Physics of Plasmas

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Cite as: Phys. Plasmas **26**, 022506 (2019); https://doi.org/10.1063/1.5080271
Submitted: 07 November 2018 . Accepted: 16 January 2019 . Published Online: 05 February 2019

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Physics of Plasmas ARTICLE scitation.org/journal/php

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Submitted: 7 November 2018 · Accepted: 16 January 2019 ·
Published Online: 5 February 2019

Wenlong Huang,[1] Haijun Ren,[2,3,a)] and X. Q. Xu[3]

AFFILIATIONS

[1]School of Computer Science and Technology, Anhui University of Technology, Ma'anshan, Anhui 243002, China
[2]CAS Key Laboratory of Geospace Environment and Department of Engineering and Applied Physics, University of Science and Technology of China, Hefei, Anhui 230026, China
[3]Lawrence Livermore National Laboratory, Livermore, California 94550, USA

[a)]E-mail: hjren@ustc.edu.cn

ABSTRACT

The geodesic acoustic mode (GAM) is analytically investigated by taking into account the finite-orbit-width (FOW) resonance effect to the second order and the finite β effect. The general dispersion relation is derived from the gyro-kinetic equations in the presence of nonzero $\delta A_\parallel$, the parallel component of the perturbed magnetic vector potential. Transparent and concise expressions for the GAM frequency and Landau damping rate in the presence of the second order FOW effect and finite β effect are first presented. It is clearly shown that the $m = \pm 2$ harmonics dominant $\delta A_\parallel$ and the kinetic expression of $\delta A_\parallel$ have the same form as the fluid one. For the real frequency, the electromagnetic effect introduces a term on the order of $q^2\beta$, which is comparable to the second order electrostatic terms, namely, the terms introduced by the second order FOW resonance effect. While for the collisionless damping rate, $\delta A_\parallel$ does not directly introduce β–dependent terms, but affects the damping rate via modifying the real frequency. Besides, our analytical result shows good agreement with the numerical examinations.

I. INTRODUCTION

The geodesic acoustic mode (GAM) naturally exists in tokamak plasmas.[1,2] It plays an important role in moderating the plasma turbulence and turbulent transport by affecting the radial electric field.[3–5] Generally, GAM is treated as a sort of electrostatic fluctuations with a finite radial wavenumber and zero poloidal and toroidal wavenumber. As the high frequency branch of zonal flows, GAM has been extensively studied by experimental observations (see, for example, Refs. 6–8) and theoretical analyses[9,10] as well as numerical simulation.[11] For example, Storelli *et al.* reported comprehensive comparisons of GAM characteristics and dynamics between Tore Supra experiments and gyrokinetic simulations.[12]

To identify GAM in the experiments or simulations, three characteristics are often unitized. First, there are toroidal and poloidal symmetrical structures $m = 0$ and $n = 0$ for the radial electric field, or namely, the electrostatic potential. Meanwhile, the perturbed density and pressure has sine structure with $m = 1$. Finally, a typical frequency of GAM performed in the ideal magnetohydrodynamic (MHD) model is $\omega^2_{\mathrm{MHD}} = \frac{5}{6}(1+\tau)(2+q^{-2})v^2_{\mathrm{Ti}}/R^2$, where R is the major radius of tokamaks, q is the safety factor, $v_{\mathrm{Ti}} = \sqrt{2T_i/m_i}$ is the ion thermal velocity and τ denotes T_e/T_i with the ion mass m_i, electron and ion temperatures T_e and T_i. The adiabatic index has been assumed to be 5/3. This dispersion relation is justified in a low-β plasma, where β is the ratio of plasma thermal pressure to the magnetic pressure defined as $\gamma p_0\mu_0/B^2$. Or else, β will modify the GAM frequency.[13,14] On the other hand, the kinetic theory predicts $\omega^2_{\mathrm{K}} = (7/4+\tau)v^2_{\mathrm{Ti}}/R^2(1+\mathcal{O}(q^{-2}))$.[15,16] It has been fruitfully illustrated that such a discrepancy between the fluid and kinetic results can be eliminated by taking into account the anisotropic ion parallel viscosity in the two-fluid equations,[17–20] the 16-momentum equations,[21] and a reduced two-fluid mode combining Chew-Goldberger-Low (CGL)-like equations and the effect of electron dynamics.[22]

In recent years, the magnetic perturbations associated with GAM have attracted much attention. Using drift kinetic equations, Zhou investigated the magnetic perturbations of GAM and found that a perpendicular magnetic perturbation with poloidal

wave number $m=2$ was generated due to the $m=2$ parallel return current.[23] Such a poloidal structure of the magnetic field was also predicted theoretically by Wahlberg in a study of the global structure of the corresponding eigenmode within ideal MHD,[24] and then discussed in detail in Ref. 25. Wang *et al.* (hereinafter W2011) numerically evaluated the structure of magnetic field perturbation of the kinetic GAM by considering the finite β effect.[14] They also found that the lowest order magnetic perturbation of the kinetic GAM was the second harmonic and the next was the first harmonic. Experimental observation and numerical simulation confirmed the structure of δB_θ.[26] Such magnetic GAM structures were also directly observed in the JT-60U tokamak device[27] and in the TCV tokamak.[28] A comprehensive review of the magnetic perturbations of GAM in the ideal MHD framework was presented in an excellent paper.[29] In recent years, theorists found that the electromagnetic effect not only can introduce a second order modification of the GAM dispersion relation but also is one possible mechanism to excite the global structure of GAM, which is observed in recent tokamak experiments.[31] To explain the related numerical simulation, Zhou and Wang developed a formalism to describe the global structure of GAM based on the ideal MHD frame.[32]

Starting from the drift kinetic equations, Bashir *et al.* derived the dispersion relation of GAM including the electromagnetic effect.[30] However, only $m=1$ harmonics of perturbations are considered. In the present work, we investigate the electromagnetic GAM by using gyro-kinetic equations. The finite-Larmor-radius (FLR) and finite-orbit-width (FOW) effects are taken into account to the second order, meaning that $m=2$ harmonics of $\delta\phi$ and $\delta A_\parallel$ are both considered for self-consistency. Transparent and concise expressions for the GAM frequency and Landau damping rate are presented. It is worth noting that our analytical results show that the obtained dependence of GAM frequency on β is in good agreement with Refs. 33 and 34. Besides, our results show that $m=2$ harmonic of $\delta A_\parallel$ is larger than $m=1$ harmonic by one order and hence dominates the magnetic vector potential, which confirms that the magnetic perturbation of GAM has a poloidal wave number $m=2$. Compared with previous work on electromagnetic GAM, we obtain the general dispersion relation but do not use the overly strict ordering analysis. And then, our work can be extended to the high β case more easily. On the other hand, we obtain transparent analytical expressions for the GAM frequency, the collisionless damping rate, and the dominant $\delta A_\parallel$. Thus, useful information can be directly extracted to provide guidance for simulation and experimental works related to the GAM.

The rest of the paper is organized as follows. Basic equations and derivations are illustrated in Sec. II. The general dispersion relation of GAM is presented. Section III is devoted to the numerical evaluation of the dispersion relation. In Sec. IV, simplified analytical expressions for the GAM real frequency and collisionless damping rate are performed. The electrostatic and electromagnetic potential perturbations are specifically discussed and a clear ordering analysis is presented. Finally, a summary is given in Sec. V.

II. EQUATIONS AND DERIVATION

A. Governing equations

By solving the linearized gyro-kinetic equations, we obtain the perturbed distribution function δf_α

$$\delta f_\alpha = q_\alpha \frac{\partial F_{0\alpha}}{\partial E}\delta\phi + q_\alpha \frac{\partial F_{0\alpha}}{\partial E} J_0^2 \sum (-1)^{m+l} i^{m+l} \times \frac{\omega}{\omega_{t\alpha}} \frac{J_m J_l}{m+n-\dfrac{\omega}{\omega_{t\alpha}}} \overline{\delta\varphi_n} e^{i(m+n+l)\theta}, \tag{1}$$

in which

$$\delta\phi - v_\parallel \delta A_\parallel = \sum \overline{\delta\varphi_n} e^{in\theta}. \tag{2}$$

Here, $\delta\phi$ and $\delta A_\parallel = \boldsymbol{b}\cdot\delta\mathbf{A}$ are the perturbed electrostatic and parallel vector potentials, respectively. In our work, we consider a toroidally symmetric magnetic field $\mathbf{B} = I(\psi)\nabla\zeta + \nabla\zeta\times\nabla\psi$, where $\psi(r)$ and $2\pi I(\psi)$ are the magnetic flux and poloidal current, respectively. In addition, a circular cross section is assumed as usual. We work in the (r, θ, ζ) coordinate system, where θ and ζ are poloidal and toroidal angles, respectively. q_α and $\omega_{t\alpha} = v_\parallel/(qR)$ are the particle electric charge and transit frequency. $F_{0\alpha}$ has a standard Maxwellian form with a temperature T_α and an unperturbed number density $n_{0\alpha}$ ($\alpha = e, i$ for species of electrons and ions, respectively). $J_0 = J_{0\alpha}(k_r\rho_\alpha)$ and $J_m = J_{m\alpha}\left(k_r \frac{v_{d\alpha}}{\omega_{t\alpha}}\right)$ are the Bessel functions of zero order and m order, where k_r, $\rho_\alpha = v_{t\alpha}/\omega_c$, $\omega_c = q_{\alpha B}/m_\alpha$, and $v_{d\alpha} = 2v_\parallel^2 + v_\perp^2/(2\omega_c R)$ are the radial wave number, gyro-radius, gyro-frequency, and magnetic drift velocity, respectively; R is the major radius.

