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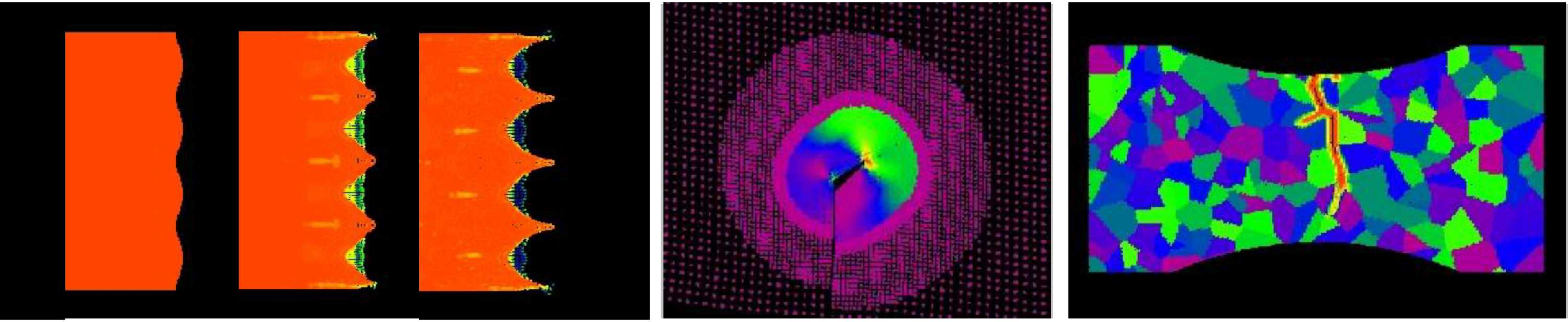

Waves in Peridynamic Media

Stewart Silling
Multiscale Science Department
Sandia National Laboratories, Albuquerque, New Mexico

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Outline

- Linear waves
    - Dispersion
    - Attenuation
- Nonlinear waves
    - Solitons
    - Shocks

Dynamic linear waves: Dispersion

- General peridynamic equation of motion:

$$\rho(x)\ddot{u}(x,t) = \int_{-\delta}^{\delta} f(\eta,\xi)\, d\xi + b(x,t), \qquad \xi = q - x, \qquad \eta = u(q,t) - u(x,t).$$

- Linear microelastic material:

$$f(\eta,\xi) = C(\xi)\eta, \qquad C(\xi) = \text{micromodulus.}$$

- Assume a wave of the form $u(x,t) = e^{i(kx-\omega t)}$. Condition on ω:

$$-\rho\omega^2 = \int_{-\infty}^{\infty} C(\xi)\left(e^{ik\xi} - 1\right)\, d\xi.$$

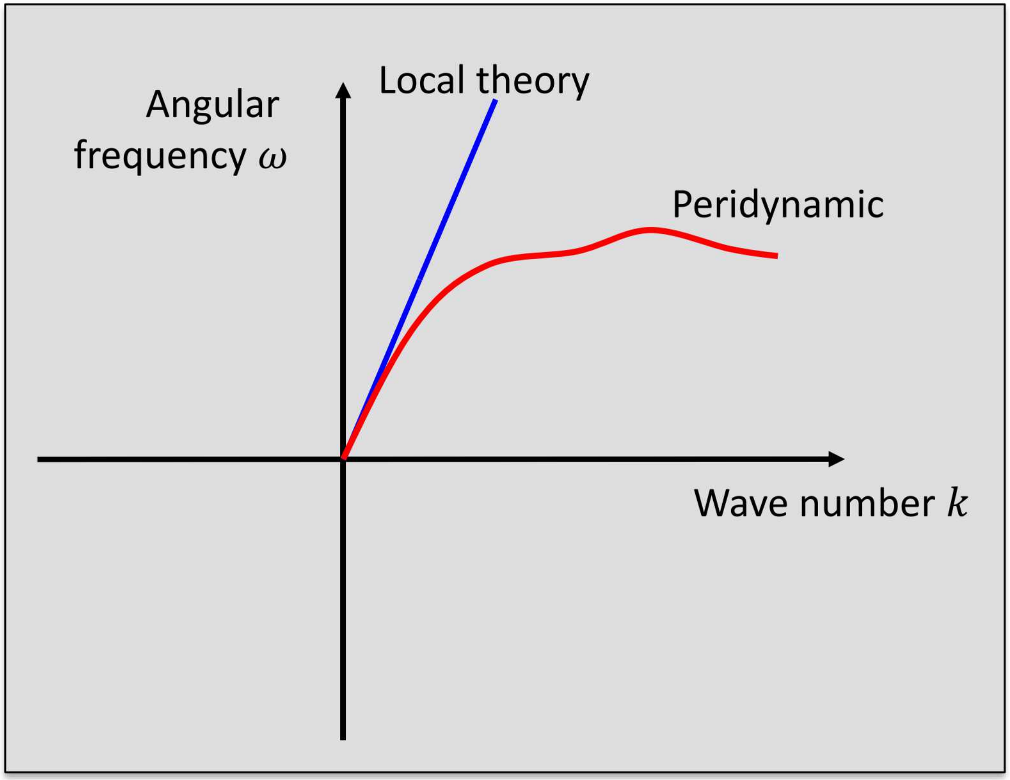
- Therefore the dispersion relation is

$$\omega(k) = \sqrt{\frac{P - \bar{C}(k)}{\rho}}, \qquad P = \bar{C}(0)$$

where $\bar{C}(k)$ is the Fourier transform of $C(\xi)$.

