Polynomial Chaos Based Uncertainty Quantification

SAND2015-0639PE

Lecture 3: Characterization of Random Variables and Processes

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DATE – LOCATION

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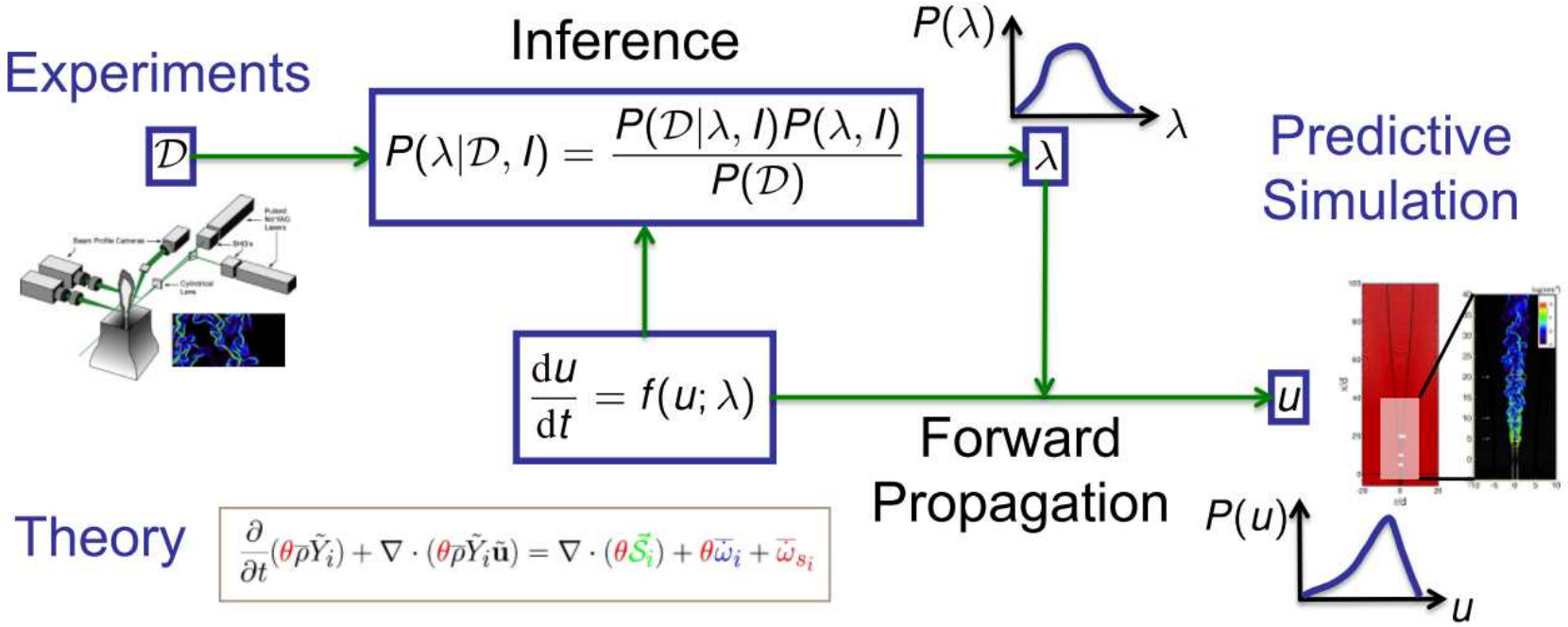
Goals of This Tutorial

- 4 Lectures
  - Lecture 1: Context and Fundamentals
  - Lecture 2: Forward Propagation
  - Lecture 3: Characterization
    - Sensitivity Analysis
    - Representing Arbitrary Random Variables
    - Representing Random Fields
  - Lecture 4: Bayesian Inference – UQ Software

Outline

Sensitivity analysis gives insight into key sources of uncertainty



- Obtaining global sensitivity analysis from PCEs
  - Identify dominant sources of uncertainty
  - Attribution

PC Postprocessing: global sensitivity information is readily obtained from PCE

$$g(\xi_1, \ldots, \xi_d) = \sum_{k=0}^{P} c_k \Psi_k(\boldsymbol{\xi})$$

- Global sensitivity analysis $\equiv$ Variance decomposition
- Total variance

$$Var[g(\boldsymbol{\xi})] = \sum_{k>0} c_k^2 ||\Psi_k||^2$$

PC Postprocessing: Main Effect and Joint Sensitivity Indices

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{\xi}|\xi_i)]}{Var[g(\boldsymbol{\xi})]} = \frac{\sum_{k\in\mathbb{I}_i} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

  - $\mathbb{I}_i$ is the set of bases with only ξ_i involved
  - S_i is the uncertainty contribution that is due to i-th parameter only

- Joint sensitivity indices

$$S_{ij} = \frac{Var[\mathbb{E}(g(\boldsymbol{\xi}|\xi_i,\xi_j)]}{Var[g(\boldsymbol{\xi})]} - S_i - S_j = \frac{\sum_{k\in\mathbb{I}_{ij}} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

  - $\mathbb{I}_{ij}$ is the set of bases with only ξ_i and ξ_j involved
  - S_{ij} is the uncertainty contribution that is due to (i,j) parameter pair

PC Postprocessing: Total Effect Sensitivity Indices

- Total effect sensitivity indices

$$T_i = 1 - \frac{Var[\mathbb{E}(g(\boldsymbol{\xi}|\xi_{-i})]}{Var[g(\boldsymbol{\xi})]} = \frac{\sum_{k \in \mathbb{I}_i^T} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

  - The notation ξ_{-i} indicates terms that do not have ξ_i in them
  - $\mathbb{I}_i^T$ is the set of bases with ξ_i involved, including all its interactions
  - The sum of all T_i is usually > 1

Sensitivity indices are directly computable from PC

$$g(\xi) = \sum_{k=0}^{P} c_k \Psi_k(\xi)$$

Consider dimensionality $d = 3$, total order $p = 2$,
number of PC terms $P + 1 = (d + p)!/(d!p!) = 10$.

$$g(\xi_1, \xi_2, \xi_3) = c_0 + c_1\psi_1(\xi_1) + c_2\psi_1(\xi_2) + c_3\psi_1(\xi_3) +$$

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Variance contributions

$$Var(g) = 0 + c_1^2\langle\psi_1^2\rangle + c_2^2\langle\psi_1^2\rangle + c_3^2\langle\psi_1^2\rangle +$$

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Main effect sensitivities ξ_1 ξ_2 ξ_3

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Joint sensitivities (ξ_1, ξ_2) (ξ_1, ξ_3) (ξ_2, ξ_3)

PC Postprocessing: Sampling-Based Approaches

$$g(\xi_1, \ldots, \xi_d) = \sum_{k=0}^{P} c_k \Psi_k(\boldsymbol{\xi})$$

- In some cases, need to resort to Monte-Carlo estimation, e.g.
  - Piecewise-PC with irregular subdomains
  - Output transformations, e.g. build PC for $\log g(\boldsymbol{\xi})$, but inquire sensitivity with respect to $g(\boldsymbol{\xi})$
- A brute-force sampling of $Var[\mathbb{E}(g(\boldsymbol{\xi}|\xi_i)]$ is extremely inefficient.

PC Postprocessing: Sampling-Based Approaches

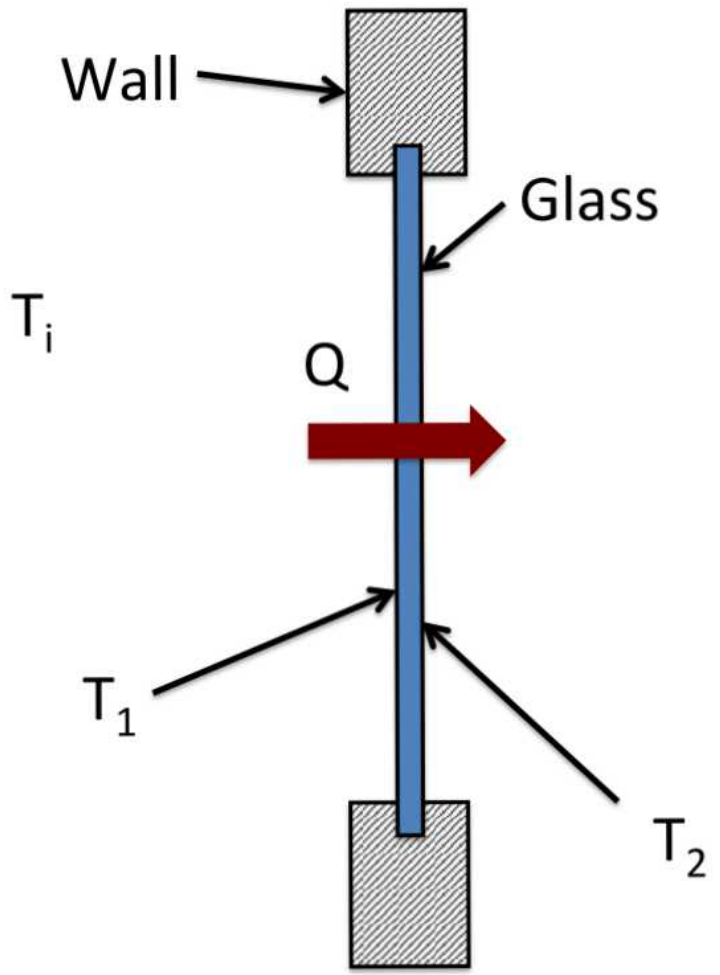
- Tricks are available, given a single set of sampled input [Saltelli, 2002]. E.g., use

$$\mathbb{E}[g(\boldsymbol{\xi}|\xi_i)^2] = \mathbb{E}[g(\boldsymbol{\xi}|\xi_i)g(\boldsymbol{\xi}'|\xi_i)] = \frac{1}{N-1}\sum_{r=1}^{N} g(\boldsymbol{\xi}^{(r)})g(\tilde{\boldsymbol{\xi}}^{(r)}),$$

  where $\tilde{\boldsymbol{\xi}}$ is $\boldsymbol{\xi}'$ with i-th element replaced by ξ_i.
- Similar formulae available for joint sensitivity indices.

- Con: as all Monte-Carlo algorithms, converges slowly.
- Pro: sampling is cheap.

Heat Transfer through a Window



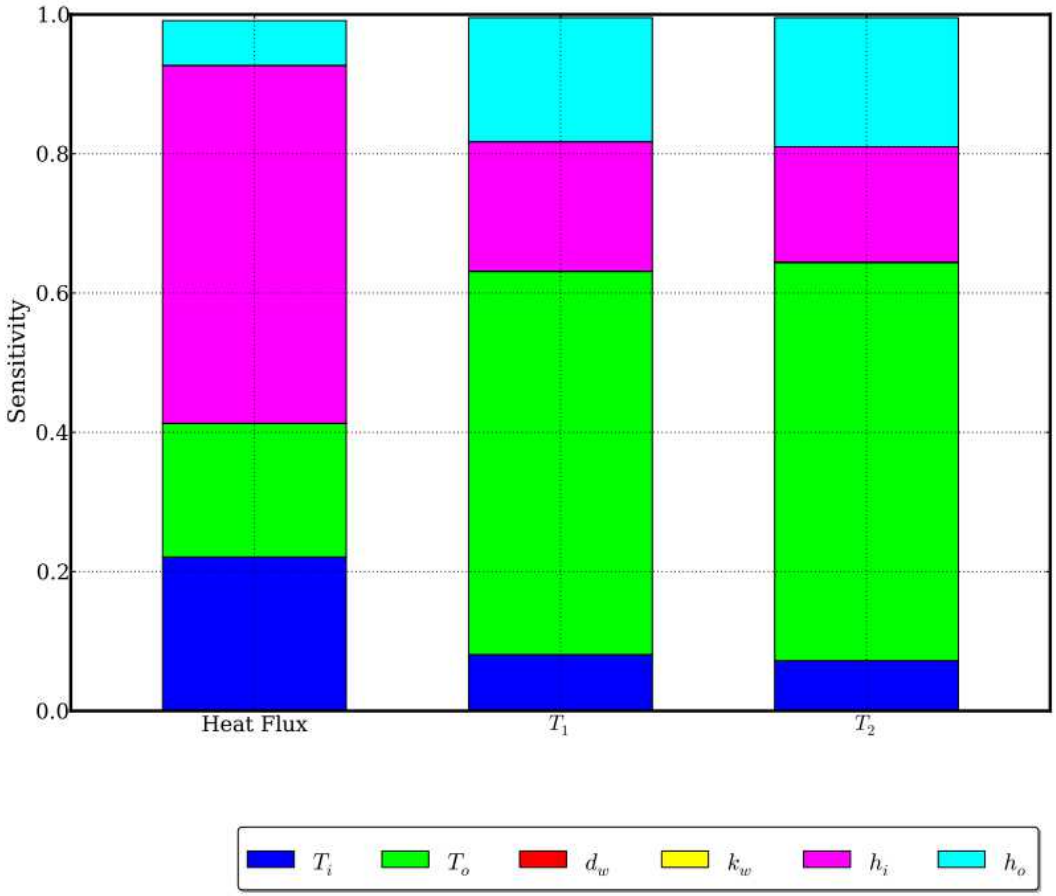
$$h_i(T_i - T_1) = k_w \frac{(T_1 - T_2)}{d_w}$$

$$k_w \frac{(T_1 - T_2)}{d_w} = h_o(T_2 - T_o)$$

6 Uncertain, Gaussian parameters

$$T_i = 293\text{K}, \sigma = 0.5\%$$
$$T_o = 273\text{K}, \sigma = 0.5\%$$
$$d_w = 0.01\text{m}, \sigma = 1\%$$
$$k_w = 1\text{W/mK}, \sigma = 5\%$$
$$h_i = 2\text{W/m}^2\text{K}, \sigma = 15\%$$
$$h_o = 6\text{W/m}^2\text{K}, \sigma = 15\%$$

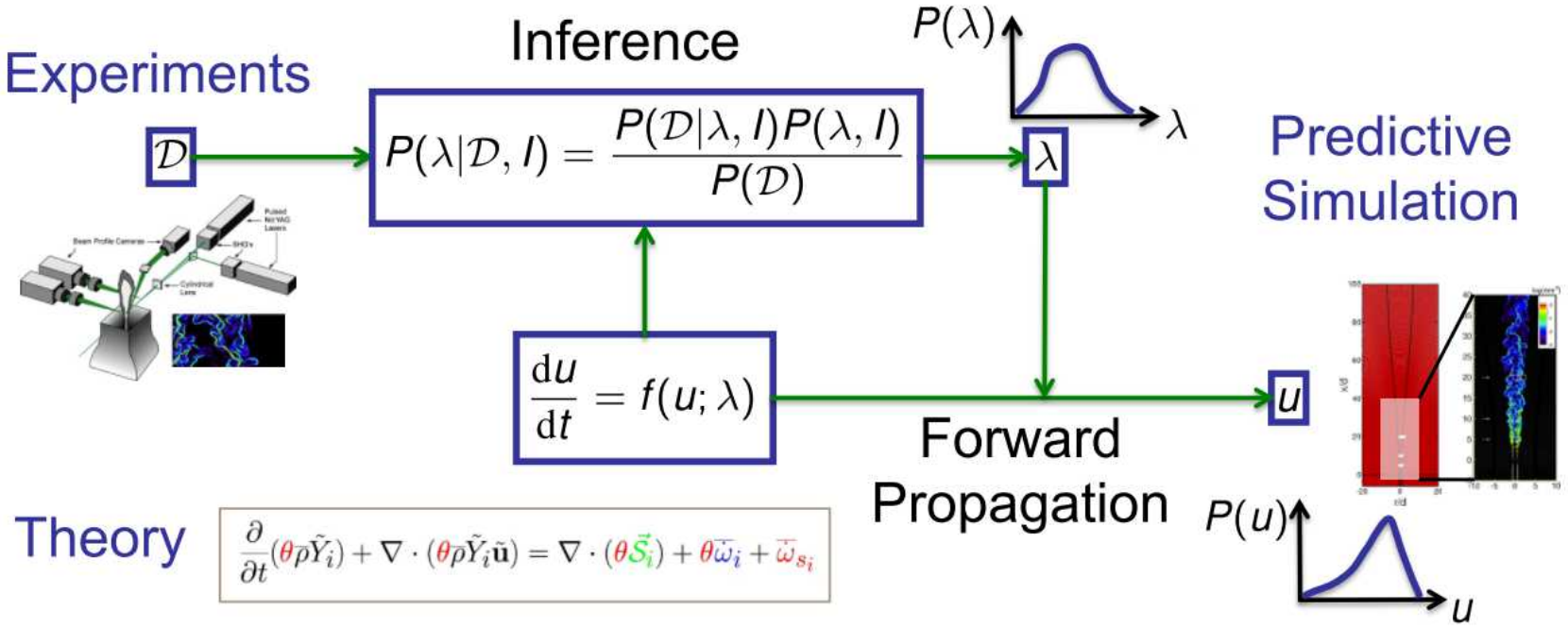
Outputs are most sensitive to ambient temperatures and convective heat transfer coefficients



- Main effect sensitivities
  - Sum to 1 only if coupling terms do not matter
- k_w has minimal contribution due to its low uncertainty

Outline

Obtaining PCEs for Uncertain Inputs

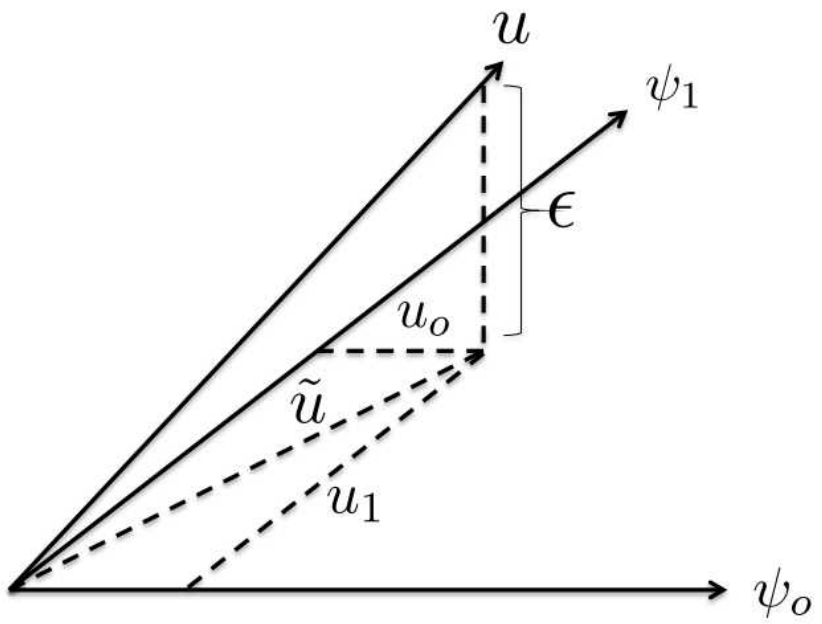


- Representation of uncertain inputs λ with PCEs

Obtaining PC coefficients for arbitrary random variables is not trivial

- Characterizing PCEs for uncertain inputs is a really difficult problem
- Inputs specified in a variety of ways, and often incomplete
  - Probability density function
  - Samples, *e.g.* from inverse problem solution
  - Expert opinion (*e.g.* “about 3.5”)
- Particular case of a random variable specified by a PDF is generally tractable

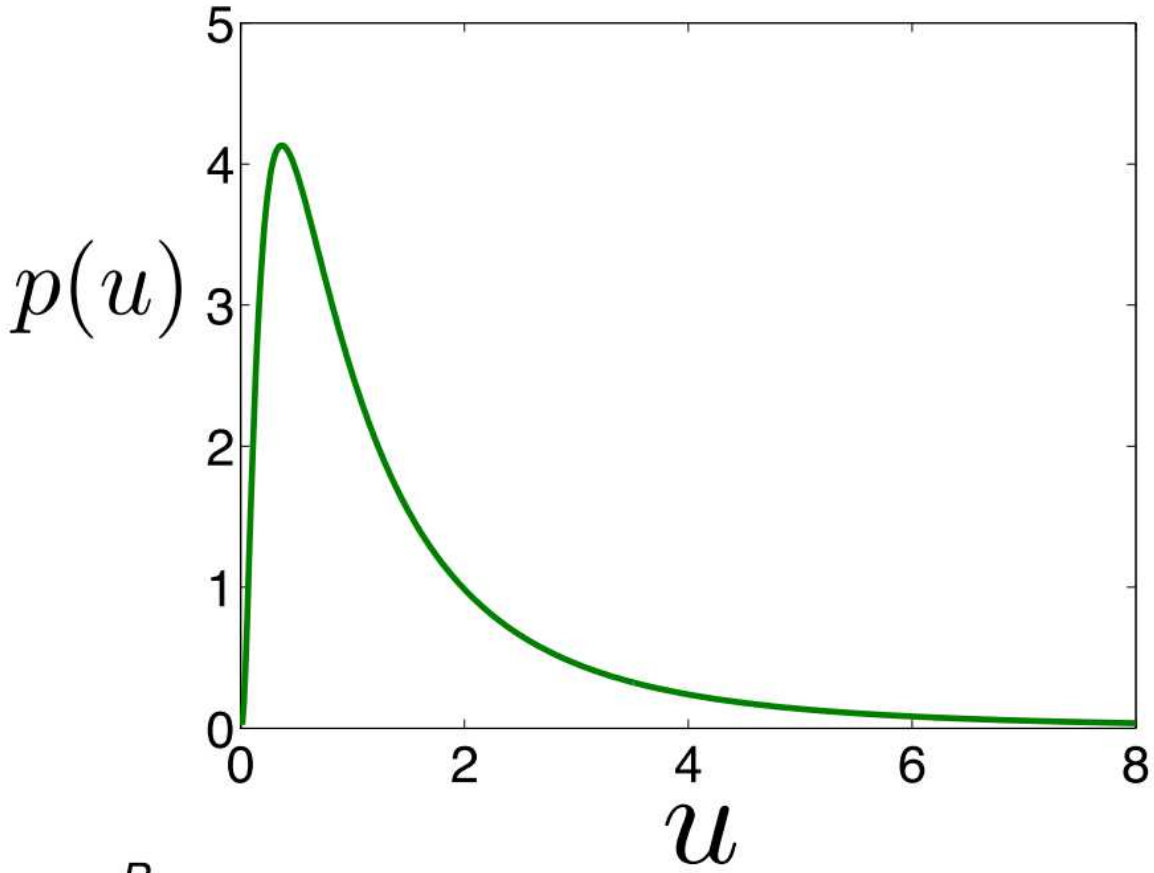
Orthogonality enables a Galerkin projection to determine the PC coefficients



$$u \approx \sum_{k=0}^{P} u_k \Psi_k \qquad \Rightarrow \qquad \langle \Psi_i u \rangle = \sum_{k=0}^{P} u_k \langle \Psi_i \Psi_k \rangle = u_i \langle \Psi_i^2 \rangle$$

$$\Rightarrow \qquad u_i = \frac{\langle u \Psi_i \rangle}{\langle \Psi_i^2 \rangle}$$

Galerkin projection requires functional relationship between random variable and germ of PCE



