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Approximate Analytical Solution for Hypersonic Near Wake Flow Field

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[Abstract] The near wake flow field associated with hypersonic blunt bodies is characterized by a complex recirculating flow field with associated base pressure and expansion-compression behavior. In spite of this complexity the near wake can be described by a simple, approximate, closed-form solution. This solution is obtained using a minimal set of linearized governing equations. Though linear, simple solutions based upon these governing equations cannot be readily constructed. As such, an approximate splitting solution procedure which is analogous to the classical method of variation of parameters from ordinary differential equation theory is identified. Application of this approach yields two solutions: a recirculating flow field in the lee of the body and a global diffusive model. The combination of these expressions is valid over the entire flow domain. The resulting solution yields qualitatively plausible flow field behavior. With access to the velocity field temperature field, Crocco-Busemann supplemented by a modified recovery factor is used to estimate the temperature field. Quantitative comparison of centerline velocity and temperature predictions to the measurements of Martellucci et. al. and Ramaswamy suggest good agreement where Mach number behavior is reflected in the effective viscosity coefficient and recovery parameter. In a manner analogous to the classical axisymmetric jet we suggest that by choosing appropriate effective viscosity values, the current model is valid for both laminar and turbulent flows. Though highly simplified, the current modeling approach offers useful insight into hypersonic near wake flow physics.

Nomenclature

h=H	=	bluff base body height or radius
M	=	Mach number
p	=	pressure (Pa)
r	=	recovery factor
R	=	Residual
u	=	streamwise velocity (m/s)
$U=U_\infty$	=	freestream velocity (m/s)
v	=	cross-stream velocity (m/s)
x	=	streamwise location (m)
y	=	normal location (m)
λ	=	eigenvalue or separation constant
ν	=	viscosity (m^2/s)
ψ	=	streamfunction (m^2/s)
ρ	=	density (kg/m^3)

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Subscripts/superscripts

0	=	constant
2d	=	two dimensional
axi	=	axisymmetric
dif	=	diffusive
lam	=	laminar
rec	=	recirculation
turb	=	turbulent

I. Introduction

Wake flow behavior for high speed reentry bodies is of considerable importance since radar effects e.g. Radar Cross Section (RCS), are intimately related to the wake flow behavior[1]. The so-called far wake, i.e. the wake flow far from base of the vehicle can often be modeled using self-similar approaches to describe the velocity defect.. The near wake, however, is complex with a localized recirculation bubble and expansion compression phenomena. The low base pressure associated in the immediate lee of the body has been the subject of numerous analytical, experimental and computational investigations. The strong adverse pressure gradient induces the flow recirculation behavior. The recirculation flow field strength, recirculation bubble length (distance from the body to the rear stagnation point) and the initial rapid decrease in the velocity defect are complex but essential to understanding the near field flow and the initial conditions in terms of velocity defect for the far wake behavior. Applications such as base antenna configuration and operation are usually immersed within the circulation bubble flow requiring accurate estimates for their application.

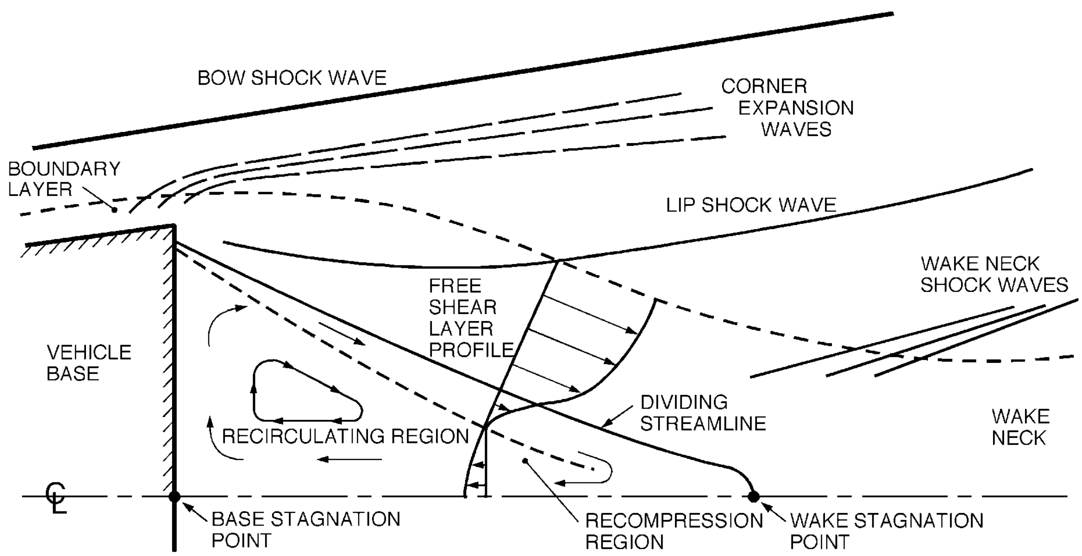

Figure 1. Schematic of hypersonic near wake flow behavior

Though complex, the necessity to be able to accurately describe the near wake flow for reentry applications drove the development of a series of early analytical models[2,3]. These models were often based upon asymptotic expansions, power-series or combined analytical numerical approaches. While undeniably clever, these approximation were often cumbersome, requiring iteration or other means to connect localized models. Modern approaches are virtually always computational simulations[4] which are usually accurate and robust but are rather opaque relative to the insight provided by analytical expressions.