Dispersion curve

- PD coincides with the local theory for long wavelengths (small k).
- Phase velocity ω/k is not constant in PD.
- Group velocity $d\omega/dk$ can be nonpositive.

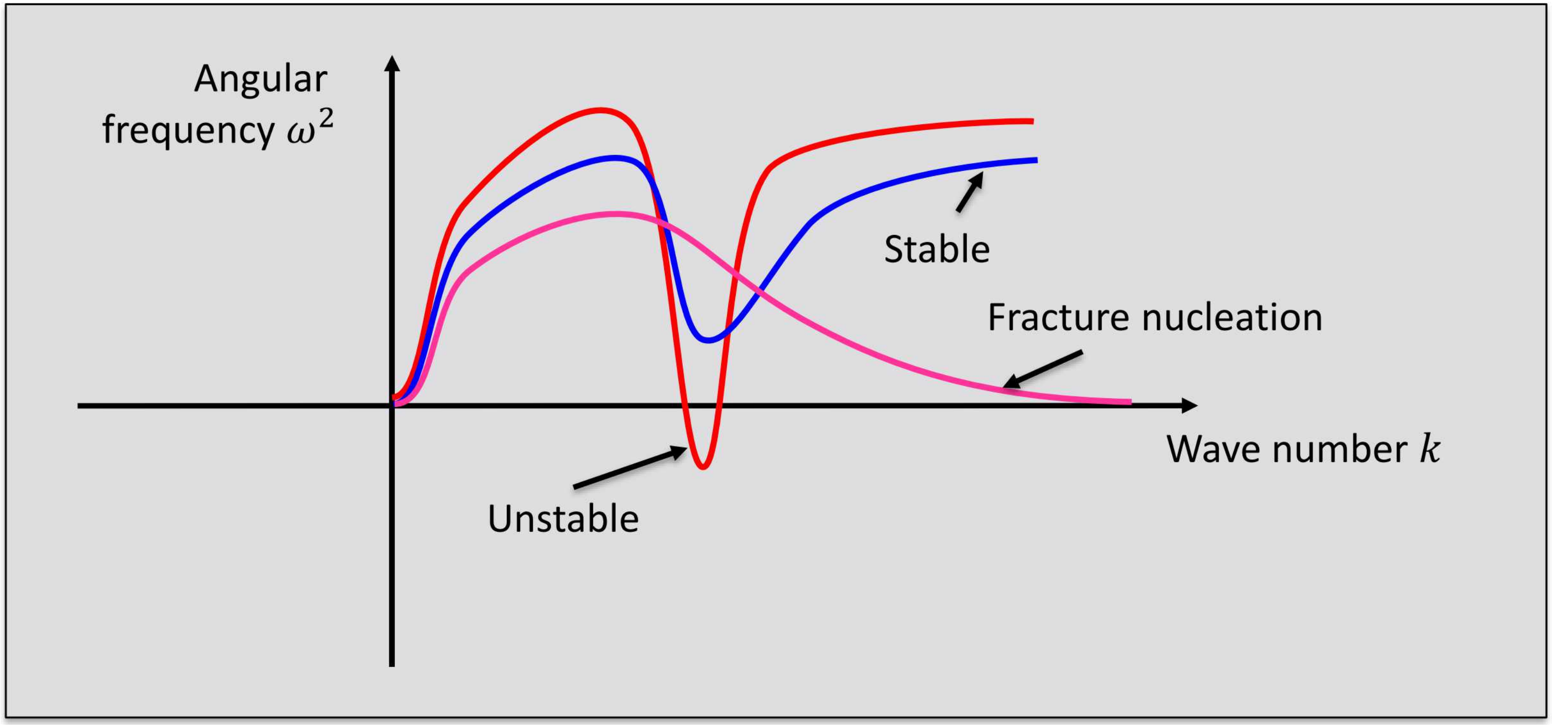


Wave dispersion in (undamped) peridynamics

- SS, *JMPS* (2000).
- Seleson, Parks, Gunzburger & Lehoucq, *Multiscale Modeling & Simulation* (2009).
- Weckner & SS, *Multiscale Computational Engineering* (2011).
- Gu, Zhang, Huang & Yv, *Engineering Fracture Mechanics* (2016).
- Butt, Timothy, & Meschke *Computational Mechanics* (2017) .

