases} 1 \text{ if } \max\{\overline{t}_{S1,ij}, \overline{t}_{S2,kl}\} < \underline{t}_{WL1,r} \\ 0 \text{ if } \underline{t}_{WL1,r} \leq \max\{\overline{t}_{S1,ij}, \overline{t}_{S2,kl}\}. \end{cases} \tag{16.14}$$

In turn, plausibility and belief for LOAS are defined by

$$Pl(\mathcal{L}_1) = \sum_{\varnothing \neq \mathcal{TM}_{ijklr} \cap \mathcal{L}_1} m_{TM}(\mathcal{TM}_{ijklr}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nSL3} \sum_{l=1}^{nSL4} \sum_{r=1}^{nWL1} \delta_{P1}(\mathcal{TM}_{ijklr}) m_{ijklr} \tag{16.15}$$

and

$$Bel(\mathcal{L}_1) = \sum_{\mathcal{TM}_{ijklr} \subset \mathcal{L}_1} m_{TM}(\mathcal{TM}_{ijklr}) = \sum_{i=1}^{nSL1} \sum_{j=1}^{nSL2} \sum_{k=1}^{nSL3} \sum_{l=1}^{nSL4} \sum_{r=1}^{nWL1} \delta_{B1}(\mathcal{T}_{ijklr}) m_{ijklr}. \tag{16.16}$$

The preceding representations for $Pl(\mathcal{L}_1)$ and $Bel(\mathcal{L}_1)$ can be (i) evaluated with nested DO loops with an embedded IF statement and (ii) extended to systems with more than 2 SL subsystems and more than 2 SLs in each SL subsystem.

16.2 WL/SL System with 2 SL Systems, Each SL System with 2 SL subsystems and a WL

As a generalization of the example in Sect. 16.1, a more complex WL/SL system is considered with (i) 2 SL systems, (ii) each SL system having 2 SL subsystems with each SL subsystem consisting of 2 SLs, (iii) failure of a SL subsystem corresponding to failure of either of its associated SLs, (iv) failure of a SL system corresponding to failure of either of its subsystems and (v) each SL system having its own WL. Two possibilities for the definition of LOAS are considered: (i) failure of either SL system before failure of its associated WL and (ii) failure of both SL systems before failure of their associated WLs.

With respect to notation, SL system 1 (i.e., S1) involves the following 4 SLs and 1 WL:

$$\text{SL 1, SL 2, SL 3, SL 4, WL 1,} \tag{16.17}$$

with (i) SL 1 and SL 2 comprising subsystem 1 (i.e., S11) of S1 and (ii) SL 3 and SL 4 comprising subsystem 2 (i.e., S12) of S1. Similarly, SL system 2 (i.e., S2) involves the following 4 SLs and 1 WL:

$$\text{SL 5, SL 6, SL 7, SL 8, WL 2,} \tag{16.18}$$

with (i) SL 5 and SL 6 comprising subsystem 1 (i.e., S21) of S2 and (ii) SL 7 and SL 8 comprising subsystem 2 (i.e., S22) of S2. The individual links are assumed to have (i) properties as defined earlier for link L and (ii) associated evidence spaces for link failure time as defined in in Eqs. (4.1)-(4.12), with the individual link names (i.e., $SL1$, $SL2$, …, $WL2$) replacing L in the definitions of link properties.

As shown in Eqs. (16.4)-(16.7) for 5 links, the link failure time evidence spaces for the 10 links SL 1, SL 2,…, WL2 can be combined to produce a product evidence space $(\mathcal{TM}, \mathbb{TM}, m_{TM})$ for link failure time as indicated below:

$$\mathcal{TM} = \mathcal{TM}_{SL1} \times \mathcal{TM}_{SL2} \times \cdots \times \mathcal{TM}_{WL2}, \tag{16.19}$$

$$\mathcal{TM}_{\mathbf{i}} = \mathcal{TM}_{SL1,i} \times \mathcal{TM}_{SL2,j} \times \cdots \times \mathcal{TM}_{WL2,s} \quad \text{for } \mathbf{i} = (i,j,k,l,m,n,p,q,r,s) \in \mathcal{I}, \tag{16.20}$$

$$\mathbb{TM} = \{\mathcal{TM}_{\mathbf{i}} : \mathbf{i} = (i,j,k,l,m,n,p,q,r,s) \in \mathcal{I}\}, \tag{16.21}$$

and

$$\begin{aligned} m_{TM}(\mathcal{TM}_{\mathbf{i}}) &= m_{SL1,t}(\mathcal{TM}_{SL1,i}) m_{SL2,t}(\mathcal{TM}_{SL2,j}) \cdots m_{WL2,t}(\mathcal{TM}_{WL2,s}) \\ &= m_{SL1}(\mathcal{T}_{SL1,i}) m_{SL2}(\mathcal{T}_{SL2,j}) \cdots m_{WL2}(\mathcal{T}_{WL2,s}) \\ &= m_{SL1,i} m_{SL2,j} \cdots m_{WL2,s} \quad \text{for } \mathbf{i} = (i,j,k,l,m,n,p,q,r,s) \in \mathcal{I} \end{aligned} \tag{16.22}$$

with $\mathcal{I} = \mathcal{I}_{SL1} \times \mathcal{I}_{SL2} \times \cdots \times \mathcal{I}_{WL2}$.