Here we derive a set of analytical expressions based upon explicit elementary functions that describe the flow field in the lee of a bluff body. Starting with a highly simplified set of linearized governing equations which can solved using an approximate splitting procedure one obtains expressions that describe both the diffusive velocity defect and the recirculation zone. Combination of these solutions yields an expression for the velocity field that is valid over the entire domain. A Crocco-Busemann energy integral[5] supplemented by a modified recovery factor is used to estimate the temperature field. The resulting solutions are qualitatively plausible flow field behavior. Quantitative comparison of centerline velocity and temperature predictions to the measurements of Martellucci et. al[6]. and Ramaswamy[7] suggest good agreement where Mach number behavior is reflected in the effective diffusion coefficient and recovery parameter. Remarkably utilization of the current analytical model for both laminar and turbulent near wake flows suggests that by choosing appropriate effective viscosity parameters that this formulation is valid for either flow regime.

II. Governing Equations

We begin by considering the linearized approximate momentum equations:

$$\begin{aligned} u_0 u_x + \frac{1}{\rho} p_x &= \nu_{eff} u_{yy} \\ u_0 v_x + \frac{1}{\rho} p_y &= 0 \end{aligned} \tag{1}$$

Incompressible continuity takes the form: $u_x + v_y = 0$. To eliminate the pressure, we "cross-differentiate" and subtract. Further introducing the stream function $u = \psi_y \quad ; \quad v = -\psi_x$ to satisfy continuity, we write:

$$u_0(\psi_{yy} + \psi_{xx})_x = \nu_{eff}\psi_{yyyy} \tag{2}$$

Equation (2) retains two basic constructs associated wake flow:

1. Velocity defect diffusion
2. Near field recirculation behavior

Though a relatively simple linear partial differential equation, equation (2) can be best analyzed by splitting it into the two elements described previously.

Equation Splitting Approximation

Splitting an equation into separate pieces and solving the associated expressions provides an avenue to understanding the approximate functional behavior of the solutions. Let's consider a trivial example for the ODE:

$$u'' + \lambda u' + \lambda^2 u = 0 \tag{3}$$

The exact solution for this expression is trivial as:

$$u = \exp(-\frac{1}{2}\lambda x)\left(c_1 \sin(\frac{\sqrt{3}}{2}\lambda x) + c_2 \cos(\frac{\sqrt{3}}{2}\lambda x) \right) \tag{4}$$

Now, let's consider an approximate solution. Let's let one of the solutions, say u_1 be governed by:

$$u_1'' + \lambda_1 u_1' = 0 \tag{5}$$

With solution: $u_1 = c_1 \exp(-\lambda_1 x) + c_1 \exp(-\lambda_1 x)$. In a similar manner, we could solve the u_2 problem:

$$u_2''+\lambda_2^2 u_2 = 0 \tag{6}$$

With solution: $u_2 = c_1\sin(\lambda_2 x) + c_2\cos(\lambda_2 x)$. By inspection one would propose that an approximate solution might be written:

$$u = \exp(-\lambda_1 x)\left(c_1\sin(\lambda_2 x) + c_2\cos(\lambda_2 x)\right) \tag{7}$$

i.e. a form related to: $u \approx u_1(x)u_2(x)$ where we have chosen Notice the remarkable similarity to the traditional method of variation of parameters utilized for inhomogeneous ODE theory[8]. Obviously, the associated constants λ_1 and λ_2 would need to be estimated by other means, e.g. a traditional Galerkin approximation[9] scheme. Indeed by introducing a trial function of the form: $u_t = \exp(-\lambda_1 x)\sin(\lambda_2 x)$ one can compute the residual $R \equiv u_t''+\lambda u_t'+\lambda^2 u_t$ and compute the constraints: $R(x=0)$ and $\int_0^\infty R(x)dx$ which can be solved to exactly recover: $\lambda_1 = \frac{1}{2}\lambda$ and $\lambda_2 = \frac{\sqrt{3}}{2}\lambda$ whereby an exact solution to the associated expression has been recovered.

Equation Splitting: Wake Flow

Let's consider a simple splitting applied to the wake flow problem. All variable are non-dimensionalized by bluff body base height, free-stream velocity and free-stream temperature. The Reynolds number is based upon: $\mathrm{Re}_h = \mathrm{Re}_H = \frac{u_\infty h}{\nu_{eff}}$.

Diffusive Wake

The obvious approach would be to model the near field problem as:

$$\psi_{xxx} = \frac{\nu_{eff}}{u_0}\psi_{yyyy} \tag{8}$$

while the defect portion is modeled via:

$$\psi_{yyx} = \frac{\nu_{eff}}{u_0}\psi_{yyyy} \tag{9}$$

Since the defect portion honors the tradition diffusive wake problem $u_x = \frac{\nu_{eff}}{u_0}u_{yy}$, the solution is easily obtained as:

$$u_{dif}(x,y) = \frac{1}{2}\left(erf\left(\frac{\mathrm{Re}_h^{1/2}(y+1)}{2(x)^{1/2}}\right) - erf\left(\frac{\mathrm{Re}_h^{1/2}(y-1)}{2(x)^{1/2}}\right)\right) \tag{10}$$