From the parallel Ampere's law, we have

$$\delta A_\parallel = \frac{\mu_0}{k_r^2}\sum q_\alpha \int v_\parallel \delta f_\alpha d^3v. \tag{3}$$

B. Calculation and derivation

In order to obtain the dispersion relation of GAM, we combine the quasi-neutrality condition and the parallel Ampere's law. Then, the following symmetries are found:

$$\delta A_0 = 0, \tag{4}$$

$$\delta A_1 = \delta A_{-1}, \tag{5}$$

$$\delta A_2 = -\delta A_{-2}, \tag{6}$$

$$\delta\phi_1 = -\delta\phi_{-1}, \tag{7}$$

$$\delta\phi_2 = \delta\phi_{-2}. \tag{8}$$

The same symmetries also can be found in Ref. 14. Based on the above symmetries, the quasi-neutrality condition and the parallel Ampere's law can be simplified as

$$\tau(C_0\delta\phi_0 - 2iC_1\delta\phi_1 + 2C_2\delta\phi_2 + 2iC_1^*\delta A_1 - 2C_2^*\delta A_2) = -2iC_{1e}^*\delta A_1, \tag{9}$$

$$i\tau C_1\delta\phi_0 + [1+\tau(D_1+I_c)]\delta\phi_1 + i\tau D_2\delta\phi_2 - (qR\omega + \tau D_1^*)\delta A_1 - i\tau D_2^*\delta A_2 = 0, \tag{10}$$

$$\tau C_2\delta\phi_0 - i\tau D_2\delta\phi_1 + \left[1+\tau Z_1\left(\frac{\zeta}{2}\right)\right]\delta\phi_2 + i\tau D_2^*\delta A_1 - \frac{1}{2}qR\omega\left[1+\tau Z_1\left(\frac{\zeta}{2}\right)\right]\delta A_2 = 0, \tag{11}$$

and

$$\frac{k_r^2\tau T_i}{\mu_0 e^2}\delta A_1 = -i(C_{1e}^* + \tau C_1^*)\delta\phi_0 - (qR\omega + \tau D_1^*)\delta\phi_1 - i\tau D_2^*\delta\phi_2 + \left[(D_{1e}^{**} - I_{ce}^{**}) + \tau(D_1^{**} - I_c^{**})\right]\delta A_1 + i(D_{2e}^{**} + \tau D_2^{**})\delta A_2, \tag{12}$$

$$\frac{k_r^2\tau T_i}{\mu_0 e^2}\delta A_2 = -\tau C_2^*\delta\phi_0 + i\tau D_2^*\delta\phi_1 - i(D_{2e}^{**} + \tau D_2^{**})\delta A_1 - \frac{1}{2}qR\omega(1+\tau Z_1(\zeta/2))\delta\phi_2 + \frac{1}{4}(qR\omega)^2(1+\tau Z_1(\zeta/2))\delta A_2, \tag{13}$$

where $\tau = \frac{T_e}{T_i}$. $\zeta_\alpha \equiv \frac{qR\omega}{v_{t\alpha}}$ is the argument of the plasma dispersion function $\mathcal{Z}_n(\zeta_\alpha) = \frac{1}{\sqrt{\pi}}\int \frac{x^n e^{-x^2}}{x-\zeta_\alpha}dx$. Hereafter, v_{ti} and ζ_i are marked as v_t and ζ, respectively. Furthermore, we define $\zeta \equiv q\Omega$ as usual, where $\Omega = \Omega_G + i\gamma_d$. $\Omega_G = \frac{\omega_t R}{v_{ti}}$ and $\gamma_d = \frac{\gamma R}{v_{ti}}$ are the normalized mode frequency and damping rate, respectively. All the coefficients like C_0 will be displayed in the Appendix unless otherwise stated. We adopt the adiabatic electron assumption in this paper for simplicity. For the electromagnetic GAM, we should reserve the complete form of δf_e. After some operations, all terms proportional to $\frac{m_e}{m_i}$ in the coefficients can be neglected due to the adiabatic electron assumption. To obtain the dispersion relation of the electromagnetic GAM, we introduce

$$\delta\phi_{1E} = \lambda\delta\hat{\phi}_{1E}\delta\phi_0 = i\tau\frac{\tau C_2 D_2 - C_1\left[1+\tau Z_1\left(\frac{\zeta}{2}\right)\right]}{[1+\tau(D_1+I_c)]\left[1+\tau Z_1\left(\frac{\zeta}{2}\right)\right] - \tau^2 D_2^2}\delta\phi_0,$$
$$\delta\phi_{2E} = \lambda^2\delta\hat{\phi}_{2E}\delta\phi_0 = \tau\frac{\tau C_1 D_2 - C_2[1+\tau(D_1+I_c)]}{[1+\tau(D_1+I_c)]\left[1+\tau Z_1\left(\frac{\zeta}{2}\right)\right] - \tau^2 D_2^2}\delta\phi_0. \tag{14}$$

Then, we rearrange Eqs. (10)–(13) as

$$\delta\phi_1 = \lambda\delta\hat{\phi}_{1E}\delta\phi_0 + H_{11}v_t\delta A_1 + i\lambda H_{12}v_t\delta A_2, \tag{15}$$
$$\delta\phi_2 = \lambda^2\delta\hat{\phi}_{2E}\delta\phi_0 + i\lambda H_{21}v_t\delta A_1 + H_{22}v_t\delta A_2, \tag{16}$$
$$v_t\delta A_1 = \frac{2\beta}{K}(\lambda M_{10} + \lambda M_{11}\delta\hat{\phi}_{1E} + \lambda^2 M_{12}\delta\hat{\phi}_{2E})\delta\phi_0, \tag{17}$$
$$v_t\delta A_2 = \frac{2\beta}{K}(M_{20} + \lambda M_{21}\delta\hat{\phi}_{1E} + \lambda^2 M_{22}\delta\hat{\phi}_{2E})\delta\phi_0, \tag{18}$$

where λ is short for $k_r\rho_i$ and $\beta = \frac{2\mu_0 n_i T_i}{B^2}$. It is found that $\delta\phi_1 \sim \mathcal{O}(\lambda)\delta\phi_0$, $\delta\phi_2 \sim \mathcal{O}(\lambda^2)\delta\phi_0$, $\delta A_1 \sim \mathcal{O}(\lambda\beta)\delta\phi_0$, and $\delta A_2 \sim \mathcal{O}(\beta)\delta\phi_0$. An assumption used here is $\beta \ll \lambda \ll 1$, which is valid in the realistic tokamak plasma parameter regime. It is worth noting that δA_1 is exactly zero both on the order of $\mathcal{O}(\lambda)$ and $\mathcal{O}(\beta)$ [see Eqs. (B7) and (17)]. And then, the dominant magnetic perturbation is the second harmonic δA_2. In our expressions, we clearly show that the electrostatic and the electromagnetic terms are separated and δA_k are now functions of $\delta\phi_{kE}$. Substituting Eqs. (15)–(18) into $\delta n_{i0} = \delta n_{e0}$, viz. Eq. (9), we have

$$\hat{H}_1\delta\phi_0 + 2i\tau\lambda^2\hat{H}_2\delta\hat{\phi}_{1E}\delta\phi_0 + 2\lambda^2\hat{H}_3\delta\hat{\phi}_{2E}\delta\phi_0 + 2i\lambda(\tau\hat{H}_2H_{11} + \hat{H}_3H_{21} + \hat{H}_4)v_t\delta A_1 - 2(\hat{H}_5 - \hat{H}_3H_{22} + \tau\lambda^2\hat{H}_2H_{12})v_t\delta A_2 = 0. \tag{19}$$

By disregarding the last two terms on the left-hand side of the equation above, the classical electrostatic GAM is recovered. Now let us focus on the electromagnetic terms. The two terms are on the order of $\mathcal{O}(\lambda^2\beta)$ and $\mathcal{O}(\beta)$, respectively. Via Taylor expansion and neglecting all the terms with higher orders, Eq. (19) is reduced to the previous dispersion relation with an adiabatic electron assumption.[14] On the other hand, the previous formulas of δA_k and $\delta\phi_k$ also can be recovered via neglecting the higher order terms in Eqs. (15)–(18). Finally, by defining $v_t\delta A_k \equiv \beta\delta\hat{A}_k\delta\phi_0$, Eq. (19) is rewritten as

$$G + \lambda^2 H + \beta Q = 0, \tag{20}$$

in which

$$G = C_0^{(2)} - \frac{2\tau\left[C_1^{(1)}\right]^2}{1+\tau Z_1}, \tag{21}$$
$$H = -\frac{4\tau C_1^{(1)}C_1^{(3)}}{1+\tau Z_1} + 2i\tau\hat{H}_2\delta\hat{\phi}_{1E} + 2\hat{H}_3\delta\hat{\phi}_{2E}, \tag{22}$$
$$Q = 2i\lambda(\tau\hat{H}_2H_{11} + \hat{H}_3H_{21} + \hat{H}_4)\delta\hat{A}_1 - 2(\hat{H}_5 - \hat{H}_3H_{22} + \tau\lambda^2\hat{H}_2H_{12})\delta\hat{A}_2. \tag{23}$$

III. NUMERICAL RESULTS

In this section, we numerically evaluated the dispersion relation Eq. (20), and compare our result with the referenced result with adiabatic electrons in W2011. When the electromagnetic effect is neglected by setting $\beta = 0$, the dispersion relation of the classical GAM is recovered. The electrostatic case is first shown in Fig. 1, illustrating the dependence of the normalized frequency and damping rate on λ^2 for different q.

