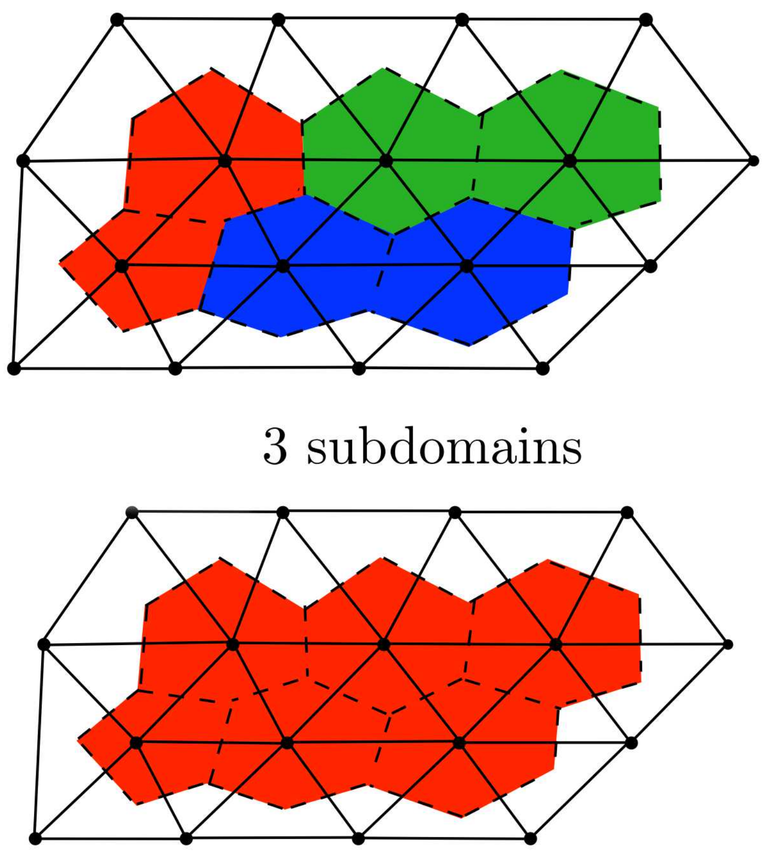
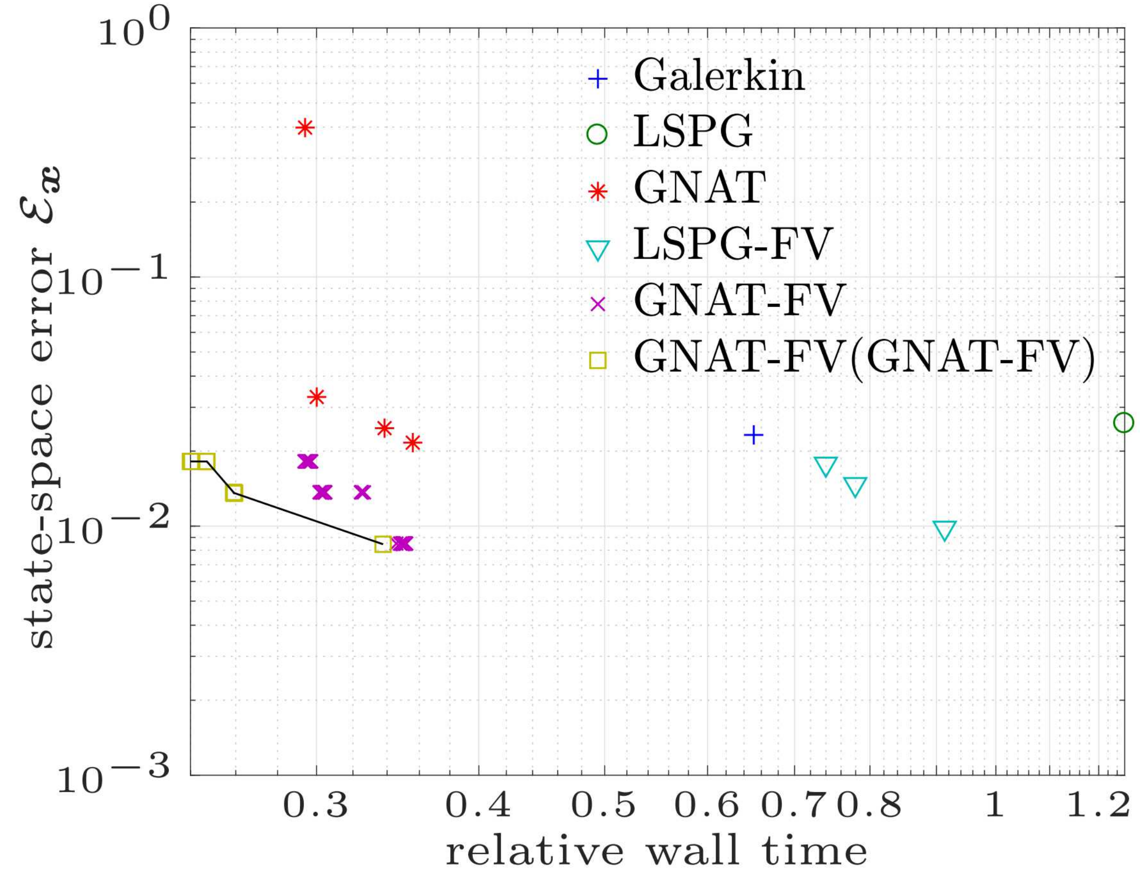
SAND2018-3548C