Where $\mathrm{Re}_h = \frac{u_0}{\nu_{eff}}$ and appropriate constants have been used to enforce a zero flow condition in the lee of the bluff body. Using the streamfunction formulation we can readily compute the "v" velocity as $v = -(\int u dy)_x = -\psi_x$:

$$v(x,y) = \frac{1}{\sqrt{\pi \mathrm{Re}_h}} \left(\exp\left(\frac{\mathrm{Re}_h (y-1)^2}{4x} \right) - \exp\left(\frac{\mathrm{Re}_h (y+1)^2}{4x} \right) \right) \tag{11}$$

Axisymmetric Solution

The preceding solution was based upon a 2-d formulation where a simple analytical solution is well-established. Unfortunately a similar model for an axi-symmetric formulation is not available. To partially remedy this issue we propose to retain the cross-stream i.e. "y" variation of equation (10) but modify the streamwise "x" variation. Indeed, by simply proposing a solution of the form: $u(x,y) = u_{2d}(x,y) g(x)$ and substituting into $\frac{\partial u}{\partial x} = \frac{1}{y} \frac{\partial}{\partial y} (\frac{y}{\mathrm{Re}_h} \frac{\partial u}{\partial y})$ evaluated for y=0, one obtains a differential equation for g(x) that can be solved to give:

$$g(x) = erf\left(\frac{\mathrm{Re}_h^{1/2}}{2(x)^{1/2}} \right) \tag{12}$$

Whereby:

$$u_{dif\_axi}(x,y) = \frac{1}{2} erf\left(\frac{\mathrm{Re}_h^{1/2}}{2(x)^{1/2}} \right) \left(erf\left(\frac{\mathrm{Re}_h^{1/2}(y+1)}{(2x)^{1/2}} \right) - erf\left(\frac{\mathrm{Re}_h^{1/2}(y-1)}{(2x)^{1/2}} \right) \right) \tag{13}$$

There is value in ascertaining the behavior associated with the velocity fields described by equations (10) 2-d versus equation (13) axi-symmetric. The asymptotic behavior for these two expressions can be written:

$$\begin{aligned} u_{dif}(x,0) &\propto x^{-1/2} \\ u_{dif\_axi}(x,0) &\propto x^{-1} \end{aligned} \tag{14}$$

Which is good agreement with standard diffusive wake theory.

Recirculating Wake

We now consider the second portion of the equation splitting problem. The governing equation takes the form:

$$\psi_{xxx} = \frac{\nu_{eff}}{u_0} \psi_{yyyy} \tag{15}$$

Equation (15) is best analyzed by introducing a separable solution as: $\psi = f(y) g(x)$ yielding the separable expressions:

$$g'''-\lambda^3 g = 0$$
$$f''''+\frac{u_0}{\nu_{eff}}\lambda^3 f = 0 \tag{16}$$

Solution to these expressions is straightforward. The streamwise expression $g'''-\lambda^3 g = 0$ can be solved to give (we retain a single solution) $g \propto \exp(-\frac{1}{2}\lambda x)\sin(\frac{\sqrt{3}}{2}\lambda x)$. In a similar manner that cross stream expression $f''''+k^4 f = 0$ can be solved to give (again retaining a single solution) $f \propto \sin(ky)\cosh(ky)$ where $k^4 = \mathrm{Re}_h\,\lambda^3$, where $\frac{u_0}{\nu_{eff}} = \mathrm{Re}_h$. While this solution would be valid for y<<1 the $\cosh(ky)$ term is inherently unbounded. A local version of this expression follows by approximating the term: $k^4 f$ as $k^4 f \approx k^2 f''$. The governing equation then becomes: $f''''+k^2 f''=0$ The solution to the stream function is then simply: $f \propto \sin(ky)$ where now $k^2 = \mathrm{Re}_h\,\lambda$.

The associated streamfunction is then written:

$$\psi(x,y) \propto -\sin(ky)\exp(-\frac{1}{2}\lambda x)\sin(\frac{\sqrt{3}}{2}\lambda x) \tag{17}$$

There are effectively two degrees of freedom in this procedure:

1. Value for λ
2. Value for ν_{eff}

The cross-stream expression then requires that: $k=\pi$ and $k=\pi \rightarrow \lambda = \frac{\pi^2}{\mathrm{Re}_h}$. In a similar manner, we can introduce a new length scale, i.e. the recirculation zone have a length, say, L_{re_0} where we require that $\frac{\sqrt{3}}{2}\lambda L_{re\_0} = \pi \rightarrow \mathrm{Re}_h = \frac{\sqrt{3}}{2}\pi L_{re\_0}$. Notice that we can write this in the form: $L_{re\_0} = \frac{\sqrt{3}}{2}\pi\,\mathrm{Re}_h \approx 0.367\,\mathrm{Re}_h$. We emphasize that the recirculation length model L_{re_0} estimate is provisional in that it is based upon a single component of the velocity field. As we shall see subsequently, the global velocity field will exhibit a significantly shorter recirculation length. We examine the recirculation length model subsequently.

