Eigenspace-Based Uncertainty Characterization in Large-Eddy Simulation of Turbulent Flow

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SIAM – Uncertainty Quantification
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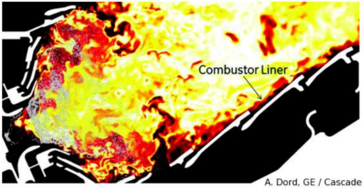
High-Fidelity Turbulent Flow Calculations
Large-Eddy Simulation

- RANS is the primary tool for modeling turbulence in engineering applications
- However, **LES** is becoming popular ... and this trend will continue to increase

Present-day example



A. Dord, GE / Cascade

LARGE-EDDY SIMULATIONS
OF COMBUSTOR LINER FLOWS

Future: NASA CFD Vision 2030 Study

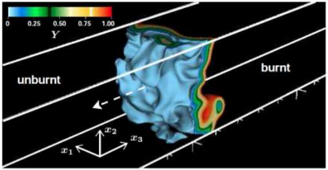
The use of CFD in the aerospace design process is severely limited by the inability to accurately and reliably predict turbulent flows with significant regions of separation. Advances in Reynolds-averaged Navier-Stokes (RANS) modeling alone are unlikely to overcome this deficiency, while the use of Large-eddy simulation (LES) methods will remain impractical for various important applications for the foreseeable future, barring any radical advances in algorithmic technology. Hybrid RANS-LES and wall-modeled LES offer the best prospects for overcoming this obstacle although significant modeling issues remain to be addressed here as well.

- Some of the reasons:
  - Provide information of transient turbulent flow structures
  - Tremendous growth in available computational power

Challenging SGS Stress Modeling

Turbulent Deflagration Example: O'Brien et al. 2014

- Problem: deflagration propagating through forced turbulence



Sample snapshot from DNS

- Model: Boussinesq (eddy viscosity)

$$\mathcal{T}_{ij} - \frac{2}{3}\overline{\rho}k_{SGS}\delta_{ij} = -2\overline{\rho}\nu_t\left(\tilde{S}_{ij} - \frac{\tilde{\Delta}_v}{3}\delta_{ij}\right)$$

... contracting with $\tilde{S}_{ij}$ yields

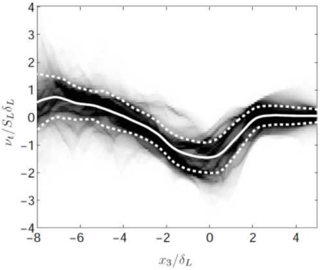
$$\nu_t = \frac{\epsilon_{SGS} + (2/3)\overline{\rho}k_{SGS}\tilde{\Delta}_v}{2\overline{\rho}\left(|\tilde{S}|^2 - \tilde{\Delta}_v^2/3\right)}$$

- Filtered kinetic energy equation

$$\frac{\partial\overline{\rho}k}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\tilde{u}_i k\right) = \alpha_\Pi + \alpha_{SGS} + \alpha_v + \Pi - \epsilon_v - \epsilon_{SGS}$$

with $\epsilon_{SGS} = -\mathcal{T}_{ij}\tilde{S}_{ij}$

- Turbulent viscosity conditioned on x_3



PDF contours, mean & σ

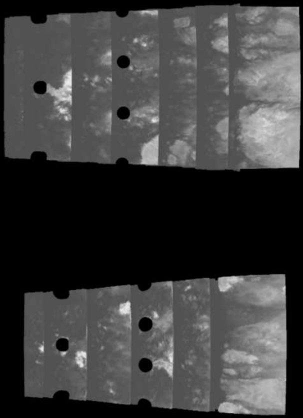
LES Complex Turbulent Flow

Combustor Example: Masquelet et al. 2017

Rich-Dome Aviation Gas Turbine: Temperature field

Intermittent, hot "bursts" on liners caused by unsteady JICF wakes

Combustion chamber

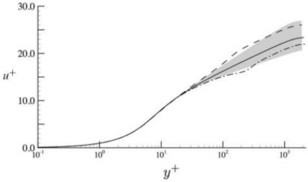


midplane

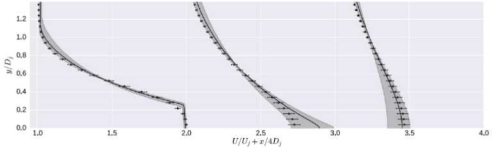
Structural UQ Turbulence models

Framework Introduction

- We propose a **framework** for the systematic estimation of **structural uncertainty**
  - **Independent of the initial model form**
  - **Computationally efficient**
  - **Suitable to general LES solvers**
- Feeds from methodology developed in RANS modeling
  - e.g., Emory et al. 2013, Gorlé & Iaccarino 2013, Mishra & Iaccarino 2017



