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A Second-Order Consistent Multilevel-Multifidelity Optimization Scheme

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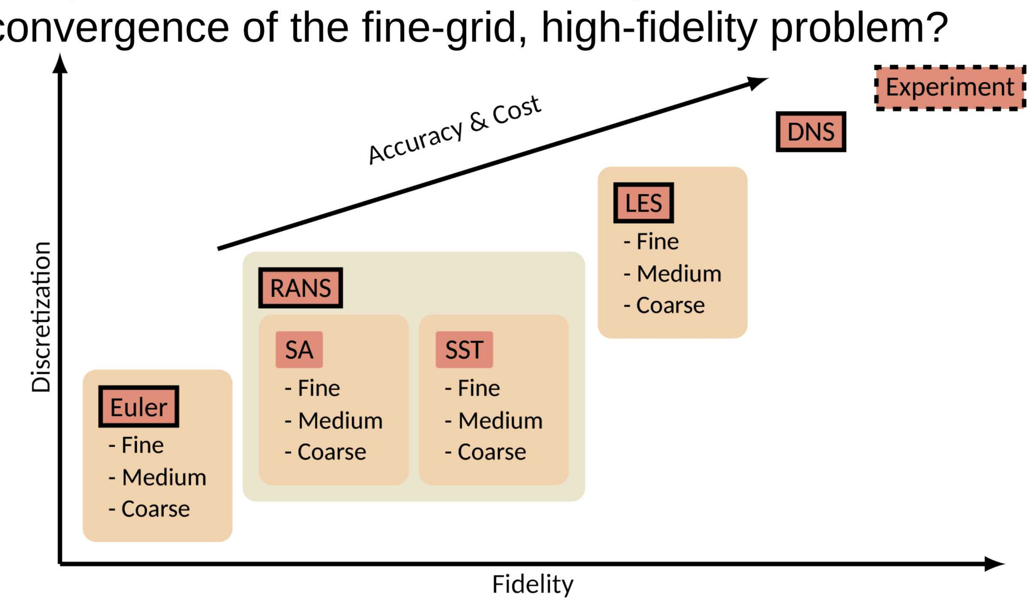
Background & Problem

Many engineering design problems can be formulated in the framework of partial differential equation (PDE) constrained optimization. Often there are multiple discretization levels and related model forms available.

Research Question:
How do we exploit the lower cost of coarse-grid and reduced fidelity models to speed up convergence of the fine-grid, high-fidelity problem?



Approach

There are numerous potential choices for navigating the hierarchy of discretization levels and model forms. This research attempts to take advantage of the close relationship between different discretizations of the same model.

Algorithm 1 Multilevel-Multifidelity Optimization

procedure MLMFOPT
 ▷ Initialize optimization at a lower fidelity and/or level
 Partially solve: $x_n^{(K,J)} = \arg\min_x \quad f^{(K,J)}(x))$
 $x_{n+1} = \prod_{k=K-1}^{1} P_{k,1} \prod_{j=J-1}^{1} P_{K,j} x_n^{(K,J)}$
 repeat
  $x_n = x_{n+1}$
  $x_{n+1} = \text{MFOPT}(1,\ x_n,\ f^{(1,1)}(x))$
 until convergence
 return x_{n+1}
end procedure

The approach builds off of the multigrid optimization ideas of Lewis & Nash [1]. Algorithm 1 is the outer optimization. Algorithms 2 & 3 are nested recursive algorithms and are very similar in structure to the Lewis & Nash multigrid method. At each fidelity level, Algorithm 3 takes advantage of multiple discretization levels to accelerate convergence at a single fidelity. Algorithm 2 uses the multilevel optimization to accelerate convergence to the highest fidelity.

Algorithm 2 Multifidelity Optimization

procedure MFOPT$(k,\ x_0^{(k,1)},\ f_{\text{CORR.}}^{(k,1)}(x))$
 if $k = N_{\text{fid.}}$ **then**
  $x_1^{(k,1)} = \text{MLOPT}(k,\ 1,\ x_0^{(k,1)},\ f_{\text{corr.}}^{(k,1)}(x))$
  return $x_1^{(k,1)}$
 else
  Partially solve: $x_1^{(k,1)} = \text{MLOPT}(k,\ 1,\ x_0^{(k,1)},\ f_{\text{corr.}}^{(k,1)}(x))$
  $x_1^{(k+1,1)} = R_{k,1}\left[x_1^{(k,1)}\right]$
  $f_{\text{corr.}}^{(k+1,1)}(x) = \text{CORRECTION}(x_1^{(k,1)},\ R_{k,1},\ f_{\text{corr.}}^{(k,1)}(x),\ f^{(k+1,1)}(x))$
  $x_2^{(k+1,1)} = \text{MFOPT}(k+1,\ x_1^{(k+1,1)},\ f_{\text{corr.}}^{(k+1,1)}(x))$
  $e = P_{k,1}\left[x_2^{(k+1,1)} - x_1^{(k+1,1)}\right]$
  $x_2^{(k,1)} = \text{LINESEARCH}(x_1^{(k,1)},\ f_{\text{corr.}}^{(k,1)}(x),\ e)$
  return $x_2^{(k,1)}$
 end if
end procedure