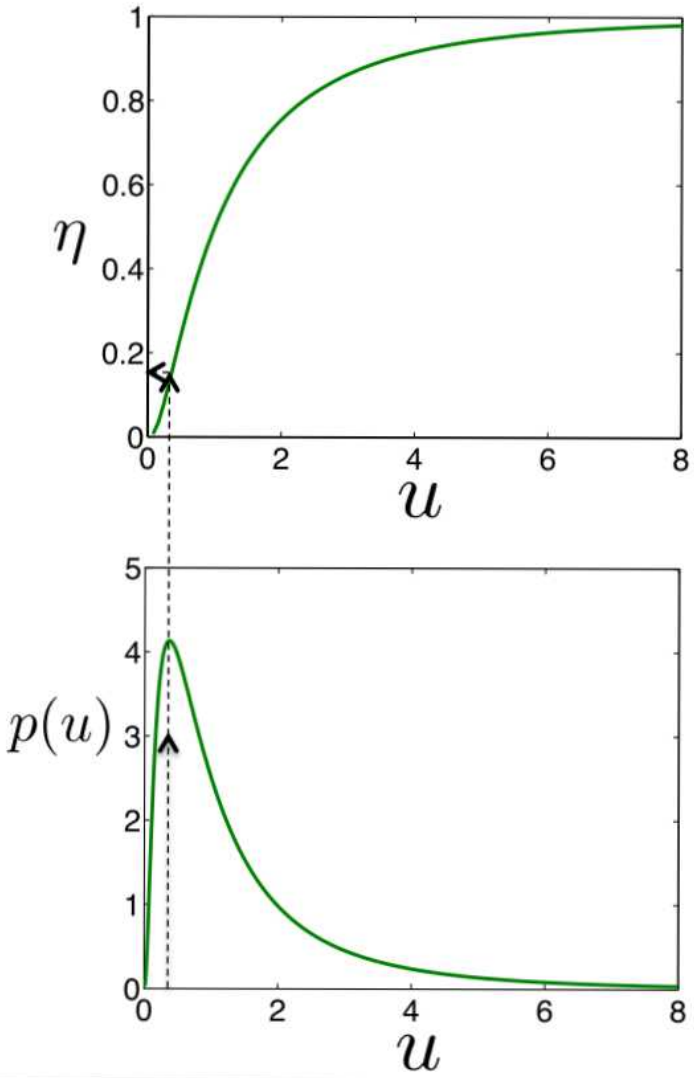
$$u = \sum_{k=0}^{P} u_k \Psi_k(\xi) \qquad \Rightarrow \qquad u_i = \frac{\langle u\, \Psi_i(\xi) \rangle}{\langle \Psi_i^2 \rangle}$$

Cumulative Distribution Function (CDF) maps arbitrary random variable to a uniform random variable

- Consider u with PDF $p(u)$
- CDF of u is given by

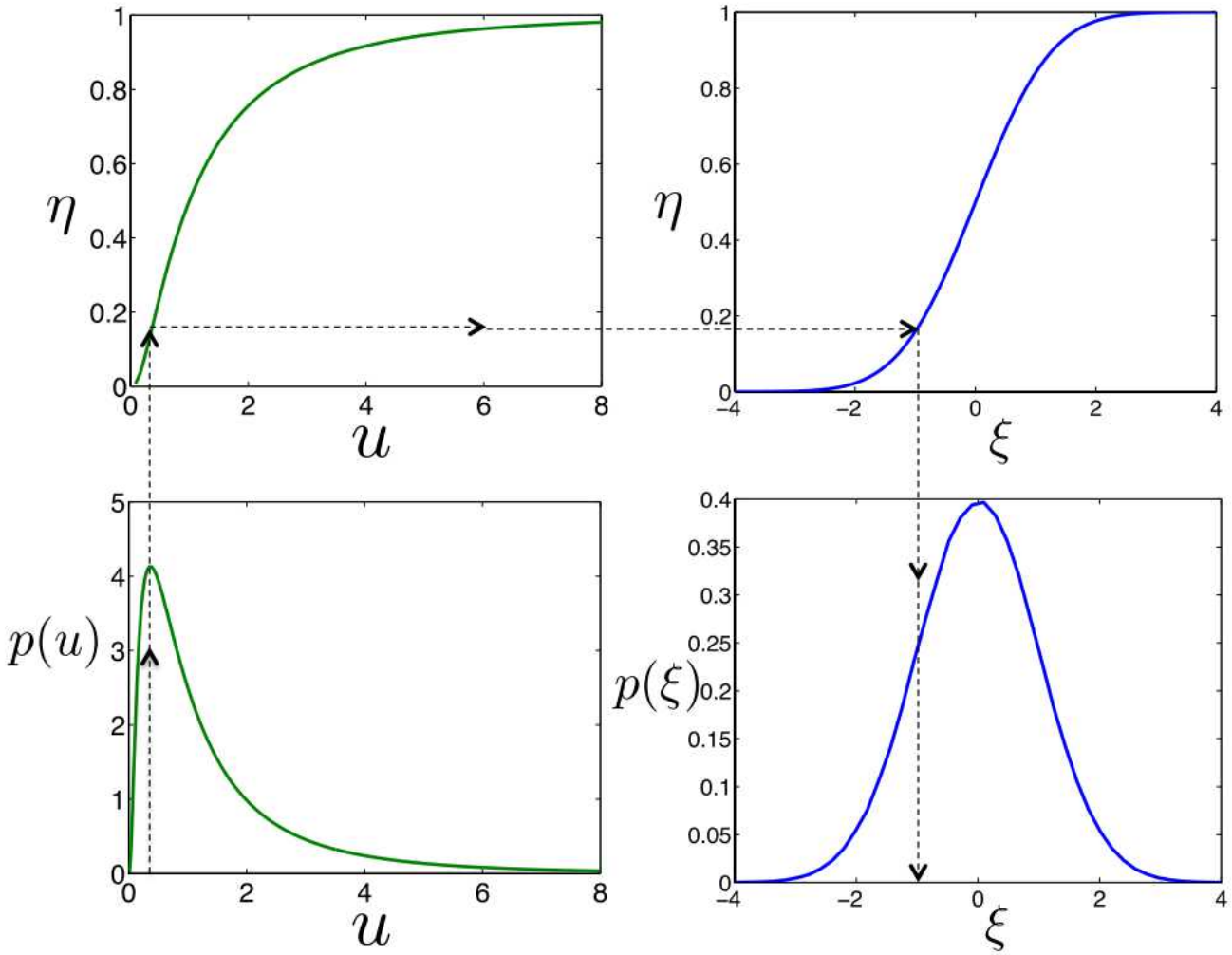
$$F(u) = \int_{-\infty}^{u} p(s)ds$$

- $F(u)$ maps u to η, uniform on $[0, 1]$



Inverse CDF mapping enables Galerkin Projection

- $\eta = F(u)$
- $\eta = \Phi(\xi)$ maps uniform η to normal RV ξ
- $u = F^{-1}(\Phi(\xi))$

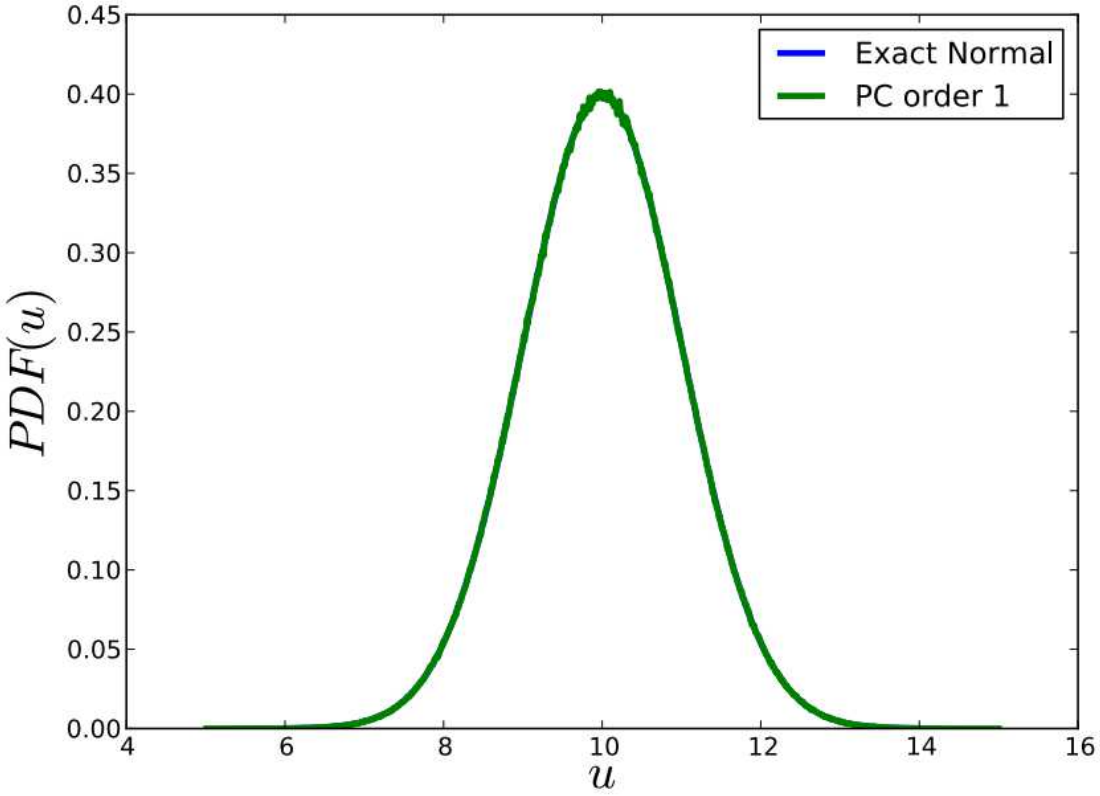


$$\langle u\, \Psi_i(\xi) \rangle = \int \underbrace{F^{-1}(\Phi(\xi))}_{u} \Psi_i(\xi) w(\xi) d\xi$$

PC Illustration: PC Expansion for a Normal RV

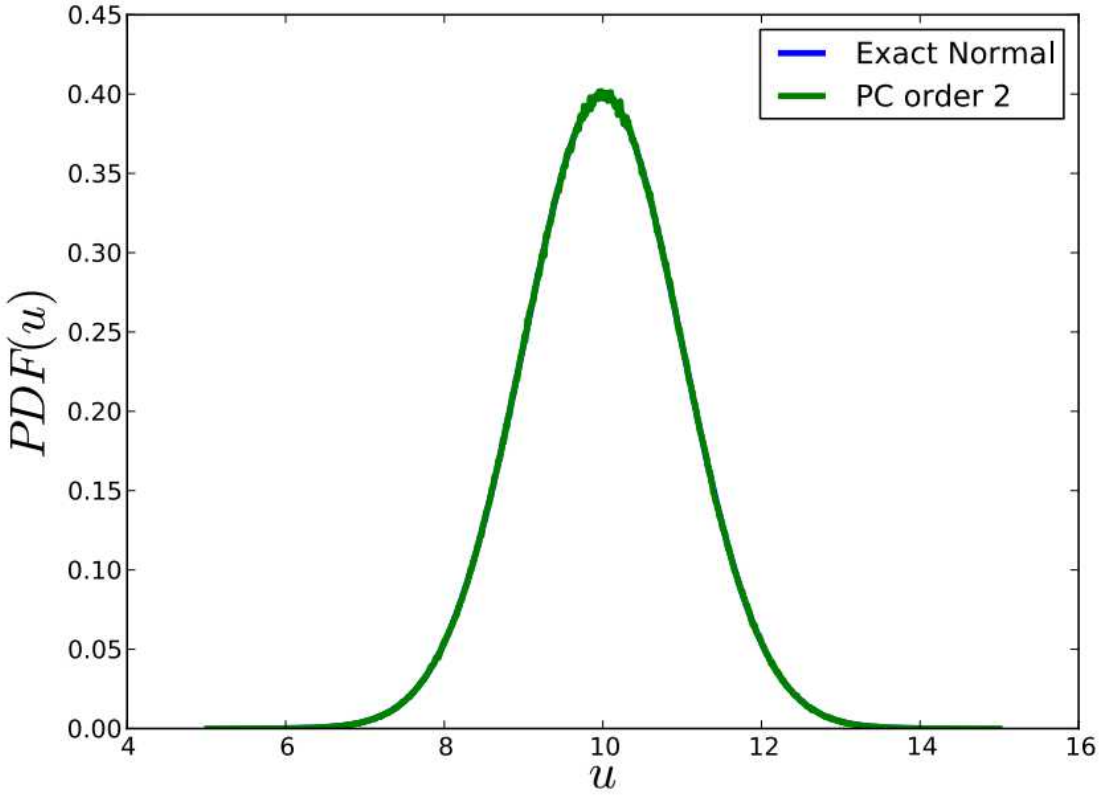
- Wiener-Hermite PCE constructed for a Normal RV
- PCE-sampled PDF superposed on true PDF
- Order = 1

$$
\begin{aligned}
u &= \sum_{k=0}^{P} u_k \Psi_k(\xi) \\
&= u_0 + u_1 \xi
\end{aligned}
$$



PC Illustration: PC Expansion for a Normal RV

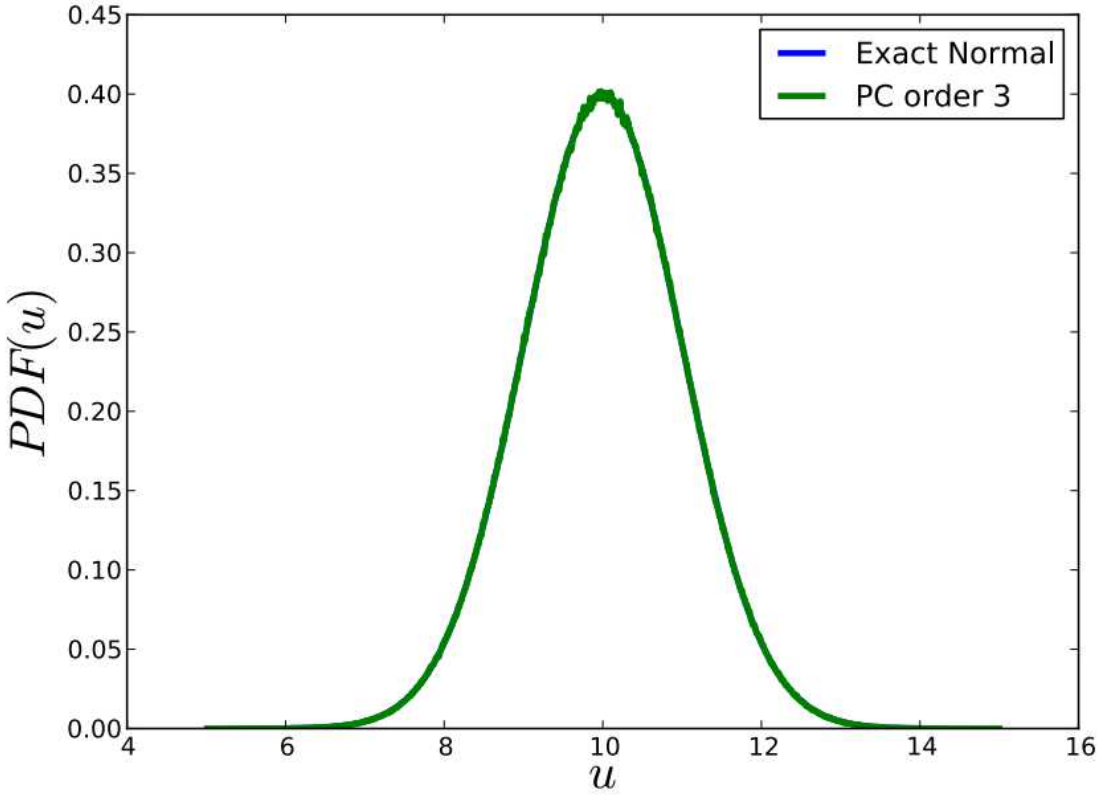
- Wiener-Hermite PCE constructed for a Normal RV
- PCE-sampled PDF superposed on true PDF
- Order = 2



$$
\begin{aligned}
u &= \sum_{k=0}^{P} u_k \Psi_k(\xi) \\
&= u_0 + u_1 \xi + u_2(\xi^2 - 1)
\end{aligned}
$$

PC Illustration: PC Expansion for a Normal RV

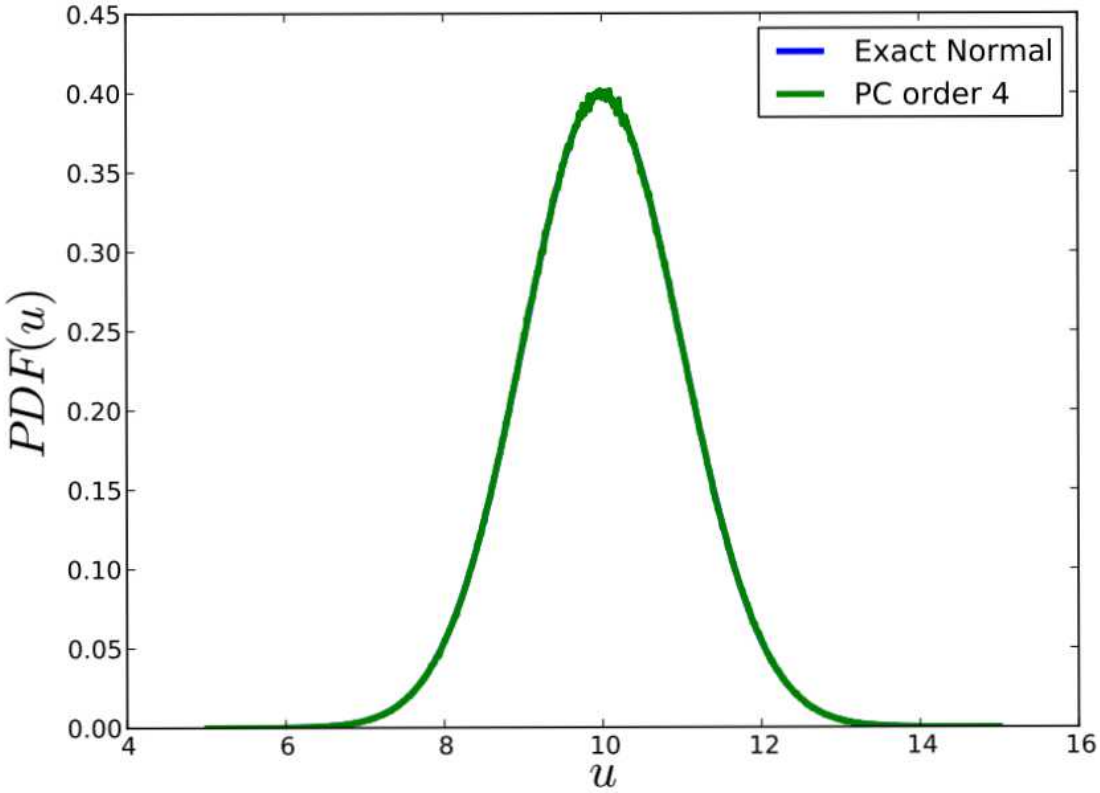
- Wiener-Hermite PCE constructed for a Normal RV
- PCE-sampled PDF superposed on true PDF
- Order = 3



$$u = \sum_{k=0}^{P} u_k \Psi_k(\xi)$$
$$= u_0 + u_1\xi + u_2(\xi^2 - 1) + u_3(\xi^3 - 3\xi)$$

PC Illustration: PC Expansion for a Normal RV

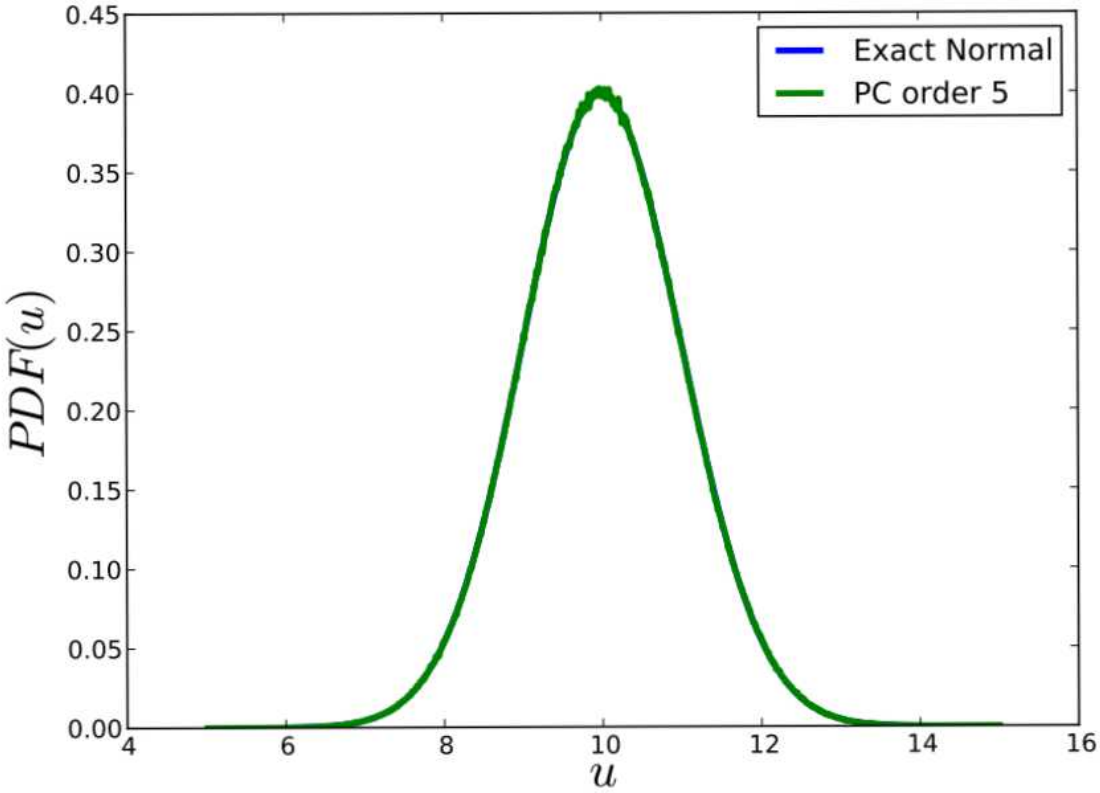
- Wiener-Hermite PCE constructed for a Normal RV
- PCE-sampled PDF superposed on true PDF
- Order = 4



$$u = \sum_{k=0}^{P} u_k \Psi_k(\xi)$$
$$= u_0 + u_1\xi + u_2(\xi^2 - 1) + u_3(\xi^3 - 3\xi) + u_4(\xi^4 - 6\xi^2 + 3)$$

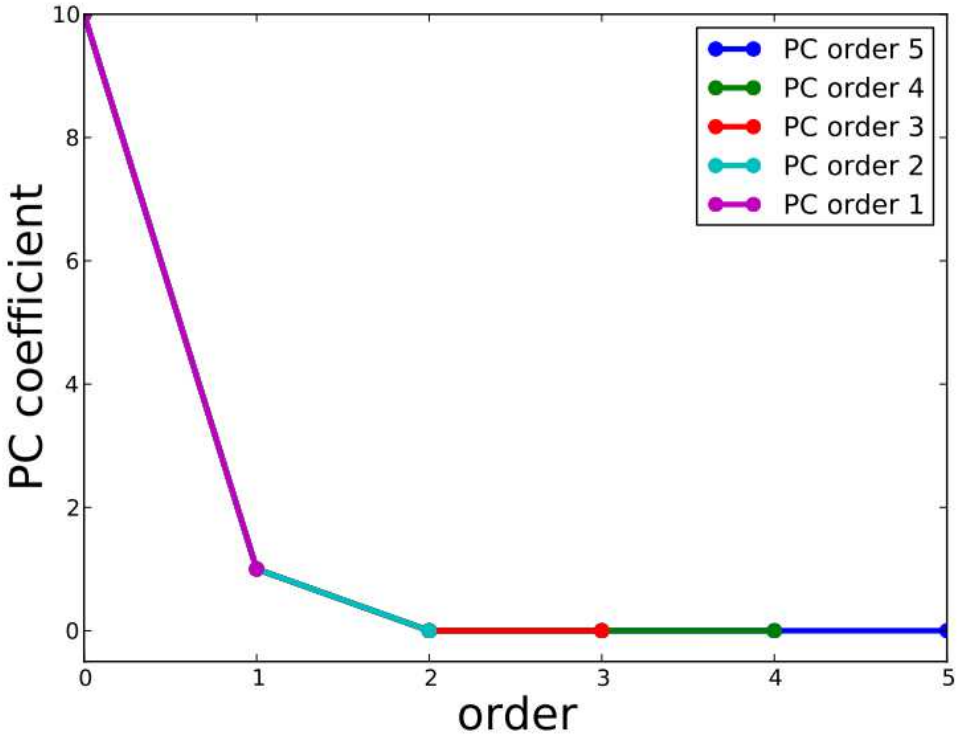
PC Illustration: PC Expansion for a Normal RV