Stream. velocity channel flow



Axial velocity radial profiles jet flow

- **Requires revisiting mathematical derivation in LES**, Jofre et al. 2018
  - Perturb decomposed SGS tensor within range of physically plausible bounds

Large-Eddy Simulation Equations
Closure Problem & Notation

- Filtered Navier-Stokes equations

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial(\overline{u}_i\overline{u}_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu\frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

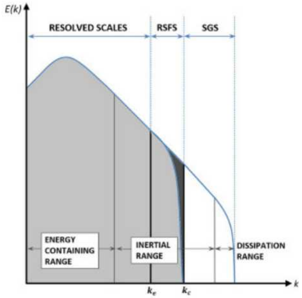
with SGS tensor $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i\overline{u}_j$

$$\longrightarrow \tau_{ij}^{SGS} \approx \tau_{ij}$$

- Mathematical notation

- $G_{ij} = \frac{\partial u_i}{\partial x_j}, \quad \phi = \overline{\phi} + \phi'$
- $S_{ij} = \frac{1}{2}\left(G_{ij} + G_{ji}\right), \quad \Omega_{ij} = \frac{1}{2}\left(G_{ij} - G_{ji}\right)$
- $P_G = -G_{ii} = 0, \quad Q_G = -\frac{1}{2}G_{ij}G_{ji},$
  $R_G = -\frac{1}{3}G_{ij}G_{jk}G_{ki}$

- Reduction modeling small scales



Resolved & modeled energy

Nonlinear Filtered Advection Term

Tensor Eigendecomposition

- $\overline{u_i u_j}$ decomposed into factors introducing the **anisotropy tensor**

$$\overline{a}_{ij} = \frac{\overline{u_i u_j}}{\overline{u_k u_k}} - \frac{1}{3}\delta_{ij} = \overline{v}_{in}\overline{\Lambda}_{nl}\overline{v}_{jl}$$

with eigenvalues ordered such that $\overline{\lambda}_1 \geq \overline{\lambda}_2 \geq \overline{\lambda}_3$

- Allows reformulating the tensor in the form $\overline{u_i u_j} = \overline{u_k u_k}\left(\overline{v}_{in}\overline{\Lambda}_{nl}\overline{v}_{jl} + \frac{1}{3}\delta_{ij}\right)$
  - **Magnitude** (trace): $\overline{u_k u_k}$
  - **Shape** (eigenvalues): $\overline{\Lambda}_{nl}$
  - **Orientation** (eigenvectors): $\overline{v}_{in}$

- Imposing **realizability conditions** bounds $\overline{a}_{ij}$ as

$$-1/3 \leq \overline{a}_{\alpha\alpha} \leq 2/3 \quad \text{for} \quad \alpha \in \{1,2,3\}$$
$$-1/2 \leq \overline{a}_{\alpha\beta} \leq 1/2 \quad \text{for} \quad \alpha \neq \beta$$

Structural UQ Framework
Perturbation Approach

- Strategy: **inject controlled perturbations** into τ_{ij}^{SGS} to assess impact on QoIs
- Step 1: separate $\overline{u_i u_j}$ into resolved and modeled parts as

$$\overline{u_i u_j} = \overline{u_k u_k}\left(a_{ij}^{res} + a_{ij}^{SGS} + \frac{1}{3}\delta_{ij}\right), \quad a_{ij}^{SGS} = \frac{1}{\overline{u_k u_k}}\left(\tau_{ij}^{SGS} - \frac{\tau_{kk}^{SGS}}{3}\delta_{ij}\right) = v_{in}^{SGS}\Lambda_{nl}^{SGS}v_{jl}^{SGS}$$

- Step 2: define perturbations (indicated with *) as

$$\overline{u_i u_j}^* = \overline{u}_i\overline{u}_j + \tau_{ij}^{SGS*}$$

with $\overline{u_k u_k}^* = \overline{u}_k\overline{u}_k + \tau_{kk}^{SGS*}$ and $a_{ij}^{SGS*} = v_{in}^{SGS*}\Lambda_{nl}^{SGS*}v_{jl}^{SGS*}$

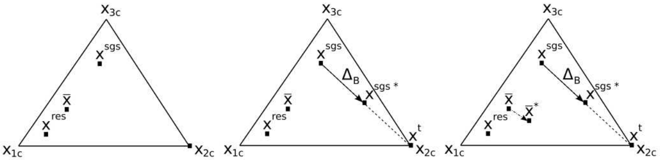
- Thus, perturbations are applied to the subgrid scales and are specified in terms of
  - Magnitude: $\tau_{kk}^{SGS*} = \tau_{kk}^{SGS} + \Delta\tau_{kk}^{SGS}$
  - Shape: diagonal matrix Λ_{nl}^{SGS*} of λ_l^*
  - Orientation: $v_{ij}^{SGS*} = q_{in}v_{nj}^{SGS}$