Given the link system failure definitions for the WL/SL system under consideration, LOAS occurs for elements of the sets

$$\begin{aligned} \mathcal{L}_2 = \{ &(t_{SL1}, t_{SL2} t_{SL3}, t_{SL4}, t_{SL5}, t_{SL6} t_{SL7}, t_{SL8}, t_{WL1}, t_{WL2}) \in \mathcal{TM} \text{ with} \\ &\min\{t_{SL1}, t_{SL2}, t_{SL3}, t_{SL4}\} < t_{WL1} \text{ or } \min\{t_{SL5}, t_{SL6}, t_{SL7}, t_{SL8}\} < t_{WL2} \} \end{aligned} \tag{16.23}$$

with LOAS corresponding to failure of either SL system before failure of its associated WL and

$$\begin{aligned} \mathcal{L}_3 = \{ &(t_{SL1}, t_{SL2} t_{SL3}, t_{SL4}, t_{SL5}, t_{SL6} t_{SL7}, t_{SL8}, t_{WL1}, t_{WL2}) \in \mathcal{TM} \text{ with} \\ &\min\{t_{SL1}, t_{SL2}, t_{SL3}, t_{SL4}\} < t_{WL1} \text{ and } \min\{t_{SL5}, t_{SL6}, t_{SL7}, t_{SL8}\} < t_{WL2} \} \end{aligned} \tag{16.24}$$

with LOAS corresponding to failure of both SL systems before failure of their associated WLs.

For SL system S1, the times

$$\underline{t}_{S11,ij}, \underline{t}_{S12,kl}, \overline{t}_{S11,ij}, \overline{t}_{S12,kl} \tag{16.25}$$

are defined the same as the times

$$\underline{t}_{S1,ij}, \underline{t}_{S2,kl}, \overline{t}_{S1,ij}, \overline{t}_{S2,kl} \tag{16.26}$$

in Eqs. (16.9)-(16.12) for SL 1, SL 2, SL 3 and SL 4. For SL system S2, the times

$$\underline{t}_{S21,mn}, \underline{t}_{S22,pq}, \overline{t}_{S21,mn}, \overline{t}_{S22,pq} \tag{16.27}$$

are also defined the same as the times in Eqs. (16.9)-(16.12) but for SL 5, SL 6, SL 7 and SL 8.

The following indicator functions are used in the definition of plausibility and belief for $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\begin{aligned}\delta_{P2}(\mathcal{TM}_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{I}) &= \begin{cases} 1 \text{ if } \mathcal{TM}_{\mathbf{i}} \cap \mathcal{L}_2 \neq \varnothing \\ 0 \text{ if } \mathcal{TM}_{\mathbf{i}} \cap \mathcal{L}_2 = \varnothing \end{cases} \\ &= \begin{cases} 1 \text{ if } \min\{\underline{t}_{S11,ij}, \underline{t}_{S12,kl}\} < \overline{t}_{WL1,r} \text{ or } \min\{\underline{t}_{S21,mn}, \underline{t}_{S22,pq}\} < \overline{t}_{WL2,s} \\ 0 \text{ if } \overline{t}_{WL1,r} \leq \min\{\underline{t}_{S11,ij}, \underline{t}_{S12,kl}\} \text{ and } \overline{t}_{WL2,s} \leq \min\{\underline{t}_{S21,mn}, \underline{t}_{S22,pq}\}, \end{cases}\end{aligned} \tag{16.28}$$

$$\begin{aligned}\delta_{P3}(\mathcal{TM}_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{I}) &= \begin{cases} 1 \text{ if } \mathcal{TM}_{\mathbf{i}} \cap \mathcal{L}_3 \neq \varnothing \\ 0 \text{ if } \mathcal{TM}_{\mathbf{i}} \cap \mathcal{L}_3 = \varnothing \end{cases} \\ &= \begin{cases} 1 \text{ if } \min\{\underline{t}_{S11,ij}, \underline{t}_{S12,kl}\} < \overline{t}_{WL1,r} \text{ and } \min\{\underline{t}_{S21,mn}, \underline{t}_{S22,pq}\} < \overline{t}_{WL2,s} \\ 0 \text{ if } \overline{t}_{WL1,r} \leq \min\{\underline{t}_{S11,ij}, \underline{t}_{S12,kl}\} \text{ or } \overline{t}_{WL2,s} \leq \min\{\underline{t}_{S21,mn}, \underline{t}_{S22,pq}\}, \end{cases}\end{aligned} \tag{16.29}$$

$$\begin{aligned}\delta_{B2}(\mathcal{TM}_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{I}) &= \begin{cases} 1 \text{ if } \mathcal{TM}_{\mathbf{i}} \subset \mathcal{L}_2 \\ 0 \text{ if } \mathcal{TM}_{\mathbf{i}} \not\subset \mathcal{L}_2 \end{cases} \\ &= \begin{cases} 1 \text{ if } \min\{\overline{t}_{S11,ij}, \overline{t}_{S12,kl}\} < \underline{t}_{WL1,r} \text{ or } \min\{\overline{t}_{S21,mn}, \overline{t}_{S22,pq}\} < \underline{t}_{WL2,s} \\ 0 \text{ if } \underline{t}_{WL1,r} \leq \min\{\overline{t}_{S11,ij}, \overline{t}_{S12,kl}\} \text{ and } \underline{t}_{WL2,s} \leq \min\{\overline{t}_{S21,mn}, \overline{t}_{S22,pq}\}, \end{cases}\end{aligned} \tag{16.30}$$

$$\delta_{B3}(\mathcal{TM}_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{I}) = \begin{cases} 1 \text{ if } \mathcal{TM}_{\mathbf{i}} \subset \mathcal{L}_3 \\ 0 \text{ if } \mathcal{TM}_{\mathbf{i}} \not\subset \mathcal{L}_3 \end{cases}$$
$$= \begin{cases} 1 \text{ if } \min\{\overline{t}_{S11,ij}, \overline{t}_{S12,kl}\} < \underline{t}_{WL1,r} \text{ and } \min\{\overline{t}_{S21,mn}, \overline{t}_{S22,pq}\} < \underline{t}_{WL2,s} \\ 0 \text{ if } \underline{t}_{WL1,r} \le \min\{\overline{t}_{S11,ij}, \overline{t}_{S12,kl}\} \text{ or } \underline{t}_{WL2,s} \le \min\{\overline{t}_{S21,mn}, \overline{t}_{S22,pq}\}. \end{cases} \quad (16.31)$$