Conservative model reduction for finite-volume models



3 subdomains

1 (global) subdomain



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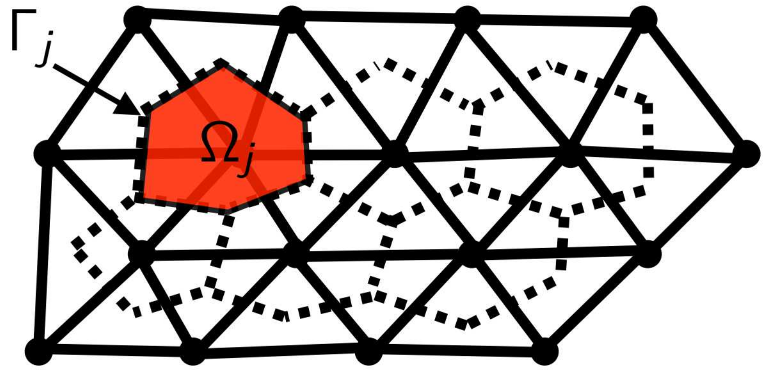
Motivation

- **Finite-volume method** widely used to discretize **conservation laws**
- Conservation is the primary problem structure for these models
- Existing reduced-order models do not enforce conservation
- **Goal**: construct reduced-order models that are conservative

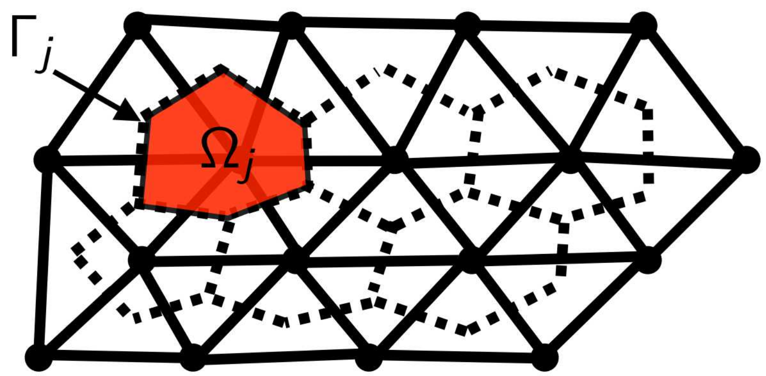
Finite-volume method

$$\text{ODE:} \quad \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; t)$$



Finite-volume method

ODE: $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; t)$



$$x_{\mathcal{I}(i,j)}(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} u_i(\vec{x}, t)\, \mathrm{d}\vec{x}$$

- average value of conserved variable *i* over control volume *j*

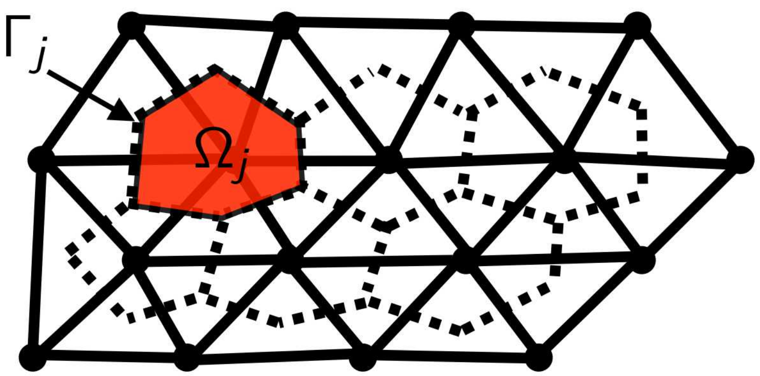
$$f_{\mathcal{I}(i,j)}(\mathbf{x}, t) = -\frac{1}{|\Omega_j|} \int_{\Gamma_j} \underbrace{\mathbf{g}_i(\mathbf{x}; \vec{x}, t)}_{\text{flux}} \cdot \mathbf{n}_j(\vec{x})\, \mathrm{d}\vec{s}(\vec{x}) + \frac{1}{|\Omega_j|} \int_{\Omega_j} \underbrace{s_i(\mathbf{x}; \vec{x}, t)}_{\text{source}}\, \mathrm{d}\vec{x}$$

- flux and source of conserved variable *i* within control volume *j*

$$r_{\mathcal{I}(i,j)} = \frac{dx_{\mathcal{I}(i,j)}}{dt}(t) - f_{\mathcal{I}(i,j)}(\mathbf{x}, t)$$

- **rate of conservation violation** of variable *i* in control volume *j*

Finite-volume method



$$\text{ODE:} \quad \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; t)$$

$$x_{\mathcal{I}(i,j)}(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} u_i(\vec{x}, t)\, d\vec{x}$$