To gain a sense of the behavior associated with this model, let's notionally define: L_e=4H. which is approximately twice the length of the physically observed. In spite of being longer than expected, this estimate is appropriate since the complete solution will involve reductions in length due to the diffusive behavior of the complete solution. With L_{re}=4 we estimate that $\mathrm{Re}_h = \frac{\sqrt{3}}{2}\pi L_{re} \approx 11$ which is grossly consistent with Reynolds numbers associated with turbulent far-field wakes e.g. Re_{2d}=12.5 and Re_{axi}=14.1.

Global Wake Solution

The preceding expressions can be used to build a globally valid flow field. The streamwise velocity u(x,y) can be constructed as a sum of the near wake solution and the global difussive

$$
\begin{aligned}
u_{glob}(x,y) &= (1+\psi_y)u_{def}(x,y) \\
v_{glab}(x,y) &= -\psi_x u_{def}(x,y)
\end{aligned}
\tag{18}
$$

Where:

$$\psi(x,y) \propto -\sin(\pi y)\exp(-\frac{1}{2}\lambda x)\sin(\frac{\sqrt{3}}{2}\lambda x)\,;\ \lambda = \frac{\pi^2}{\mathrm{Re}_h}$$

$$u_{def}(x,y) = \frac{1}{2}\left(erf\left(\frac{\mathrm{Re}_h^{1/2}(y+1)}{2(x)^{1/2}}\right) - erf\left(\frac{\mathrm{Re}_h^{1/2}(y-1)}{2(x)^{1/2}}\right)\right)$$

While the axi-symmetric modification is:

$$u_{def}(x,y) = \frac{1}{2} erf\left(\frac{\mathrm{Re}_h^{1/2}}{2(x)^{1/2}}\right)\left(erf\left(\frac{\mathrm{Re}_h^{1/2}(y+1)}{2(x)^{1/2}}\right) - erf\left(\frac{\mathrm{Re}_h^{1/2}(y-1)}{2(x)^{1/2}}\right)\right)$$

We are particularly interested in the centerline velocity and can write an explicit formula for it as:

$$
\begin{aligned}
u_{cent\_2d}(x) &= 1-(1+\sin(\frac{\sqrt{3}}{2}\left(\frac{\pi^2}{\mathrm{Re}_h}\right)x)\exp(-\frac{1}{2}\left(\frac{\pi^2}{\mathrm{Re}_h}\right)x))erf(\frac{\mathrm{Re}_h^{1/2}}{2x^{1/2}}) \\
u_{cent\_axi}(x) &= 1-(1+\sin(\frac{\sqrt{3}}{2}\left(\frac{\pi^2}{\mathrm{Re}_h}\right)x)\exp(-\frac{1}{2}\left(\frac{\pi^2}{\mathrm{Re}_h}\right)x))erf^2(\frac{\mathrm{Re}_h^{1/2}}{2x^{1/2}})
\end{aligned}
\tag{19}
$$

Though the 2d and axisymmetric solutions behave in a similar manner the axisymmetric flow has a decreased recirculation length and increase in recovery of the defect velocity as compared to the 2d flow which is consistent with equation (14). We compare the two solutions in figure 2.

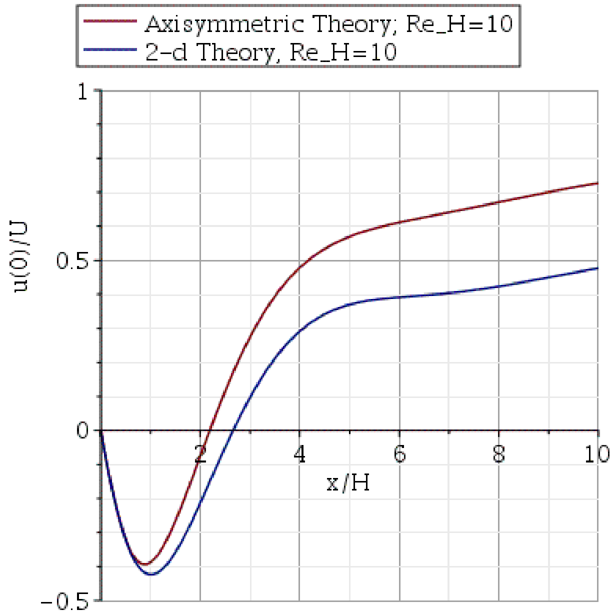

Figure 2. Comparison between axisymmetric and 2d centerline velocity for Re_h=10. Notice the decreased recirculation length and more rapid recovering of the velocity defect.

The centerline velocity expressions developed in equation (19) include within them estimates for the location of the rear stagnation point i.e. the recirculation zone length L_{re}. Indeed, L_{re} is simply the x location which yields a zero velocity: $u_{cent}(x = L_{re}) = 0$. While formally described by this expression the actual inversion of equation (19) is tedious. A simpler procedure is to simply compute roots of equation (19) for a family of Reynolds numbers and then fit an expression to the curve suing simple regression. The result of this procedure for the axisymmetric model in equation (19) is depicted in figure 2. As the figure suggests, a simple proportionality relationship $L_{re} = 0.228\,\mathrm{Re}_h$ provides a good description of the recirculation bubble lengths.