In turn,

$$Pl(\mathcal{L}_s) = \sum_{\varnothing \neq \mathcal{TM}_{\mathbf{i}} \cap \mathcal{L}_s} m_{TM}(\mathcal{TM}_{\mathbf{i}}) = \sum_{\mathbf{i} \in \mathcal{I}} \delta_{Ps}(\mathcal{TM}_{\mathbf{i}}) m_{\mathbf{i}} \quad (16.32)$$

and

$$Bel(\mathcal{L}_s) = \sum_{\mathcal{TM}_{\mathbf{i}} \subset \mathcal{L}_s} m_{TM}(\mathcal{TM}_{\mathbf{i}}) = \sum_{\mathbf{i} \in \mathcal{I}} \delta_{Bs}(\mathcal{TM}_{\mathbf{i}}) m_{\mathbf{i}}. \quad (16.33)$$

for $s = 2, 3$.

If desired, similar results can be obtained for more than 2 SL systems, more than 2 SL subsystems in a SL system, and more than 2 SLs in a SL subsystem. However, increasing system complexity also increases the complexity of the associated notation.

17. Sampling-Based Calculation of Belief and Plausibility

As shown in prior sections, determination of belief and plausibility for results associated with WL/SL systems is reasonably straight forward when the only uncertainties are the failure temperatures for the individual links. However, the calculation of belief and plausibility becomes much more difficult when a potentially complex function maps a large number of epistemically uncertain quantities into an analysis result of interest. As briefly described, the preceding situation often requires the use of sampling-based procedures to estimate belief and plausibility for the analysis results of interest.

The following discussion considers an analysis involving

(i) input quantities $x_1, x_2, ..., x_{nX}$ with corresponding evidence spaces $(\mathcal{X}_i, \mathbb{X}_i, m_i)$ for $i = 1, 2, ..., nX$ characterizing the epistemic uncertainty associated with each x_i,

(ii) the product evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ constructed from the evidence spaces $(\mathcal{X}_i, \mathbb{X}_i, m_i)$, $i = 1, 2, ..., nX$, with focal elements $\mathcal{E}_k, k = 1, 2, ..., nE,$ and each element of $\mathcal{X}$ corresponding to a vector $\mathbf{x} = [x_1, x_2, ..., x_{nX}]$, and

(iii) a function $f(\mathbf{x})$ that maps each $\mathbf{x} \in \mathcal{X}$ into an analysis result y, and

(iv) the set $\mathcal{Y} = \{y : y = f(\mathbf{x}) \text{ for } \mathbf{x} \in \mathcal{X}\}$.

In concept, an evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$ exists for the possible values for y, but this evidence space is difficult to determine when $f(\mathbf{x})$ corresponds to a complex, computationally demanding calculational procedure.

Two computationally similar results are possible for $\mathcal{Y}$: (i) belief and plausibility for a specific subset $\mathcal{S}$ of $\mathcal{Y}$ and (ii) the CBF, CCBF, CPF and CCPF for $\mathcal{Y}$. The estimation of both possibilities starts with a large random or Latin hypercube sample [84; 85]

$$\mathbf{x}_r = [x_{1r}, x_{2r}, ..., x_{nX,r}], r = 1, 2, ..., nR, \tag{17.1}$$

of size nR from $\mathcal{X}$ generated in a manner so that the focal elements for each x_i are well-covered (i.e., sampled). Probably the best way to define the sampling distribution is to (i) define a uniform density function $d_{ij}(x_i)$ on each of the $j = 1, 2, ..., nF_i$ focal elements $\mathcal{X}_{ij}$ for the evidence space $(\mathcal{X}_i, \mathbb{X}_i, m_i)$, (ii) define the density function for sampling from $\mathcal{X}_i$ by

$$d_i(x_i) = \sum_{j=1}^{nF_i} m_i(\mathcal{X}_{ij}) d_{ij}(x_i) \tag{17.2}$$

and (iii) then generate the sample in Eq. (17.1) by sampling each x_i from the distribution defined by the density function defined in Eq. (17.2).

Next, the function $f(\mathbf{x})$ is evaluated for each element of the sample in Eq. (17.1) to create a mapping

$$y_r = f(\mathbf{x}_r), r = 1, 2, ..., nR, \tag{17.3}$$

from $\mathcal{X}$ to $\mathcal{Y}$ that will be used in determining belief and plausibility. However, a very real possibility is that evaluation of $f(\mathbf{x})$ may be too computationally demanding to permit the evaluation of all elements of a sample that is large enough to adequately cover all the focal elements for the x_i's. In this situation, it is necessary to use nonparametric regression or some other appropriate procedure (e.g., [86]) to construct a surrogate model that approximates $f(\mathbf{x})$ and then use this surrogate model in the generation of the mapping in Eq. (17.3).