Full details in Jofre et al. Flow Turbul. Combust., 100(2):341-363, 2018

Structural UQ Framework
Example: SGS Anisotropy Perturbation

- Different strategies can be defined for Λ_{nl}^{SGS*} based on $\lambda_l^{SGS*} = \mathbf{B}^{-1}\mathbf{x}^{SGS*}$ where $\mathbf{x} = \mathbf{B}\lambda_l = \mathbf{x}_{1c}(\lambda_1 - \lambda_2) + 2\mathbf{x}_{2c}(\lambda_2 - \lambda_3) + \mathbf{x}_{3c}(3\lambda_3 + 1)$
- **We characterize uncertainty** by **direction** $\mathbf{x}^t - \mathbf{x}^{SGS}$ and **rel. distance** $\Delta_B \in [0, 1]$

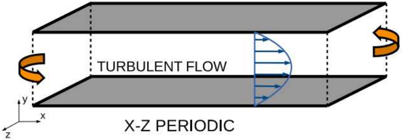
$$\mathbf{x}^{SGS*} = \mathbf{x}^{SGS} + \Delta_B\left(\mathbf{x}^t - \mathbf{x}^{SGS}\right) \quad \longrightarrow \quad \lambda_l^{SGS*} = (1 - \Delta_B)\,\lambda_l^{SGS} + \Delta_B\lambda_l^t$$



- Graphical representation

TURBULENT CHANNEL FLOW $Re_\tau = 395$
PROBLEM CONFIGURATION

- Investigate framework performance on LES of turbulent flow
- LES Channel flow $Re_\tau = 395$:
  - $\Delta x^+ = 38.8$, $\Delta z^+ = 12.9$, $\Delta y^+ = [0.5 - 15.1]$; size $64 \times 128 \times 96$



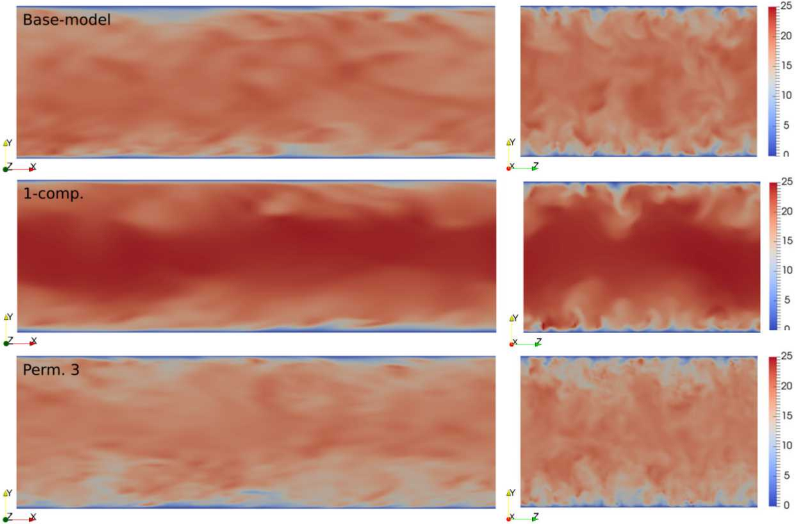
- Results:

1. Sensitivity to individual homogeneous perturbations WALE[1] (base-model)
   - Magnitude: $\Delta\tau_{kk}^{sgs} < 0$ or $\Delta\tau_{kk}^{sgs} > 0$
   - Shape: 1-comp., 2-comp., or 3-comp.
   - Orientation: perm. 1, perm. 2, or perm. 3

[1] F. Nicoud & F. Ducros. Flow Turbul. Comb. 62 (1999)

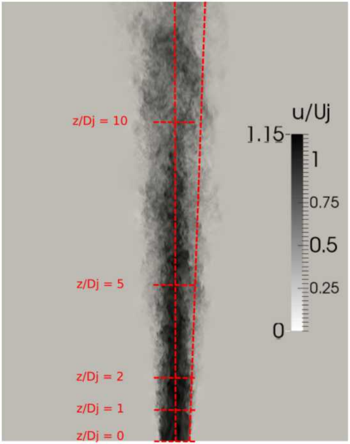
Turbulent Channel Flow $Re_\tau = 395$
Flow visualization