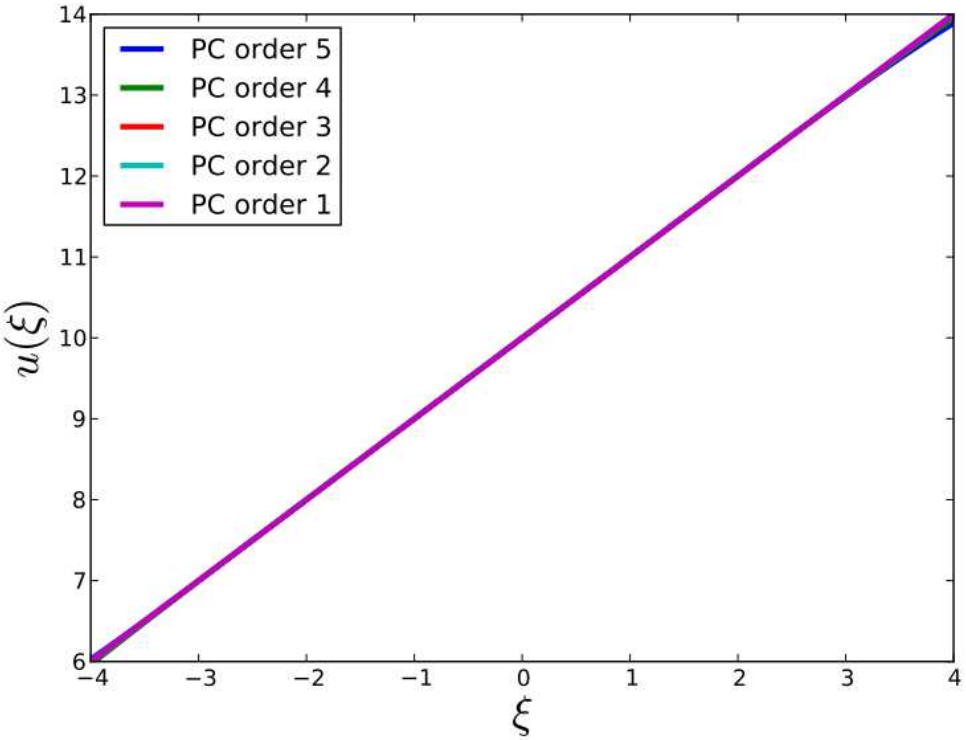
- Wiener-Hermite PCE constructed for a Normal RV
- PCE-sampled PDF superposed on true PDF
- Order = 5



$$
\begin{aligned}
u &= \sum_{k=0}^{P} u_k \Psi_k(\xi) \\
&= u_0 + u_1\xi + u_2(\xi^2 - 1) + u_3(\xi^3 - 3\xi) + u_4(\xi^4 - 6\xi^2 + 3) \\
&\quad + u_5(\xi^5 - 10\xi^3 + 15\xi)
\end{aligned}
$$

PC Illustration: WH PCE for a Normal RV



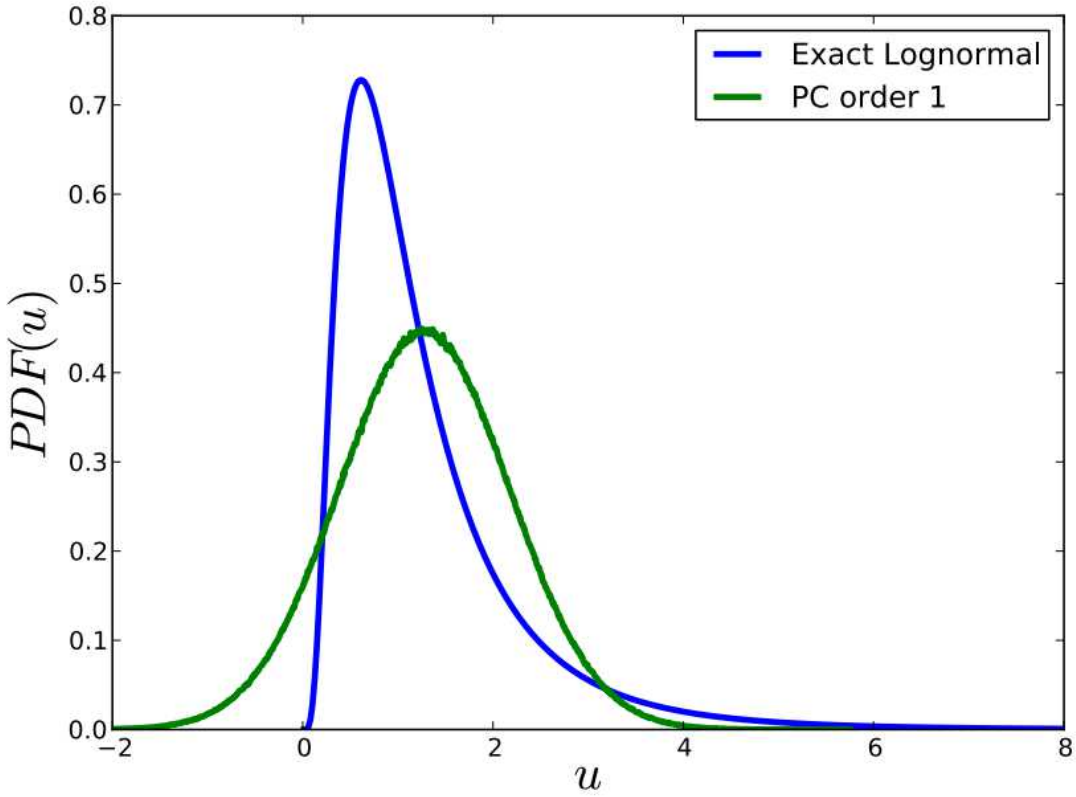


- First order Wiener-Hermite PCE exact for a normal RV
- Linear function of ξ
- Higher order terms are negligible

PC Illustration: WH PCE for a Lognormal RV

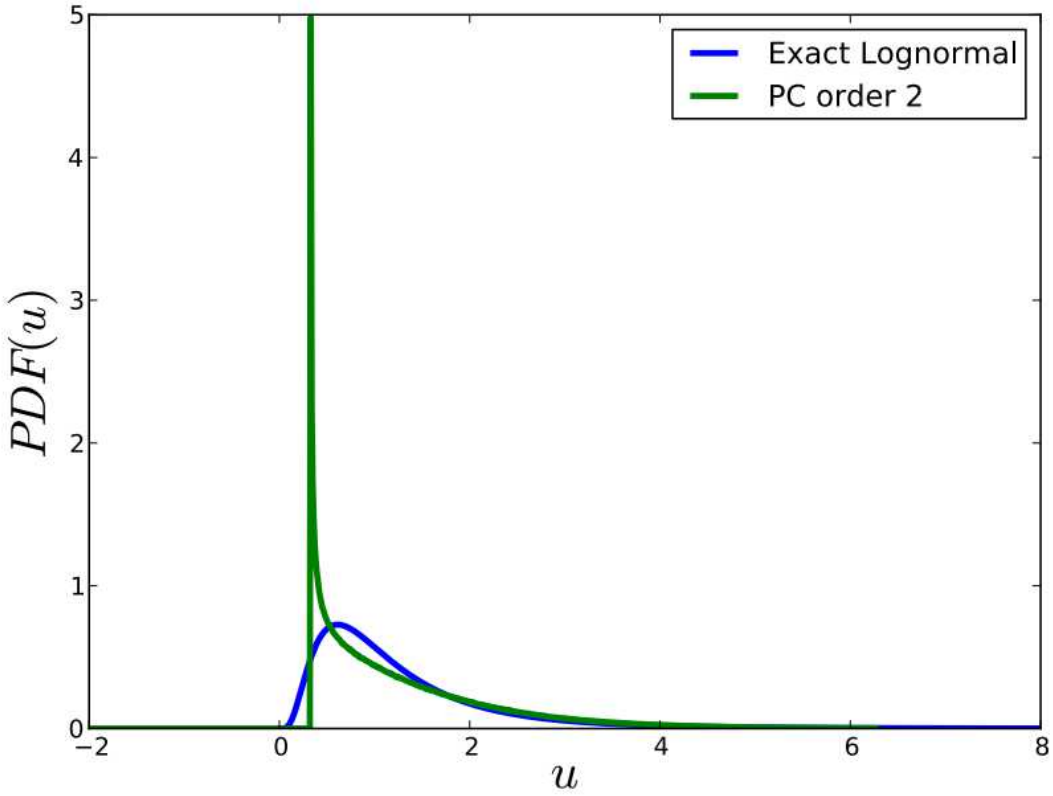
- Wiener-Hermite PCE constructed for a Lognormal RV
- PCE-sampled PDF superposed on true PDF
- Order = 1

$$
\begin{aligned}
u &= \sum_{k=0}^{P} u_k \Psi_k(\xi) \\
&= u_0 + u_1 \xi
\end{aligned}
$$



PC Illustration: WH PCE for a Lognormal RV

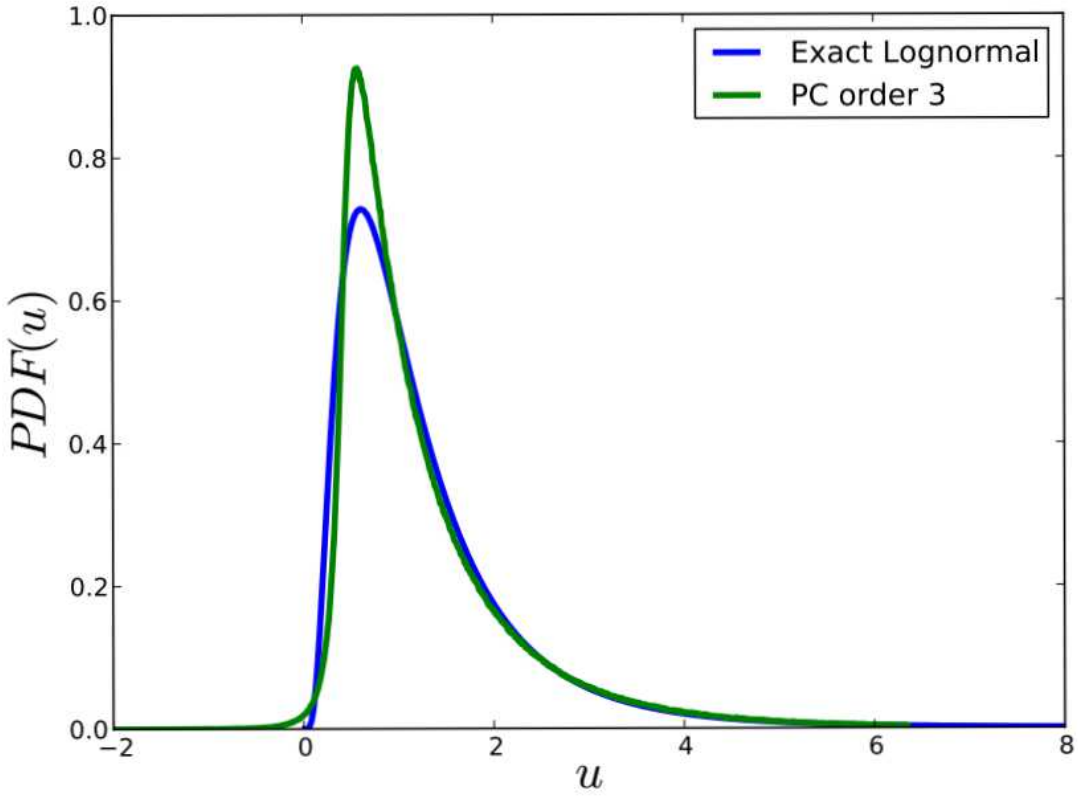
- Wiener-Hermite PCE constructed for a Lognormal RV
- PCE-sampled PDF superposed on true PDF
- Order = 2



$$u = \sum_{k=0}^{P} u_k \Psi_k(\xi)$$
$$= u_0 + u_1\xi + u_2(\xi^2 - 1)$$

PC Illustration: WH PCE for a Lognormal RV

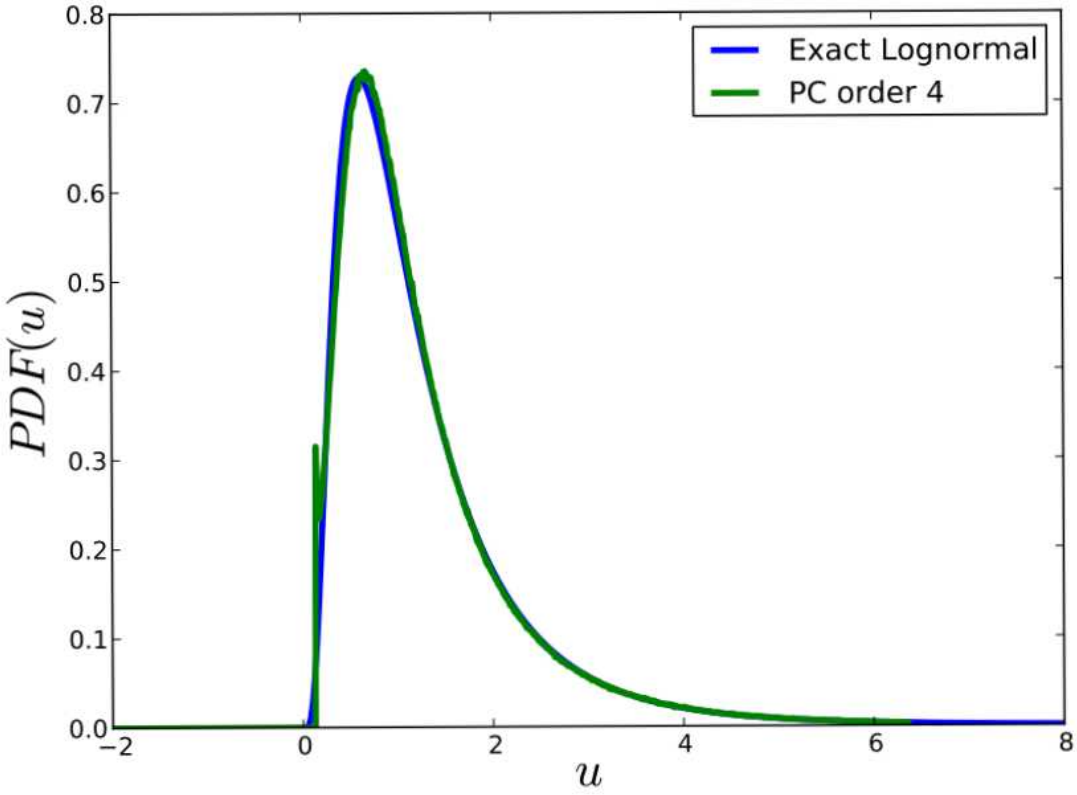
- Wiener-Hermite PCE constructed for a Lognormal RV
- PCE-sampled PDF superposed on true PDF
- Order = 3



$$u = \sum_{k=0}^{P} u_k \Psi_k(\xi)$$
$$= u_0 + u_1\xi + u_2(\xi^2 - 1) + u_3(\xi^3 - 3\xi)$$

PC Illustration: WH PCE for a Lognormal RV

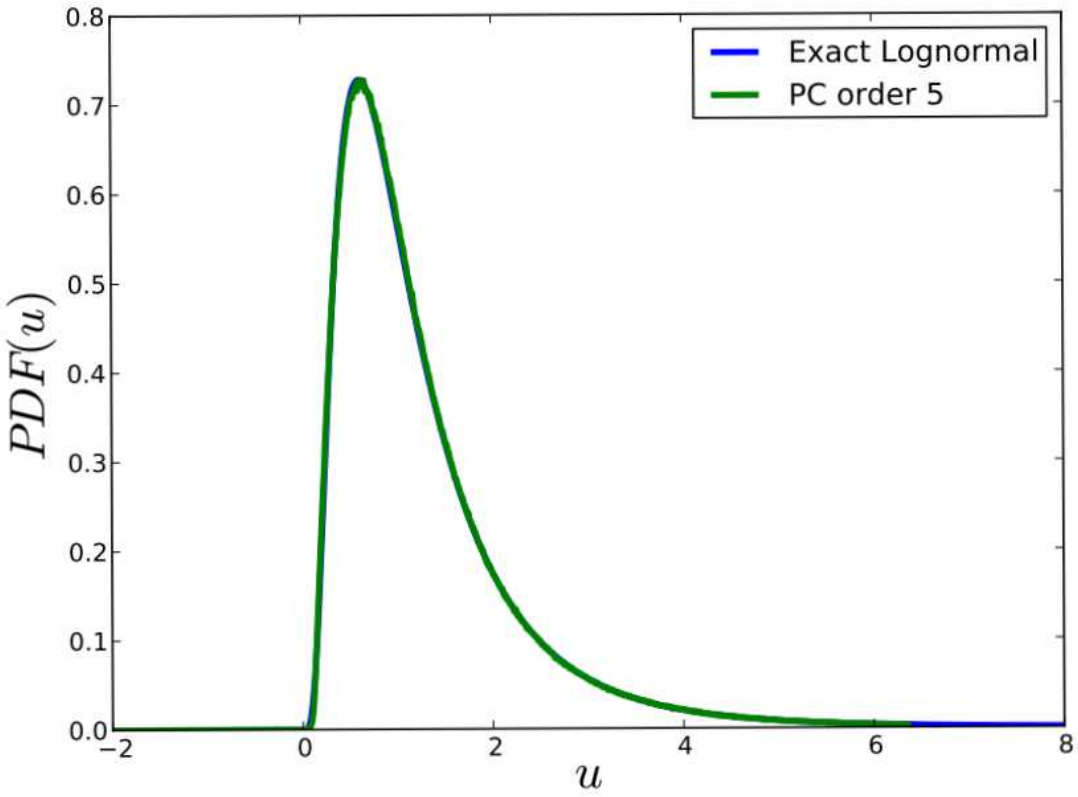
- Wiener-Hermite PCE constructed for a Lognormal RV
- PCE-sampled PDF superposed on true PDF
- Order = 4



$$u = \sum_{k=0}^{P} u_k \Psi_k(\xi)$$
$$= u_0 + u_1\xi + u_2(\xi^2 - 1) + u_3(\xi^3 - 3\xi) + u_4(\xi^4 - 6\xi^2 + 3)$$

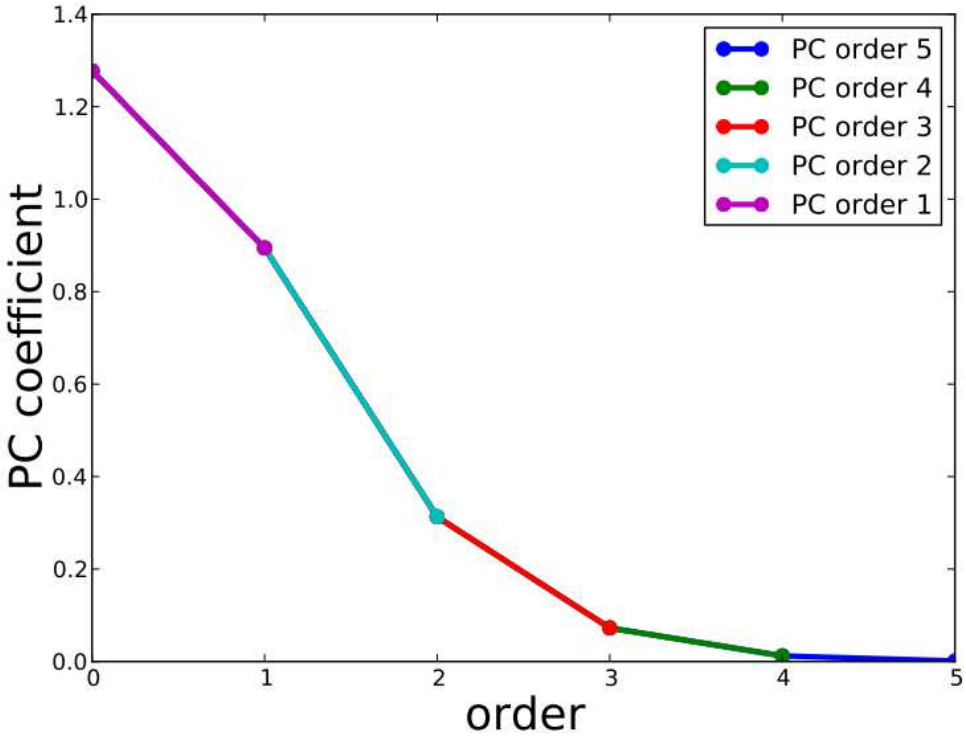
PC Illustration: WH PCE for a Lognormal RV

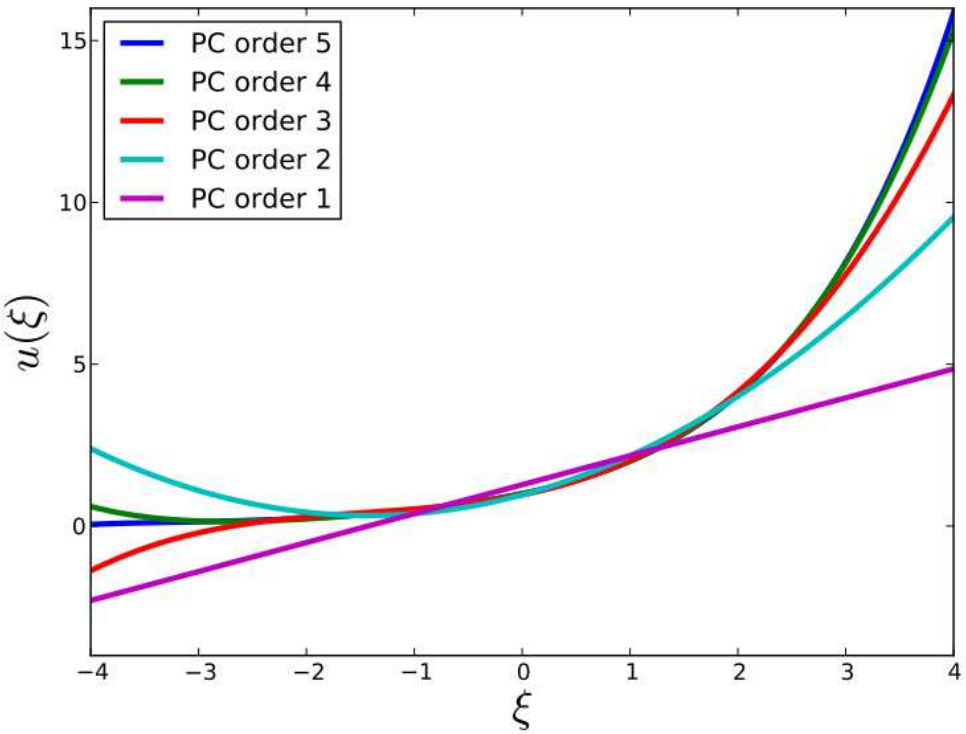
- Wiener-Hermite PCE constructed for a Lognormal RV
- PCE-sampled PDF superposed on true PDF
- Order = 5



$$
\begin{aligned}
u &= \sum_{k=0}^{P} u_k \Psi_k(\xi) \\
&= u_0 + u_1\xi + u_2(\xi^2 - 1) + u_3(\xi^3 - 3\xi) + u_4(\xi^4 - 6\xi^2 + 3) \\
&\quad + u_5(\xi^5 - 10\xi^3 + 15\xi)
\end{aligned}
$$