Use of the mapping in Eq. (17.3) to estimate belief and plausibility for a specific subset $\mathcal{S}$ of $\mathcal{Y}$ is now described. To facilitate this description, $Bel_Y(\mathcal{U})$ and $Pl_Y(\mathcal{U})$ are used to represent belief and plausibility for the evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$ and subsets $\mathcal{U}$ of $\mathcal{Y}$. Similarly, $Bel_X(\mathcal{U})$ and $Pl_X(\mathcal{U})$ are used to represent belief and plausibility for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ and subsets $\mathcal{U}$ of $\mathcal{X}$. The following additional notation is also needed:

$$\hat{\mathcal{S}} = \{y : y = f(\mathbf{x}_r) \in \mathcal{S}\},\ \hat{\mathcal{S}}^c = \{y : y = f(\mathbf{x}_r) \notin \mathcal{S}\} \tag{17.4}$$

and

$$\hat{\mathcal{X}} = f^{-1}(\hat{\mathcal{S}}) = \{\mathbf{x}_r : y = f(\mathbf{x}_r) \in \mathcal{S}\},\ \hat{\mathcal{X}}^c = f^{-1}(\hat{\mathcal{S}}^c) = \{\mathbf{x}_r : y = f(\mathbf{x}_r) \notin \mathcal{S}\}. \tag{17.5}$$

The approximation of $Pl_Y(\mathcal{S})$ with use of the sample in Eq. (17.3) is given by

$$Pl_Y(\mathcal{S}) \cong Pl_Y(\hat{\mathcal{S}}) = Pl_X\left(f^{-1}(\hat{\mathcal{S}})\right) = Pl_X(\hat{\mathcal{X}}) = \sum_{\varnothing \neq \hat{\mathcal{X}} \cap \mathcal{E}_k} m_X(\mathcal{E}_k) \tag{17.6}$$

with the indicated sum over the *nE* focal elements for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$.

The approximation of $Pl_Y(\mathcal{S})$ in Eq. (17.6) is straight forward as a result of plausibility being defined on the basis of set intersection (i.e., $\varnothing \neq \hat{\mathcal{X}} \cap \mathcal{E}_k$). The approximation of $Bel_Y(\mathcal{S})$ is not as straight forward because belief is defined on the basis of subsets (i.e., $\mathcal{E}_k \subseteq \hat{\mathcal{X}}$). Specifically, the relationship $\mathcal{E}_k \subseteq \hat{\mathcal{X}}$ cannot hold with $\mathcal{E}_k$ containing an infinite number of values and $\hat{\mathcal{X}}$ containing a finite number of values. Fortunately, the relationship

$$Bel_Y(\mathcal{S}) = 1 - Pl_Y(\mathcal{S}^c) \tag{17.7}$$

previously stated in Eq. (2.24) provides a solution to this problem by providing a way to convert the approximation of $Bel_Y(\mathcal{S})$ to a problem in the approximation of $Pl_Y(\mathcal{S}^c)$. Specifically, $Bel_Y(\mathcal{S})$ can be approximated by

$$\begin{aligned} Bel_Y(\mathcal{S}) &= 1 - Pl_Y(\mathcal{S}^c) \\ &\cong 1 - Pl_Y(\hat{\mathcal{S}}^c) \\ &= 1 - Pl_X\left(f^{-1}(\hat{\mathcal{S}}^c)\right) \\ &= 1 - Pl_X(\hat{\mathcal{X}}^c) \text{ with } \hat{\mathcal{X}}^c = f^{-1}(\hat{\mathcal{S}}^c) \\ &= 1 - \sum_{\varnothing \neq \hat{\mathcal{X}}^c \cap \mathcal{E}_k} m_X(\mathcal{E}_k) \end{aligned} \tag{17.8}$$

with the indicated sum over the *nE* focal elements for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$.

Formal representations for the CBF, CPF, CCBF and CCPF for the evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$ are defined by the sets of points

$$\mathcal{CBF} = \{[y, Bel(\underline{\mathcal{Y}}_y)] : y \in \mathcal{Y}\}, \quad \mathcal{CPF} = \{[y, Pl(\underline{\mathcal{Y}}_y)] : y \in \mathcal{Y}\} \tag{17.9}$$

and

$$\mathcal{CCBF} = \{[y, Bel(\overline{\mathcal{Y}}_y))] : y \in \mathcal{Y}\}, \quad \mathcal{CCPF} = \{[y, Pl(\overline{\mathcal{Y}}_y)] : y \in \mathcal{Y}\} \tag{17.10}$$

with

$$\underline{\mathcal{Y}}_y = \{\underline{y} : \underline{y} \in \mathcal{Y} \text{ with } \overline{y} \le y\} = \overline{\mathcal{Y}}_y^c \text{ and } \overline{\mathcal{Y}}_y = \{\overline{y} : \overline{y} \in \mathcal{Y} \text{ with } y < \overline{y}\} = \underline{\mathcal{Y}}_y^c. \tag{17.11}$$