Dispersion curve and stability

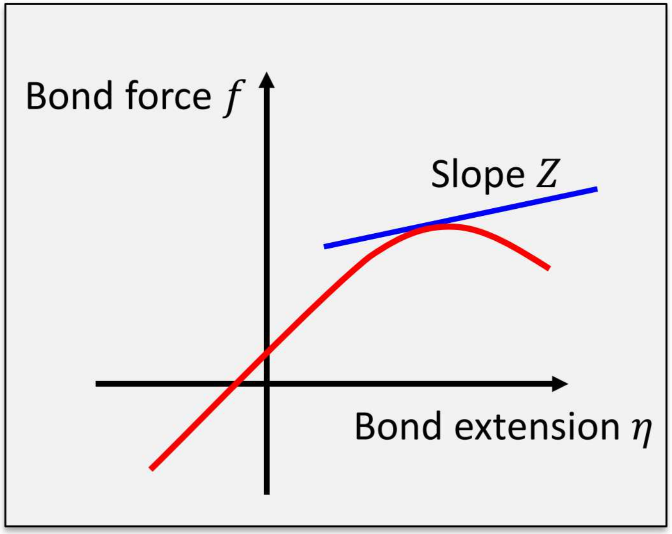
- Nonpositive $\omega(k)$ means waves can grow unboundedly over time.
- "Imaginary wave speed" since $\mathrm{V} = \omega(k)/k$.
- If $\omega(\infty) \to 0$, a small discontinuity can grow: fracture nucleation*.

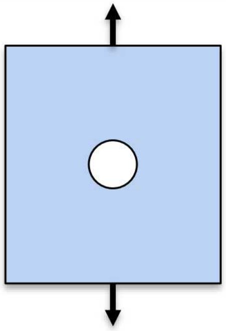


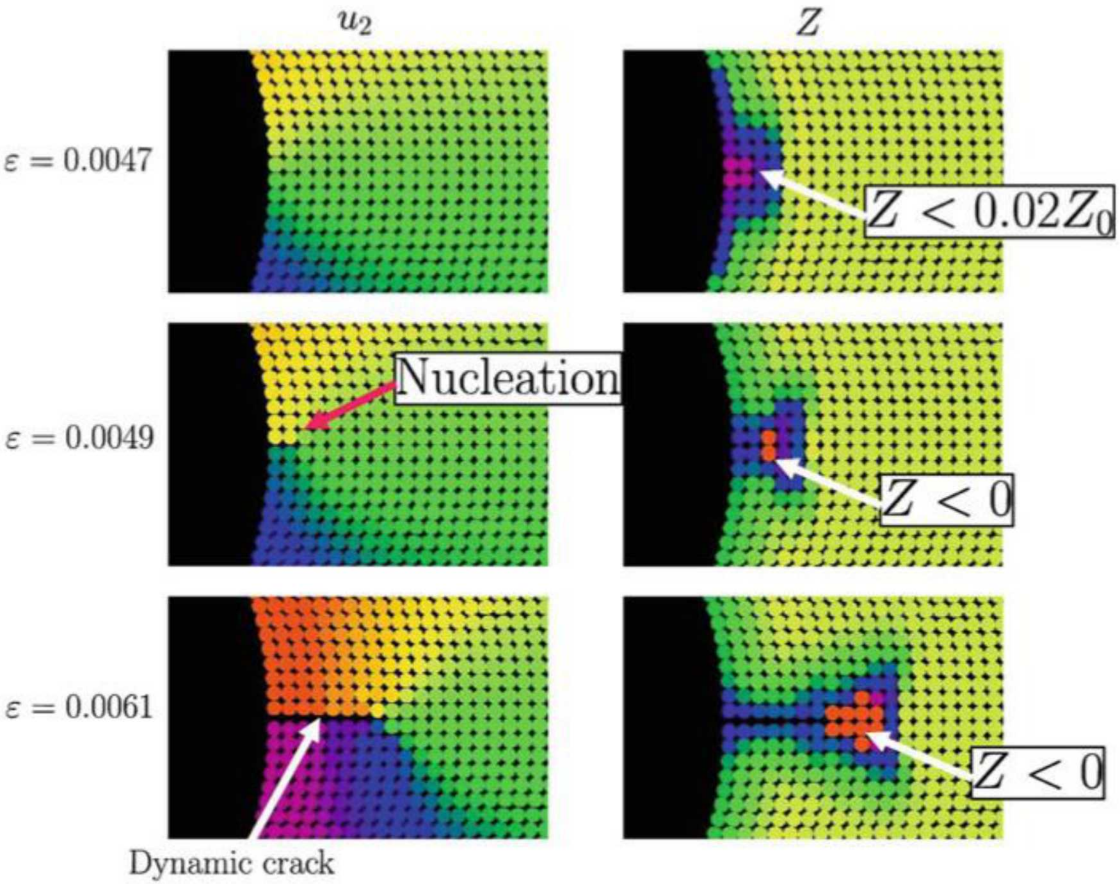
- *Silling, S. A., Weckner, O., Askari, E., & Bobaru, F. (2010) International Journal of Fracture, 162, 219-227.
- Lipton, R. (2014) Journal of Elasticity, 117, 21-50.