PC Illustration: WH PCE for a Lognormal RV





- Fifth-order Wiener-Hermite PCE represents the given Lognormal well
- Higher order terms are negligible

Constructing an *n*D PCE for a RV with a Given PDF

- Given RV $z \in \mathbb{R}$ with PDF: $g(z)$, define:

$$z = \sum_{i=0}^{P} z_i \Psi_i(\xi_1, \xi_2, \ldots, \xi_n), \qquad P + 1 = \frac{(n+p)!}{n!p!}$$

- No general procedure
- Construct PCE as model choice to represent what is known about RV
- Can choose $\{n, p\}$ and the mode strengths by ensuring
  accurate capture of
  - the PDF $g(z)$
  - select moments of z
  - some observable of interest $\phi(z)$

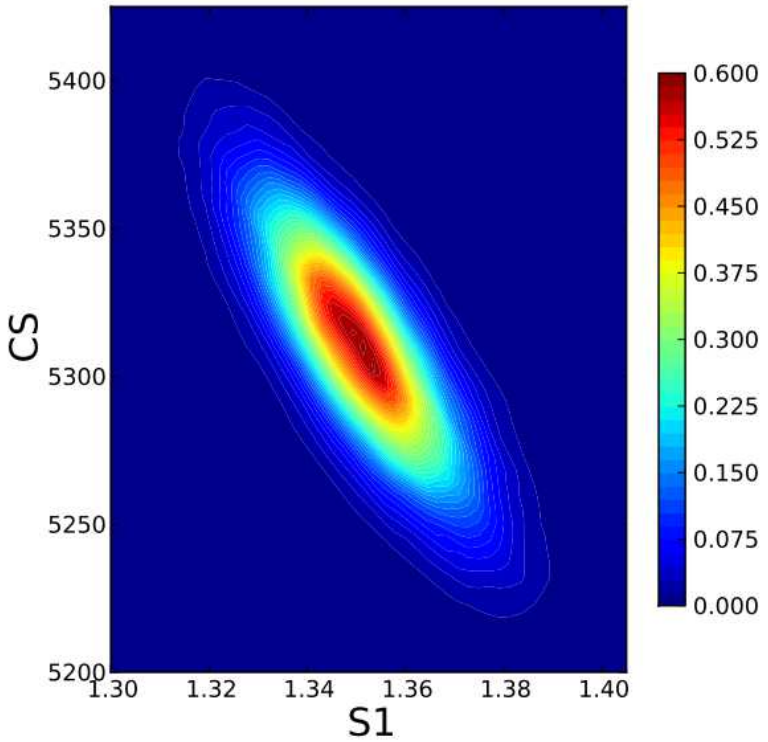
Multivariate Normal Approximation (MVN)

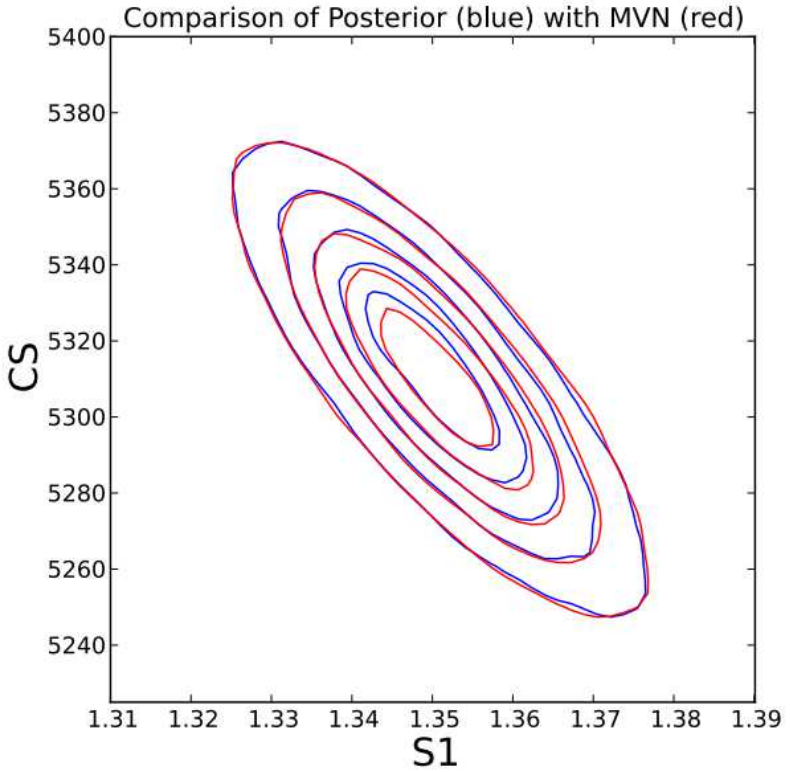
- Many distributions are unimodal and somewhat shaped like Gaussians
- MultiVariate Normal approximations capture mean and correlation structure of the random variables
- Easy to extract from a set of samples
  - In 1D: just compute mean and standard deviation: $u = u_0 + u_1 \xi$
  - Multi-D: Cholesky factorization of covariance

$$C = LL^T$$
$$u = L\xi$$

```
# Compute mean parameter values
par_mean = numpy.mean(samples,axis=0)
# Compute the covariance
par_cov = numpy.cov(samples,rowvar=0)
# Compute the Cholesky Decomposition
chol_lower = numpy.linalg.cholesky(par_cov)
```

MVN Approximation of Distribution from Samples





$$S_1 = 1.351 + 0.01367\xi_1$$
$$CS = 5310 - 26.25\xi_1 + 20.26\xi_2$$

Rosenblatt Transformation for Multi-D RVs

- Assume samples of multi-D RVs are (*e.g.* from MCMC sampling of posterior parameter distribution)
- Rosenblatt transformation maps any (not necessarily independent) set of random variables $(\lambda_1, \ldots, \lambda_d)$ to uniform i.i.d.'s $\{\eta_i\}_{i=1}^d$ (Rosenblatt, 1952).

$$\begin{aligned} \eta_1 &= F_1(\lambda_1) \\ \eta_2 &= F_{2|1}(\lambda_2|\lambda_1) \\ &\vdots \\ \eta_d &= F_{d|d-1,\ldots,1}(\lambda_d|\lambda_{d-1}, \ldots, \lambda_1) \end{aligned}$$

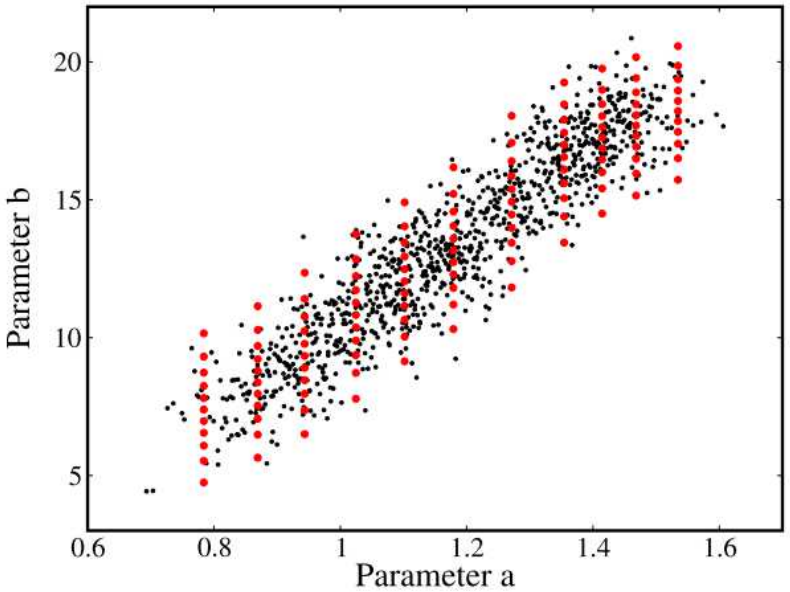
- Rosenblatt transformation is a multi-D generalization of 1D CDF mapping.
- Conditional CDFs are harder to evaluate in high dimensions

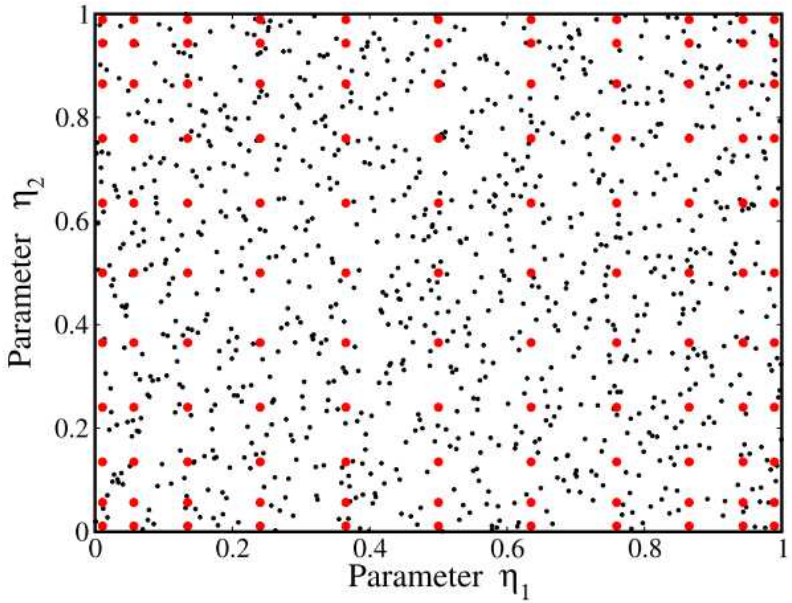
Rosenblatt transformation of a given sample set requires KDE

- Given samples $\{\boldsymbol{\lambda}^{(n)}\}_{n=1}^{N}$ of the random variable $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$
- Kernel Density Estimation (KDE) is needed to compute conditional CDFs

$$
\begin{aligned}
\eta_k &= F_{k|k-1,\ldots,1}(\lambda_k|\lambda_{k-1},\ldots,\lambda_1) = \\
&= \int_{\infty}^{\lambda_k} p_{k|k-1,\ldots,1}(\lambda_k^{'}|\lambda_{k-1},\ldots,\lambda_1)d\lambda_k^{'} = \int_{\infty}^{\lambda_k} \frac{p_{k,\ldots,1}(\lambda_k^{'},\lambda_{k-1},\ldots,\lambda_1)}{p_{k-1,\ldots,1}(\lambda_{k-1},\ldots,\lambda_1)} d\lambda_k^{'} \\
&\approx \frac{1}{h\sqrt{2\pi}} \int_{\infty}^{\lambda_k} \frac{\sum_{n=1}^{N} \exp\left(-\frac{(\lambda_1-\lambda_1^{(n)})^2+\cdots+(\lambda_k^{'}-\lambda_k^{(n)})^2}{2h^2}\right)}{\sum_{n=1}^{N} \exp\left(-\frac{(\lambda_1-\lambda_1^{(n)})^2+\cdots+(\lambda_{k-1}-\lambda_{k-1}^{(n)})^2}{2h^2}\right)} d\lambda_k^{'} \\
&= \int_{\infty}^{\lambda_k} \frac{\sum_{n=1}^{N} \exp\left(-\frac{(\lambda_1-\lambda_1^{(n)})^2+\cdots+(\lambda_{k-1}-\lambda_{k-1}^{(n)})^2}{2h^2}\right) \times \frac{1}{h\sqrt{2\pi}} \exp\left(-\frac{(\lambda_k^{'}-\lambda_k^{(n)})^2}{2h^2}\right)}{\sum_{n=1}^{N} \exp\left(-\frac{(\lambda_1-\lambda_1^{(n)})^2+\cdots+(\lambda_{k-1}-\lambda_{k-1}^{(n)})^2}{2h^2}\right)} d\lambda_k^{'} \\
&= \frac{\sum_{n=1}^{N} \exp\left(-\frac{(\lambda_1-\lambda_1^{(n)})^2+\cdots+(\lambda_{k-1}-\lambda_{k-1}^{(n)})^2}{2h^2}\right) \times \Phi\left(\frac{\lambda_k-\lambda_k^{(n)}}{h}\right)}{\sum_{n=1}^{N} \exp\left(-\frac{(\lambda_1-\lambda_1^{(n)})^2+\cdots+(\lambda_{k-1}-\lambda_{k-1}^{(n)})^2}{2h^2}\right)},
\end{aligned}
$$

Rosenblatt transformation enables Galerkin projection





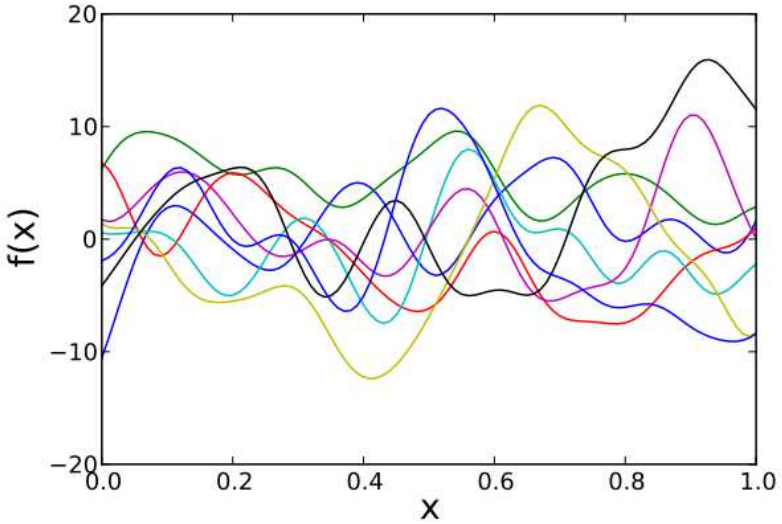
$(a, b) = R^{-1}(\xi_1, \xi_2)$ ensures a well-defined quadrature integration

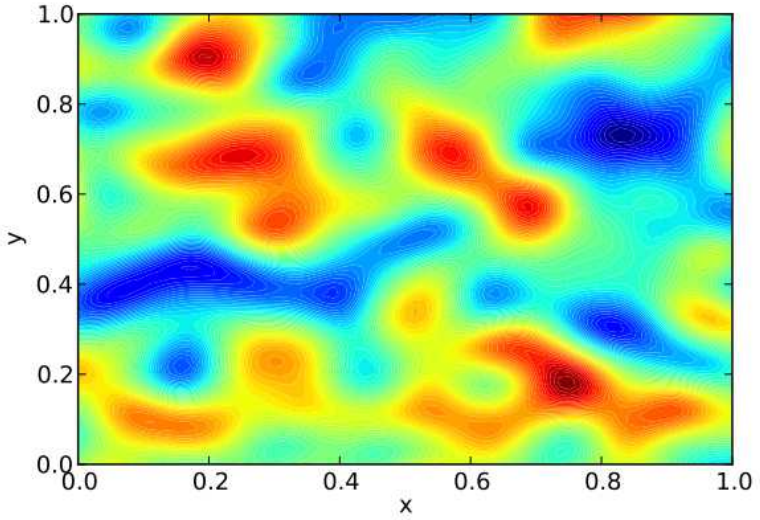
$$a = \sum_{k=0}^{P} a_k \Psi_k(\xi) \qquad a_k \propto \int \underbrace{R_a^{-1}(\xi)}_{a} \Psi_k(\xi) w(\xi) d\xi$$

$$b = \sum_{k=0}^{P} b_k \Psi_k(\xi) \qquad b_k \propto \int \underbrace{R_b^{-1}(\xi)}_{b} \Psi_k(\xi) w(\xi) d\xi$$

Outline

Compact Representations of Random Fields





- Sometimes an uncertain input is a random field (also called stochastic process)
  - Friction coefficient along a wall
  - Thermal conductivity in heterogeneous material
  - Permeability in groundwater flow simulations
- Have infinitely many degrees of freedom
- To model effect of random fields on computations, need compact representation that captures key aspects of random field

Karhunen-Loève (KL) Expansions

- Assume stochastic process $F(x,\theta) : D \times \Theta \to \mathbb{R}$ an L^2 random field on D
- With covariance function $Cov(x,y)$
- F can be written as

$$F(x,\theta) = \langle F(x,\theta) \rangle_\theta + \sum_{k=1}^{\infty} \sqrt{\lambda_k} f_k(x) \xi_k$$

- $f_k(x)$: eigenfunctions of $Cov(x,y)$
- λ_k: corresponding eigenvalues, all positive
- ξ_k: uncorrelated random variables, unit variance
- Samples obtained by projecting realizations of F onto f_k
- Generally not independent
  - For Gaussian F, ξ_k are i.i.d. normal random variables
- The KLE is *optimal*: of all possible orthonormal bases for $L^2(\Theta \times D)$ the $\{f(x)\}$ minimize the mean-square error in a finite linear representation of $F(\cdot)$

KL Expansions - Numerical Approach - 1

- Covariance Matrix, $Cov(x, y) = \langle F(x, \theta)F(y, \theta)\rangle_\theta$:
  - specified analytically
  - estimated from samples
- Estimate eigenvalues and eigenvectors for the Fredholm equation of second kind:

$$\int Cov(x, y)f(y)dy = \lambda f(x)$$

KL Expansions - Numerical Approach - 2

- Nystrom algorithm for Fredholm equation

$$\sum_{i=1}^{N_p} w_i Cov(x, y_i) f(y_i) = \lambda f(x)$$

where w_i are the weights for the quadrature rule that uses N_p points y_i where realizations are provided.

- Further manipulation leads to the eigenvalue problem

$$Ag = \lambda g$$

where $A = WKW$ and $g = Wf$, with W being the diagonal matrix $W_{ii} = \sqrt{w_i}$ and $K_{ij} = Cov(x_i, y_j)$. Solutions consist of pairs of eigenvalues λ_k and eigenmodes $f_k = W^{-1} g_k$.

KL Expansions - Numerical Approach - 3

- Samples of random variables ξ_k are obtained by projecting realizations of the random process F on the eigenmodes f_k

$$\xi_k|_{\theta_l} = \langle F(x,\theta_l) - \langle F(x,\theta)\rangle_\theta , f_k(x)\rangle_x / \sqrt{\lambda_k}$$

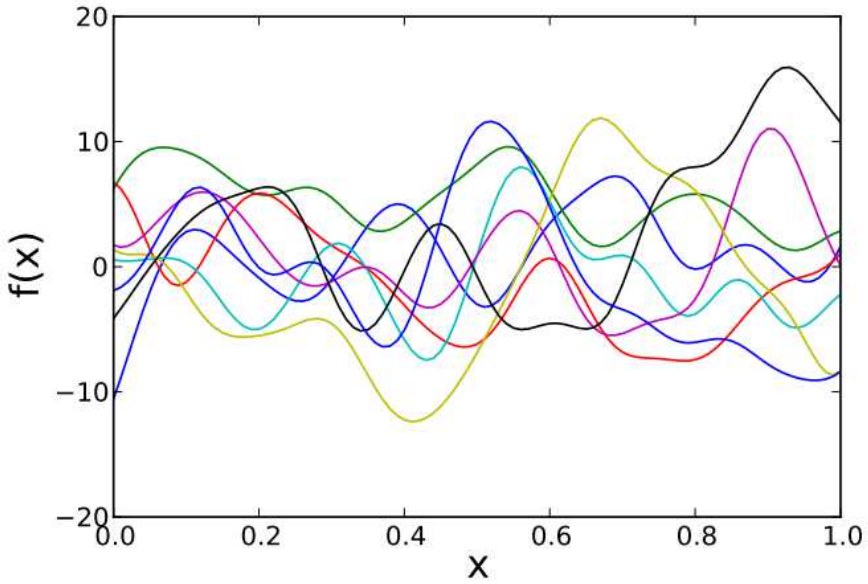
  - ... or numerically

$$\xi_k|_{\theta_l} = \sum_{i=1}^{N_p} w_i \left(F(x_i,\theta_l) - \langle F(x_i,\theta)\rangle_\theta\right) f_k(x_i)/\sqrt{\lambda_k}$$

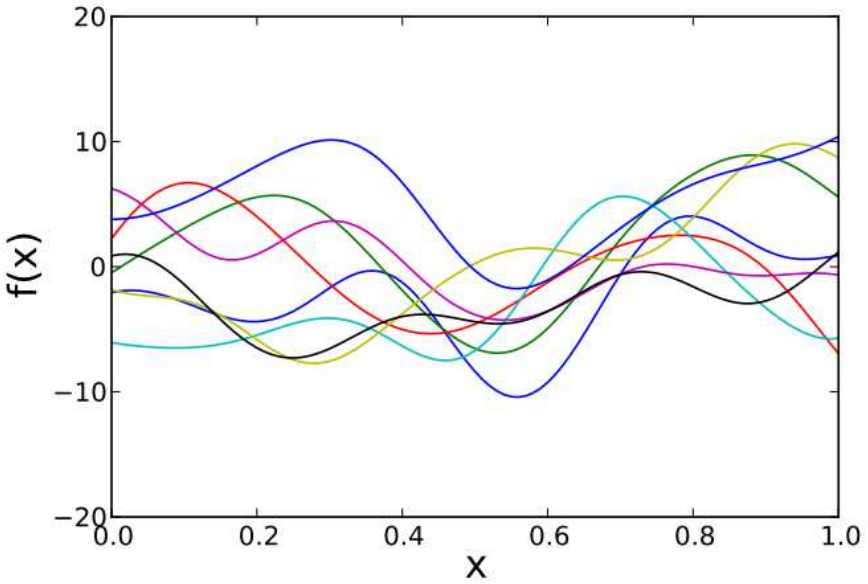
- If Gaussian process: automatically have first order WH PCE
- If not, same approaches as for converting RVs to PCEs applied to KL RVs