Construction of approximations to $\mathcal{CBF}$, $\mathcal{CPF}$, $\mathcal{CCBF}$ and $\mathcal{CCPF}$ requires (i) defining an increasing sequence

$$y_1 < y_2 < \cdots < y_n \tag{17.12}$$

of points from $\mathcal{Y}$ and (ii) approximating $Bel(\underline{\mathcal{Y}}_{y_i})$, $Pl(\underline{\mathcal{Y}}_{y_i})$, $Bel(\overline{\mathcal{Y}}_{y_i})$ and $Pl(\overline{\mathcal{Y}}_{y_i})$ for $i = 1, 2, ..., n$ as indicated in Eqs. (17.6) and (17.8). In turn, the following approximations to $\mathcal{CBF}$, $\mathcal{CPF}$, $\mathcal{CCBF}$ and $\mathcal{CCPF}$ result:

$$\mathcal{CBF} \cong \{[y_i, Bel(\underline{\mathcal{Y}}_{y_i})] : i = 1, 2, ..., n\}, \quad \mathcal{CPF} \cong \{[y_i, Pl(\underline{\mathcal{Y}}_{y_i})] : i = 1, 2, ..., n\} \tag{17.13}$$

and

$$\mathcal{CCBF} \cong \{[y_i, Bel(\bar{\mathcal{Y}}_{y_i})]: i = 1,2,...,n\}, \quad \mathcal{CCPF} \cong \{[y_i, Pl(\bar{\mathcal{Y}}_{y_i})]: i = 1,2,...,n\}. \quad (17.14)$$

Examples of sampling-based approximations to belief and plausibility are presented in Refs. [1; 87-89]. Sampling-based analyses of the type summarized in this section can become computationally impractical when the product evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ has a very large number of focal elements. For example, if $(\mathcal{X}, \mathbb{X}, m_X)$ is constructed from 10 evidence spaces and each of these evidence spaces has 10 focal elements, then $(\mathcal{X}, \mathbb{X}, m_X)$ has $nE = 10^{10}$ focal elements. This number of focal elements makes the evaluation of the summations in Eqs. (17.6) and (17.8) impractical. Refs. [88] and [89] discuss computational strategies to deal with this problem.

18. References

1. Helton J.C., J.D. Johnson and W.L. Oberkampf. 2004. An Exploration of Alternative Approaches to the Representation of Uncertainty in Model Predictions. *Reliability Engineering and System Safety* 85:39-71.
2. Wagner R.L. 2003. Science, Uncertainty and Risk: The Problem of Complex Phenomena. *APS News* 12:8.
3. Oberkampf W.L., S.M. DeLand, B.M. Rutherford, K.V. Diegert and K.F. Alvin. 2002. Error and Uncertainty in Modeling and Simulation. *Reliability Engineering and System Safety* 75:333-357.
4. NCRP (National Council on Radiation Protection and Measurements). 1996. A Guide for Uncertainty Analysis in Dose and Risk Assessments Related to Environmental Contamination, National Council on Radiation Protection and Measurements, NCRP Commentary No. 14. Bethesda, MD.
5. NAS/NRC (National Academy of Science/National Research Council). 2008. *Evaluation of Quantification of Margins and Uncertainties for Assessing and Certifying the Reliability of the Nuclear Stockpile*. Washington, DC: National Academy Press.
6. NRC (National Research Council). 1994. Science and Judgment in Risk Assessment, National Academy Press, Washington, DC.
7. NRC (National Research Council). 1993. *Issues in Risk Assessment*. Washington, DC: National Academy Press.
8. U.S. EPA (U.S. Environmental Protection Agency). 1993. An SAB Report: Multi-Media Risk Assessment for Radon, Review of Uncertainty Analysis of Risks Associated with Exposure to Radon. *Rpt. EPA-SAB-RAC-93-014*, U.S. Environmental Protection Agency, Washington, DC.
9. IAEA (International Atomic Energy Agency). 1989. Evaluating the Reliability of Predictions Made Using Environmental Transfer Models. *Rpt. Safety Series No. 100*, International Atomic Energy Agency, Vienna.
10. Beck M.B. 1987. Water-Quality Modeling: A Review of the Analysis of Uncertainty. *Water Resources Research* 23:1393-1442.
11. Helton J.C. and D.E. Burmaster. 1996. Guest Editorial: Treatment of Aleatory and Epistemic Uncertainty in Performance Assessments for Complex Systems. *Reliability Engineering and System Safety* 54:91-94.
12. Fine T.L. 1973. Theories of Probability: An Examination of Foundations, Academic Press, New York, NY.
13. Hacking I. 1975. *The Emergence of Probability: A Philosophical Study of Early Ideas About Probability, Induction and Statistical Inference*. London; New York: Cambridge University Press.
14. Weatherford R. 1982. Philosophical Foundations of Probability Theory, Routledge & Kegan Paul, London, Boston.
15. Stigler S.M. 1986. The History of Statistics: The Measurement of Uncertainty Before 1900, Harvard University Press, Cambridge, MA.
16. Howson C. and P. Urbach. 1993. *Scientific Reasoning: The Bayesian Approach*. Chicago, IL: Open Court.
17. Hacking I. 2001. An Introduction to Probability and Inductive Logic, Cambridge University Press, Cambridge, New York.