Figure 1 shows that both the real frequency and damping rate increase with λ. With fixed λ, both the real frequency and damping rate decrease with q. It can be clearly explained by the analytical theory[35] and also can be seen directly from the analytical expression of the GAM frequency in Sec. IV. Furthermore, to the leading order, it is well known that the GAM frequency and Landau damping rate can be approximated as[16]

$$\Omega_G^2 \simeq \left(\frac{7}{4}+\tau\right)\left[1+\frac{1}{q^2}\frac{46+32\tau+8\tau^2}{(4\tau+7)^2}\right], \tag{24}$$
$$\gamma_d \simeq -\frac{\sqrt{\pi}}{2}q^5\frac{\Omega_G^6}{\tau+\frac{7}{4}}e^{-q^2\Omega_G^2}. \tag{25}$$

Hence, the analytical frequency decreases with q due to the $1/q^2$ term, and the damping rate decreases with q in the large q regime.

The dependence of the normalized frequency and damping rate on β for different τ in the large q regime ($q = 3.5$) is

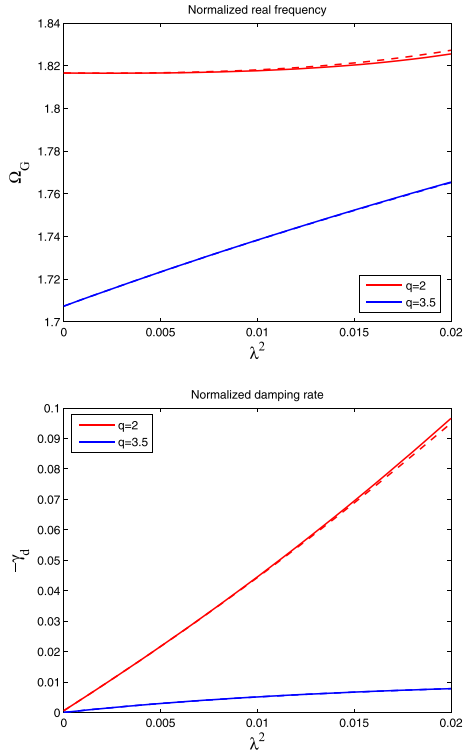

FIG. 1. The normalized GAM frequency and damping rate of GAM versus λ^2 with $\beta = 0$ and $\tau = 2$ for different q. The solid curves are plotted according to Eq. (20) and the dashed ones are plotted according to Eq. (23) in W2011.

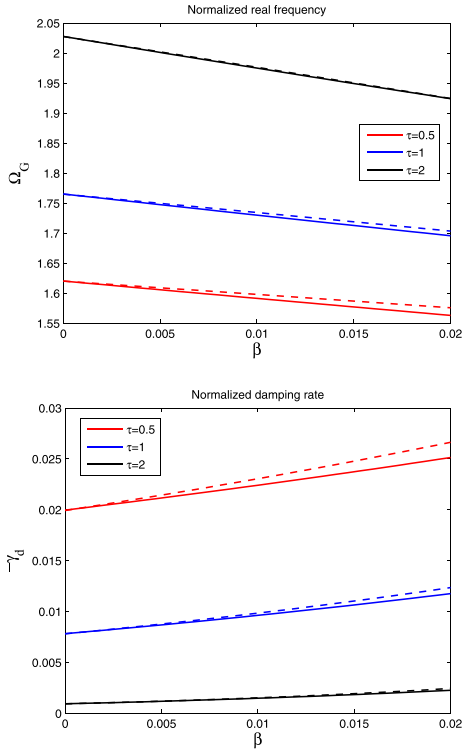

FIG. 2. The normalized GAM frequency and damping rate versus β with $\lambda^2 = 0.02$ and $q = 3.5$ for different τ. The solid curves are plotted according to Eq. (20) and the dashed ones are plotted according to Eq. (23) in W2011.

illustrated in Fig. 2. In Fig. 2, the frequency and damping rate decrease and increase with β, respectively. Besides, the real frequency increases with τ for fixed β. It is worth noting that the damping rate decreases with τ for fixed β in the large q regime. While in the small q regime ($q = 2$), the damping rate does not monotonically decrease with τ for fixed β again, as detailedly shown and further discussed in Figs. 3 and 4. From the analytical results for large q as below, we find that a $-q^2\beta$-dependent term is introduced by the electromagnetic effect in the approximated frequency Eq. (24). It explains that the frequency decreases with β and increases with τ for fixed β. According to Eq. (25), the damping rate turns to increase with β and decrease with τ for fixed β, as displayed in Fig. 2.

As for the small q case, the competition between Ω_G^6 and $e^{-q^2\Omega_G^2}$ may be different. To make a comparison, we plotted Fig. 3. All the parameters are the same as Fig. 2 except that $q = 2$ now. We note that similar to the large q case, the real frequency decreases with β, and there is no noticeable difference between our results and W2011. However, the damping rates show a remarkable difference to those in Fig. 2. It is seen that the referenced damping rate still increases with β, whereas our damping rate decreases with β now. The discrepancy between our damping rate and W2011 increases with β. We also point out that our dispersion relation can be reduced to W2011's one with the help of Taylor expansion when neglecting the higher order terms. Then, such a remarkable discrepancy can be attributed to the higher order effects, which will be discussed, displayed, and confirmed by Fig. 6. In addition, we find that the damping rate for $q = 2$ is not a monotonic function of τ, as shown in detail in Fig. 4.

Figure 4 is shown to study the dependence of the normalized frequency and damping rate on τ for different q. All four cases show that the mode frequency increases with τ. The damping rate monotonically decreases with τ for large q (such as $q = 4$), which is consistent with Fig. 2. In the small q regime (such as $q = 2$), the Landau damping rate is no longer a monotonic function of τ, as confirmed in Fig. 3. The damping rate increases with τ at first until it reaches its maximum value. When τ is larger about 1.3, the damping rate decreases with τ.

Furthermore, the real and imaginary parts of Ω versus the safety factor q for different β are illustrated in Fig. 5. It can be seen that the real frequency decreases with q. As for the damping rate, it is not a monotonic function of q. The negative γ_d decreases first as q increases and then increases with q. As q continues to climb up, the damping rate decreases again as q increases due to the second order FOW resonant effect.[16,35] Since the 3rd FOW effect is not considered here, there is only one fluctuation on the curve, appearing at about $q = 1.5$ with the peak at about $q = 2$. As illustrated in Sec. IV, the electromagnetic effect does not introduce new terms in the damping rate so that Fig. 5 is similar to the pure electrostatic case.[35] It also can be seen that for given q, our damping rate decreases with β when q

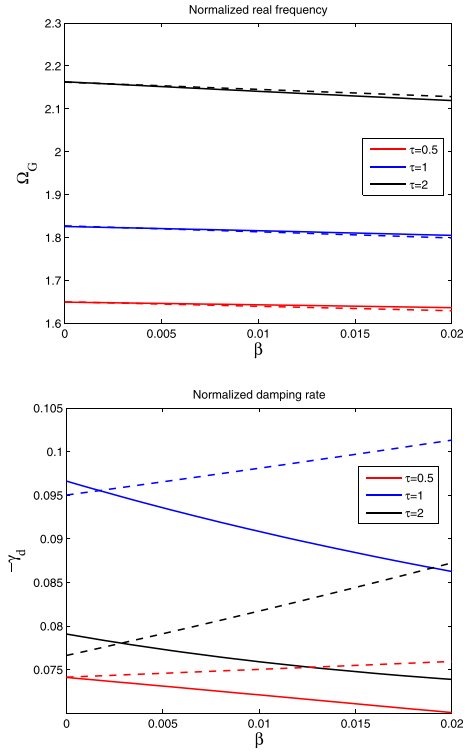

FIG. 3. The normalized GAM frequency and damping rate versus β with $\lambda^2 = 0.02$ and $q = 2$ for different τ. The solid curves are plotted according to Eq. (20) and the dashed ones are plotted according to Eq. (23) in W2011.

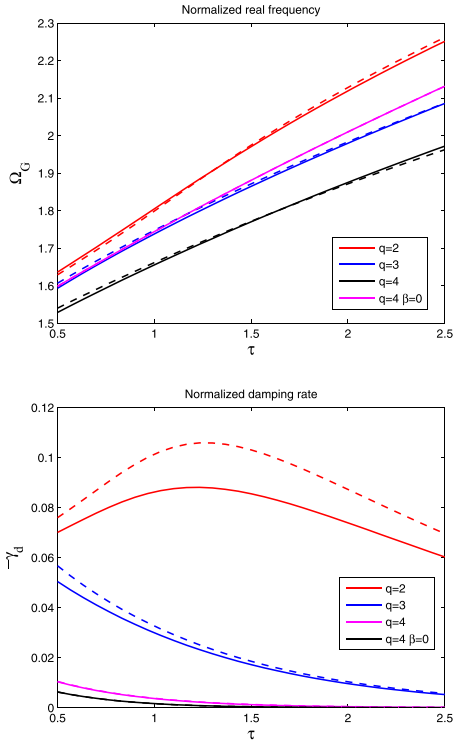

FIG. 4. The normalized frequency and damping rate of GAM versus τ with $\lambda^2 = 0.02$ and $\beta = 0.02$ for different q. The solid curves are plotted according to Eq. (20) and the dashed ones are plotted according to Eq. (23) in W2011.

is small, or specifically, $1.7 \le q \le 2.6$. When q is larger than 2.6, the damping rate increases with β for given q. On the other hand, the damping rate in W2011 increases with β for arbitrary q. These features are consistent with Figs. 2 and 3.