1D Gaussian Process: Realizations

$\delta = 0.1$



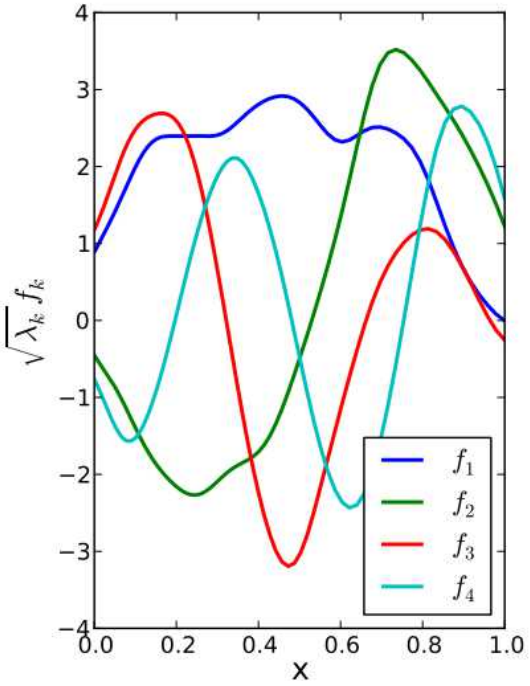
$\delta = 0.2$



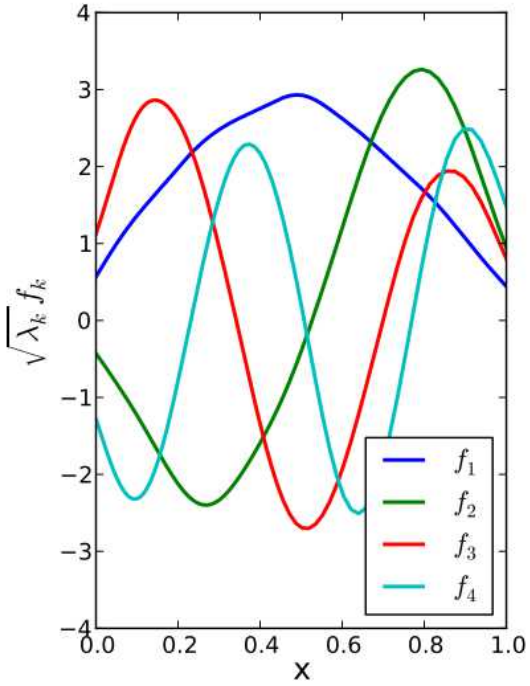
- Covariance $Cov(x_1, x_2) = exp(-(x_1 - x_2)^2/\delta^2)$
- Sample realizations are noisier as correlation length decreases

1D Gaussian Process: KL Modes for $\delta = 0.10$

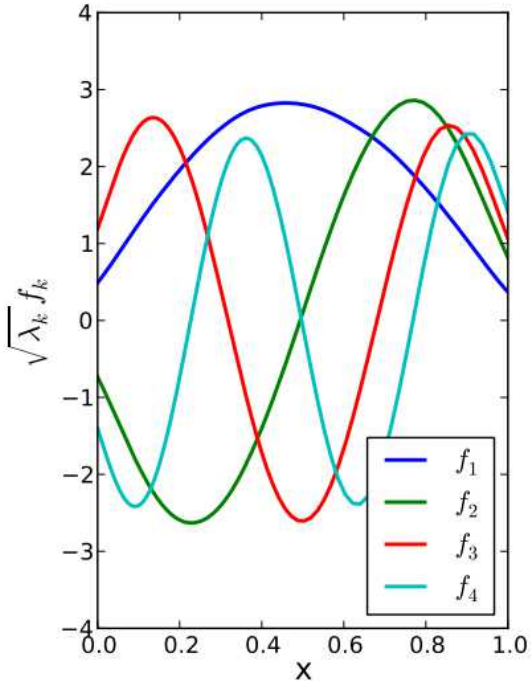
2^9 realizations



2^{13} realizations



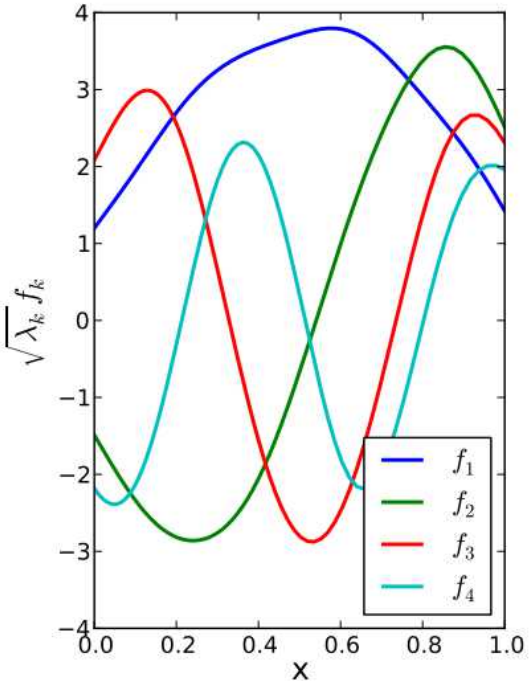
2^{17} realizations



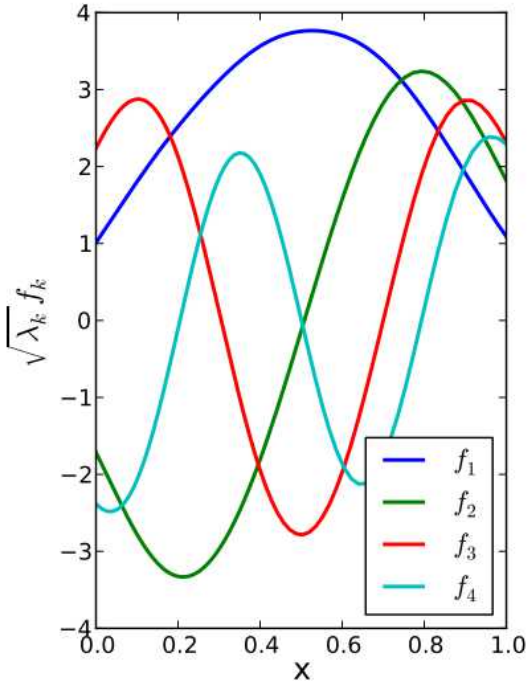
- Eigenmodes of the covariance matrix
  - Data covariance matrix constructed from $2^9 = 512$, $2^{13} = 8192$, and $2^{17} = 131072$ Gaussian process realizations
- Higher modes are more oscillatory

1D Gaussian Process: KL Modes for $\delta = 0.20$

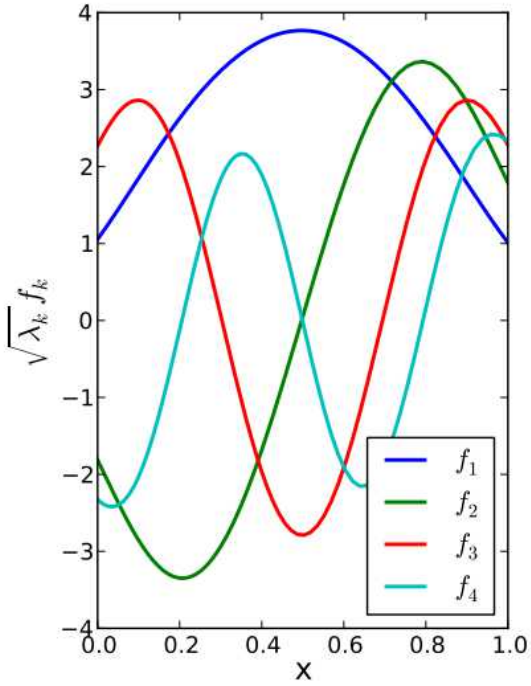
2^9 realizations



2^{13} realizations



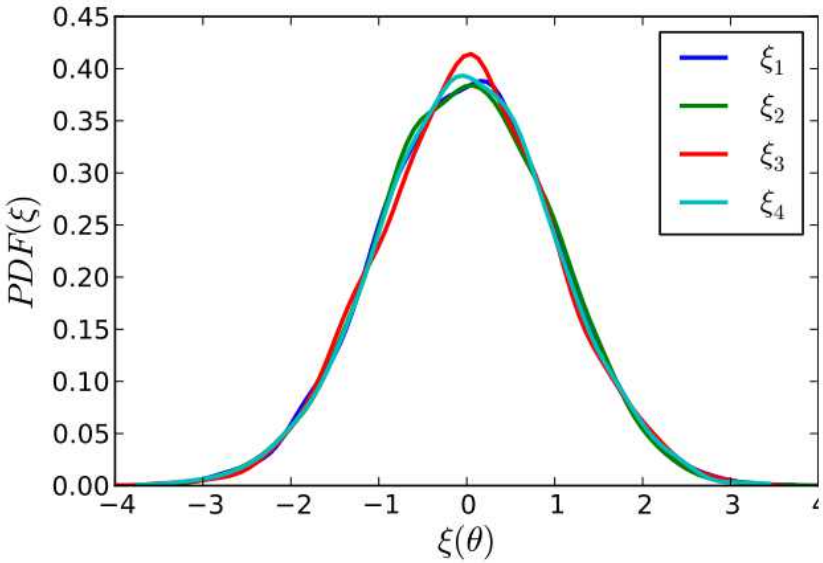
2^{17} realizations



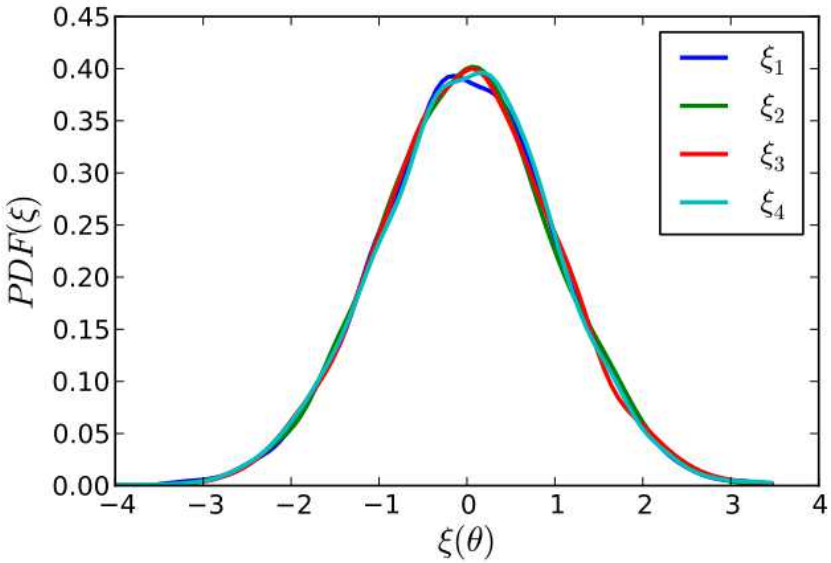
- Eigenmodes of the covariance matrix
  - Data covariance matrix constructed from $2^9 = 512$, $2^{13} = 8192$, and $2^{17} = 131072$ Gaussian process realizations
- Higher modes are more oscillatory

1D Gaussian Process: KL Random Variables

$\delta = 0.1$

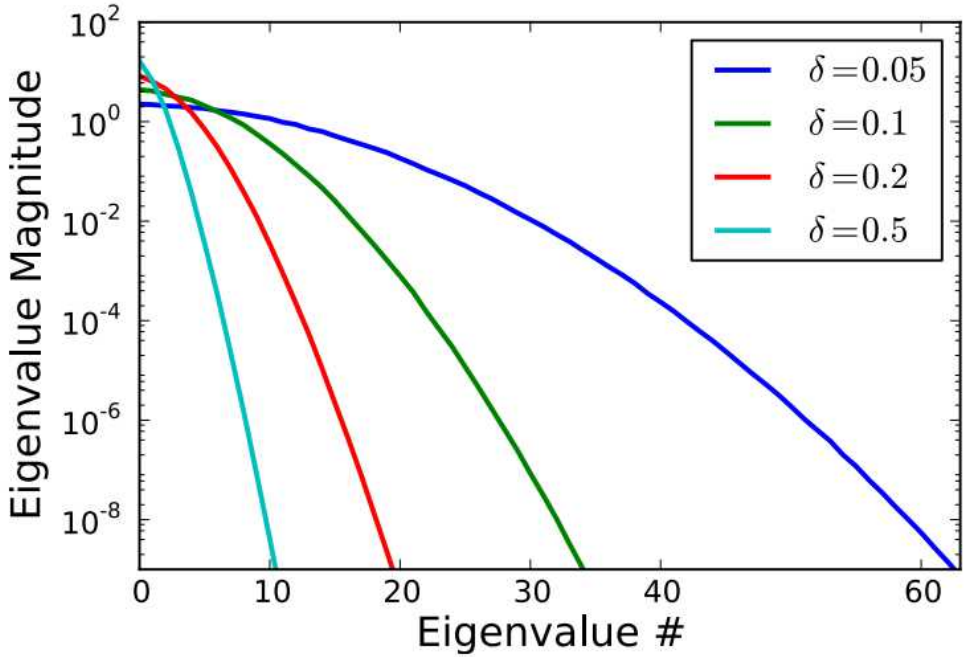


$\delta = 0.2$



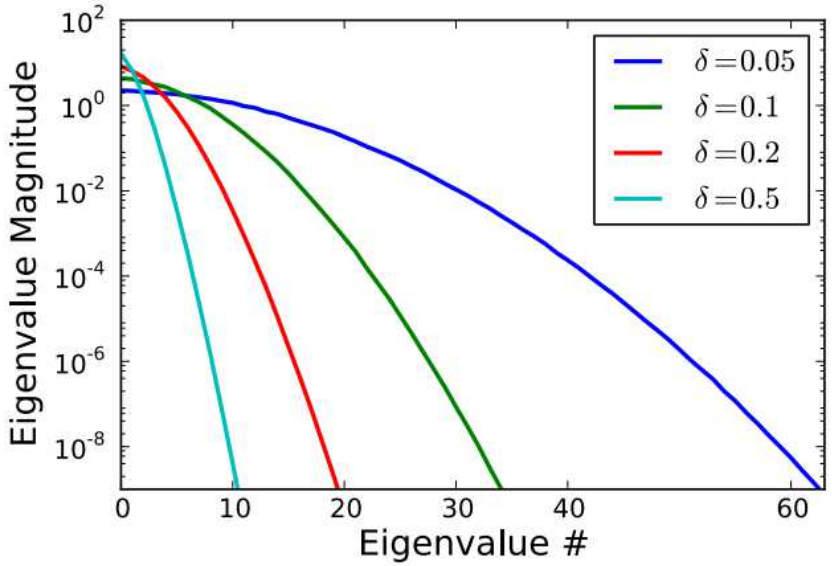
- Random variables obtained by projecting realizations onto KL modes
- Uncorrelated by construction
  - Also independent due to nature of Gaussian Process

1D Gaussian Process: Eigenvalue Spectrum



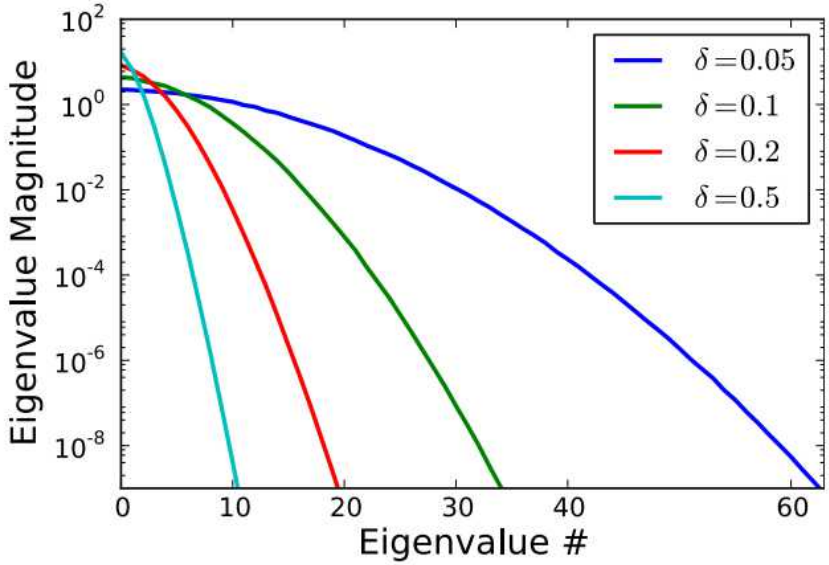
- Eigenvalue spectrum decays more slowly as correlation length decreases
  - More oscillatory modes needed to represent fluctuations in x
- KL expansion generally is truncated after enough modes are included to capture a specified fraction of the total variance

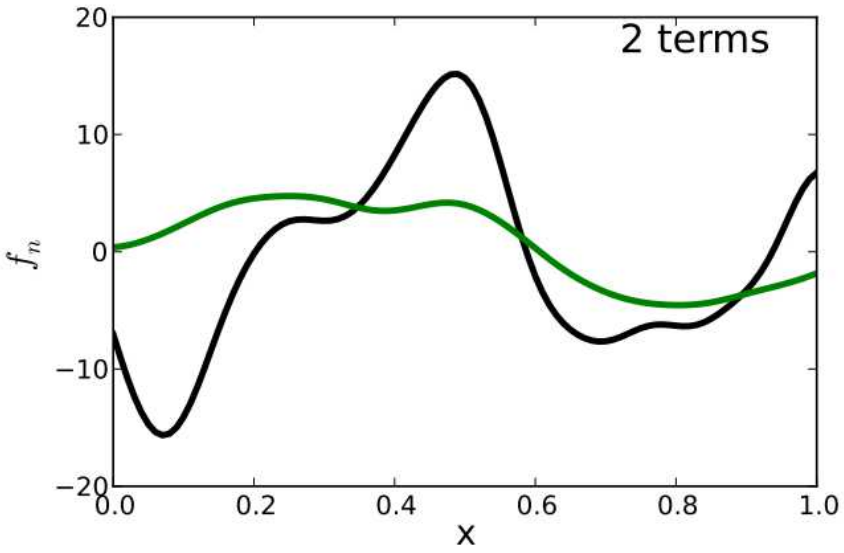
1D Gaussian Process: Reconstructed Realizations



- Large scale features can be resolved with small number of modes
- Smaller scale features require higher modes

1D Gaussian Process: Reconstructed Realizations





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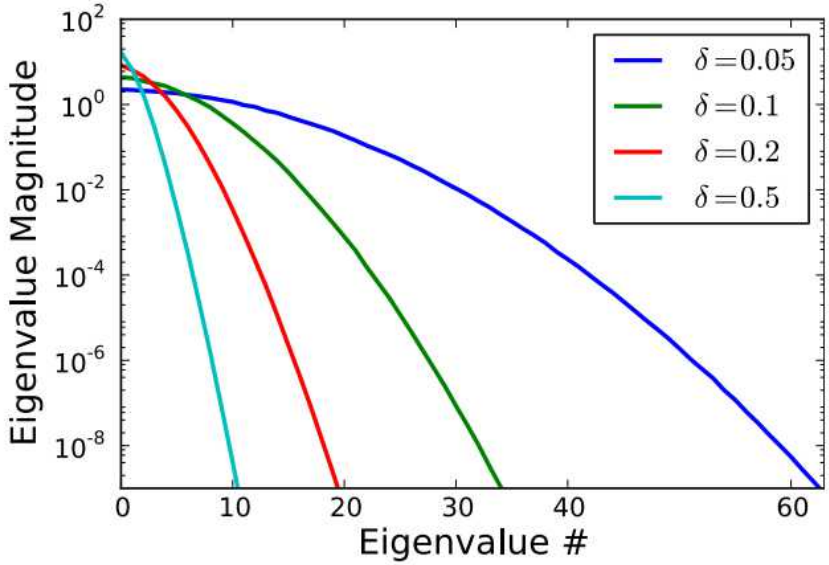

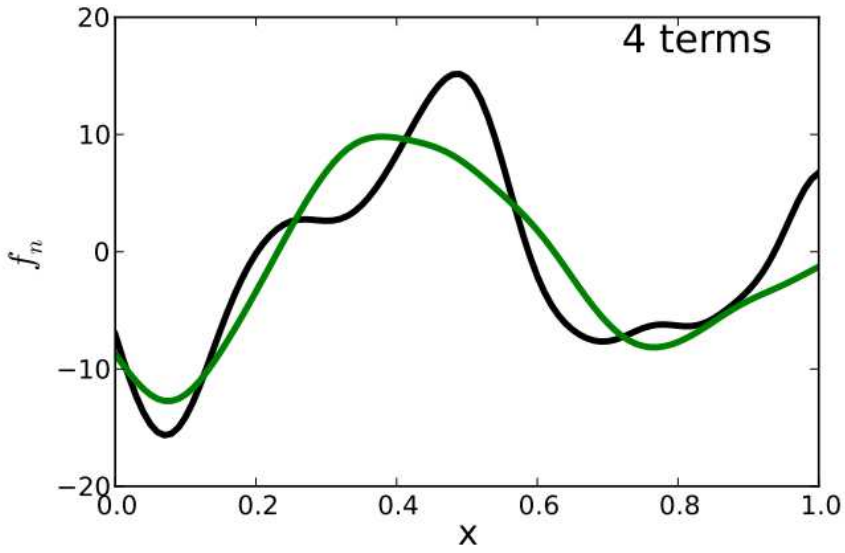

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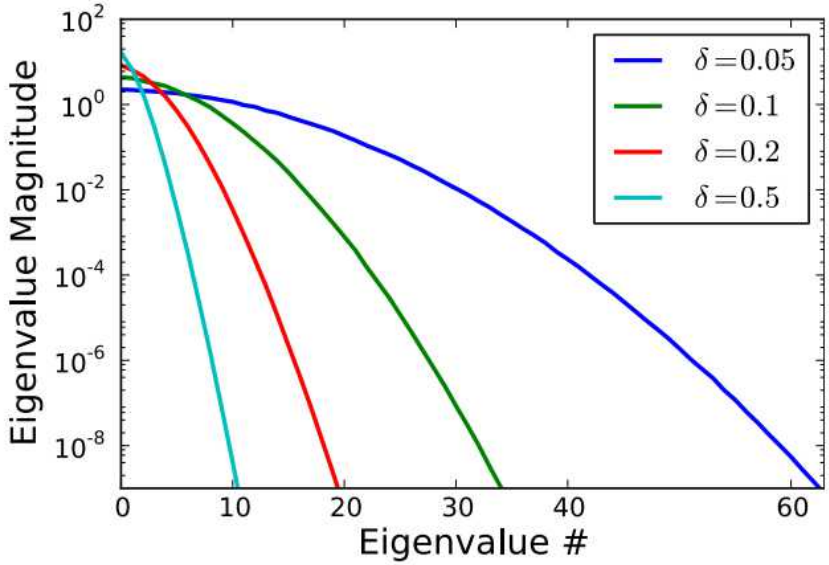

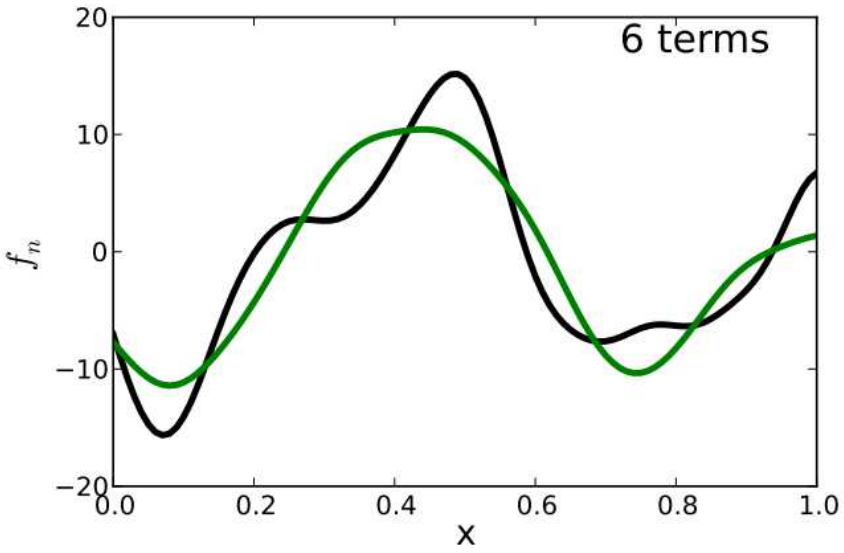

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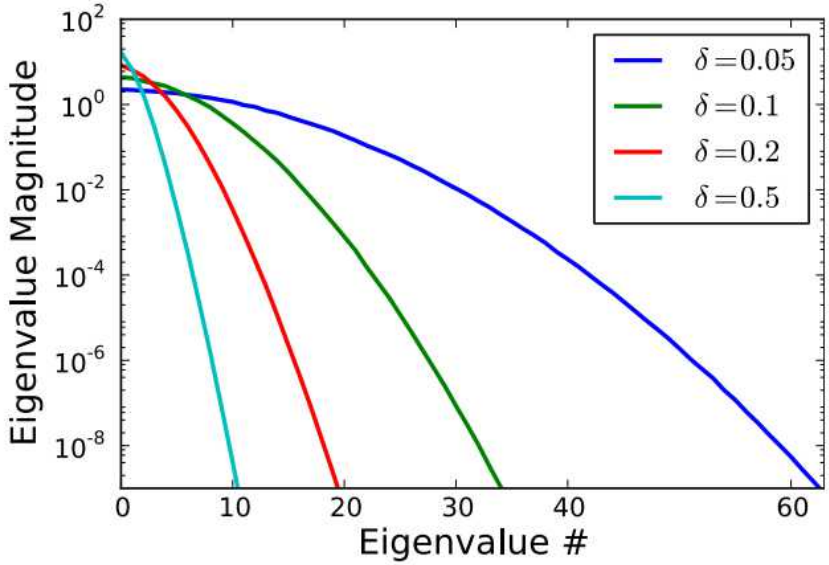