18. Parry G.W. and P.W. Winter. 1981. Characterization and Evaluation of Uncertainty in Probabilistic Risk Analysis. *Nuclear Safety* 22:28-42.
19. Apostolakis G. 1990. The Concept of Probability in Safety Assessments of Technological Systems. *Science* 250:1359-1364.
20. Dempster A.P. 1967. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Statistics* 38:325-339.
21. Dempster A.P. 1967. Upper and Lower Probability Inferences Based on a Sample from a Finite Univariate Population. *Biometrika* 54:515-528.
22. Dempster A.P. 1968. A Generalization of Bayesian Inference. *Journal of The Royal Statistical Society, Series B* 30:205-247.
23. Shafer G. 1976. A Mathematical Theory of Evidence, Princeton Univ. Press, Princeton, NJ.
24. Wasserman L. 1988. Belief Functions and Likelihood, Technical Report 420, Department of Statistics, Carnegie-Mellon University, Pittsburgh, PA.
25. Wasserman L.A. 1990. Belief Functions and Statistical Inference. *The Canadian Journal of Statistics* 18:183-196.
26. Guan J.W. and D.A. Bell. 1991. Evidence Theory and Its Applications, North-Holland, Vols. 1, 2. New York, NY.
27. Halpern J.Y. and R. Fagin. 1992. Two Views of Belief: Belief as Generalized Probability and Belief as Evidence. *Artificial Intelligence* 54:275-317.
28. Yager R.R., J. Kacprzyk and M. Fedrizzi (eds). 1994. *Advances in the Dempster-Shafer Theory of Evidence*. New York, NY: Wiley.
29. Kramosil I. 2001. Probabilistic Analysis of Belief Functions, Kluwer Academic/Plenum Publishers, New York.
30. Zadeh L.A. 1978. Fuzzy Sets as a Basis for a Theory of Possibility. *Fuzzy Sets and Systems* 1:3-28.
31. Dubois D. and H. Prade. 1998. Possibility Theory: Qualitative and Quantitative Aspects. In *Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol. 1*, ed. DM Grabbay, P Smets, pp. 169-226. Boston, MA: Kluwer Academic Publishers.
32. Dubois D. and H. Prade. 2006. Possibility Theory and Its Applications: A Retrospective and Prospective View. *CISM Courses and Lectures* 482:89-109.
33. Dubois D. 2006. Possibility Theory and Statistical Reasoning. *Computational Statistics & Data Analysis* 51:47-69.
34. Dubois D. and H. Prade. 1988. Possibility Theory: An Approach to Computerized Processing of Uncertainty, Plenum, New York, NY.
35. deCooman G., D. Ruan and E.E. Kerre, eds. 1995. *Foundations and Applications of Possibility Theory, Proceedings of FAPT '95, Ghent, Belgium, December 13-15, 1995*. River Edge, NJ: World Scientific Publishing Co.
36. deCooman G. 1997. Possibility Theory Part I: Measure- and Integral-theoretic Groundwork; Part II: Conditional Possibility; Part III: Possibilistic Independence. *International Journal of General Systems* 25:291-371.
37. Moore R.E. 1966. Interval Analysis, Prentice Hall, Englewood Cliffs, NJ.
38. Moore R.E. 1979. Methods and Applications of Interval Analysis, Society for Industrial and Applied Mathematics, Philadelphia, PA.
39. Neumaier A. 1990. *Interval Methods for Systems of Equations*. Cambridge: Cambridge University Press.

40. Hammer R., M. Hocks, U. Kulisch and D. Ratz. 1993. Numerical Toolbox for Verified Computing. I. Basic Numerical Problems, Springer Verlag, New York, NY.
41. Kearfott R.B. and V. Kreinovich. 1996. Applications of Interval Computations, Kluwer Academic Publishers, Boston, MA.
42. Jaulin L., M. Kieffer, O. Didrit and E. Walter. 2001. *Applied Interval Analysis*. New York, NY: Springer-Verlag.
43. Helton J.C. and J.D. Johnson. 2011. Quantification of Margins and Uncertainties: Alternative Representations of Epistemic Uncertainty. *Reliability Engineering and System Safety* 96:1034-1052.
44. Dubois D. and H. Prade. 1989. Fuzzy Sets, Probability and Measurement. *European Journal of Operational Research* 40:135-154.
45. Wu J.S., G.E. Apostolakis and D. Okrent. 1990. Uncertainty in System Analysis: Probabilistic Versus Nonprobabilistic Theories. *Reliability Engineering and System Safety* 30:163-181.
46. Smets P. 1998. Probability, Possibility, Belief: Which and Where? In *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, ed. DM Grabbay, P Smets, pp. 1-24. Boston, MA: Kluwer Academic Publishers.
47. Dubois D., H.T. Nguyen and H. Prade. 2000. Possibility Theory, Probability Theory and Fuzzy Sets: Misunderstandings, Bridges and Gaps. In *Fundamentals of Fuzzy Sets*, ed. D Dubois, H Prade, pp. 343-438. Boston, MA: Kluwer Academic Publishers.
48. Klir G.J. 2000. Measures of Uncertainty and Information. In *Fundamentals of Fuzzy Sets*, ed. D Dubois, H Prade:439-457. Boston, MA: Kluwer Academic Publishers. Number of 439-457 pp.
49. Ross T.J., J.M. Booker and W.J. Parkinson (eds.). 2002. *Fuzzy Logic and Probability Applications: Bridging the Gap*. Philadelphia, PA: Society for Industrial and Applied Mathematics.
50. Helton J.C. 1997. Uncertainty and Sensitivity Analysis in the Presence of Stochastic and Subjective Uncertainty. *Journal of Statistical Computation and Simulation* 57:3-76.
51. Paté-Cornell M.E. 1996. Uncertainties in Risk Analysis: Six Levels of Treatment. *Reliability Engineering and System Safety* 54:95-111.
52. Winkler R.L. 1996. Uncertainty in Probabilistic Risk Assessment. *Reliability Engineering and System Safety* 54:127-132.
53. Hofer E. 1996. When to Separate Uncertainties and When Not to Separate. *Reliability Engineering and System Safety* 54:113-118.
54. Parry G.W. 1988. On the Meaning of Probability in Probabilistic Safety Assessment. *Reliability Engineering and System Safety* 23:309-314.
55. Hoffman F.O. and J.S. Hammonds. 1994. Propagation of Uncertainty in Risk Assessments: The Need to Distinguish Between Uncertainty Due to Lack of Knowledge and Uncertainty Due to Variability. *Risk Analysis* 14:707-712.
56. Helton J.C. 1994. Treatment of Uncertainty in Performance Assessments for Complex Systems. *Risk Analysis* 14:483-511.
57. Apostolakis G. 1999. The Distinction between Aleatory and Epistemic Uncertainties is Important: An Example from the Inclusion of Aging Effects into PSA In *Proceedings of the International Topical Meeting on Probabilistic Safety Assessment, PSA '99: Risk Informed and Performance-Based Regulation in the New Millennium* 1:135-142. La Grange Park, IL: American Nuclear Society. Number of 135-142 pp.