To clearly show the discrepancy between our numerical damping rate and W2011, the q scan of the damping rate for different β is re-plotted in Fig. 6. It is found that the discrepancy increases with β, as shown in Figs. 3 and 5. To take a further step, we separate Eq. (20) as basic terms, higher order electrostatic terms, δA_1 related higher order terms, and δA_2 related higher order terms. The basic terms constitute the dispersion relation in W2011. It is found that the higher order electrostatic terms have little effect on the damping rate. Furthermore, such a discrepancy vanishes when we drop both the higher order electrostatic terms and the δA_1 related terms in Eq. (20). It confirms that such a discrepancy is introduced by the δA_1 related terms.

IV. ANALYTICAL RESULTS

A. Perturbed potentials

In this section, we present the analytical expressions of the frequency and damping rate. The perturbed electrostatic potential and parallel electromagnetic vector potential are discussed first. As we know, $\delta\phi_1 \sim \mathcal{O}(\lambda)\delta\phi_0$, $\delta\phi_2 \sim \mathcal{O}(\lambda^2)\delta\phi_0$, $\delta A_1 \sim \mathcal{O}(\lambda\beta)\delta\phi_0$, and $\delta A_2 \sim \mathcal{O}(\beta)\delta\phi_0$. By Taylor expansion and neglecting higher order terms, we rewrite δA_k and $\delta\phi_k$ as

$$v_t\delta A_1 = 2\beta\left[-i\lambda\hat{C}_1' + i\lambda\hat{C}_2'\frac{\tau C_1'^{(1)}}{1+\tau\mathcal{Z}_1}\right]\delta\phi_0, \tag{26}$$

$$v_t\delta A_2 = -\frac{2\beta}{v_t}\left[\left(C_2^{*\prime} - \frac{1}{2}\zeta v_t C_2'\right) - \frac{\tau C_1^{(1)\prime}}{1+\tau\mathcal{Z}_1}\left(D_2^{*\prime} - \frac{1}{2}\zeta v_t D_2'\right)\right]\delta\phi_0, \tag{27}$$

$$\delta\phi_1 = \lambda\delta\hat{\phi}_{1E}^{(1)}\delta\phi_0 + \lambda^3\delta\hat{\phi}_{1E}^{(3)}\delta\phi_0 + \zeta v_t\delta A_1 + i\tau\frac{D_2^{*\prime} - \frac{1}{2}\zeta v_t D_2'}{1+\tau\mathcal{Z}_1}\lambda\delta A_2, \tag{28}$$

$$\delta\phi_2 = \lambda^2\delta\hat{\phi}_{2E}\delta\phi_0 + \frac{1}{2}\zeta v_t\delta A_2, \tag{29}$$

in which

$$\lambda\delta\hat{\phi}_{1E}^{(1)} = -i\lambda\frac{\tau C_1'^{(1)}}{1+\tau\mathcal{Z}_1}, \tag{30}$$

$$\lambda^3\delta\hat{\phi}_{1E}^{(3)} = -i\lambda^3\frac{\tau C_1'^{(3)}}{1+\tau\mathcal{Z}_1} - \frac{\tau D_c'}{1+\tau\mathcal{Z}_1}\lambda^3\delta\hat{\phi}_{1E}^{(1)} - \frac{i\tau D_2'}{1+\tau\mathcal{Z}_1}\lambda^3\delta\hat{\phi}_{2E}, \tag{31}$$

$$\lambda\delta\hat{\phi}_{1E} = \lambda\delta\hat{\phi}_{1E}^{(1)} + \lambda^3\delta\hat{\phi}_{1E}^{(3)}, \tag{32}$$

$$\lambda^2\delta\hat{\phi}_{2E} = -\lambda^2\frac{\tau C_2'}{1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)} + \frac{i\tau D_2'}{1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)}\lambda^2\delta\hat{\phi}_{1E}^{(1)}. \tag{33}$$

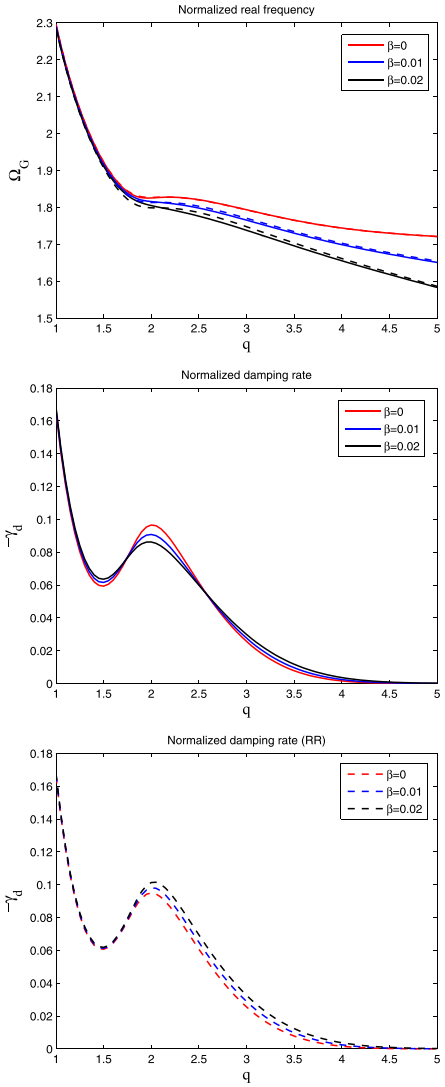

FIG. 5. The real and imaginary parts of Ω versus q with $\lambda^2 = 0.02$ and $\tau = 1$ for different β. The solid curves are plotted according to Eq. (20) and the dashed ones are plotted according to Eq. (23) in W2011.

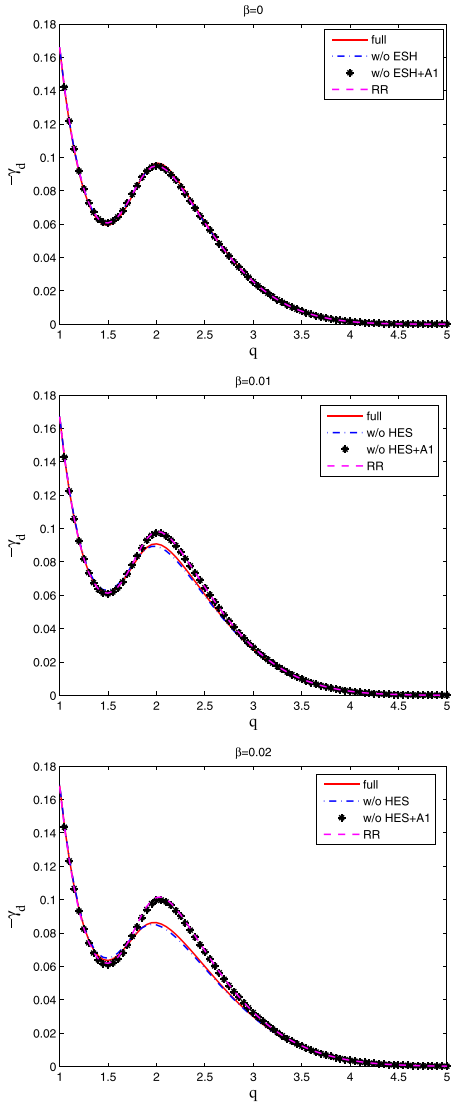

FIG. 6. The q scan of normalized damping rate with $\lambda^2 = 0.02$ and $\tau = 1$ for different β. Our numerical damping rate (full) and the corresponding reference ones are plotted as red solid curves and maroon dashed curves according to Eqs. (20) and (23) in W2011, respectively. The blue dotted-dashed curves are plotted according to Eq. (20) without the higher order electrostatic terms (w/o HES). The black stars are plotted according to Eq. (20) without both the higher order electrostatic terms and the δA_1 related higher order terms (w/o HES+A1).