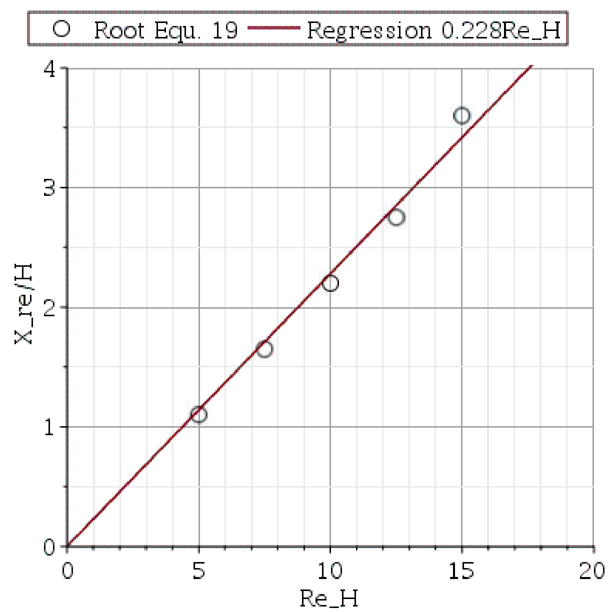

Figure 3. Recirculation bubble length as function of Reynolds number for axi-symmetric wakes using equation (19). Regression of lengths gives $L_{re} = 0.228\,\mathrm{Re}_h$ which is consistent with localized theory for $L_{r\bar{e}0}$ but is shorter.

The temperature behavior is also of interest. As a first approximation we can estimate the temperature field behavior by utilizing the Crocco-Busemann relationship as:

$$\frac{T}{T_\infty} = 1 + r\frac{(\gamma - 1)}{2} M_\infty^2 \left(1 - \left(\frac{u}{u_\infty}\right)^2\right) \tag{20}$$

Where "r" is the recovery factor. While one would typically correlate the recovery factor with a Prandtl number based expression, e.g. $r = \mathrm{Pr}^{1/2}$ we recognize that a reduced recovering is more correctly associated with local heat transfer in the lee of the bluff body whereby we propose that the recovery is on the order of 2/3 though we will permit the use of an empirical estimate derived from the magnitude of the temperature in the immediate lee of the bluff body. Martellucci et. al[10]. show a total temperature ratio of 0.6 for x<<1 for their M=6 measurements which is consistent with our approximations.

Laminar Versus Turbulent

To this point in our development we have not explicitly stated whether the model approach is valid for a mean turbulent flow or a laminar flow. We contend that the near wake formulation is largely the same whether the flow is laminar or turbulent with the exception of the magnitude of the supporting parameters e.g. Re_h. The concept that the mean turbulent flow for a problem and the laminar flow could be governed by equivalent governing equations is not unique to this flow class. An example where laminar and mean turbulent solution forms are analogous would be the classical axi-symmetric jet problem where the flow solutions are precisely the same except for the supporting coefficients. For example, the laminar flow solution is shown to be[5]:

$$u_{lam} \propto x^{-1}\left(1 + C_{lam}^2 \frac{y/x}{4}\right)^{-2} \quad ; \quad C_{lam} = \left(\frac{3J}{16\rho\pi\nu_{lam}^2}\right)^{1/2} = 0.22\,\mathrm{Re}_d \tag{21}$$

Where $J = 2\pi\rho\int_0^\infty u^2 r dr$ and Re_d for transition is O(10-20) while the turbulent solution is:

$$u_{turb} \propto x^{-1}\left(1 + C_{turb}^2 \frac{\eta_{turb}^2}{4}\right)^{-2} \quad ; \quad C_{turb} = \sqrt{15.2} \approx 3.9 \tag{22}$$

Notice that the turbulent solution is of course, Reynolds number invariant. These two expressions for both laminar and turbulent axisymmetric jets are both structurally similar and supported by comparable Reynolds number parameters.

Although we will discuss comparison of the analytical models with data in the next section, there is value in examining the near wake centerline velocity for a slender cone Mach 8 cone flow as measured by Schmidt and Cresci[11]. Their measurements are notable in that they were able to control Reynolds number for their flow such that for one wake flow was laminar while the second, due to turbulent boundary layer on the cone body, had a turbulent near wake. The resulting measurements and simulations where the Reynolds number has been varied are presented in figure 4. As can be seen in the figure, the flow behavior is similar between the laminar and turbulent flows with the difference being one of rate of recovery of the velocity deficit The turbulent flow recovers more quickly relative to the laminar flow due its more energetic behavior. In a similar manner, the analytical solutions from equation (19) are successful in describing these behaviors using the same solution with modified parameters. Thus, we cautiously suggest that the current near field model is a viable model for both laminar and turbulent mean flows where the effective Reynolds number for the turbulent flows is approximately ½ of the laminar value. We emphasize, however, that the coincidence between laminar and turbulent flow solutions is valid for the near wake only. Far wake behavior is characterized by a marked difference in length scales associated with laminar and turbulent flows.