Material instability can be a good thing

- There is a small elastically unstable region surrounding the tip of a growing crack.







- *Silling, S. A., Weckner, O., Askari, E., & Bobaru, F. (2010) International Journal of Fracture, 162, 219-227.
- Lipton, R. (2014) Journal of Elasticity, 117, 21-50.

Microviscoelastic material

- General peridynamic equation of motion:

$$\rho(x)\ddot{u}(x,t) = \int_{\mathcal{H}} f(q,x,t)\,dq + b(x,t).$$

- Bond-based linear microviscoelastic material:

$$f(\eta,\dot{\eta},\xi) = C(\xi)\eta + D(\xi)\dot{\eta}$$

where C=micromodulus, D=damping modulus.

- Requirement for linear momentum conservation:

$$C(-\xi) = C(\xi) \qquad D(-\xi) = D(\xi).$$

- C and D can have different horizons (cutoff distances).
- Second law of thermodynamics implies

$$D(\xi) \geq 0.$$

Viscoelasticity in peridynamics

- Weckner & Mohamed, *Applied Mathematics and Computation* (2013).
- Mitchell, SAND2011-8064 (2011).
- Madenci & S. Oterkus. *Engineering Fracture Mechanics* (2017).
- Nadimi, Miscovic & McLennan, *Journal of Petroleum Science and Engineering* (2016).

Fourier transform of the equation of motion

- Transformed equation of motion leads to the following condition on $\omega(k)$:

$$\omega^2(k) + 2ir(k)\omega(k) - \omega_0^2(k) = 0,$$

$$r(k) := \frac{Q - \bar{D}(k)}{2\rho}, \qquad \omega_0(k) := \sqrt{\frac{P - \bar{C}(k)}{\rho}}.$$

- r and ω_0^2 depend only on the material properties.
- For an undamped wave, $D \equiv 0 \implies \omega(k) = \pm\omega_0(k)$.
- Otherwise, $\omega(k)$ is in general complex (for real k).

Attenuated steady waves

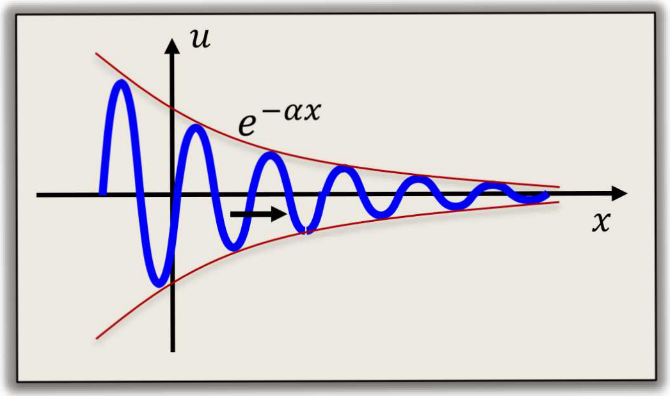
- Seek an attenuated wave solution of the form

$$u(x,t) = e^{-\alpha(k_0)x} e^{i(k_0 x - \omega(k_0)t)}, \qquad k_0, \omega \text{ real}$$

where $\alpha(k_0)$ is the *attenuation coefficient* (real).