- Streamwise instantaneous velocity u^+



Turbulent Axisymmetric Jet $Re_D = 21000$
Problem Configuration

- Parameters jet $Re_D = 21000$:
  - Near-field exp. data Amielh et al. (1996)
  - $D_e/D_j = 10$, $L/D_j = 40$, $U_j/U_e = 13$
- Computational set-up:
  - Axisymmetric mesh ≈ 200M elements
  - $\Delta/\eta \sim 1$ at shear layers
  - 2^{nd}-order, low-dissipation finite-volume
    Nalu: low-Mach number flow solver[2]
- Averaging & filtering operators:
  - Time & axisymmetric ensemble average
  - 2 temporal snapshots (at present)
  - Gaussian filter: $\overline{\phi} = \phi + \frac{\bar{\Delta}^2}{24}\nabla^2\phi$

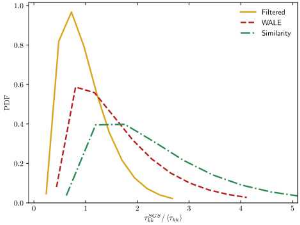


Jet centerline & $r/r_{1/2} = 1$ profile

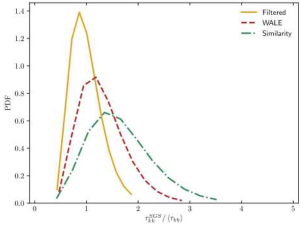
[2] S. P. Domino, Tech. Rep. SAND2015-3107W, SNL 2015

TURBULENT AXISYMMETRIC JET $Re_D = 21000$

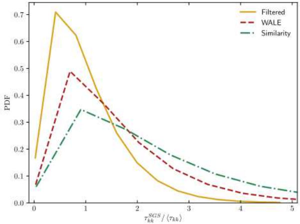
PDF MODELED τ_{kk}^{SGS} BIAS ($\bar{\Delta}/\Delta = 4$)



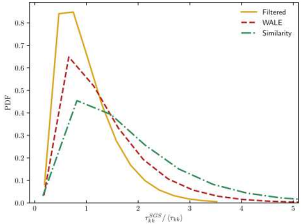
$$z/D_j = 1, \quad r/r_{1/2} = 0, \quad \langle \tau_{kk} \rangle / \tau_{kk}^{ref} = 1$$



$$z/D_j = 10, \quad r/r_{1/2} = 0, \quad \langle \tau_{kk} \rangle / \tau_{kk}^{ref} \approx 3$$



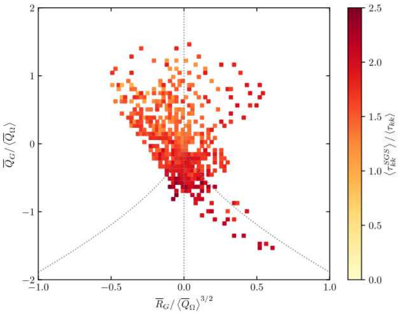
$$z/D_j = 1, \quad r/r_{1/2} = 1, \quad \langle \tau_{kk} \rangle / \tau_{kk}^{ref} \approx 10$$



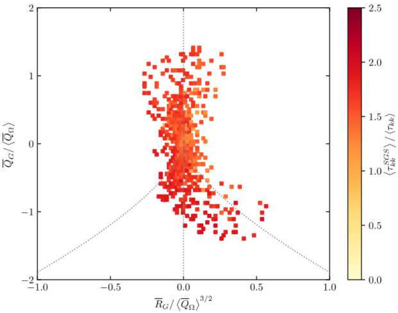
$$z/D_j = 10, \quad r/r_{1/2} = 1, \quad \langle \tau_{kk} \rangle / \tau_{kk}^{ref} \approx 3$$

TURBULENT AXISYMMETRIC JET $Re_D = 21000$

CONDITIONAL JPDF τ_{kk}^{SGS} BIAS ON R_G, Q_G $(\bar{\Delta}/\Delta = 4)$



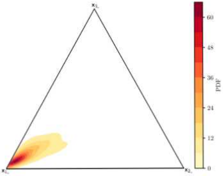
WALE, $z/D_j = 1$, $r/r_{1/2} = 0$



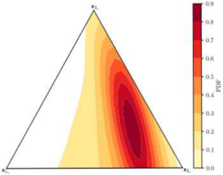
WALE, $z/D_j = 1$, $r/r_{1/2} = 1$

TURBULENT AXISYMMETRIC JET $Re_D = 21000$

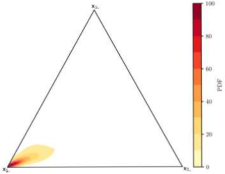
PDF $\overline{\lambda}_l$ BIAS ($\bar{\Delta}/\Delta = 4$), $r/r_{1/2} = 1$