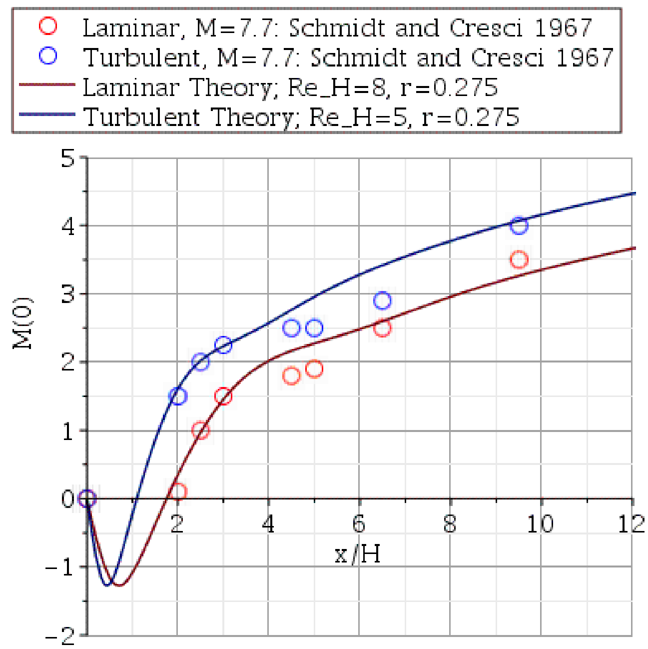

Figure 4. Centerline velocity for a slender cone Mach 8 cone flow as measured by Schmidt and Cresci 1967. Laminar and turbulent flows are successfully modeled using the same analytical solution, e.g. equation (19) but with different parameters.

III. Results

The preceding analytical development provides flow field and temperature field estimates for near wake flows. The overall flow field associated with equation (18) with Red=13 is given in figure 5:

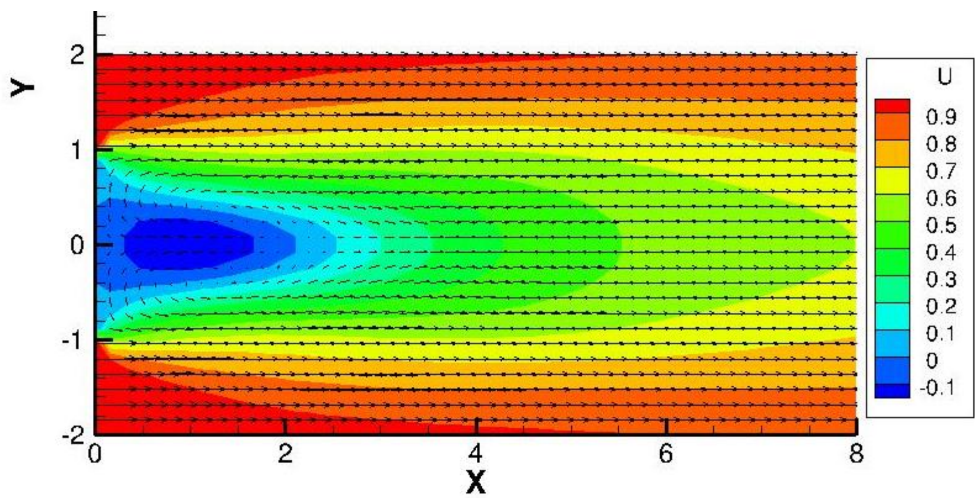

Figure 5. Axisymmetric wake flow field, Re_h=13

Quantitative comparisons between the simple model developed here and near field wake measurements is readily performed by considering data for centerline velocity and temperature fields. Martellucci 1966 provides both Mach 8 and Mach 10 results as described by figures 6 and 7:

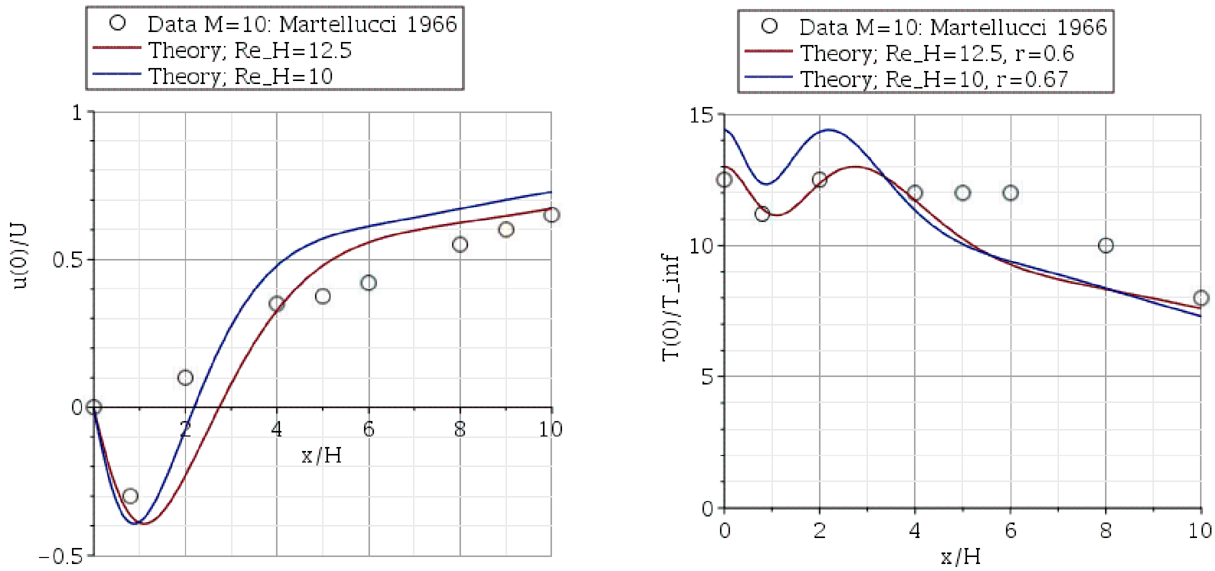

Figure 6. Turbulent centerline velocity for 10 degree (half angle) cone, M=10 from Martellucci et. al. 1966. Analytical solutions using equation (19) with Re_h=12.5 and a recovery factor r=0.6 yield overall reasonable conmparison.