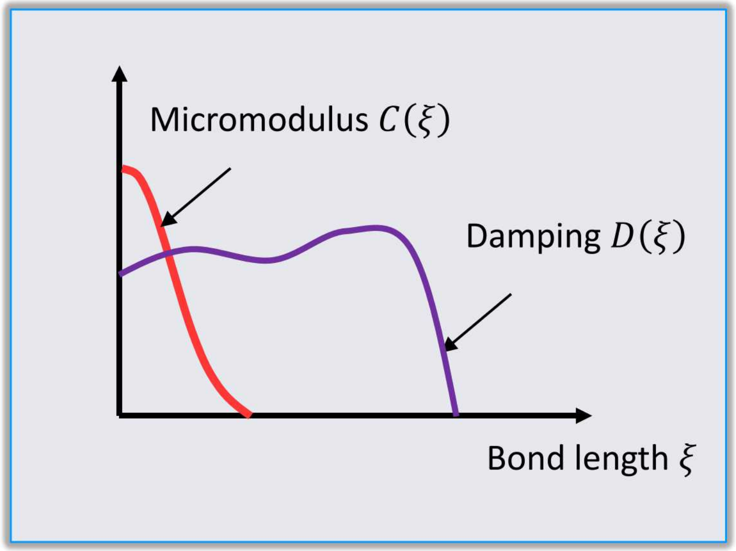
- Condition on α turns out to be

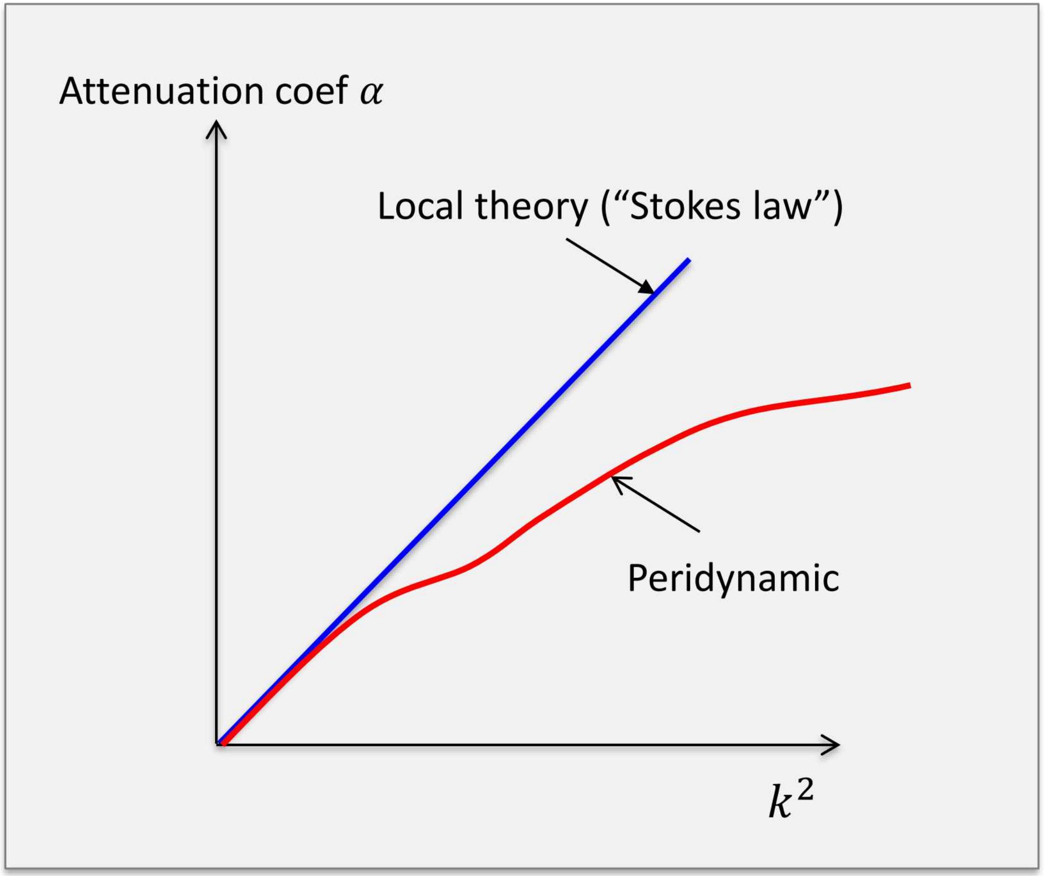
$$\text{Im}\left\{ -ir(k+i\alpha) + \sqrt{\omega_0^2(k+i\alpha) - r^2(k+i\alpha)} \right\} = 0$$