The approximated formulas of δA_k and $\delta\phi_k$ are consistent with W2011. To take a further step, we asymptotically expand the plasma dispersion function $\mathcal{Z}(\zeta) = i\sqrt{\pi}exp(-\zeta^2) - \zeta^{-1}(1+\zeta^{-2}/2 +3\zeta^{-4}/4+\cdots)$ as usual and neglect all terms of orders higher than $\mathcal{O}(\zeta^{-4})$, and then, we find

$$v_t\delta A_1 = 2\lambda\beta\left[-\sqrt{\pi}e^{-\frac{1}{4}q^2\Omega^2}\frac{q^6\Omega^3}{512}(q^2\Omega^2+8\tau+6) + iq\left(\frac{3}{8}+\frac{1}{4}\tau\right) - \frac{iq}{\Omega^2}\left(\frac{\tau^2}{8}+\frac{7\tau}{8}+\frac{27}{16}\right) - \frac{i}{q\Omega^4}\left(\frac{5\tau^3}{16}+\frac{145\tau^2}{64}+\frac{233}{32}\tau+\frac{783}{64}\right)\right]\delta\phi_0, \quad (34)$$

$$v_t\delta A_2 = 2\beta\left[\frac{q}{\Omega}\left(\frac{7}{32}+\frac{\tau}{8}\right)+\frac{1}{q\Omega^3}\left(\frac{1}{16}\tau^2+\frac{1}{4}\tau+\frac{23}{64}\right)\right]\delta\phi_0, \quad (35)$$

$$\delta\phi_1 = \lambda\delta\hat{\phi}_{1E}^{(1)}\delta\phi_0 + \lambda^3\delta\hat{\phi}_{1E}^{(3)}\delta\phi_0 + q\Omega v_t\delta A_1 - i\left[\frac{1}{4}q\tau + \frac{1}{\Omega^2}\left(\frac{\tau^2+2\tau}{8q}\right)+\frac{1}{\Omega^4}\left(\frac{5\tau^2+9\tau}{16q^3}\right)\right]v_t\lambda\delta A_2, \quad (36)$$

$$\delta\phi_2 = \lambda^2\delta\hat{\phi}_{2E}\delta\phi_0 + \frac{1}{2}q\Omega v_t\delta A_2, \quad (37)$$

in which

$$\delta\hat{\phi}_{1E}^{(1)} = \frac{1}{2}\tau q^3\Omega^2\sqrt{\pi}e^{-q^2\Omega^2} + \frac{i}{2\Omega}\tau + \frac{i}{\Omega^3}\left(\frac{\tau^2}{4q^2} + \frac{\tau}{2q^2}\right),$$

$$\delta\hat{\phi}_{1E}^{(3)} = \sqrt{\pi}e^{-\frac{1}{4}q^2\Omega^2}\frac{\tau q^5\Omega^2}{256}\left(q^2\Omega^2 + \frac{13\tau+12}{2}\right) - \frac{i}{\Omega}\frac{2\tau^2+3\tau}{8} + \frac{i}{\Omega^3}\frac{2\tau^3+14\tau^2+27\tau}{16},$$

$$\delta\phi_{2E} = i\sqrt{\pi}e^{-\frac{1}{4}q^2\Omega^2}\frac{q^3\Omega\tau}{64}(q^2\Omega^2+6\tau+4) - \frac{1}{\Omega^2}\left(\frac{1}{4}\tau^2 + \frac{7}{16}\tau\right) - \frac{1}{\Omega^4}\left(\frac{5}{8q^2}\tau^3 + \frac{23}{8q^2}\tau^2 + \frac{161}{32q^2}\tau\right).$$

Let us focus on the dominant terms in δA_2 and $\lambda\delta\hat{\phi}_{1E}^{(1)}$

$$v_t\delta A_2 = \frac{q\beta}{4\Omega}\left(\tau + \frac{7}{4}\right)\delta\phi_0, \tag{38}$$

$$\lambda\delta\phi_{1E}^{(1)} = \lambda\delta\hat{\phi}_{1E}^{(1)}\delta\phi_0 = \frac{i\tau\lambda}{2\Omega}\delta\phi_0, \tag{39}$$

$$\left|\frac{v_t\delta A_2}{\lambda\delta\phi_{1E}^{(1)}}\right| = \frac{q\beta}{2\lambda}\left(1 + \frac{7}{4\tau}\right). \tag{40}$$

From the equations above, we find that $v_t\delta A_2$ is proportional to β. With fixed β, $v_t\delta A_2$ increases with τ. On the other hand, $|v_t\delta A_2/(\lambda\delta\phi_{1E}^{(1)})|$ increases with β and decreases with τ for fixed β. For large q, $v_t\delta A_2$ and $\lambda\delta\phi_{1E}^{(1)}$ can be of the same order, i.e., $|v_t\delta A_2/(\lambda\delta\phi_{1E}^{(1)})| \sim 1$. These features are consistent with the previous result [see Fig. 4 in W2011].

According to Eqs. (34) and (38) and recalling that $\delta A_{-2} = -\delta A_2$ and $\delta A_0 = 0$, we can write

$$\delta A_\parallel = 2i\sin(2\theta)\delta A_2 = i\frac{1}{2}\Omega^2\frac{q\beta}{\omega R}\delta\phi_0\sin(2\theta). \tag{41}$$

While the ideal MHD model gives[36]

$$\delta A_\parallel = i\frac{1}{2}\Omega_{MHD}^2\frac{q\beta}{\omega_{MHD}R}\delta\phi_0\sin(2\theta). \tag{42}$$

The two formulas have an identical form to each other. Only the difference lies in the frequency, noting that $\Omega \simeq \sqrt{\tau+7/4}$ and then $\omega = \Omega v_t/R$ whereas $\Omega_{MHD} = \sqrt{2+1/q^2}$ and $\omega_{MHD} = \Omega_{MHD}c_s/R$ with $c_s = \sqrt{5(1+\tau)/6}v_t$ being the sound speed in the MHD framework. Equation (41) indicates that the dominant magnetic vector has poloidal wave number $m = 2$. Hence, the corresponding perturbed magnetic field with $m = 2$ is dominant, as observed in the experiment[28] and theoretical predicted.[24]

B. Analytical dispersion relation

To obtain a transparent expression of the dispersion relation, we have to simplify G, H, and Q in Eq. (19) by neglecting all the higher order terms and expanding $\mathcal{Z}_n$ as usual. After some extremely redundant but not technically tricky calculation, the simplified analytical dispersion relation of GAM is derived as

$$G + \lambda^2 H + \beta Q = 0, \tag{43}$$

in which

$$G = \frac{i}{2}\sqrt{\pi}q^5\Omega^3e^{-q^2\Omega^2} + \frac{1}{2} - \frac{1}{\Omega^2}\left(\frac{1}{2}\tau + \frac{7}{8}\right) - \frac{1}{\Omega^4}\left(\frac{\tau^2}{4q^2} + \frac{\tau}{q^2} + \frac{23}{16q^2}\right),$$

$$H = i\sqrt{\pi}\frac{q^7\Omega^3}{1024}e^{-\frac{1}{4}q^2\Omega^2}(q^2\Omega^2+8+8\tau) - \frac{3}{16} + \frac{1}{\Omega^2}\left(\frac{1}{4}\tau^2 + \frac{3}{4}\tau + \frac{13}{16}\right) - \frac{1}{\Omega^4}\left(\frac{\tau^3}{8} + \frac{35\tau^2}{32} + \frac{481\tau}{128} + \frac{747}{128}\right),$$

$$Q = \frac{q^2}{\Omega^2}\left(\frac{\tau^2}{16} + \frac{7\tau}{32} + \frac{49}{256}\right) + \frac{1}{\Omega^4}\left(\frac{\tau^3}{16} + \frac{23\tau^2}{64} + \frac{51\tau}{64} + \frac{161}{256}\right).$$

Our electrostatic dispersion relation above is exactly consistent with previous results.[37,38] The real part of Eq. (43) yields the real frequency of the electromagnetic GAM to the order of $1/q^2$, λ^2, and β as

$$\Omega_G^2 = \left(\tau + \frac{7}{4}\right)\left[1 + \frac{1}{q^2}\frac{46+32\tau+8\tau^2}{(4\tau+7)^2} - \lambda^2\frac{\tau^2 - \frac{3}{4}\tau + \frac{31}{8}}{4\tau+7} + \lambda^2\frac{747+334\tau+28\tau^2}{(8\tau+14)^2} - \beta\frac{q^2}{2}\frac{\tau^2+\frac{7}{2}\tau+\frac{49}{16}}{4\tau+7} - \beta\frac{8\tau^3+46\tau^2+102\tau+\frac{161}{2}}{(8\tau+14)^2}\right]. \tag{44}$$

We can see that the electromagnetic effect modifies the GAM frequency on the order of $q^2\beta$; the dominant electromagnetic term $-\beta\frac{q^2}{2}\frac{\tau^2+\frac{7}{2}\tau+\frac{49}{16}}{4\tau+7}$ and the electrostatic term $\frac{1}{q^2}\frac{46+32\tau+8\tau^2}{(4\tau+7)^2}$ can be of the same order for large q.

By setting $\tau = 1$ and $\lambda = 0$, our analytical formula Eq. (44) can be reduced to

$$\frac{(\omega_{GAM})^2}{(\omega_G^0)^2} \approx 1 + \frac{0.7}{q^2} - 0.5\beta - 0.3\beta q^2, \tag{45}$$

where $\beta = \frac{v_{ti}^2}{V_A^2}$ and $\omega_G^0 = \sqrt{\frac{11}{4}}\frac{v_{ti}}{R}$. $v_{ti} = \sqrt{\frac{2T_i}{m_i}}$ and V_A are the thermal velocity and Alfvén velocity, respectively. By assuming $q \gg 1$, $q^2\beta \ll \sqrt{2}$, and $\omega^2 \gg k_{m,n}^2c_s^2$, the analytical expression of GAM frequency in Refs. 33 and 34 can be approximated as

$$\frac{(\omega_{GAM}^{FKY})^2}{(\omega_G^{WJD})^2} \approx 1 + \frac{1}{2q^2} - \frac{1}{8}\beta_s - \frac{1}{4}\beta_s q^2, \tag{46}$$

where $\beta_s = \frac{c_s^2}{V_A^2}$ and $\omega_G^{WJD} = \sqrt{2}\frac{c_s}{R_0}$. $c_s = \sqrt{\gamma p/\rho} = \sqrt{2\gamma T_i/m_i}$ is the sound velocity. The GAM frequency in our work and in Refs. 33 and 34 is normalized by the thermal velocity and sound velocity, respectively. We should also note that β and β_s are defined by v_{ti} and c_s, respectively. We can see that the forms of Eqs. (45) and (46) are very similar except the coefficients due to the difference between the gyro-kinetic theory and MHD frame. Then, we can conclude that the dependence of our analytical GAM frequency on β is in good agreement with Refs. 33 and 34.