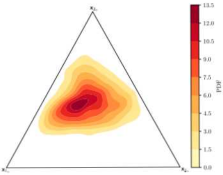
Filtered, $z/D_j = 1$



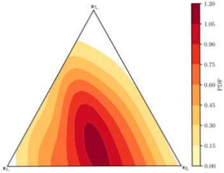
WALE, $z/D_j = 1$



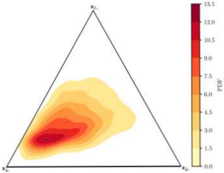
Similarity, $z/D_j = 1$



Filtered, $z/D_j = 10$



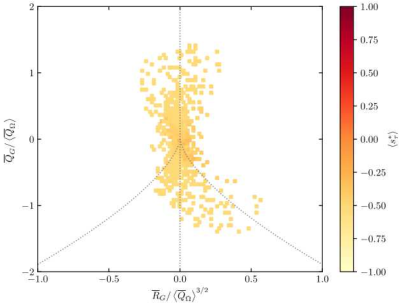
WALE, $z/D_j = 10$



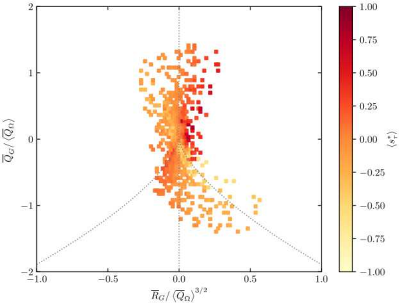
Similarity, $z/D_j = 10$

TURBULENT AXISYMMETRIC JET $Re_D = 21000$

CONDITIONAL JPDF s^*_τ BIAS ON R_G, Q_G ($\bar{\Delta}/\Delta = 4$)

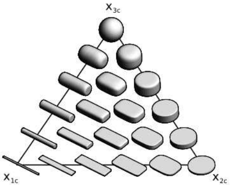


Filtered, $z/D_j = 1, \quad r/r_{1/2} = 1$



WALE, $z/D_j = 1, \quad r/r_{1/2} = 1$

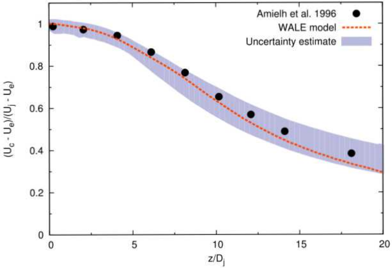
- Parameter $s^* = -3\sqrt{6}\,\lambda_1\lambda_2\lambda_3/\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)^{3/2}$
  - $s = -1$ axisymmetric expansion
  - $s = 0$ two-component limit
  - $s = 1$ axisymmetric contraction



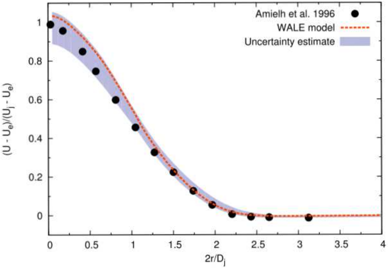
Turbulent Axisymmetric Jet $Re_D = 21000$

Uncertainty Estimates: Magnitude & Anisotropy Perturbations

- *A posteriori* LES uncertainty estimates:
  - Axisymmetric mesh $\approx$ 3M elements ($64\times$ coarser)
  - WALE base-model + $\Delta\tau_{kk}^{SGS}$ & λ_l^{SGS*}



Mean axial velocity along jet axis



Mean axial velocity radial profile $z/D_j = 5$

Conclusions & Future Work

- Presented framework to estimate structural uncertainty in LES closures[3]
  - Independent of initial model form
  - Computationally efficient and suitable to general solvers
  - Uncertainty in terms of magnitude, shape & orientation
  - Physically reasonable bounds derived for each degree of freedom
- Performance tested by computing LES of canonical flows
  - Perturbation toward $\mathbf{x}_{1c}$ laminarizes flow
  - Permutation 3 may increase turbulence through backscatter
  - Combined perturbations produce different variability
  - SGS tensor shape bias depends on flow topology
- Ongoing & future work
  - Development of strategies for inhomogeneous perturbations
  - Focus on combination of different perturbations
  - Test framework on complex flows, e.g., two-phase flow, combustion processes

[3] Jofre et al. A framework for characterizing structural uncertainty in large-eddy simulation closures. Flow Turbulence Combust. 100(2):341-363, 2018.