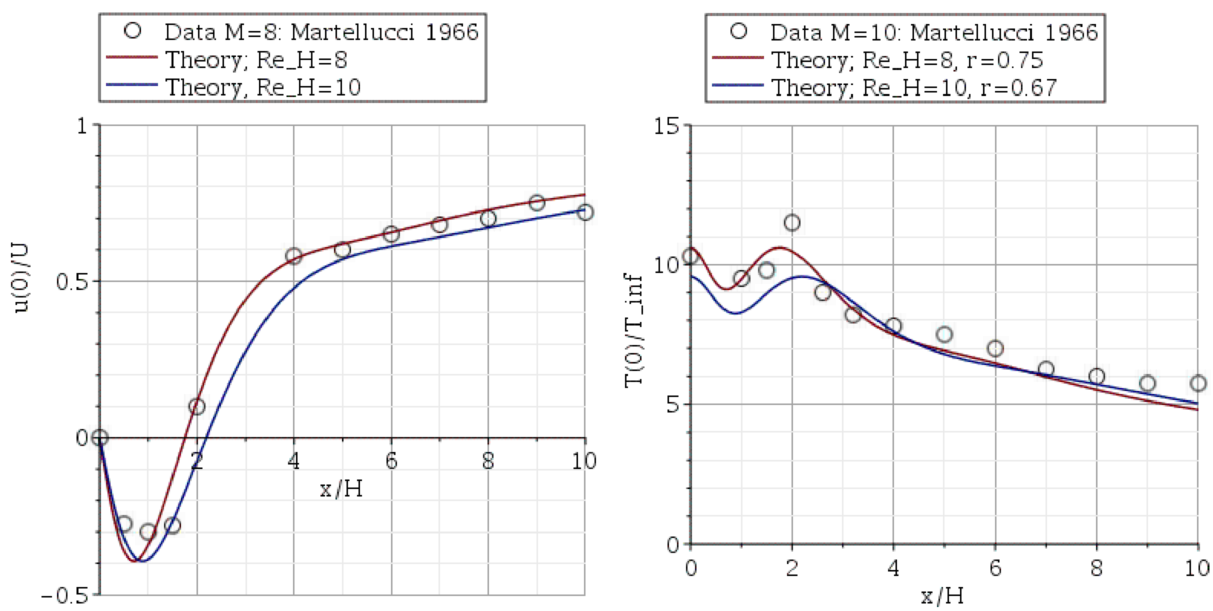

Figure 7. Turbulent centerline velocity for 10 degree (half angle) cone, M=8 from Martellucci et. al. 1966. Analytical solutions using equation (19) with Re_h=12.5 and a recovery factor r=0.6 yield overall reasonable conmparison.

Data for a 2-d circular cylinder is obtained by Ramaswamy[7]. The centerline velocity for a turbulent circular cylinder is described in figure 8.

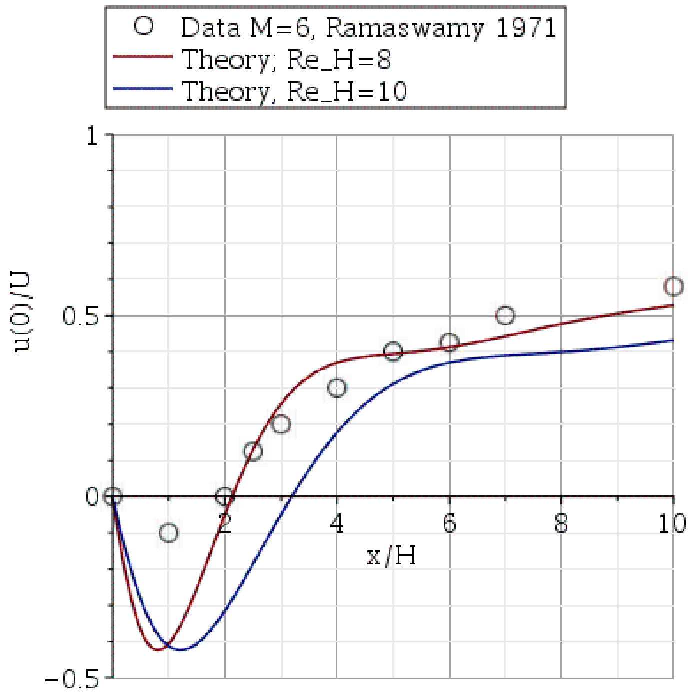

Figure 8. Turbulent centerline velocity for cicular cylinder, M=6 from Ramaswamy 1971. Analytical solutions using equation (19) with Re_h=8 suggesting resonable comparion. Notice that the analytical model significantly over predicts the strength of the recirculation zone due to the lake of a well defined recirculation zone for a cylindrical body.

Add here additional comparison with computational simulation results. Use computational models to better correlate Re_h and recovery r with flow Mach number and geometry.