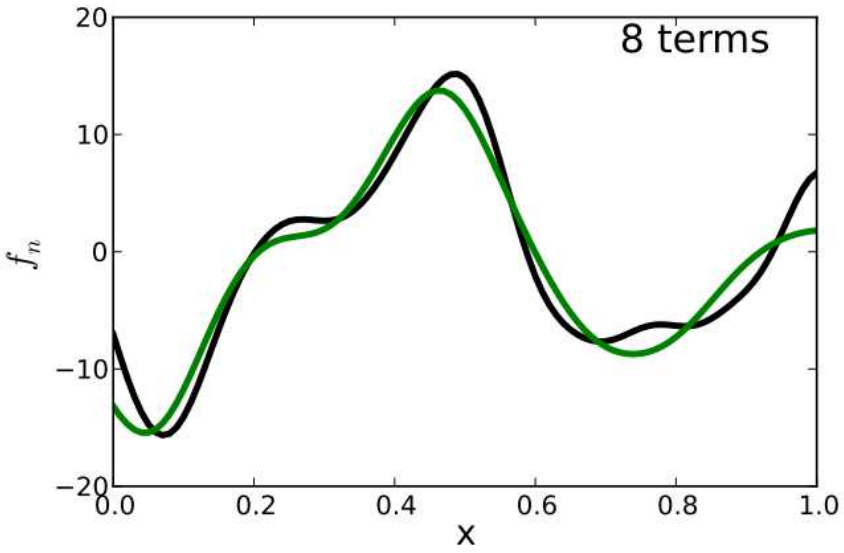

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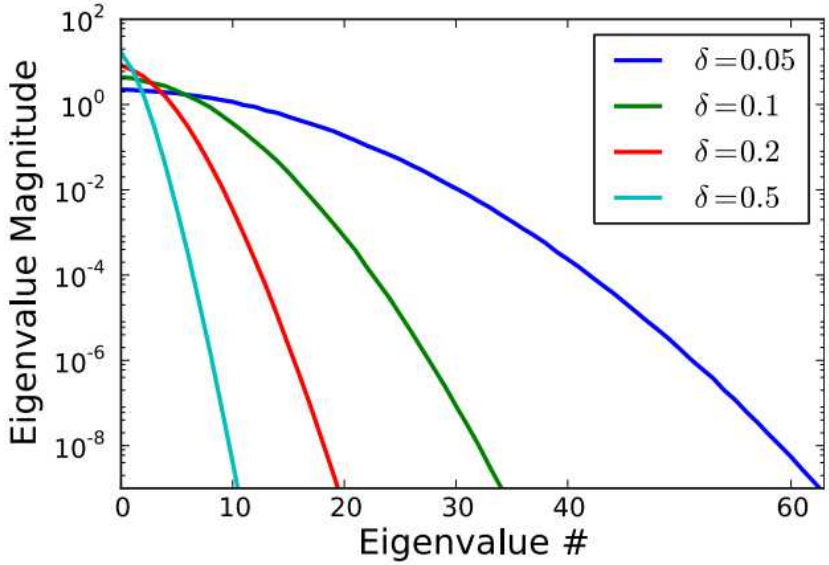

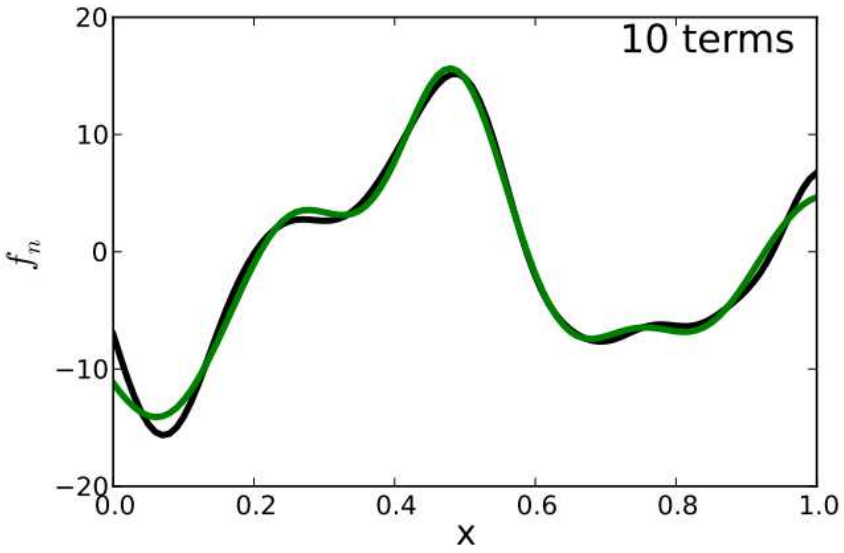

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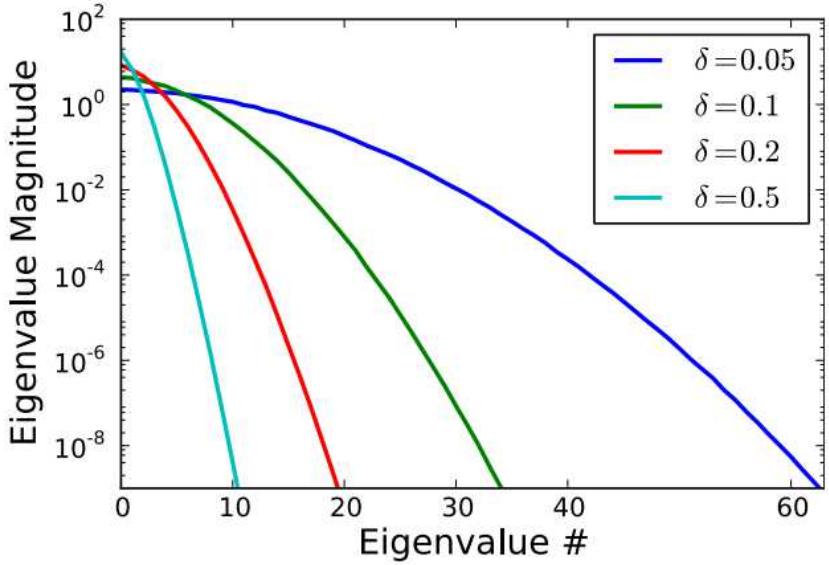

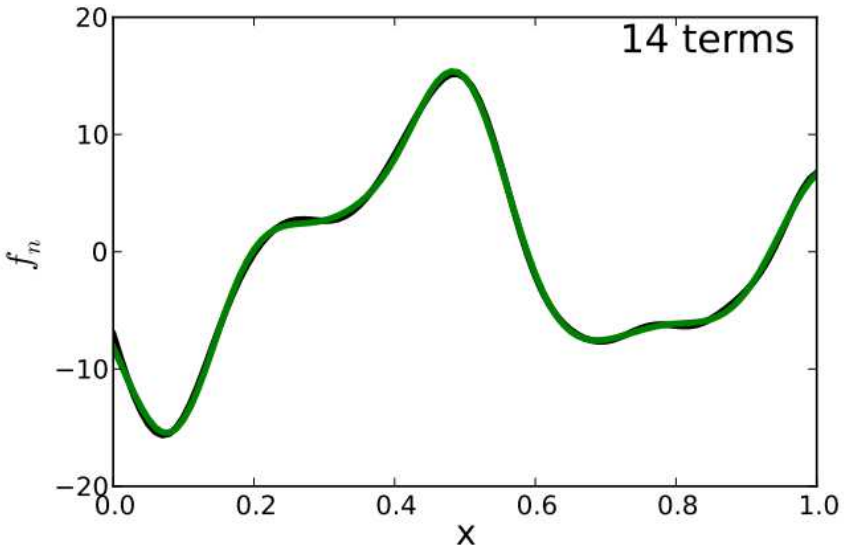

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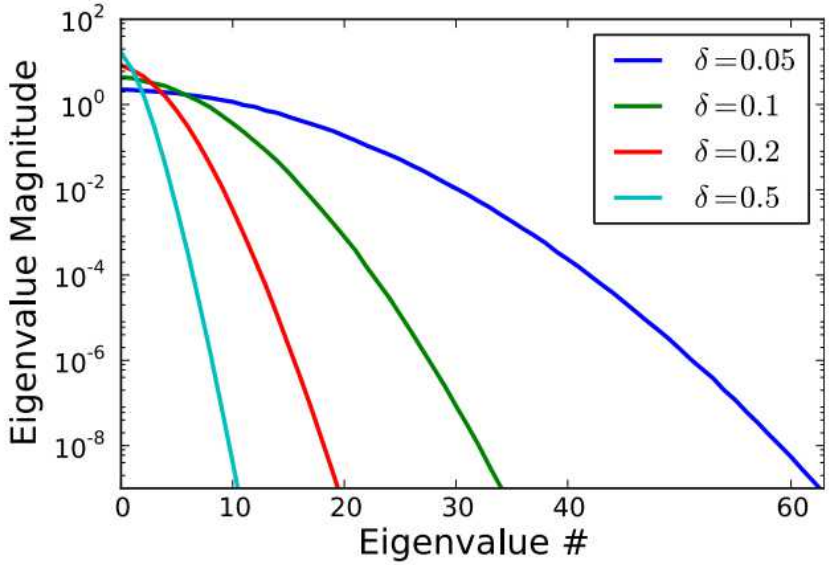

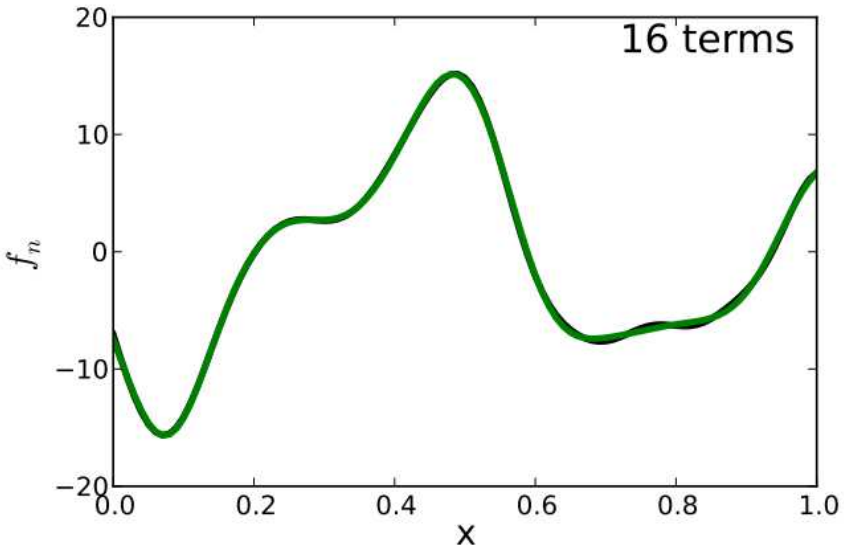

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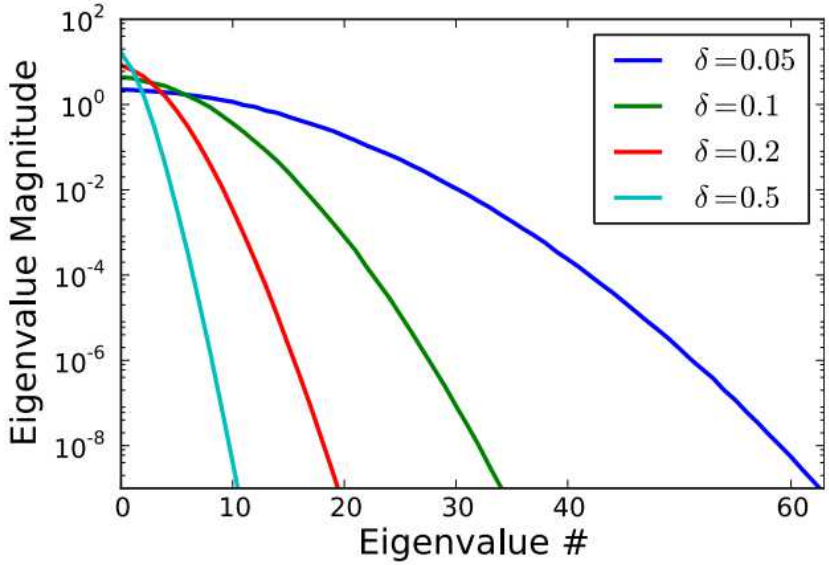

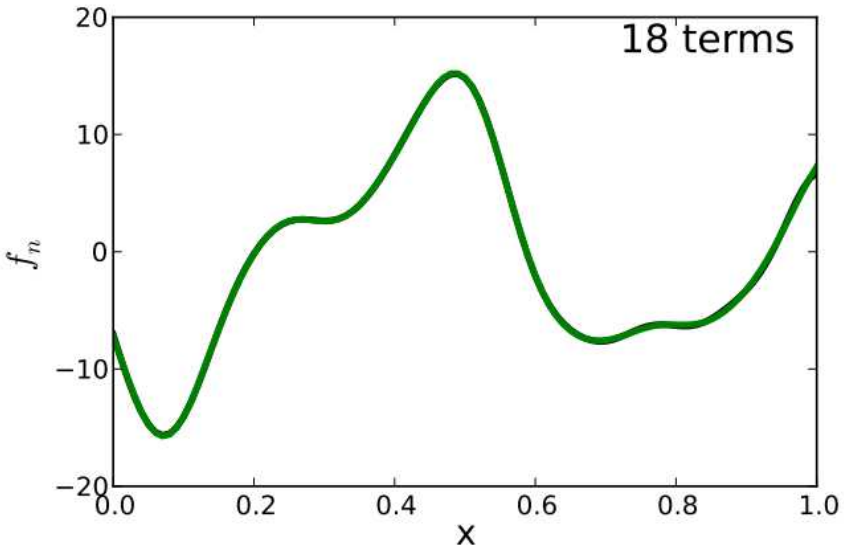

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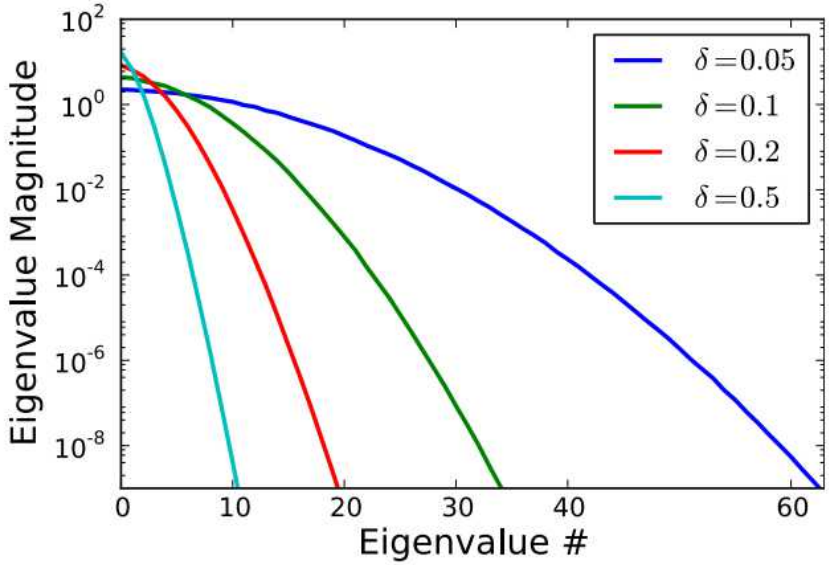

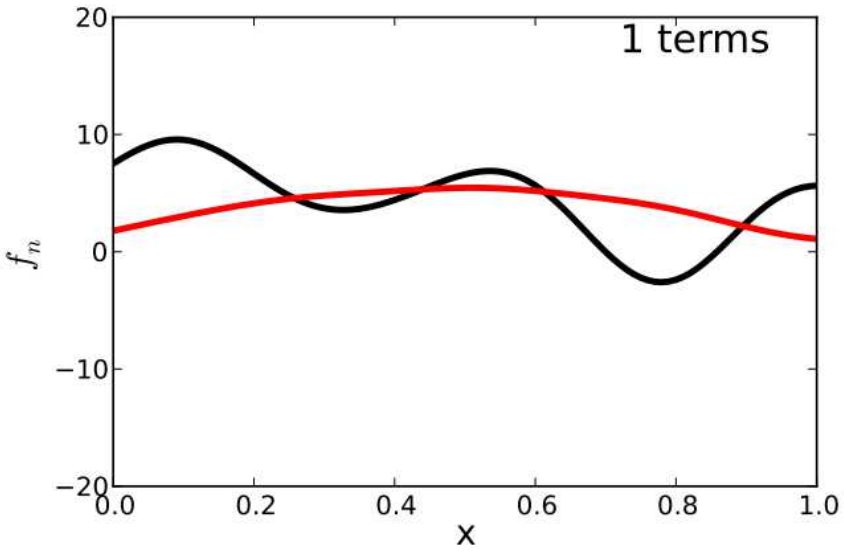

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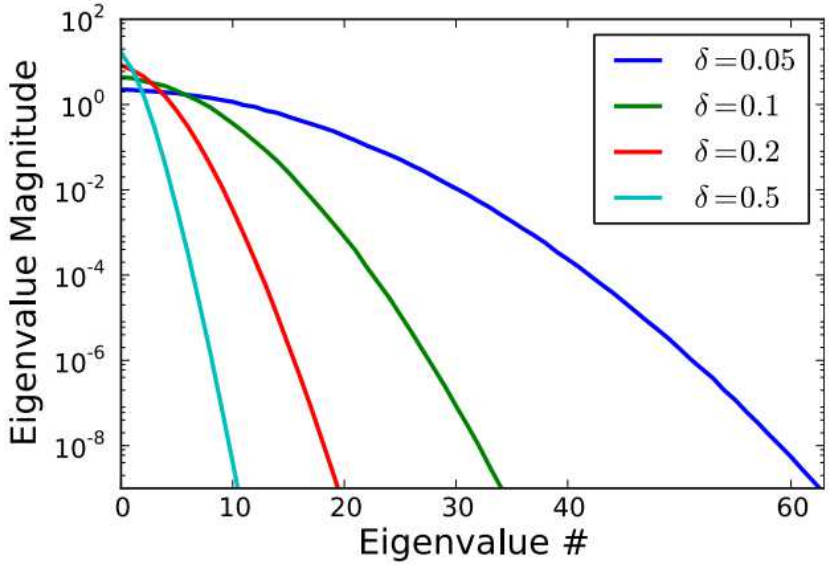

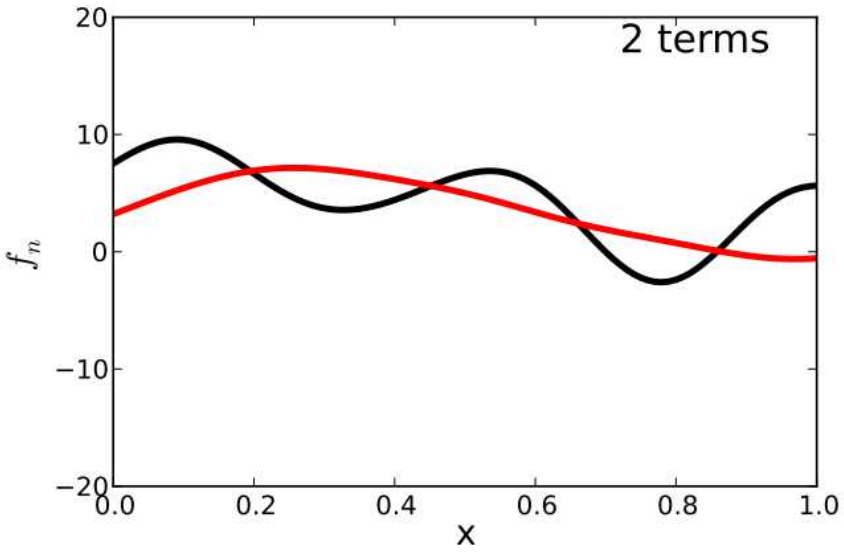

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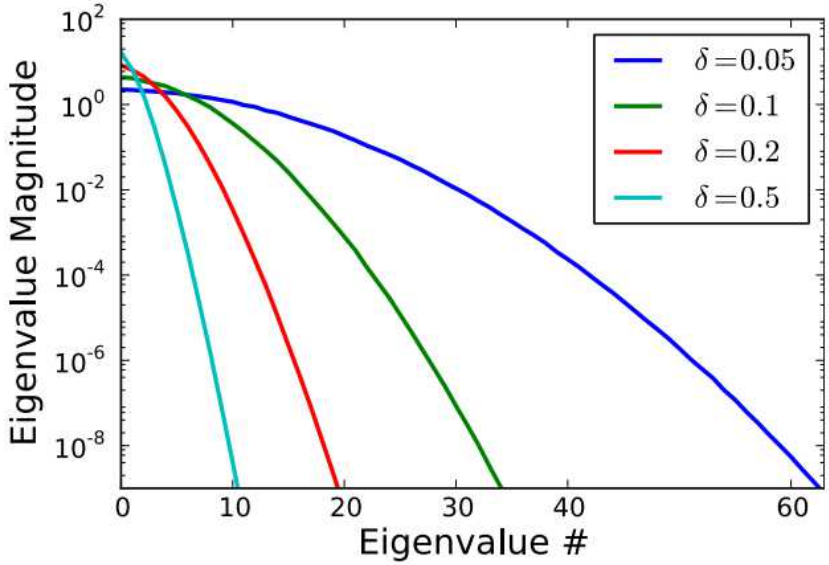