58. Der Kiureghian A. and O. Ditlevsen. 2009. Aleatory or Epistemic? Does It Matter? . *Structural Safety* 31:105-112.
59. U.S. NRC (U.S. Nuclear Regulatory Commission). 1990-1991. Severe Accident Risks: An Assessment for Five U.S. Nuclear Power Plants. *Rpt. NUREG-1150, Vols. 1-3*, U.S. Nuclear Regulatory Commission, Office of Nuclear Regulatory Research, Division of Systems Research, Washington, DC.
60. Breeding R.J., J.C. Helton, E.D. Gorham and F.T. Harper. 1992. Summary Description of the Methods Used in the Probabilistic Risk Assessments for NUREG-1150. *Nuclear Engineering and Design* 135:1-27.
61. Helton J.C. and R.J. Breeding. 1993. Calculation of Reactor Accident Safety Goals. *Reliability Engineering and System Safety* 39:129-158.
62. Helton J.C., M.G. Marietta (eds). 2000. Special Issue: The 1996 Performance Assessment for the Waste Isolation Pilot Plant. *Reliability Engineering and System Safety* 69:1-451.
63. Helton J.C., C.W. Hansen, P.N. Swift (eds). 2014. Special Issue: Performance Assessment for the Proposed High-Level Radioactive Waste Repository at Yucca Mountain, Nevada. *Reliability Engineering and System Safety* 122:1-456.
64. Plummer D.W. and W.H. Greenwood. 1998. The History of Nuclear Weapon Safety Devices. In *AIAA/ASME/SAE/ASEE 34th Joint Propulsion Conference and Exhibit, Cleveland, OH, July 13-15, 1998*. Reston, VA: American Institute of Aeronautics and Astronautics.
65. Caldwell M. and P.E. D'Antonio. 1998. A Study of Using Electronics for Nuclear Weapon Detonation Safety. In *AIAA/ASME/SAE/ASEE 34th Joint Propulsion Conference and Exhibit, Cleveland, OH, July 13-15, 1998*. Reston, VA: American Institute of Aeronautics and Astronautics.
66. Winter W.L., J.M. Covan and L.J. Dalton. 1998. Passive Safety in High-Consequence Systems. *Computer* 31:35-37.
67. D'Antonio P.E. 1998. Surety Principles Development and Integration for Nuclear Weapons. In *High Consequence Operations Safety Symposium II, SAND98-1557*, ed. D Isbell, pp. 141-149. Albuquerque, NM: Sandia National Laboratories.
68. Demmie P.N. 1998. A First-Principle Based Approach to Quantitative Assessment of Nuclear-Detonation Safety. In *High Consequence Operations Safety Symposium II, SAND98-1557*, ed. D Isbell, pp. 325-341. Albuquerque, NM: Sandia National Laboratories.
69. Jones M. 1998. The Role of Microelectronics and Software in a Very High Consequence System. In *High Consequence Operations Safety Symposium II, SAND98-1557*, ed. D Isbell, pp. 411-429. Albuquerque, NM: Sandia National Laboratories.
70. Harris J.W. and H. Stocker. 1998. *Handbook of Mathematics and Computational Science*. New York, NY: Springer-Verlag.
71. Helton J.C., J.D. Johnson and W.L. Oberkampf. 2007. Verification of the Calculation of Probability of Loss of Assured Safety in Temperature Dependent Systems with Multiple Weak and Strong Links. *Reliability Engineering and System Safety* 92:1363-1373.
72. Helton J.C., J.D. Johnson and W.L. Oberkampf. 2007. Verification Test Problems for the Calculation of Probability of Loss of Assured Safety in Temperature Dependent Systems with Multiple Weak and Strong Links. *Reliability Engineering and System Safety* 92:1374-1387.