The Landau damping rate of GAM is

$$\gamma_d = \gamma_c\left[1 + \frac{q^2\lambda^2}{512}e^{\frac{3}{4}q^2\Omega_G^2}(8\tau+8+q^2\Omega_G^2)\right], \tag{47}$$

where $\gamma_c = -\frac{\sqrt{\pi}}{2} q^5 \frac{\Omega_G^6}{\tau+\frac{7}{4}} e^{-q^2\Omega_G^2}$. Different from the frequency above, the electromagnetic effect does not directly introduce β-dependent terms into the damping rate. It is believed that the electromagnetic effect does not provide second order resonant terms. Mathematically, $\beta e^{-\frac{1}{4}q^2\Omega_G^2}$ is introduced by the term $\mathcal{Z}_n\left(\frac{\zeta}{2}\right)$. However, $\mathcal{Z}_n\left(\frac{\zeta}{2}\right)$ related terms are exactly canceled by each other in $C_2^{*\prime} - \frac{1}{2}qR\omega C_2'$ and $D_2^{*\prime} - \frac{1}{2}qR\omega D_2'$, which constitute the electromagnetic terms in the dispersion relation. The Landau damping rate is different from the electrostatic one due to the fact that the real frequency is modified by finite β while γ_d is sensitive to Ω_G. The above analytical formulas can be used to explain our numerical results. It also should be noted that the third order FOW effect is not considered here. According to the previous study,[35] the 3rd order FOW introduced a term $\propto \lambda^4 e^{-\frac{1}{4}q^2\Omega_G^2}$ in the damping rate.

Figure 7 illustrates the mode frequency and damping rate versus β with $\lambda = 0.05$, $\tau = 1$, and $q = 4$. The analytical results agree well with the numerical solutions. We note that the finite FOW effect almost can be neglected since $\lambda = 0.05$. Now, we can focus on the finite β effect on the mode frequency and damping rate. We also plot the q scan of the frequency and damping rate in Fig. 8. For the real frequency, the analytical result agrees well with the numerical solution. While for the damping rate, good agreement is found between the analytical and numerical

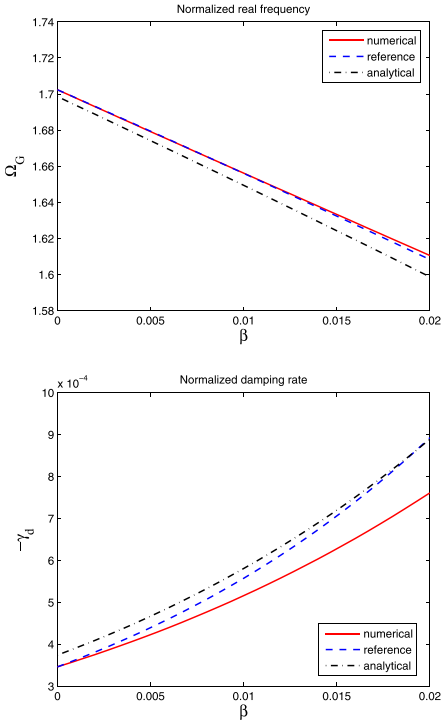

FIG. 7. The normalized frequency and damping rate of GAM as functions of β with $\lambda = 0.05$, $\tau = 1$ and $q = 4$. The red solid line, blue dashed line, and black dotted-dashed line are numerical results, reference results and analytical results, respectively.

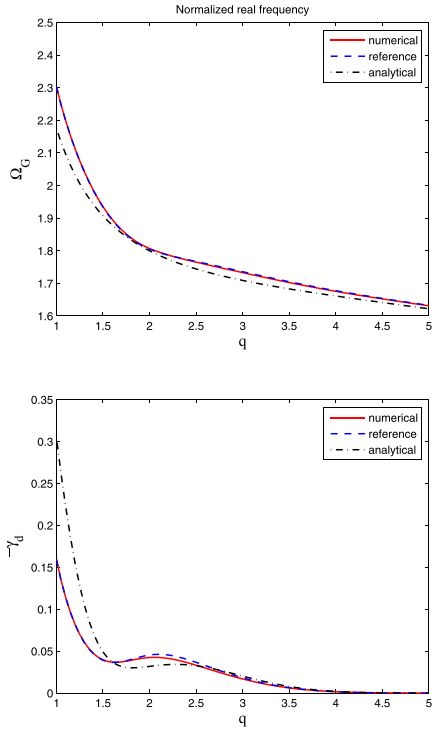

FIG. 8. The normalized frequency and damping rate of GAM as functions of q with $\lambda^2 = 0.01$, $\tau = 1$, and $\beta = 0.01$. The red solid line, blue dashed line, and black dotted-dashed line are numerical results, reference results, and analytical results, respectively.

results except for small q, or more precisely, for $q < 2.5$. As discussed in Ref. 35, such a discrepancy in the small q regime is introduced by the asymptotic expansion of the plasma dispersion function $\mathcal{Z}(\zeta)$, which requires $\zeta \gg 1$, namely, $q \gg 1$, recalling that ζ is about $q\sqrt{7/4+\tau}$.

V. SUMMARY AND DISCUSSION

Using gyro-kinetic equations in the presence of perturbed parallel magnetic vector potential, we theoretically investigated the geodesic acoustic mode (GAM) by taking into account 2nd order FOW and finite β effects. The general dispersion relation is analytically derived with the assumption of $\beta \ll \lambda \ll 1$ and adiabatic electrons. The dispersion relation is numerically evaluated and displayed in Figs. 1–6 to show the dependence of the real frequency and damping rate on λ, β, τ, and q, respectively. Transparent and concise expressions for the real frequency and damping rate are analytically obtained. Good agreement is found between analytical results and the exact numerical solution of Eq. (20), as displayed in Figs. 7 and 8.

As a comparison, the referenced result with the adiabatic electron assumption in W2011 is plotted. It is worth noting that our general dispersion relation can be reduced to W2011 by Taylor expansion and neglecting the higher order terms. There is negligible discrepancy between our numerical frequency and W2011. While for damping rate, our numerical results are qualitatively different from W2011 in the small q regime. Further

analysis shows that the discrepancy can be attributed to the higher order electromagnetic effect.

By disregarding the higher order terms and asymptotically expanding $\mathcal{Z}_n(\zeta)$ as usual, we first obtain the analytical expressions of electrostatic potentials $\delta\phi_k$ and electromagnetic vector potentials δA_k. For electrostatic potentials, our results show that $\delta\phi_k \sim \mathcal{O}(\lambda^k)\delta\phi_0$ for $k=0$, 1, 2,..., which is consistent with previous work.[39] For the electromagnetic potentials, δA_0 is exactly zero. The first harmonic δA_1 and the second harmonic δA_2 are of the order of $\mathcal{O}(\lambda\beta)\delta\phi_0$ and $\mathcal{O}(\beta)\delta\phi_0$, respectively. One can find that the dominant magnetic perturbation is the second harmonic δA_2, leading to the expression of $\delta A_\parallel$ [see Eq. (41)], which has the same form with the previous one derived from the ideal MHD model. As a result, the parallel vector potential $\delta A_\parallel$ and the corresponding perturbed magnetic field $\delta\mathbf{B}$ are of the order $\beta\delta\phi_0$ with poloidal wave number $m=2$ to the leading order, as theoretically predicted by Refs. 14, 23, 24, and 40 and numerically simulated and experimentally detected by Refs. 26–28. In addition, the analytical results show that the dominant electromagnetic potential δA_2 can be able to compare with $\delta\phi_1$ for large q, as numerically demonstrated in W2011.

The analytical expressions of the GAM frequency and Landau damping rate are presented in Eqs. (44) and (47), respectively. The analytical results can qualitatively explain and agree well with the numerical solutions. The real frequency is modified by the electromagnetic effect on the order of $q^2\beta$. Besides, the dependence of our analytical GAM frequency on β shows good agreement with the result in Refs. 33 and 34 except some coefficients. Meanwhile, the electromagnetic effect does not introduce an additional resonant mechanism in the second order, leading to the analytical damping rate (47) the same as the electrostatic one. The damping rate depends on the finite β via the real frequency Ω_G. The analytical expressions of the perturbed potentials, mode frequency, and damping rate are useful to understand recent GAM-related simulation and experimental works.[26–28]

ACKNOWLEDGMENTS

This work was supported by the China National Magnetic Confinement Fusion Energy Research Project under Grant No. 2015GB120005, the National Natural Science Foundation of China under Grant No. 11675175, the Young Elite Scientists Sponsorship Program by CAST under Grant No. 2017QNRC001, and the Innovative Program of Development Foundation of Hefei Center for Physical Science and Technology under Grant No. 2018CXFX009.