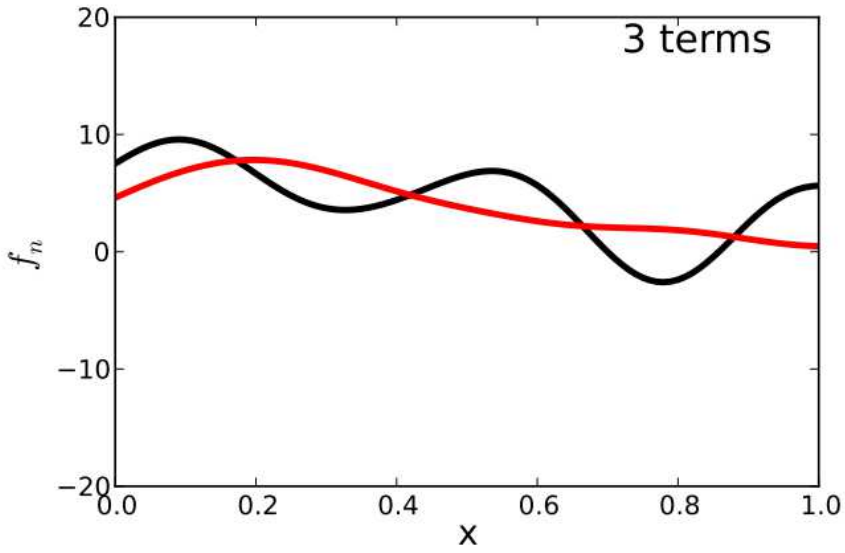

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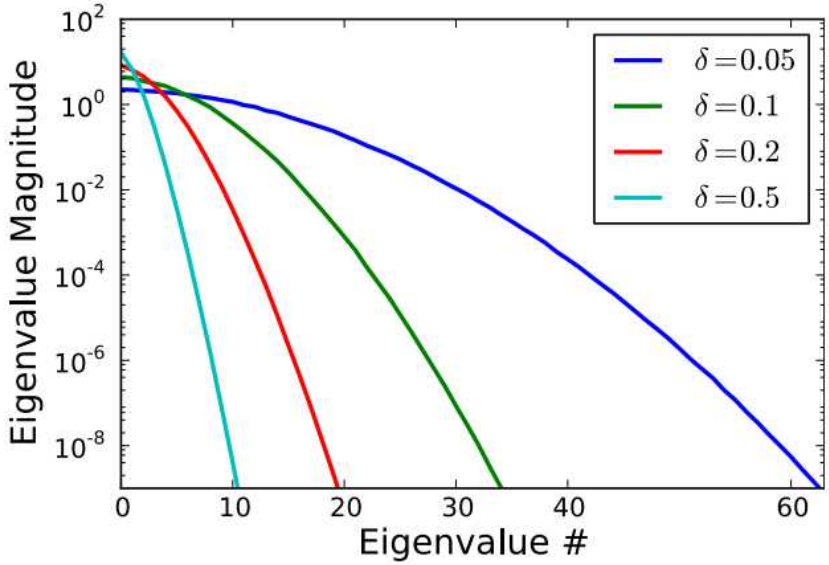

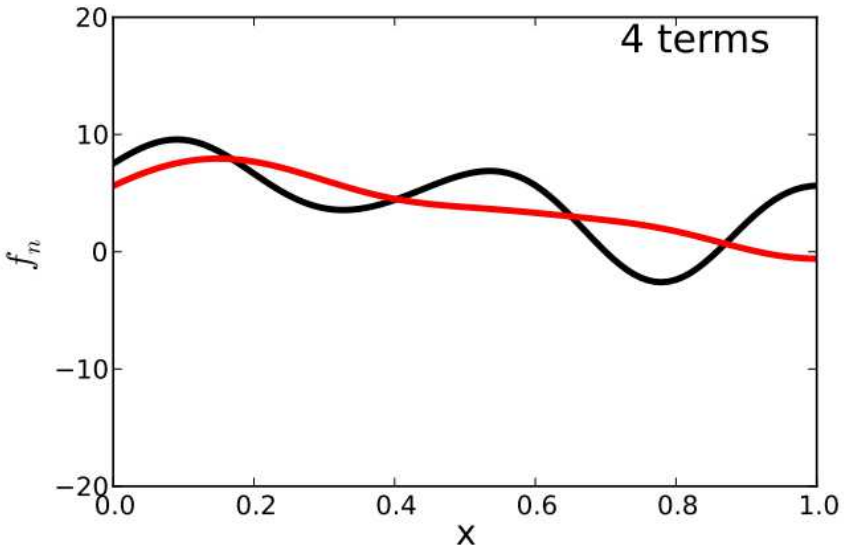

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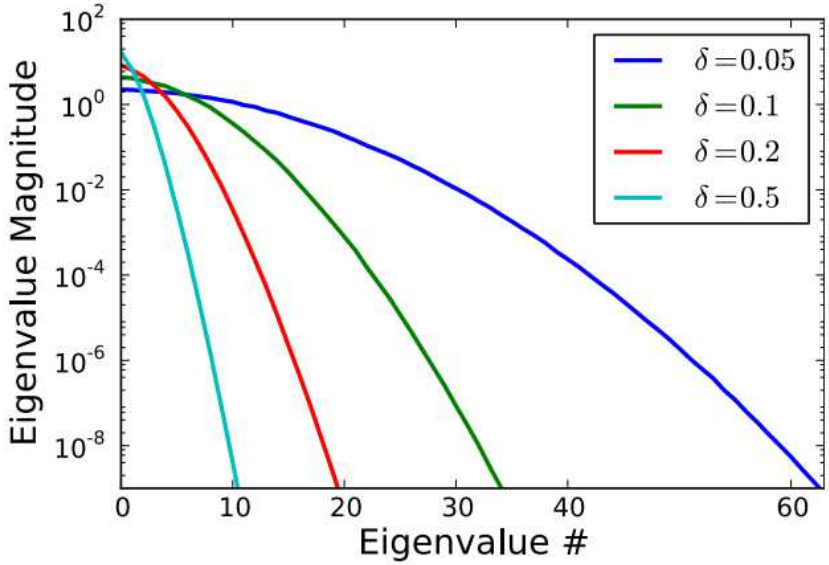

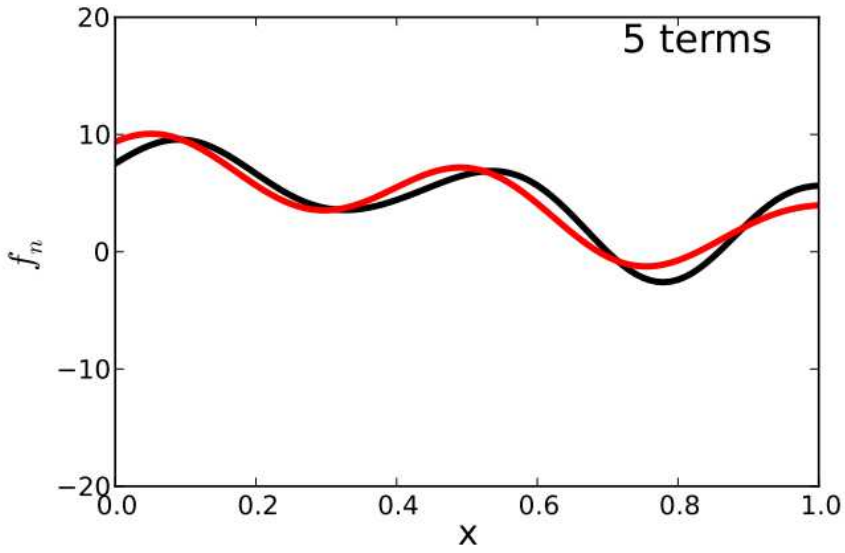

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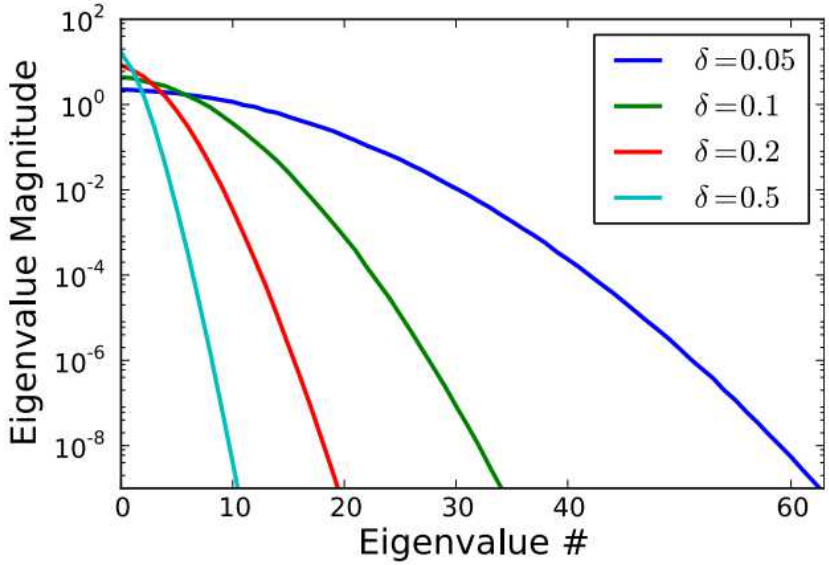

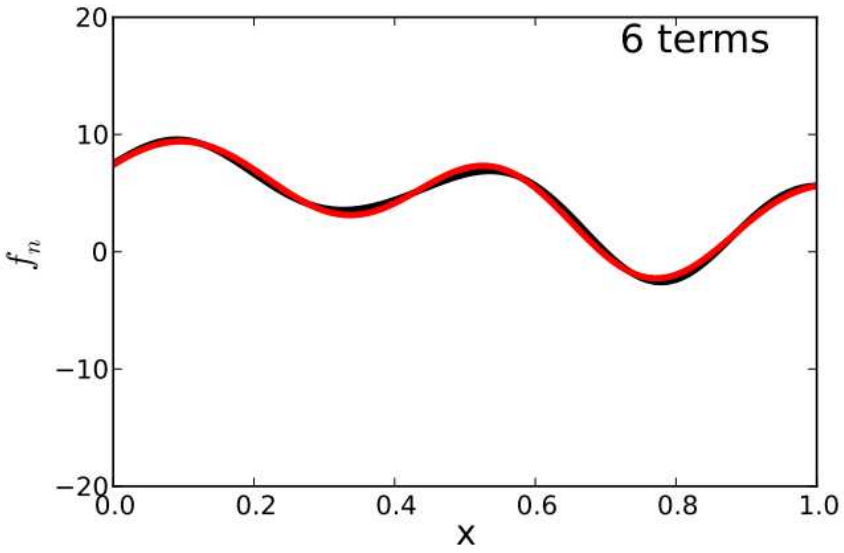

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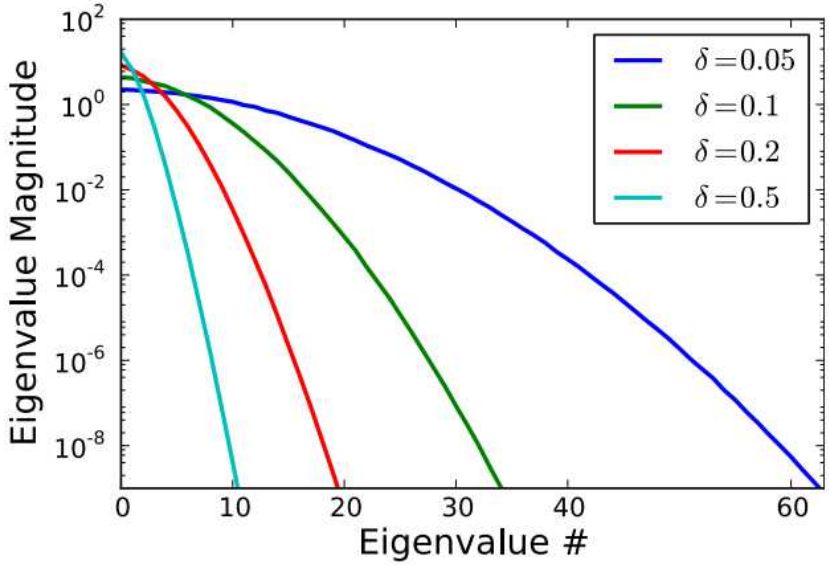

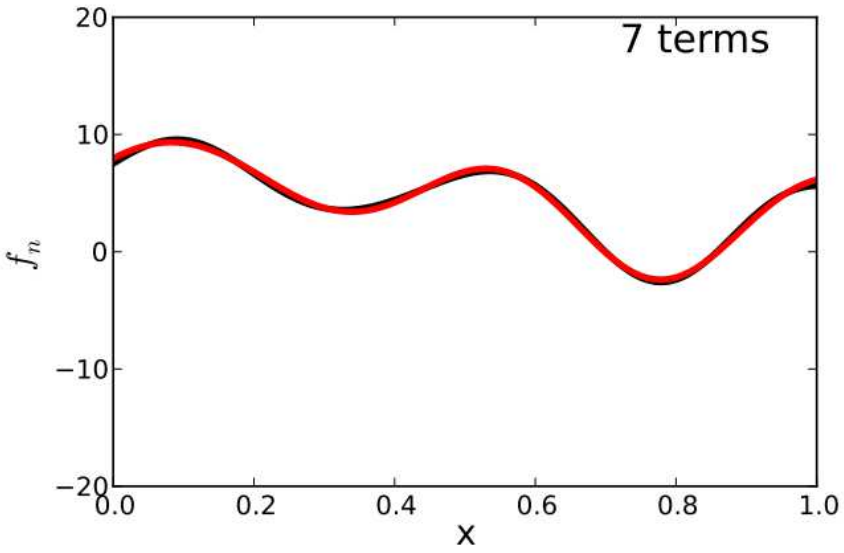

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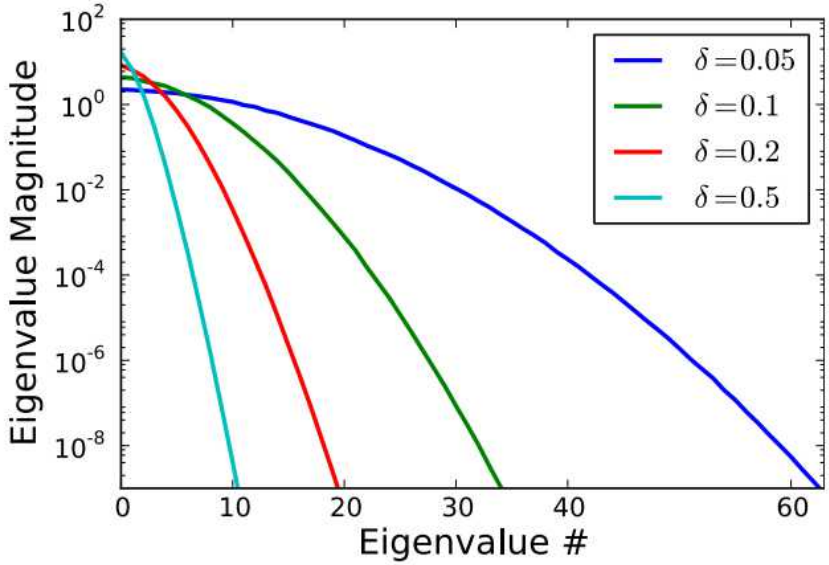

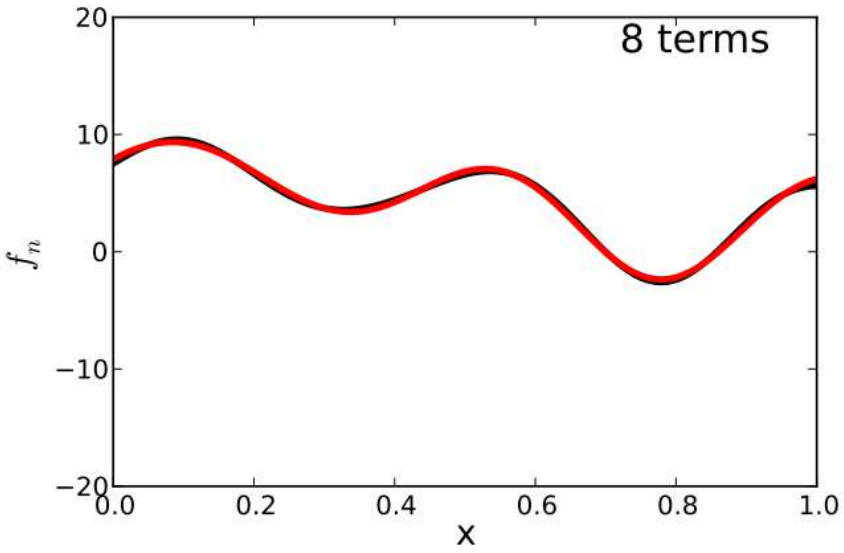

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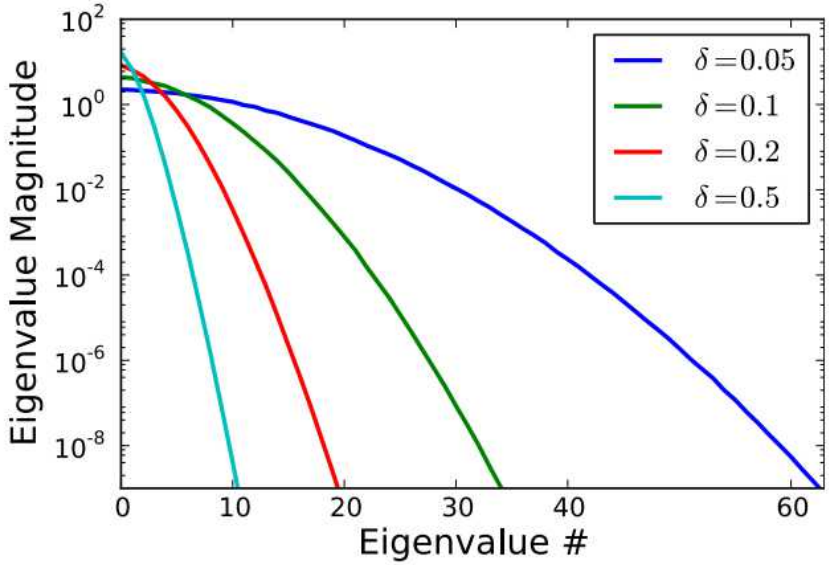

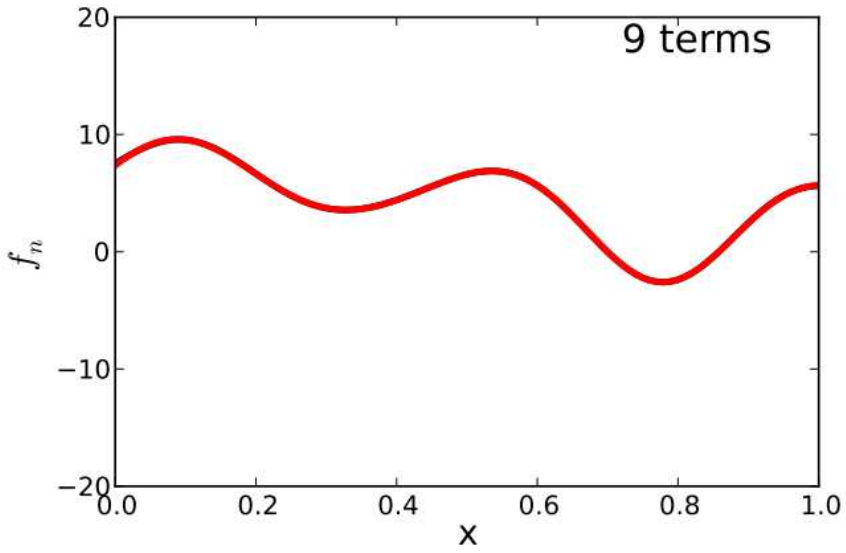

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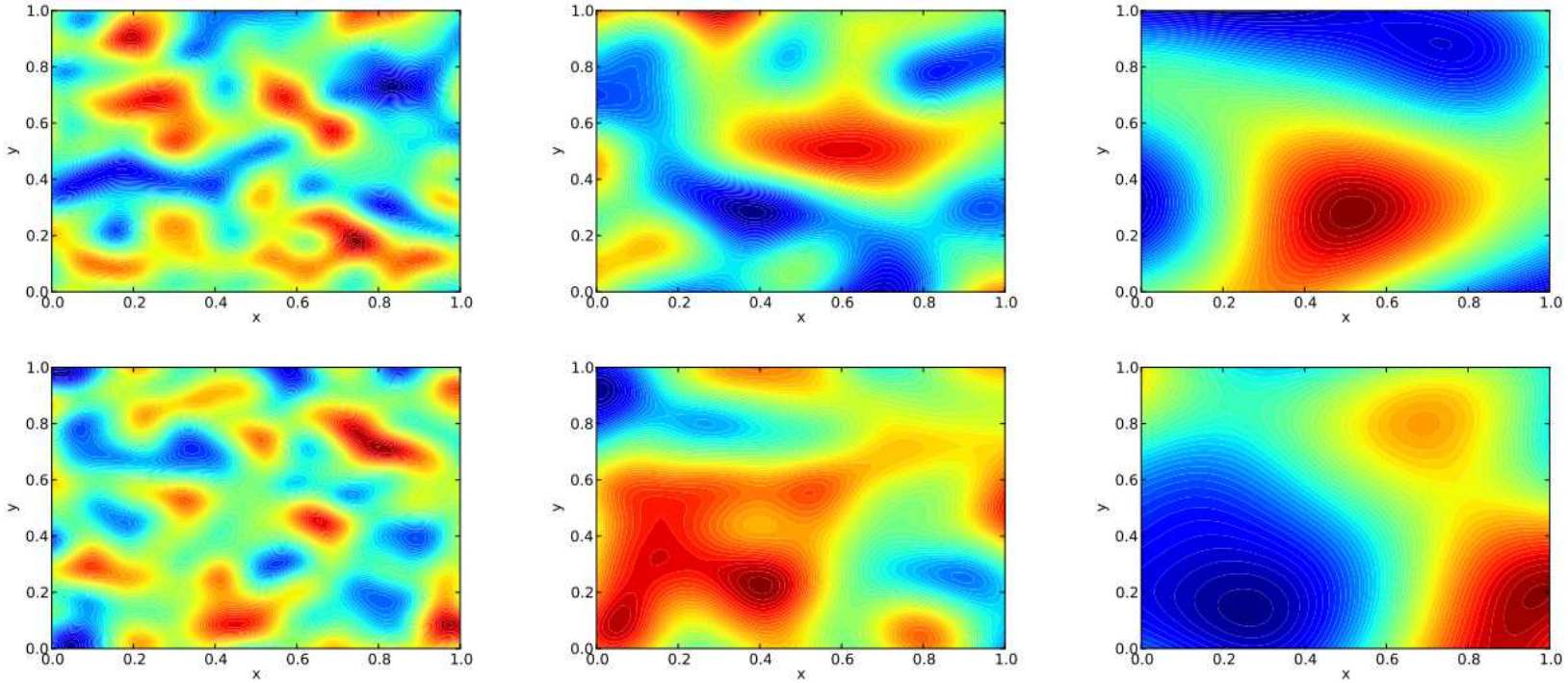
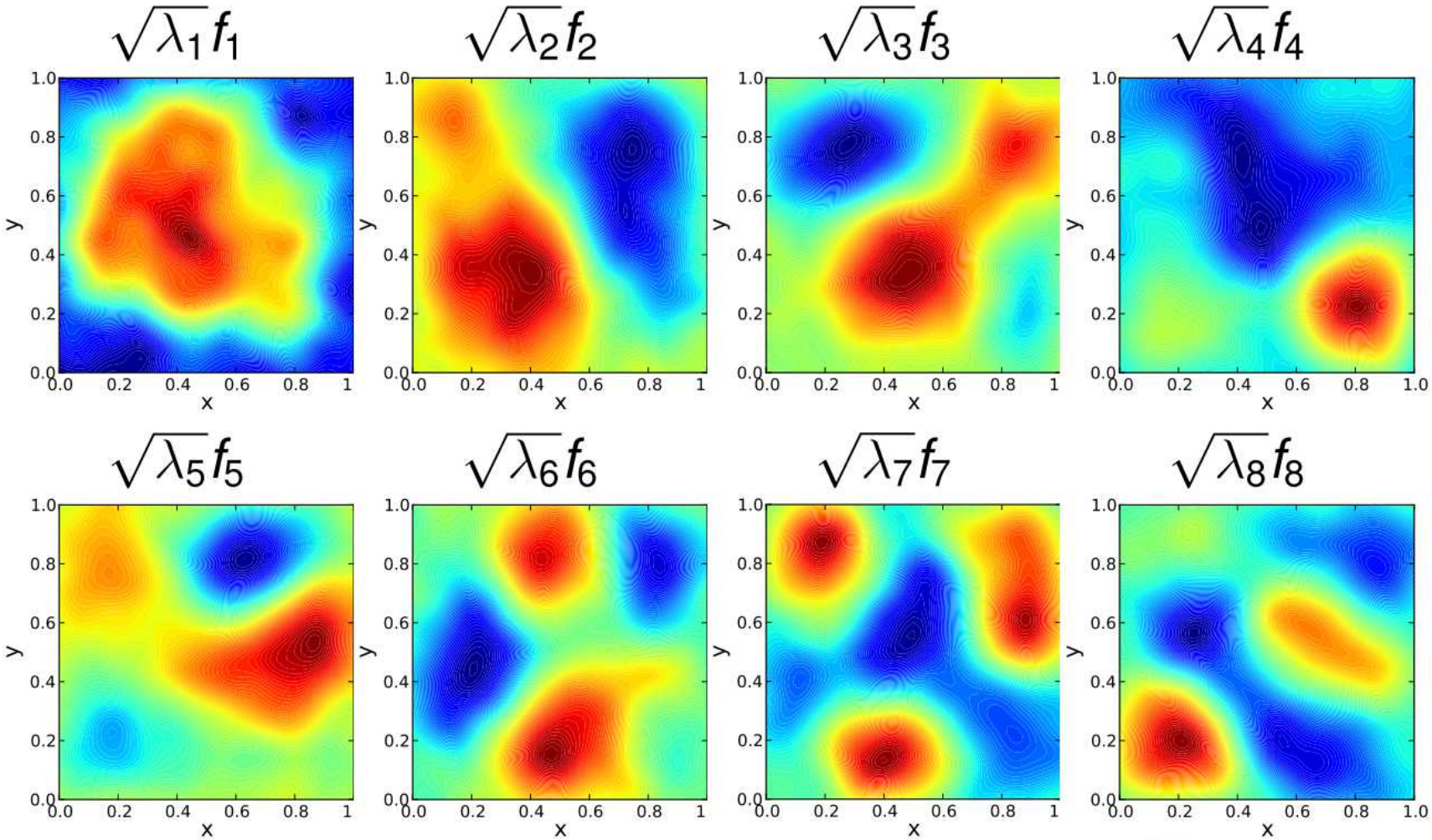
KL of 2D Gaussian Process