Nonlinear Filtered Advection Term

Realizability Conditions

- In RANS, Reynolds stresses R_{ij} must be symmetric & positive semi-definite
  - Physical interpretation: kinetic energy is non-negative & real
  - Equivalent to the conditions of realizability[4]

$$\begin{aligned} R_{\alpha\alpha} &\geq 0 \quad \text{for} \quad \alpha \in \{1,2,3\} \\ R_{\alpha\beta}^2 &\leq R_{\alpha\alpha} R_{\beta\beta} \quad \text{for} \quad \alpha \neq \beta \\ \det(R_{ij}) &\geq 0 \end{aligned}$$

- In **LES** context, should $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$ be realizable?
  - Generally assumed to be, but not a physical requirement $\rightarrow$ modeling choice
  - In fact, the conditions are not satisfied for nonpositive filters ... implicit filtering?
  - We **choose** to impose $\overline{u_i u_j}$ realizable, viz. total filtered kinetic energy ≥ 0

$$\begin{aligned} \overline{u_\alpha u_\alpha} &\geq 0 \quad \text{for} \quad \alpha \in \{1,2,3\} \\ \overline{u_\alpha u_\beta}^2 &\leq \overline{u_\alpha u_\alpha}\,\overline{u_\beta u_\beta} \quad \text{for} \quad \alpha \neq \beta \\ \det(\overline{u_i u_j}) &\geq 0 \end{aligned}$$

[4] The summation convention is adopted for Latin, but not for Greek indices

Nonlinear Filtered Advection Term

Tensor Eigendecomposition

- $\overline{u_i u_j}$ decomposed into factors introducing the **anisotropy tensor**

$$\overline{a}_{ij} = \frac{\overline{u_i u_j}}{\overline{u_k u_k}} - \frac{1}{3}\delta_{ij} = \overline{v}_{in}\overline{\Lambda}_{nl}\overline{v}_{jl}$$

  with eigenvalues ordered such that $\overline{\lambda}_1 \geq \overline{\lambda}_2 \geq \overline{\lambda}_3$

- Allows reformulating the tensor in the form $\overline{u_i u_j} = \overline{u_k u_k}\left(\overline{v}_{in}\overline{\Lambda}_{nl}\overline{v}_{jl} + \frac{1}{3}\delta_{ij}\right)$
  - **Magnitude** (trace): $\overline{u_k u_k}$
  - **Shape** (eigenvalues): $\overline{\Lambda}_{nl}$
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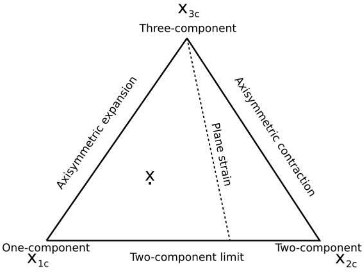
$$-1/2 \leq \overline{a}_{\alpha\beta} \leq 1/2 \quad \text{for} \quad \alpha \neq \beta$$

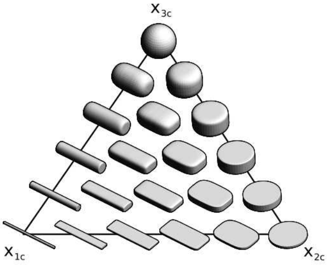
Nonlinear Filtered Advection Term

Barycentric Map

- Tensor anisotropy: **Barycentric map** (linear projection)

$$\mathbf{x} = \mathbf{x}_{1c}\left(\lambda_1 - \lambda_2\right) + 2\mathbf{x}_{2c}\left(\lambda_2 - \lambda_3\right) + \mathbf{x}_{3c}\left(3\lambda_3 + 1\right)$$



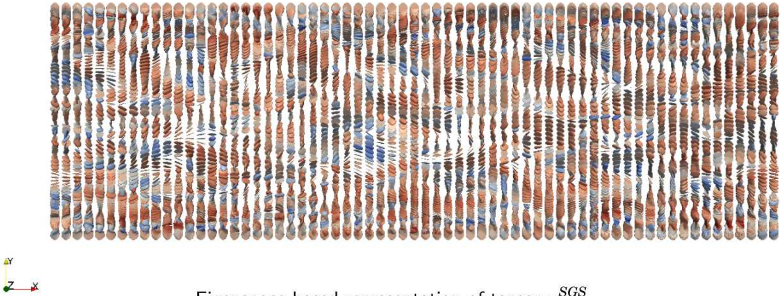


- Limiting states positive semi-definite second-order tensor:
  - One-component (rod-like): $2/3 = \lambda_1 > \lambda_2 = \lambda_3 = -1/3$
  - Two-component (disk-like): $1/6 = \lambda_1 = \lambda_2 > \lambda_3 = -1/3$
  - Three-component (spherical): $\lambda_1 = \lambda_2 = \lambda_3 = 0$