Nonlocal attenuation curve

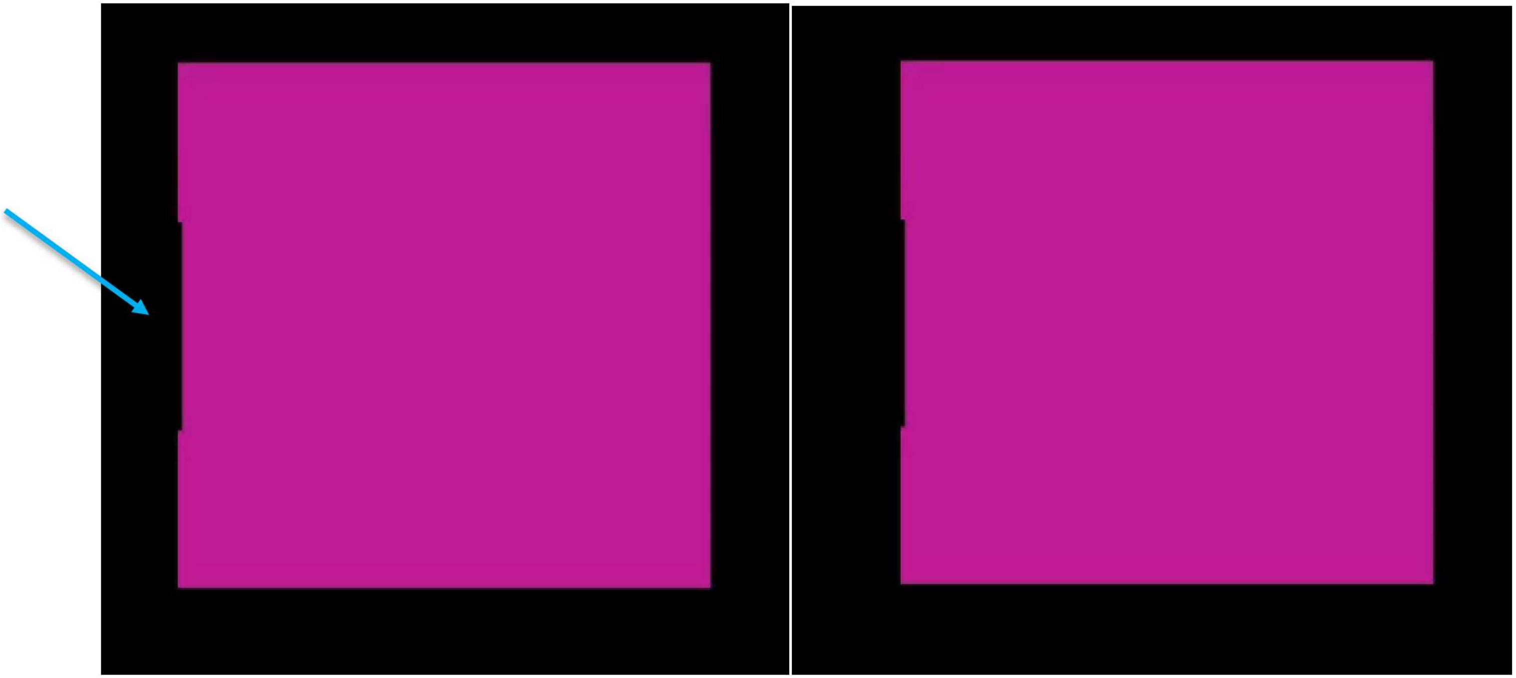
- Choose $C(\xi), D(\xi)$.
- Solve numerically the equation for $\alpha(k)$.
- Attenuation curve can be more complex than in the local theory.





Linear vs. nonlinear waves: Example

VIDEOS

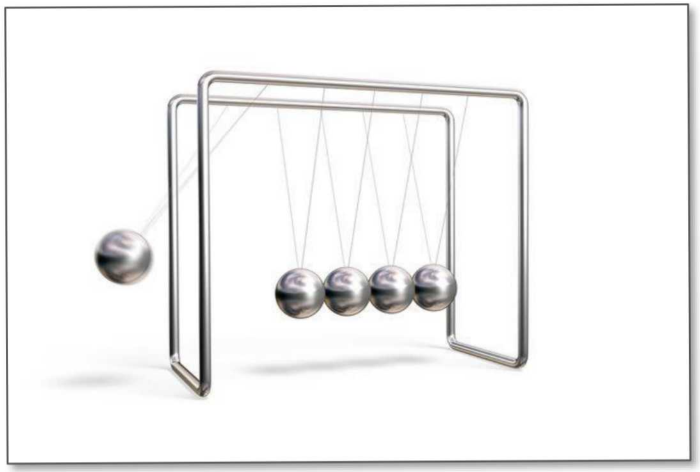


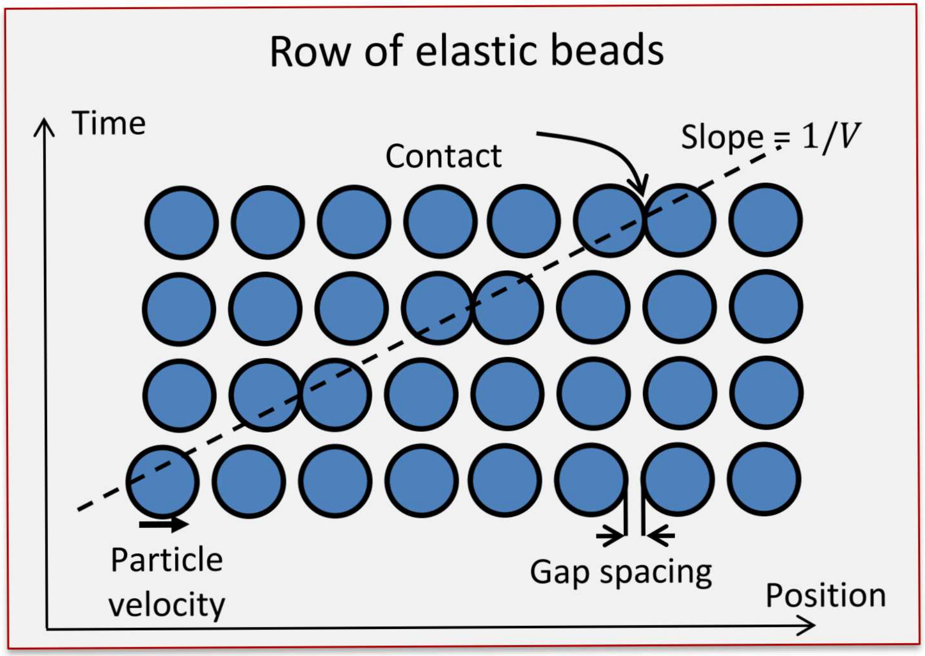
Step function for displacement at boundary

Linear

Nonlinear

What are solitary waves?