APPENDIX A: QUANTITIES IN THE MOMENT OF PERTURBED DISTRIBUTION FUNCTIONS

Here, we introduce the coefficients in the moment of perturbed distribution functions used in the procedure of derivation. For ion

$$C_0 = \frac{1}{2}\lambda^2\left(1-\frac{3}{8}\lambda^2\right) + \frac{q^2\lambda^2}{2\zeta}\left(\frac{1}{2}\mathcal{Z}_0 + \mathcal{Z}_2 + \mathcal{Z}_4\right) - \frac{q^2\lambda^4}{2\zeta}\left(\frac{1}{2}\mathcal{Z}_4 + \mathcal{Z}_2 + \frac{3}{4}\mathcal{Z}_0\right) - \frac{q^4\lambda^4}{4\zeta^3}\left\{\frac{3}{4}[\mathcal{Z}_0 - 2\bar{\mathcal{Z}}_0] + \frac{3}{2}[\mathcal{Z}_2 - 2\bar{\mathcal{Z}}_2] + \frac{3}{2}[\mathcal{Z}_4 - 2\bar{\mathcal{Z}}_4] + [\mathcal{Z}_6 - 2\bar{\mathcal{Z}}_6] + \frac{1}{2}[\mathcal{Z}_8 - 2\bar{\mathcal{Z}}_8]\right\}, \tag{A1}$$

$$C_1 = \frac{q\lambda}{4}(\mathcal{Z}_0 + 2\mathcal{Z}_2) - \frac{q^3\lambda^3}{16\zeta^2}\left\{\frac{3}{2}[\mathcal{Z}_0 - 2\bar{\mathcal{Z}}_0] + 3[\mathcal{Z}_2 - 2\bar{\mathcal{Z}}_2] + 3[\mathcal{Z}_4 - 2\bar{\mathcal{Z}}_4] + 2[\mathcal{Z}_6 - 2\bar{\mathcal{Z}}_6]\right\} - \frac{q\lambda^3}{4}(\mathcal{Z}_0 + \mathcal{Z}_2), \tag{A2}$$

$$\frac{C_1^*}{v_t} = \frac{q\lambda}{4}(\mathcal{Z}_1 + 2\mathcal{Z}_3) - \frac{q^3\lambda^3}{16\zeta^2}\left\{\frac{3}{2}[2\mathcal{Z}_1 - 2\bar{\mathcal{Z}}_1] + 3[2\mathcal{Z}_3 - 2\bar{\mathcal{Z}}_3] + 3[2\mathcal{Z}_5 - 2\bar{\mathcal{Z}}_5] + 2[2\mathcal{Z}_7 - 2\bar{\mathcal{Z}}_7]\right\} - \frac{q\lambda^3}{4}(\mathcal{Z}_1 + \mathcal{Z}_3), \tag{A3}$$

$$C_2 = \frac{q^2\lambda^2}{4\zeta}\left\{\frac{1}{2}[\mathcal{Z}_0 - \bar{\mathcal{Z}}_0] + [\mathcal{Z}_2 - \bar{\mathcal{Z}}_2] + [\mathcal{Z}_4 - \bar{\mathcal{Z}}_4]\right\}, \tag{A4}$$

$$\frac{C_2^*}{v_t} = \frac{q^2\lambda^2}{4\zeta}\left\{\frac{1}{2}[\mathcal{Z}_1 - \bar{\mathcal{Z}}_1] + [\mathcal{Z}_3 - \bar{\mathcal{Z}}_3] + [\mathcal{Z}_5 - \bar{\mathcal{Z}}_5]\right\}, \tag{A5}$$

$$\frac{C_2^{**}}{v_t^2} = \frac{q^2\lambda^2}{4\zeta}\left\{\frac{1}{2}[\mathcal{Z}_2 - \bar{\mathcal{Z}}_2] + [\mathcal{Z}_4 - \bar{\mathcal{Z}}_4] + [\mathcal{Z}_6 - \bar{\mathcal{Z}}_6]\right\}, \tag{A6}$$

$$D_1 = 1 + \zeta\mathcal{Z}_0\left(1-\frac{1}{2}\lambda^2\right) - 2C_2, \tag{A7}$$

$$\frac{D_1^*}{v_t} = \zeta Z_1\left(1-\frac{1}{2}\lambda^2\right) - 2\frac{C_2^*}{v_t}, \tag{A8}$$

$$\frac{D_1^{**}}{v_t^2} = \zeta Z_2\left(1-\frac{1}{2}\lambda^2\right) - 2\frac{C_2^{**}}{v_t^2}, \tag{A9}$$

$$D_2 = \frac{q\lambda}{4}\left[(\mathcal{Z}_0 - \bar{\mathcal{Z}}_0) + 2(\mathcal{Z}_2 - \bar{\mathcal{Z}}_2)\right], \tag{A10}$$

$$\frac{D_2^*}{v_t} = \frac{q\lambda}{4}\left[(\mathcal{Z}_1 - \bar{\mathcal{Z}}_1) + 2(\mathcal{Z}_3 - \bar{\mathcal{Z}}_3)\right], \tag{A11}$$

$$\frac{D_2^{**}}{v_t^2} = \frac{q\lambda}{4}\left[(\mathcal{Z}_2 - \bar{\mathcal{Z}}_2) + 2(\mathcal{Z}_4 - \bar{\mathcal{Z}}_4)\right], \tag{A12}$$

$$I_c = \frac{q^2\lambda^2}{4\zeta}\left[\frac{1}{2}\mathcal{Z}_0 + \mathcal{Z}_2 + \mathcal{Z}_4\right], \tag{A13}$$

$$\frac{I_c^*}{v_t} = 0, \tag{A14}$$

$$\frac{I_c^{**}}{v_t^2} = \frac{q^2\lambda^2}{4\zeta}\left[\frac{1}{2}\mathcal{Z}_2 + \mathcal{Z}_4 + \mathcal{Z}_6\right], \tag{A15}$$

where $y_\alpha = \frac{v_\parallel}{v_{t\alpha}}$ and $\bar{\mathcal{Z}}_n = \mathcal{Z}_n\left(\frac{\zeta}{2}\right)$. For the adiabatic electron, we have $D_{1e} = 1$, $\frac{C_{1e}^*}{v_t} = -\frac{1}{2}\tau q\lambda$, $\frac{D_{1e}^*}{v_t} = \zeta$, $\frac{C_{2e}^{**}}{v_t^2} = \frac{7}{32}q^2\lambda^2\tau^2$, $\frac{D_{1e}^{**}}{v_t^2} = \zeta^2 - \frac{7}{16}q^2\lambda^2\tau^2$, $\frac{D_{2e}^{**}}{v_t^2} = -\frac{1}{4}\tau q\lambda\zeta$, $\frac{I_{ce}^{**}}{v_t^2} = \frac{7}{16}q^2\lambda^2\tau^2$.

For simplicity, we mark coefficients like C_0 as $C_0 = \lambda^2 C_0^{(2)} + \lambda^4 C_0^{(4)}$, $C_1 = \lambda C_1' = \lambda C_1^{(1)} + \lambda^3 C_1^{(3)}$, $C_2 = \lambda^2 C_2'$, $D_c = \lambda^2 D_c'$, $D_2 = \lambda D_2'$, $C_2^* = \lambda^2 C_2^{*\prime}$, $D_c^* = \lambda^2 D_c^{*\prime}$, and $D_2^* = \lambda D_2^{*\prime}$, in which

$$\lambda^2 C_0^{(2)} = \frac{1}{2}\lambda^2 + \frac{q^2\lambda^2}{2\zeta}\left(\frac{1}{2}\mathcal{Z}_0 + \mathcal{Z}_2 + \mathcal{Z}_4\right), \tag{A16}$$

$$\lambda^4 C_0^{(4)} = -\frac{3}{16}\lambda^4 - \frac{q^2\lambda^4}{2\zeta}\left(\frac{1}{2}\mathcal{Z}_4 + \mathcal{Z}_2 + \frac{3}{4}\mathcal{Z}_0\right) - \frac{q^4\lambda^4}{4\zeta^3}\left[\frac{3}{4}(\mathcal{Z}_0 - 2\bar{\mathcal{Z}}_0) + \frac{3}{2}(\mathcal{Z}_2 - 2\bar{\mathcal{Z}}_2) + \frac{3}{2}(\mathcal{Z}_4 - 2\bar{\mathcal{Z}}_4) + (\mathcal{Z}_6 - 2\bar{\mathcal{Z}}_6) + \frac{1}{2}(\mathcal{Z}_8 - 2\bar{\mathcal{Z}}_8)\right], \tag{A17}$$

$$\lambda C_1^{(1)} = \frac{q\lambda}{4}(\mathcal{Z}_0 + 2\mathcal{Z}_2), \tag{A18}$$

$$\lambda^3 C_1^{(3)} = -\frac{q^3\lambda^3}{16\zeta^2}\left[\frac{3}{2}(\mathcal{Z}_0 - 2\bar{\mathcal{Z}}_0) + 3(\mathcal{Z}_2 - 2\bar{\mathcal{Z}}_2) + 3(\mathcal{Z}_4 - 2\bar{\mathcal{Z}}_4) + 2(\mathcal{Z}_6 - 2\bar{\mathcal{Z}}_6)\right] - \frac{q\lambda^3}{4}(\mathcal{Z}_1 + \mathcal{Z}_3), \tag{A19}$$

$$D_c = -\frac{1}{2}\zeta\mathcal{Z}_0\lambda^2 - 2C_2 + I_c, \tag{A20}$$

$$D_c^* = -\frac{1}{2}v_t\zeta\mathcal{Z}_1\lambda^2 - 2C_2^*. \tag{A21}$$

APPENDIX B: DETAILS OF THE DERIVATION OF THE ELECTROMAGNETIC GAM

By substituting Eqs. (15)–(18) into $\delta n_{i0} = \delta n_{e0}$ [Eq. (9)], we have

$$\hat{H}_1\delta\phi_0 + 2i\tau\lambda\hat{H}_2\delta\phi_1 + 2\hat{H}_3\delta\phi_2 + 2i\lambda\hat{H}_4 v_t\delta A_1 - 2\hat{H}_5 v_t\delta A_2 = 0, \tag{B1}$$

where

$$\lambda^2\hat{H}_1 = C_0 - \frac{2\tau C_1^2}{1+\tau\mathcal{Z}_1}, \tag{B2}$$

$$\lambda^3\hat{H}_2 = \frac{C_1 D_c}{1+\tau\mathcal{Z}_1}, \tag{B3}$$

$$\lambda^2\hat{H}_3 = C_2 - \frac{\tau C_1 D_2}{1+\tau\mathcal{Z}_1}, \tag{B4}$$

$$\lambda^3 v_t\hat{H}_4 = C_1^* - \zeta v_t C_1 - \frac{1}{2}q\lambda v_t - \frac{\tau C_1 D_c^*}{1+\tau\mathcal{Z}_1}, \tag{B5}$$

$$\lambda^2 v_t\hat{H}_5 = C_2^* - \frac{\tau C_1 D_2^*}{1+\tau\mathcal{Z}_1}. \tag{B6}$$