Tensor Eigendecomposition

SGS Stress Tensor Example

- WALE (eddy-viscosity) model: $\tau_{ij}^{SGS} - \frac{\tau_{kk}}{3}\delta_{ij} = -2\nu_t\overline{S}_{ij}$
- LES channel flow $Re_\tau = 395$



Eigenspace-based representation of tensor τ_{ij}^{SGS}

Structural UQ Framework

Perturbation Approach

- Strategy: **inject controlled perturbations** into τ_{ij}^{SGS} to assess impact on QoIs
- Step 1: separate $\overline{u_i u_j}$ into resolved and modeled parts as

$$\overline{u_i u_j} = \overline{u_k u_k}\left(a_{ij}^{res} + a_{ij}^{SGS} + \frac{1}{3}\delta_{ij}\right), \quad a_{ij}^{SGS} = \frac{1}{\overline{u_k u_k}}\left(\tau_{ij}^{SGS} - \frac{\tau_{kk}^{SGS}}{3}\delta_{ij}\right) = v_{in}^{SGS}\Lambda_{nl}^{SGS}v_{jl}^{SGS}$$

- Step 2: define perturbations (indicated with *) as

$$\overline{u_i u_j}^* = \overline{u}_i\overline{u}_j + \tau_{ij}^{SGS^*}$$

with $\overline{u_k u_k}^* = \overline{u}_k\overline{u}_k + \tau_{kk}^{SGS^*}$ and $a_{ij}^{SGS^*} = v_{in}^{SGS^*}\Lambda_{nl}^{SGS^*}v_{jl}^{SGS^*}$

- Thus, perturbations are applied to the subgrid scales and are specified in terms of
  - Magnitude: $\tau_{kk}^{SGS^*} = \tau_{kk}^{SGS} + \Delta\tau_{kk}^{SGS}$
  - Shape: diagonal matrix $\Lambda_{nl}^{SGS^*}$ of λ_l^*
  - Orientation: $v_{ij}^{SGS^*} = q_{in}v_{nj}^{SGS}$

Structural UQ Framework
SGS Magnitude Perturbation

- Plausible lower and upper bounds for $\Delta\tau_{kk}^{SGS}$ based on $\overline{u_k u_k} = \overline{u}_k \overline{u}_k + \tau_{kk}^{SGS}$
  1. $\overline{u_k u_k} = \overline{u}_k \overline{u}_k + \tau_{kk}^{SGS} \geq 0$ due to the restriction that $\overline{u_i u_j}$ is realizable
  2. $\overline{u}_k \overline{u}_k = \overline{u_k u_k} - \tau_{kk}^{SGS} \geq 0$ by construction independently of the filter

  - **Interval** of **magnitude discrepancy** results in

$$-\overline{u}_k \overline{u}_k - \tau_{kk}^{SGS} \leq \Delta\tau_{kk}^{SGS} \leq \overline{u_k u_k} - \tau_{kk}^{SGS}$$

  ! τ_{kk}^{SGS} is not typically considered ... but models exist[5]

- Graphical representation

$\Delta\tau_{kk}^{SGS} < 0$

$\Delta\tau_{kk}^{SGS} = 0$

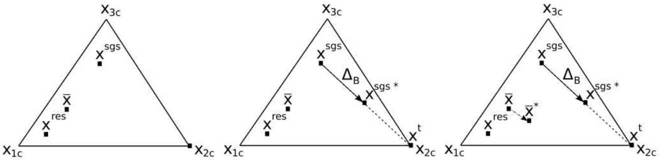
$\Delta\tau_{kk}^{SGS} > 0$

[5] e.g., Yoshizawa. Phys. Fluids 29 (1986), Moin et al. Phys. Fluids 3 (1991)

Structural UQ Framework
SGS Anisotropy Perturbation

- Different strategies can be defined for Λ_{nl}^{SGS*} based on $\lambda_l^{SGS*} = \mathbf{B}^{-1}\mathbf{x}^{SGS*}$ where $\mathbf{x} = \mathbf{B}\lambda_l = \mathbf{x}_{1c}(\lambda_1 - \lambda_2) + 2\mathbf{x}_{2c}(\lambda_2 - \lambda_3) + \mathbf{x}_{3c}(3\lambda_3 + 1)$
- **We characterize uncertainty** by **direction** $\mathbf{x}^t - \mathbf{x}^{SGS}$ and **rel. distance** $\Delta_B \in [0, 1]$

$$\mathbf{x}^{SGS*} = \mathbf{x}^{SGS} + \Delta_B\left(\mathbf{x}^t - \mathbf{x}^{SGS}\right) \quad \longrightarrow \quad \lambda_l^{SGS*} = (1 - \Delta_B)\,\lambda_l^{SGS} + \Delta_B\lambda_l^t$$