73. Oberkampf W.L. and C.J. Roy. 2010. *Verification and Validation in Scientific Computing* New York, NY: Cambridge University Press.
74. Roache P.J. 2009. *Fundamentals of Verification and Validation*. Socorro, NM: Hermosa Publishers.
75. American Society of Mechanical Engineers. 2006. Guide for Verification and Validation in Computational Solid Mechanics. *Rpt. ASME Standard V&V 10-2006*, American Society of Mechanical Engineers, New York, NY.
76. Trucano T.G., L.P. Swiler, T. Igusa, W.L. Oberkampf and M. Pilch. 2006. Calibration, Validation, and Sensitivity Analysis: What's What. *Reliability Engineering and System Safety* 91:1331-1357.
77. Babuska I. and J.T. Oden. 2004. Verification and Validation in Computational Engineering and Science: Basic Concepts. *Computer Methods in Applied Mechanics and Engineering* 193:4057-4066.
78. Oberkampf W.L., T.G. Trucano and C. Hirsch. 2004. Verification, Validation, and Predictive Capability in Computational Engineering and Physics. *Applied Mechanics Review* 57:345-384.
79. Roache P.J. 2004. Building PDE Codes to be Verifiable and Validatable. *Computing in Science & Engineering* 6:30-38.
80. Knupp P. and K. Salari. 2002. *Verification of Computer Codes in Computational Science and Engineering*. Boca Raton, FL: Chapman and Hall/CRC.
81. Oberkampf W.L. and T.G. Trucano. 2002. Verification and Validation in Computational Fluid Dynamics. *Progress in Aerospace Sciences* 38:209-272.
82. American Institute of Aeronautics and Astronautics. 1998. Guide for the Verification and Validation of Computational Fluid Dynamics Simulations. *Rpt. AIAA G-077-1998*, American Institute of Aeronautics and Astronautics, Reston, VA.
83. Helton J.C., M. Pilch and C.J. Sallaberry. 2014. Probability of Loss of Assured Safety in Systems with Multiple Time-Dependent Failure Modes: Representations with Aleatory and Epistemic Uncertainty. *Reliability Engineering and System Safety* 124:171-200.
84. McKay M.D., R.J. Beckman and W.J. Conover. 1979. A Comparison of Three Methods for Selecting Values of Input Variables in the Analysis of Output from a Computer Code. *Technometrics* 21:239-245.
85. Helton J.C. and F.J. Davis. 2003. Latin Hypercube Sampling and the Propagation of Uncertainty in Analyses of Complex Systems. *Reliability Engineering and System Safety* 81:23-69.
86. Storlie C.B., L.P. Swiler, J.C. Helton and C.J. Sallaberry. 2009. Implementation and Evaluation of Nonparametric Regression Procedures for Sensitivity Analysis of Computationally Demanding Models. *Reliability Engineering and System Safety* 94:1735-1763.
87. Helton J.C., W.L. Oberkampf and J.D. Johnson. 2005. Competing Failure Risk Analysis Using Evidence Theory. *Risk Analysis* 25:973-995.
88. Helton J.C., J.D. Johnson, W.L. Oberkampf and C.J. Sallaberry. 2006. Sensitivity Analysis in Conjunction with Evidence Theory Representations of Epistemic Uncertainty. *Reliability Engineering and System Safety* 91:1414-1434.
89. Helton J.C., J.D. Johnson, W.L. Oberkampf and C.B. Storlie. 2007. A Sampling-Based Computational Strategy for the Representation of Epistemic Uncertainty in Model

Predictions with Evidence Theory. *Computational Methods in Applied Mechanics and Engineering* 196:3980-3998.

Distribution

1 W.L. Oberkampf
5112 Hidden Springs Trail
Georgetown, TX 78633
wloconsulting@gmail.com

1 C.J. Sallaberry
7461 Davis Road
Hilliard, OH 43026
CSallaberry@emc-sq.com

1	MS0152	W.E. Gilmore	9131	wegilmo@sandia.gov
1	MS0405	J.W. Glover	9432	jwglove@sandia.gov
1	MS0405	J.L. Darby	9435	jldarby@sandia.gov
	(johnldarbycomo@gmail.com after Sept 11, 2019)			
1	MS0405	N.B. Hubbard	9432	nbhubba@sandia.gov
1	MS0405	M.F. Pasik	9432	mfpasik@sandia.gov
1	MS0405	S. Moran	9435	smoran@sandia.gov
1	MS0405	M.R. Winter	9435	mrwinte@sandia.gov
1	MS0405	T.E. Cordova	9433	tecordo@sandia.gov
1	MS0404	A.L. Moy	9462	almoy@sandia.gov
1	MS0405	K.E. Mabey	9435	kemabey@sandia.gov
1	MS0429	J.T. Bond	2430	jtbond@sandia.gov
1	MS0434	T.N. Hinnerichs	9412	tnhinne@sandia.gov
1	MS0492	J.D. Brewer	9411	jdbrewe@sandia.gov
1	MS0492	T.D. Brown	9411	tdbrown@sandia.gov
1	MS0492	T.W. Simmermacher	9411	twsimme@sandia.gov
1	MS0635	I. Brito	8254	ibrito@sandia.gov
1	MS0748	J.C. Helton	1514	jchelto@sandia.gov
1	MS0748	D.M. Brooks	8853	dbrooks@sandia.gov
1	MS0757	G.D. Wyss	5881	gdwyss@sandia.gov
1	MS0781	K.S. Johns	6652	ksjohns@sandia.gov
1	MS0789	M.K. Snell	6835	mksnell@sandia.gov
1	MS0825	S.P. Burns	1514	spburns@sandia.gov
1	MS0828	K.J. Dowding	1544	kjdowdi@sandia.gov
1	MS0828	M. Pilch	1544	mpilch@sandia.gov
1	MS0828	V.J. Romero	1544	vjromer@sandia.gov
1	MS0828	B.B. Schroeder	1544	bbschro@sandia.gov
1	MS0828	H. Silva	1514	hrsilva@sandia.gov
1	MS0828	G.E. Orient	1544	georien@sandia.gov
1	MS0829	N.N. Brown	9415	nisbrown@sandia.gov
1	MS0829	J.T. Newcomer	9136	jtnewco@sandia.gov
1	MS0829	L. Hund	9136	lbhund@sandia.gov
1	MS0836	N.D. Francis	1514	ndfranc@sandia.gov
1	MS0836	J.W. Shelton	1514	jwshelt@sandia.gov

1	MS1318	J.R. Stewart	1440	jrstewart@sandia.gov
1	MS1318	M.S. Eldred	1463	mseldre@sandia.gov
1	MS1318	L.P. Swiler	1463	lpswile@sandia.gov
1	MS9042	T.A. Johnson	8751	tajohns@sandia.gov
1	MS0899	Technical Library	9536 (electronic copy)	