$\delta = 0.1$ $\delta = 0.2$ $\delta = 0.5$



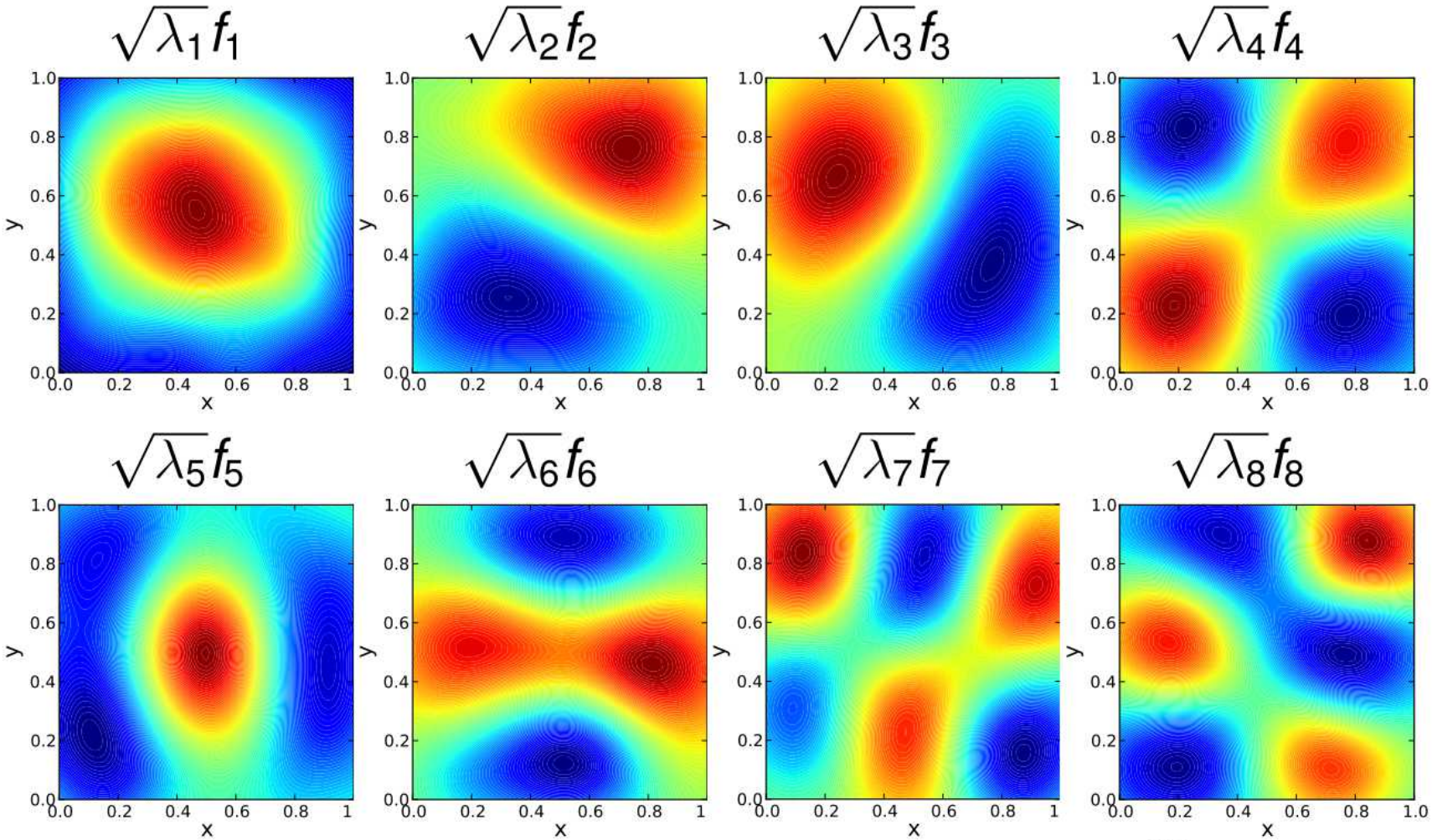
- 2D Gaussian Process with covariance: $Cov(x_1, x_2) = exp(-||x_1 - x_2||^2/\delta^2)$
- Realizations are smoother as covariance length δ increases

2D KL - Modes for $\delta = 0.1$



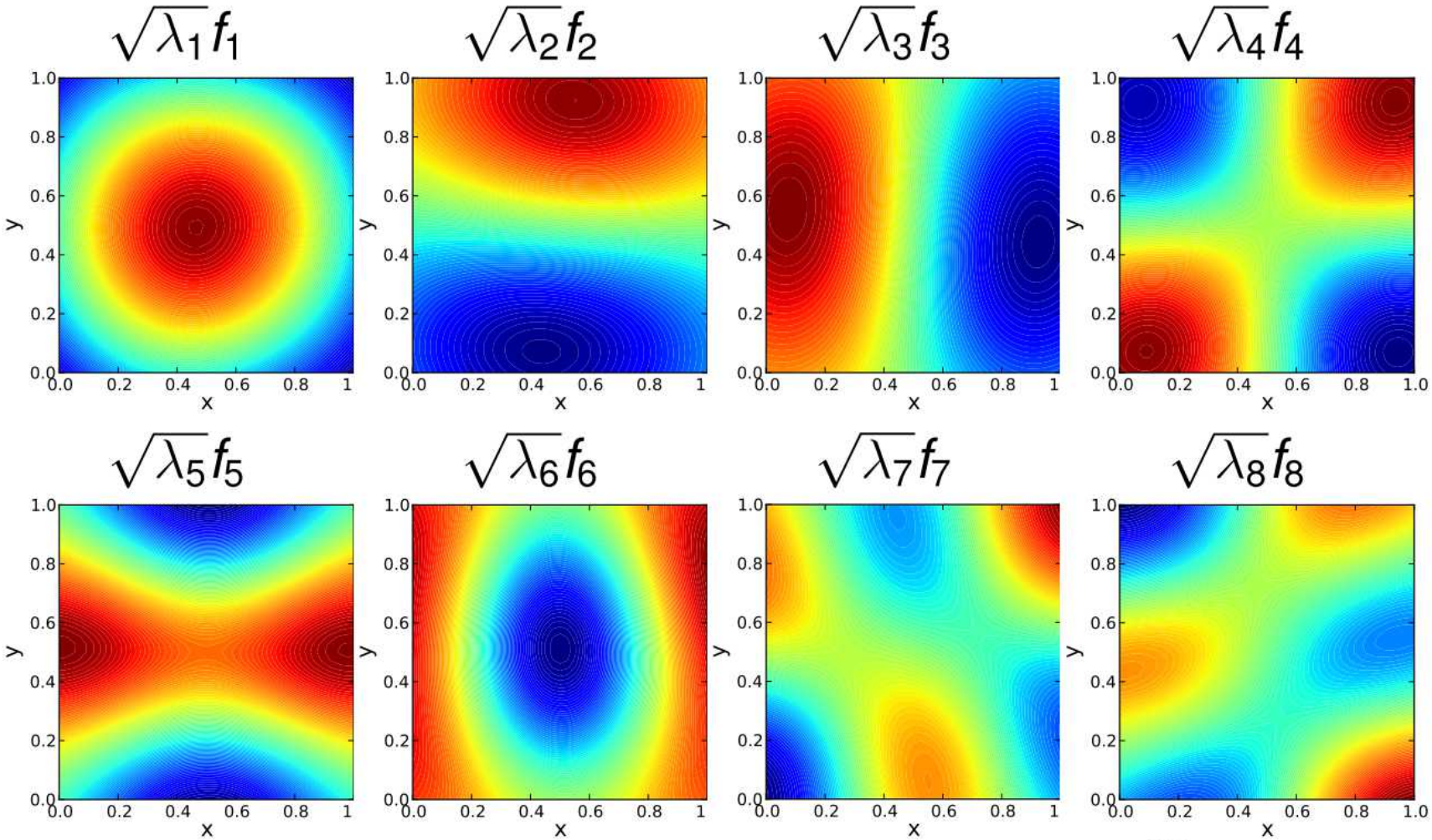
- Data covariance matrix constructed from $2^{12} = 4096$ Gaussian process realizations

2D KL - Modes for $\delta = 0.2$



- Data covariance matrix constructed from $2^{12} = 4096$ Gaussian process realizations

2D KL - Modes for $\delta = 0.5$



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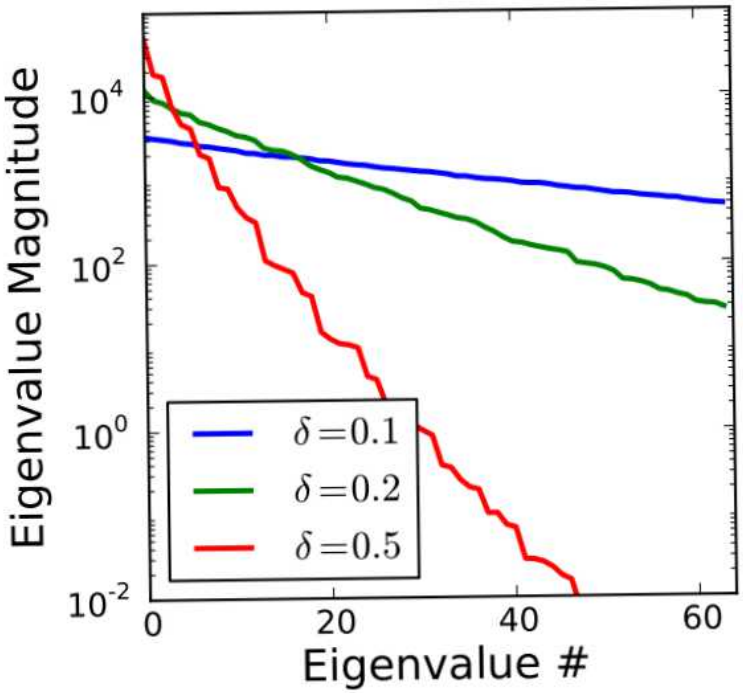
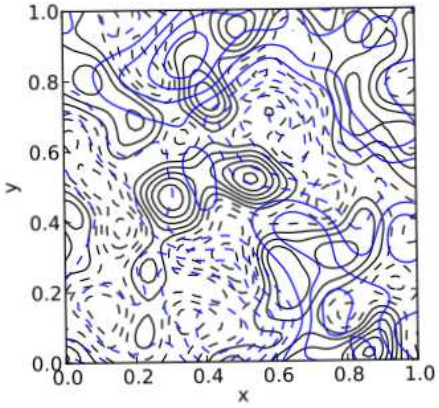
2D KL - Eigenvalue Spectrum

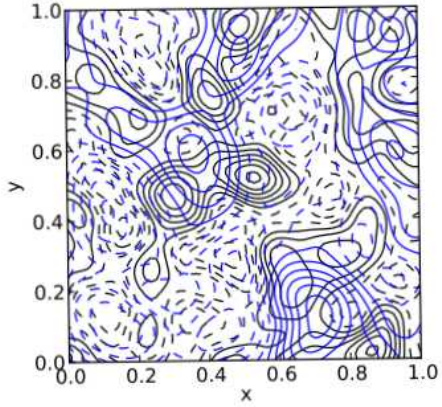


$\delta = 0.1$

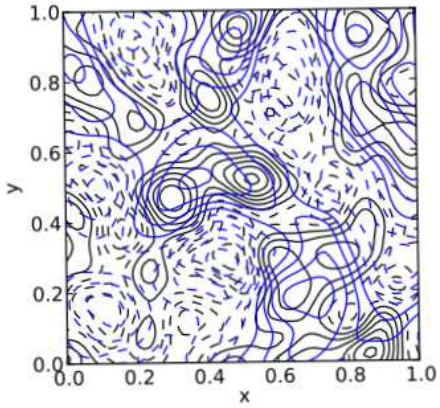
4 terms



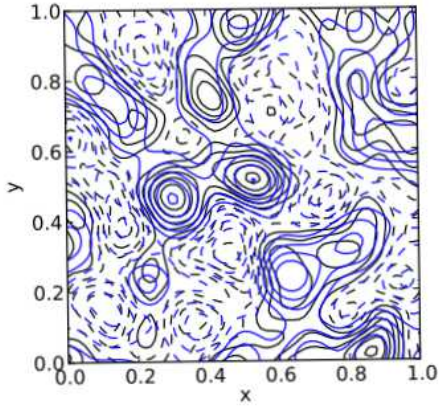
16 terms



32 terms



64 terms



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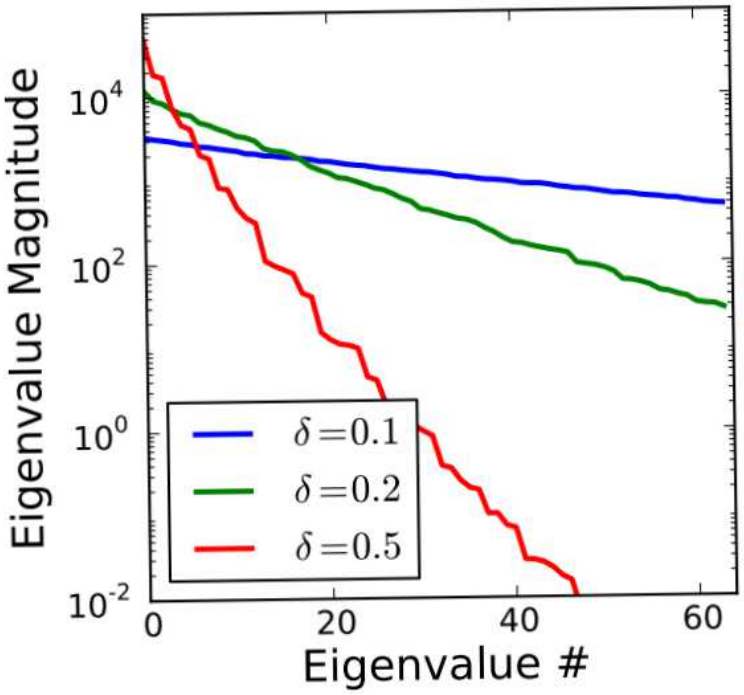

$\delta = 0.2$

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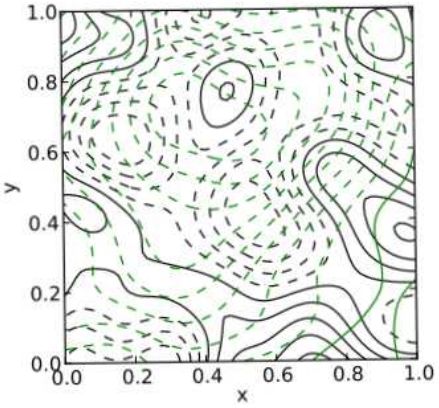

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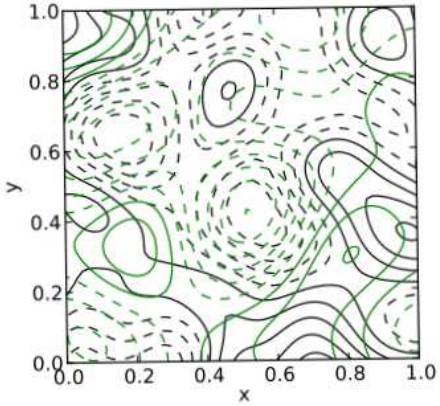

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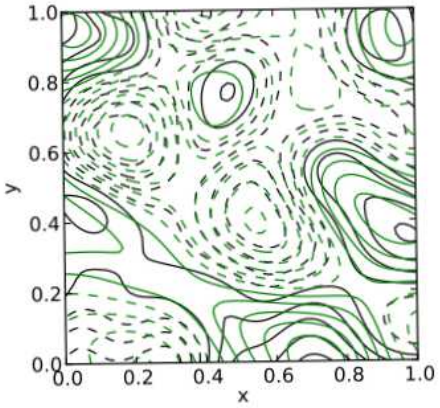

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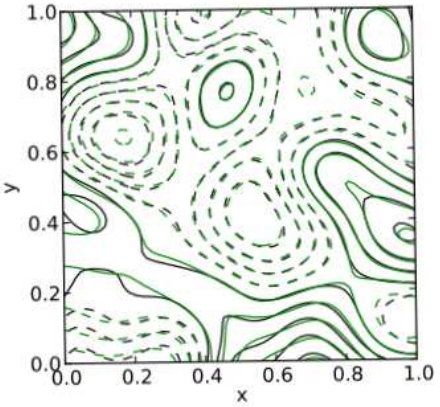

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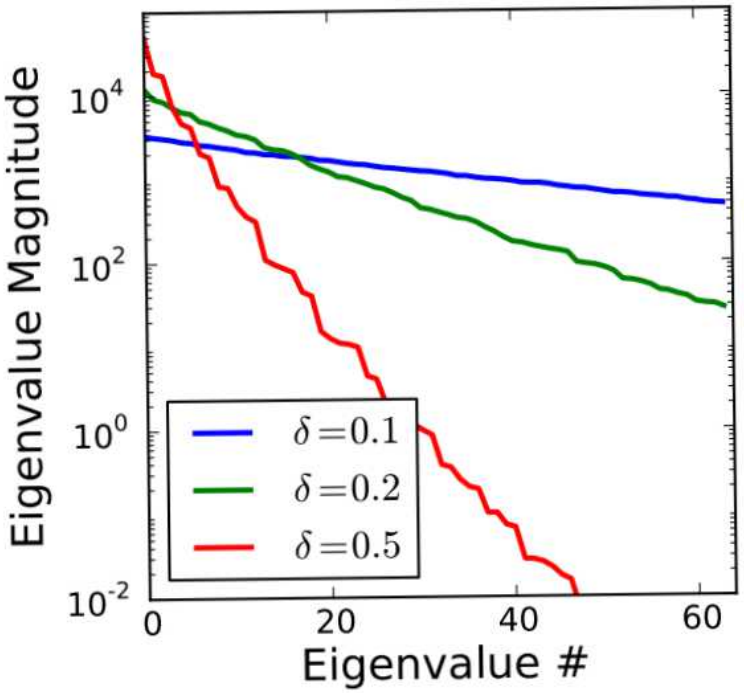
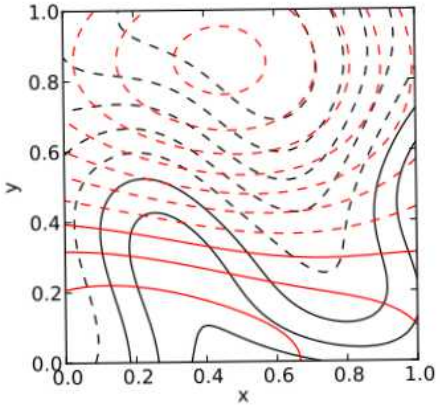


$\delta = 0.5$

4 terms



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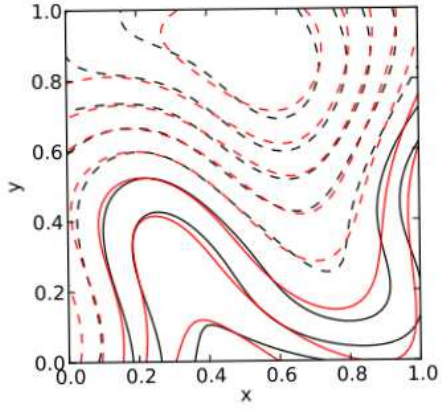

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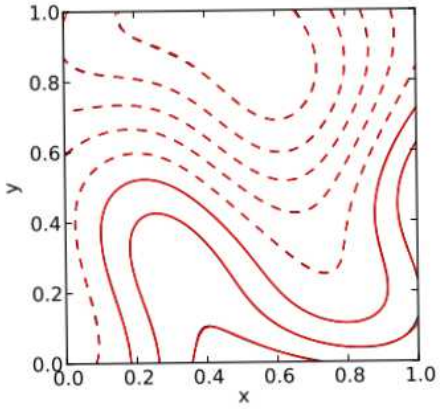

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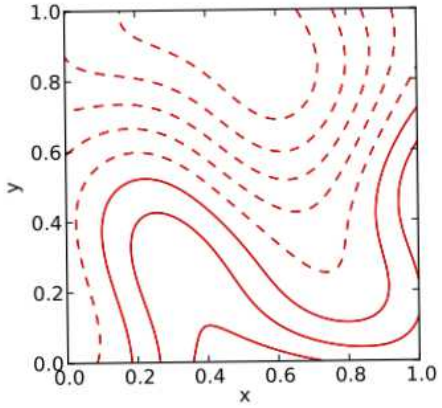

KL: Summary

- The KL expansion allows a compact, finite dimensional representation of random fields
- Dimensionality reduction is strongest if eigenvalue spectrum decays rapidly
  - Long correlation length in random field realizations
  - Relatively narrow range of feature sizes
- A potential drawback is that many samples are needed to compute the covariance matrix accurately
  - 1000's of realizations were needed to get smooth eigenmodes in 2D examples shown earlier
- Often, a covariance function is assumed and the associated eigenmodes are used to parametrize an unknown random field

Further Reading

References on Polynomial Chaos Fundamentals

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