A solitary wave is a nonlinear wave that moves

- without dispersion
- without changing shape
- without dissipation
- without changing the state of the material it passes through.

Nonlinear bond force

- Material model:

$$f(\eta, \xi) = F(s)\,\mathrm{sgn}\,(\xi), \qquad s = \frac{\eta}{\xi}, \qquad 0 < |\xi| \leq \delta$$

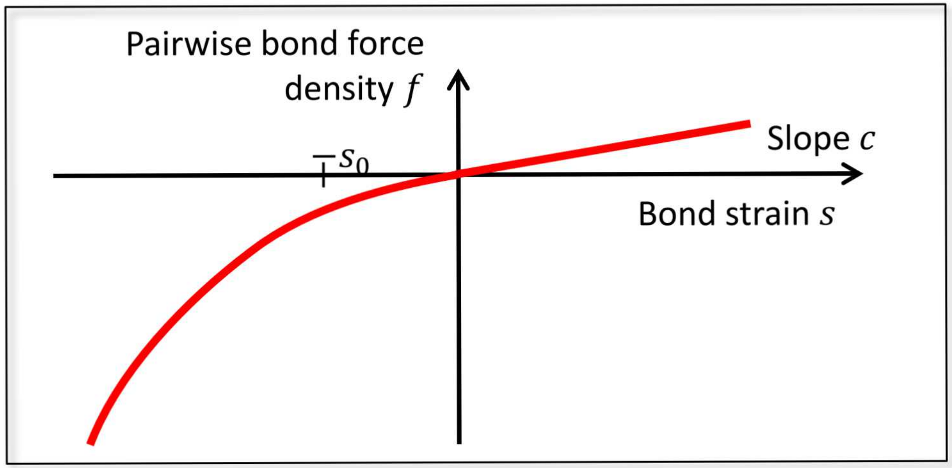
$$\eta = u(x+\xi) - u(x).$$

where F is a function and s is the bond strain.

- Consider a material model that stiffens in compression (similar to Fermi-Pasta-Ulam lattice model):

$$F(s) = \begin{cases} c(1 - s/s_0)s & \text{if } s < 0, \\ cs & \text{otherwise} \end{cases}$$

where c and s_0 are positive constants.



Steady wave assumption

- We seek steady-wave solutions of the form

$$u(x,t) = U(z), \qquad z = x - Vt$$

where V is the wave velocity (to be determined).

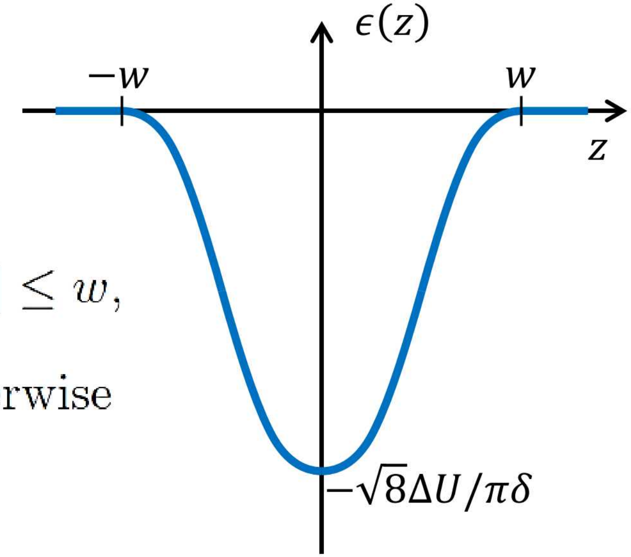
- The equation of motion becomes

$$\rho V^2 U''(z) = \int_{-\delta}^{\delta} f(\eta(z,\xi),\xi)\, d\xi$$

- Taylor expansion

$$U(z+\xi) = U(z) + U'(z)\xi + \frac{U''(z)\xi^2}{2} + \frac{U'''(z)\xi^3}{6} + \frac{U''''(z)\xi^4}{24} + O(\delta^5).$$

Exact solution to the 3rd order nonlinear ODE



- An exact solution to the ODE is

$$\epsilon(z) = \begin{cases} -\frac{\sqrt{8}\Delta U}{\pi\delta}\cos^2\left(\frac{\sqrt{2}\ z}{\delta}\right) & \text{if } |z| \le w, \\ 0, & \text{otherwise} \end{cases}$$

  where the pulse half-width is

$$w = \frac{\pi\delta}{\sqrt{8}}$$

  and ΔU is the total displacement change through the pulse.