To obtain a more clear dispersion relation of the electromagnetic GAM, we should rearrange the form of $\delta\phi_k$ and δA_k. Combining with the quasi-neutrality condition, we can rearrange the perturbed parallel vector potential [Eqs. (12) and (13)] as

$$\delta A_1 = \frac{2\beta}{v_t}\left(-i\lambda\hat{C}_1'\delta\phi_0 - \hat{C}_2'\delta\phi_1 + \hat{C}_4' v_t\delta A_1 + i\frac{\hat{C}_5'}{\lambda}v_t\delta A_2\right), \tag{B7}$$

$$\delta A_2 = \frac{2\beta}{v_t}\left(-\hat{D}_1'\delta\phi_0 + i\lambda\hat{D}_2'\delta\phi_1 - \hat{D}_3'\delta\phi_2 - i\frac{\hat{D}_4'}{\lambda}v_t\delta A_1 + \hat{D}_5' v_t\delta A_2\right), \tag{B8}$$

where

$$\lambda^3 v_t\hat{C}_1 = C_1^* - \zeta v_t C_1 - \frac{1}{2}q\lambda v_t, \tag{B9}$$

$$\lambda^2 v_t\hat{C}_2 = D_c^* - \zeta v_t D_c, \tag{B10}$$

$$\lambda v_t\hat{C}_3 = D_2^* - \zeta v_t D_2, \tag{B11}$$

$$\lambda^2 v_t^2\hat{C}_4 = D_c^{**} - \zeta v_t D_c^*, \tag{B12}$$

$$\lambda v_t^2\hat{C}_5 = D_{2e}^{**}/\tau + D_2^{**} - \zeta v_t D_2^*, \tag{B13}$$

$$\lambda^2 v_t\hat{D}_1 = C_2^* - \frac{1}{2}\zeta v_t C_2, \tag{B14}$$

$$\lambda v_t\hat{D}_2 = D_2^* - \frac{1}{2}\zeta v_t D_2, \tag{B15}$$

$$\lambda v_t^2\hat{D}_4 = D_{2e}^{**}/\tau + D_2^{**} - \frac{1}{2}\zeta v_t D_2^*, \tag{B16}$$

and

$$v_t\hat{C}_1' = v_t\hat{C}_1 - \frac{\tau C_2' v_t\hat{C}_3}{1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)}, \tag{B17}$$

$$v_t\hat{C}_2' = v_t\hat{C}_2 - \frac{\tau D_2' v_t\hat{C}_3}{1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)}, \tag{B18}$$

$$v_t^2\hat{C}_4' = v_t^2\hat{C}_4 - \frac{\tau D_2^{*\prime} v_t\hat{C}_3}{1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)}, \tag{B19}$$

$$v_t^2\hat{C}_5' = v_t^2\hat{C}_5 - \frac{1}{2}\zeta v_t^2\hat{C}_3 = 0, \tag{B20}$$

as well as

$$v_t\hat{D}_1' = v_t\hat{D}_1 - \frac{\tau C_1' v_t\hat{D}_2}{1+\tau\mathcal{Z}_1}, \tag{B21}$$

$$v_t\hat{D}_2' = -\frac{\tau D_c' v_t\hat{D}_2}{1+\tau\mathcal{Z}_1}, \tag{B22}$$

$$v_t\hat{D}_3' = -\frac{\tau D_2' v_t\hat{D}_2}{1+\tau\mathcal{Z}_1}, \tag{B23}$$

$$v_t^2\hat{D}_4' = v_t^2\hat{D}_4 - \frac{\tau\lambda^2 D_c^{*\prime} v_t\hat{D}_2}{1+\tau\mathcal{Z}_1}, \tag{B24}$$

$$v_t^2\hat{D}_5' = -\frac{\tau D_2^{*\prime} v_t\hat{D}_2}{1+\tau\mathcal{Z}_1}. \tag{B25}$$

In order to gain the final expressions of δA_k and $\delta\phi_k$, we also introduce

$$\delta\phi_{1E} = \lambda\delta\hat{\phi}_{1E}\delta\phi_0 = i\tau\frac{\tau C_2 D_2 - C_1\left[1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)\right]}{[1+\tau(D_1+I_c)]\left[1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)\right] - \tau^2 D_2^2}\delta\phi_0, \tag{B26}$$

$$\delta\phi_{2E} = \lambda^2\delta\hat{\phi}_{2E}\delta\phi_0 = \tau\frac{\tau C_1 D_2 - C_2[1+\tau(D_1+I_c)]}{[1+\tau(D_1+I_c)]\left[1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)\right] - \tau^2 D_2^2}\delta\phi_0. \tag{B27}$$

By substituting $\delta\phi_{kE}$ into the expressions of $\delta\phi_k$ and δA_k, we arrive at

$$\delta A_1 = \frac{2\beta}{K v_t}\left\{\lambda M_{10} + \lambda M_{11}\delta\hat{\phi}_{1E} + \lambda^2 M_{12}\delta\hat{\phi}_{2E}\right\}\delta\phi_0, \tag{B28}$$

$$\delta A_2 = \frac{2\beta}{K v_t}\left\{M_{20} + \lambda M_{21}\delta\hat{\phi}_{1E} + \lambda^2 M_{22}\delta\hat{\phi}_{2E}\right\}\delta\phi_0, \tag{B29}$$

$$\delta\phi_1 = \lambda\delta\hat{\phi}_{1E}\delta\phi_0 + H_{11}v_t\delta A_1 + i\lambda H_{12}v_t\delta A_2, \tag{B30}$$

$$\delta\phi_2 = \lambda^2\delta\hat{\phi}_{2E}\delta\phi_0 + i\lambda H_{21}v_t\delta A_1 + H_{22}v_t\delta A_2, \tag{B31}$$

where

$$a = 1 - 2\beta\left\{-\hat{C}_2' H_{11} + \hat{C}_4'\right\}, \tag{B32}$$

$$d = 1 - 2\beta\left\{-\lambda^2\hat{D}_2' H_{12} - \hat{D}_3' H_{22} + \hat{D}_5'\right\}, \tag{B33}$$

$$ib = -i2\beta\left\{-\lambda\hat{C}_2' H_{12} + \frac{\hat{C}_5'}{\lambda}\right\}, \tag{B34}$$

$$ic = -i2\beta\left\{\lambda\hat{D}_2' H_{11} - \lambda\hat{D}_3' H_{21} - \frac{\hat{D}_4'}{\lambda}\right\}, \tag{B35}$$

$$K = ad + bc, \tag{B36}$$

$$M_{10} = -id\hat{C}_1' + i\frac{b}{\lambda}\hat{D}_1', \tag{B37}$$

$$M_{11} = -d\hat{C}_2' + b\lambda\hat{D}_2', \tag{B38}$$

$$M_{12} = ib\hat{D}_3', \tag{B39}$$

$$M_{20} = -c\lambda\hat{C}_1' - a\hat{D}_1', \tag{B40}$$

$$M_{21} = ic\hat{C}_2' + ia\lambda\hat{D}_2', \tag{B41}$$

$$M_{22} = -a\hat{D}_3', \tag{B42}$$

and

$$v_t H_{11} = \frac{(\zeta v_t + \tau D_1^*)[1+\tau\mathcal{Z}_1(\zeta/2)]}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2} - \frac{\tau^2 D_2 D_2^*}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2}, \tag{B43}$$

$$v_t H_{12} = \frac{\tau D_2^{*\prime}[1+\tau\mathcal{Z}_1(\zeta/2)]}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2} - \frac{\frac{1}{2}\zeta v_t\left[1+\tau\mathcal{Z}_1\left(\frac{\zeta}{2}\right)\right]\tau D_2'}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2}, \tag{B44}$$

$$v_t H_{21} = \frac{\tau D_2'[\zeta v_t + \tau D_1^*]}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2} - \frac{\tau D_2^{*\prime}[1+\tau(D_1+I_c)]}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2}, \tag{B45}$$

$$v_t H_{22} = \frac{\frac{1}{2}\zeta v_t[1+\tau\mathcal{Z}_1(\zeta/2)][1+\tau(D_1+I_c)]}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2} - \frac{\tau^2 D_2 D_2^*}{[1+\tau(D_1+I_c)][1+\tau\mathcal{Z}_1(\zeta/2)] - \tau^2 D_2^2}. \tag{B46}$$

By substituting the final expression of $\delta\phi_k$ and δA_k into Eq. (B1), we obtain the general dispersion relation of the electromagnetic GAM as Eq. (20).

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