- Graphical representation

Structural UQ Framework
SGS Orientation Perturbation

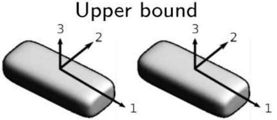
- Perturbations based on **energy transfer constraints** btw. resolved and SGS scales

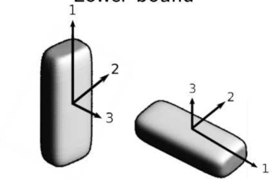
$$\frac{\partial E_f}{\partial t} + \overline{u}_j \frac{\partial E_f}{\partial x_j} - \frac{\partial}{\partial x_i}\left[\overline{u}_j \left(2\nu \overline{S}_{ij} - \tau_{ij}^d - \frac{1}{\rho}\overline{p}\,\delta_{ij}\right)\right] = -\epsilon_f - \mathcal{P}_r$$

- **Focus** on $\mathcal{P}_r = -\tau_{ij}^d \overline{S}_{ij}$ since it involves **single-point information**
  - Frobenius inner product $\longrightarrow$ depends on the alignment btw. tensors
  - Lower and upper bounds given by[6]

$$\lambda_1\gamma_3 + \lambda_2\gamma_2 + \lambda_3\gamma_1 \leq -\mathcal{P}_r \leq \lambda_1\gamma_1 + \lambda_2\gamma_2 + \lambda_3\gamma_3$$

- Upper bound $\longrightarrow$ same basis of eigenvectors
- Lower bound $\longrightarrow$ permutation btw. eigenvectors 1 and 3
- Graphical representation





[6] Lasserre. IEEE Trans. Autom. Control 40 (1995)

SGS Orientation Perturbation
Forward-Scatter & Backscatter

- Conservation filtered kinetic energy triply periodic domain

$$\frac{\partial E_f}{\partial t} + \bar{u}_j \underbrace{\frac{\partial E_f}{\partial x_j}}_{0} - \underbrace{\frac{\partial}{\partial x_i}\left[\bar{u}_j \left(2\nu \bar{S}_{ij} - \tau^d_{ij} - \frac{1}{\rho}\bar{p}\,\delta_{ij}\right)\right]}_{0} = -\epsilon_f - \mathcal{P}_r$$

- Consider τ^d_{ij}, $\bar{S}_{ij}$ with same shape and orientation

$$\mathcal{P}_r = -\tau_{kk}\|\bar{S}_{ij}\|\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) \longrightarrow \mathcal{P}_r \leq 0$$

... setting $\tau_{kk} = -2\nu_t\|\bar{S}_{ij}\|$ (eddy viscosity) $\longrightarrow \mathcal{P}_r \geq 0$ (**forward-scatter**)

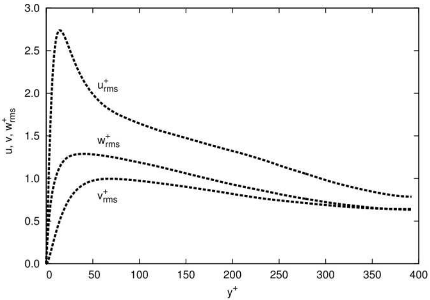
- Consider permutation of 1^{st} & 3^{rd} eigenvectors

$$\mathcal{P}_r = -\tau_{kk}\|\bar{S}_{ij}\|\left(\lambda_1^2 + 4\lambda_1\lambda_3 + \lambda_3^2\right) \quad \text{with} \quad \lambda_1\lambda_3 \leq 0 \;\; (1^{st} \text{ tensor invariant})$$

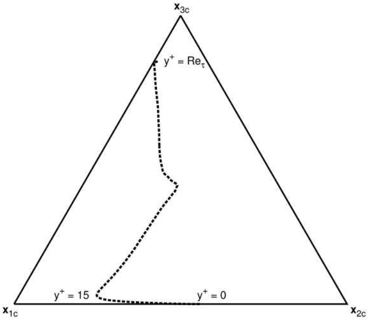
- if $|\lambda_1| \,/\, |\lambda_3| \sim 1 \longrightarrow \mathcal{P}_r \geq 0$; setting $\tau_{kk} = -2\nu_t\|\bar{S}_{ij}\| \longrightarrow \mathcal{P}_r \leq 0$ (**backscatter**)

Barycentric Map
Reynolds Stresses Example

- Anisotropy Reynolds stresses
- DNS channel flow $Re_\tau = 395$



Velocity rms fluctuations



R_{ij} barycentric map