- The wave velocity depends on the total displacement:

$$V = \pm\sqrt{\frac{2E}{3\rho}\left(1 \pm \frac{\sqrt{8}\Delta U}{\pi\delta s_0}\right)}.$$

Collisions between solitary waves

VIDEOS

- SS, Solitary waves in a peridynamic elastic solid, JMPS 96 (2016) 121-132.
- Pego, Robert L., and Truong-Son Van. "Existence of solitary waves in one dimensional peridynamics." arXiv preprint arXiv:1802.00516 (2018).

Same material model as with solitons but with nonlinear bond damping

- 1D equation of motion:

$$\rho \ddot{u}(x,t) = \int_{-\delta}^{\delta} f(\eta, \dot{\eta}, \xi)\, d\xi + b(x,t)$$

- Material with nonlinear elastic and damping terms:

$$f(\eta, \dot{\eta}, \xi) = \left(F^e(s) + F^d(\dot{s})\right) \operatorname{sgn}(\xi), \qquad s = \frac{\eta}{\xi}, \qquad 0 < |\xi| \le \delta$$

where s=bond strain, ξ=bond vector.

$$F^e(s) = \begin{cases} c(1 - s/s_0)s & \text{if } s < 0, \\ cs & \text{otherwise} \end{cases},$$

$$\boxed{F^d(\dot{s}) = \begin{cases} -D\dot{s}^2 & \text{if } \dot{s} < 0, \\ 0 & \text{otherwise} \end{cases}}$$

New

where c and D are constants.

Dissipative material model leads to a stable shock wave

- Use the same trick again to get an ODE:

$$\left[\frac{\rho V^2}{E}-1\right]\epsilon' = \left[-\frac{2\epsilon\epsilon'}{s_0}-2\beta\epsilon'\epsilon''\right]+\left[\frac{\epsilon'''}{24}-\frac{\epsilon'\epsilon''}{6s_0}-\frac{\epsilon\epsilon'''}{12s_0}-\frac{\beta\epsilon'\epsilon''''}{12}-\frac{\beta\epsilon''\epsilon'''}{6}\right]\delta^2.$$

- Ansatz:

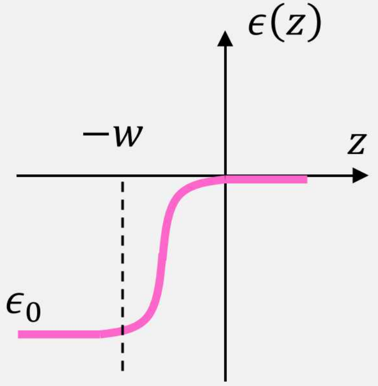
$$\epsilon(z)=\begin{cases}\frac{\epsilon_0}{2}(1-\cos kz) & \text{if } -\pi\le kz\le 0,\\ 0 & \text{otherwise.}\end{cases}$$

- This $\epsilon(z)$ satisfies the ODE with

$$V=\sqrt{\frac{E}{2\rho}\left[L+\sqrt{L\left(L-\frac{\delta^2\rho}{3Ds_0}\right)}\right]},\qquad L=1-\frac{\epsilon_0}{s_0}$$

- Shock thickness is

$$w=\frac{\pi V}{2}\sqrt{\frac{D}{c}}.$$



Shock wave structure

Nonlinear nonlocal waves: an almost completely unexplored area

- Just by choosing an appropriate material model, the basic equations without modification reproduce
  - Solitary waves
  - Phase boundaries
  - Shocks
  - Bending waves of beams and shells (O'Grady & Foster 2014, Diyaroglu et al 2015)
  - Unknown:
    - Periodic media
    - Metamaterials
    - Band gaps
    - Quantized energy levels
    - Monotonicity
    - Blow-up
    - Scattering

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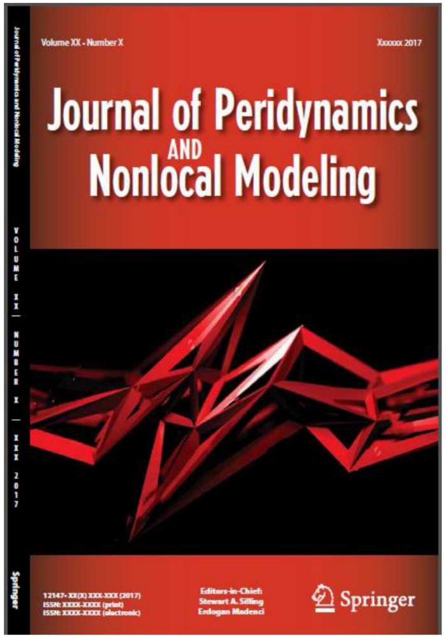

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