IV. Conclusion

Here we have developed an approximate closed form analytical near wake solution. This solution is obtained using a set of linearized momentum equations along with continuity. Though linear, explicit closed form solutions based upon these governing equations cannot be readily constructed. As such, an approximate splitting solution procedure which is analogous to the classical method of variation of parameters from ordinary differential equation theory is identified. Application of this approach yields two solutions: a recirculating flow field in the lee of the body and a global diffusive model. The combination of these expressions is valid over the entire flow domain. The resulting solution yields qualitatively plausible flow field behavior capturing buth the recirculation zone and the wake velocity defect. With access to the velocity field temperature field, Crocco-Busemann supplemented by a modified recovery factor is used to estimate the temperature field. Quantitative comparison of centerline velocity and temperature predictions to the measurements of Martellucci et. al[6]., Ramaswamy[7] and Schmidt and Cresci[11] suggest good agreement with centerline velocity and temperature data. Mach number behavior is reflected in the effective viscosity coefficient and recovery parameter used to represent these flows. In a manner analogous to the classical axisymmetric jet we suggest that by choosing appropriate effective viscosity values, the current model is valid for both laminar and turbulent flows. Though highly simplified, the current modeling approach offers useful insight into hypersonic near wake flow physics.

V. Acknowledgements

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VI. References

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VII. Appendix: A Preliminary

The analysis developed here depends on being able to obtain approximate solutions to linear partial differential equations. There is value in considering the splitting solution approach for a typical partial differential equation initial value problem. We consider the linear PDE:

$$u_x = u_{yy} + u_y \tag{A.1}$$

where the initial conditions are analogous to the wake flow problem. Though simple, equation (A.1) is not immediately solvable but requires an extension of classical heat equation methods by including a travelling wave term f(z). An exact solution flows as:

$$u = f(z)w(x,y) \quad ; \quad z = x + \alpha y \tag{A.2}$$

By substituting equation (A.2) into (A.1) we can collect a series of expressions that will help us to determine f(z), α and w(x,y). These expressions are:

$$\begin{aligned} &w_x = w_{yy} \\ &f' = \alpha^2 f'' + \alpha f' \\ &2\alpha f' + f = 0 \end{aligned} \tag{A.3}$$

These equation can be readily solved to give:

$$u = f(z)w(x,y) = \exp(-\frac{1}{4}(x+2y))\left[erf(\frac{y+1}{2x^{1/2}}) - erf(\frac{y-1}{2x^{1/2}}) \right] \tag{A.4}$$

Direct substitution reveals that equation (A.4) is indeed a solution for equation (A.1).

Solution of equation (A.1) by splitting is relatively simple. Indeed the split system can be written:

$$\begin{aligned} &u_{1x} = u_{1yy} \\ &0 = u_{2yy} + u_{2y} \end{aligned} \tag{A.5}$$

Although we usually prefer to retain the highest order derivative in the splitting procedure a third possibility for the split equation is the first order PDE: $u_{3x} = u_{3y}$

The $u_{1x} = u_{1yy}$ is obviously directly related to the w(x,y) expression with $u_1 = w(x,y)$. The second expression: $0 = u_{2yy} + u_{2y}$ is immediately integrable by introducing the change in variables: $u_2 = f(z) \quad ; \quad z = g(x) + \alpha_2 y$. The governing equation can be written: $0 = \alpha_2 f'' + f'$ whereby the solution is computed as:

$$u_2 = \exp(-\frac{z}{\alpha_2}) = \exp(g(x) - y) \tag{A.6}$$

We do not have sufficient information from this expression to compute the value for g(x), hence we turn to the third split equation as: $u_{3x} = u_{3y}$. This PDE is solvable in a number of ways. As an initial value problem one would solve:

$$\begin{aligned} &u_{3x} = 0 \rightarrow u_{30}(\xi) \\ &\frac{dy}{dx} = -1 \rightarrow y = \xi - x \rightarrow \xi = x + y \end{aligned} \tag{A.6}$$

Alternatively one could compute a separable solution as: $u_3 = \exp(-\lambda(x+y))$.

Obviously, the split equation provides a solution family that is largely analogous to the exact solution equation (A.4). A generalized (through unknown coefficients A abd B) solution takes the form:

$$u = \exp(Ax+By)w(x,y) = \exp(Ax+By)\left[erf\left(\frac{y+1}{2x^{1/2}}\right) - erf\left(\frac{y-1}{2x^{1/2}}\right)\right] \tag{A.7}$$

Notice that the w(x,y) expression is precisely the same as the exact solution. The split solution offers an approximate solution functional form. Unknown parameters can be estimated using several possible approaches e.g. Galerkin etc. Here we use an unknown coefficient approach to compute the coefficients A and B. One can substitute the expression $\exp(Ax+By)w(x,y)$ into the complete governing equation (A.1) and collect terms in w:

$$A = B^2 + B \tag{A.8}$$

and terms in w_y as:

$$0 = 2B + 1 \rightarrow B = -\frac{1}{2} \tag{A.8}$$

and B=-1/4. Thus, we have recovered the exact solution though to be fair, we have used a procedure that largely mimics the original solution procedure. That said, however, we have justified the functional form associated with the exact solution by solving simple portions of the problem.