- average value of conserved variable i over control volume j

$$f_{\mathcal{I}(i,j)}(\mathbf{x}, t) = -\frac{1}{|\Omega_j|} \int_{\Gamma_j} \underbrace{\mathbf{g}_i(\mathbf{x}; \vec{x}, t)}_{\text{flux}} \cdot \mathbf{n}_j(\vec{x})\, d\vec{s}(\vec{x}) + \frac{1}{|\Omega_j|} \int_{\Omega_j} \underbrace{s_i(\mathbf{x}; \vec{x}, t)}_{\text{source}}\, d\vec{x}$$

- flux and source of conserved variable i within control volume j

$$r_{\mathcal{I}(i,j)} = \frac{dx_{\mathcal{I}(i,j)}}{dt}(t) - f_{\mathcal{I}(i,j)}(\mathbf{x}, t)$$

- **rate of conservation violation** of variable i in control volume j

$$\text{O}\Delta\text{E:} \quad \mathbf{r}^n(\mathbf{x}^n) = 0, \ \ n = 1, \ldots, N$$

$$r^n_{\mathcal{I}(i,j)} = x_{\mathcal{I}(i,j)}(t^{n+1}) - x_{\mathcal{I}(i,j)}(t^n) + \int_{t^n}^{t^{n+1}} f_{\mathcal{I}(i,j)}(\mathbf{x}, t)dt$$

- **conservation violation** of variable i in control volume j over time step n

Reduced-order models

Galerkin

$$\frac{d\hat{\mathbf{x}}}{dt} = \boldsymbol{\Phi}^T \mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t)$$

$$\Updownarrow$$

$$\frac{d\hat{\mathbf{x}}}{dt}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t; \boldsymbol{\mu}) = \underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\arg\min} \|\mathbf{r}\left(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t\right)\|_2$$

- min. sum of squared conservation-violation **rates**

LSPG

$$\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n; \boldsymbol{\mu})^T \mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}^n) = \mathbf{0}$$

$$\Updownarrow$$

$$\hat{\mathbf{x}}^n = \underset{\hat{\mathbf{z}}\in\mathbb{R}^p}{\arg\min} \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{z}})\|_2$$

- min. sum of squared conservation violations **over time step *n***

Reduced-order models

Galerkin

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{\Phi}^T \mathbf{f}(\mathbf{\Phi}\hat{\mathbf{x}}, t)$$

$$\Updownarrow$$

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- min. sum of squared conservation violations **over time step *n***

- Neither enforces conservation!

Reduced-order models

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$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{\Phi}^T \mathbf{f}(\mathbf{\Phi}\hat{\mathbf{x}}, t)$$

$$\Updownarrow$$

$$\frac{d\hat{\mathbf{x}}}{dt}(\mathbf{\Phi}\hat{\mathbf{x}}, t; \boldsymbol{\mu}) = \arg\min_{\hat{\mathbf{v}} \in \mathbb{R}^p} \left\| \mathbf{r}\left(\mathbf{\Phi}\hat{\mathbf{v}}, \mathbf{\Phi}\hat{\mathbf{x}}, t\right) \right\|_2$$

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- min. sum of squared conservation violations **over time step *n***

- Neither enforces conservation!

Conservative Galerkin

$$\underset{\hat{\mathbf{v}} \in \mathbb{R}^p}{\text{minimize}} \left\| \mathbf{r}\left(\mathbf{\Phi}\hat{\mathbf{v}}, \mathbf{\Phi}\hat{\mathbf{x}}, t\right) \right\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}\left(\mathbf{\Phi}\hat{\mathbf{v}}, \mathbf{\Phi}\hat{\mathbf{x}}, t\right) = \mathbf{0}$$

- min. sum of squared conservation-violation **rates**
  subject to zero conservation-violation **rates** over subdomains

Conservative LSPG

$$\underset{\hat{\mathbf{z}} \in \mathbb{R}^p}{\text{minimize}} \left\| \mathbf{r}^n(\mathbf{\Phi}\hat{\mathbf{z}}) \right\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\mathbf{\Phi}\hat{\mathbf{z}}) = 0$$

- min. sum of squared conservation violations **over time step *n***
  subject to zero conservation violations **over time step *n*** over subdomains

Reduced-order models

Galerkin

$$\frac{d\hat{\mathbf{x}}}{dt} = \boldsymbol{\Phi}^T \mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t)$$

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- min. sum of squared conservation-violation **rates**

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$$\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n; \boldsymbol{\mu})^T \mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}^n) = \mathbf{0}$$

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- Neither enforces conservation!

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$$\underset{\hat{\mathbf{v}} \in \mathbb{R}^p}{\text{minimize}} \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

- min. sum of squared conservation-violation **rates** subject to zero conservation-violation **rates** over subdomains

Conservative LSPG

$$\underset{\hat{\mathbf{z}} \in \mathbb{R}^p}{\text{minimize}} \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{z}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{z}}) = 0$$

- min. sum of squared conservation violations **over time step *n*** subject to zero conservation violations **over time step *n*** over subdomains

+ Conservation enforced over subdomains!