Algorithm 3 Multilevel Optimization

procedure MLOPT$(k,\ j,\ x_0^{(k,j)},\ f_{\text{CORR.}}^{(k,j)}(x))$
 if $j = N_{\text{level}}$ **then**
  $x_1^{(k,j)} = \arg\min_x \quad f_{\text{corr.}}^{(k,j)}(x))$
  return $x_1^{(k,j)}$
 else
  Partially solve: $x_1^{(k,j)} = \arg\min_x \quad f_{\text{corr.}}^{(k,j)}(x))$
  $x_1^{(k,j+1)} = R_{k,j}\left[x_1^{(k,j)}\right]$
  $f_{\text{corr.}}^{(k,j+1)}(x) = \text{CORRECTION}(x_1^{(k,j)},\ R_{k,j},\ f_{\text{corr.}}^{(k,j)}(x),\ f^{(k,j+1)}(x))$
  $x_2^{(k,j+1)} = \text{MLOPT}(k,\ j+1,\ x_1^{(k,j+1)},\ f_{\text{corr.}}^{(k,j+1)}(x))$
  $e = P_{k,j}\left[x_2^{(k,j+1)} - x_1^{(k,j+1)}\right]$
  $x_2^{(k,j)} = \text{LINESEARCH}(x_1^{(k,j)},\ f_{\text{corr.}}^{(k,j)}(x),\ e)$
  return $x_2^{(k,j)}$
 end if
end procedure

Corrections are applied to the lower fidelity and coarser discretization models to make them up to 2nd-order consistent with the high-fidelity fine-grid model. This correction function is listed in Algorithm 4 and is based on Eldred et al. [2]

- 1st-order corrections guarantee that Algorithms 2 & 3 provide descent directions
- The linesearch in Algorithms 2 & 3 ensure provable convergence. The method reverts to the underlying optimization method at highest-fidelity and discretization if the sub-models are not informative.

Algorithm 4 Apply up to 2nd-order corrections

procedure CORRECTION$(x_c,\ R,\ f_{\text{HI}}(x),\ f_{\text{LO}}(x))$
 $A_0,\ A_1,\ A_2,\ B_0,\ B_1,\ B_2 = 0$
 if (correction order ≥ 0) **then**
  $A_0 = f_{\text{hi}}(x_c) - f_{\text{lo}}(Rx_c), \qquad B_0 = \frac{f_{\text{hi}}(x_c)}{f_{\text{lo}}(Rx_c)}$
 end if
 if (correction order ≥ 1) **then**
  $A_1 = R\left[\nabla f_{\text{hi}}(x_c)\right] - \nabla f_{\text{lo}}(Rx_c), \qquad B_1 = \frac{1}{f_{\text{lo}}(Rx_c)} R\left[\nabla f_{\text{hi}}(x_c)\right] - \frac{f_{\text{hi}}(x_c)}{f_{\text{lo}}^2(Rx_c)} \nabla f_{\text{lo}}(Rx_c)$
 end if
 if (correction order ≥ 2) **then**
  $A_2 = R\left[\nabla^2 f_{\text{hi}}(x_c)\right] R^T - \nabla^2 f_{\text{lo}}(Rx_c)$
  $B_2 = \frac{1}{f_{\text{lo}}(Rx_c)} R\left[\nabla^2 f_{\text{hi}}(x_c)\right] R^T - \frac{f_{\text{hi}}(x_c)}{f_{\text{lo}}^2(Rx_c)} \nabla^2 f_{\text{lo}}(Rx_c) + \frac{2 f_{\text{hi}}(x_c)}{f_{\text{lo}}^3(Rx_c)} \nabla f_{\text{lo}}(Rx_c) \nabla f_{\text{lo}}^T(Rx_c)$
   $- \frac{1}{f_{\text{lo}}^2(Rx_c)} \left[\nabla f_{\text{lo}}(Rx_c) \left(R \nabla f_{\text{hi}}(x_c)\right)^T + R \nabla f_{\text{hi}}(x_c) \nabla f_{\text{lo}}^T(Rx_c)\right]$
 end if
 $\alpha(\tilde{x}) = A_0 + A_1^T (\tilde{x} - Rx_c) + \frac{1}{2} (\tilde{x} - Rx_c)^T A_2 (\tilde{x} - Rx_c)$
 $\beta(\tilde{x}) = B_0 + B_1^T (\tilde{x} - Rx_c) + \frac{1}{2} (\tilde{x} - Rx_c)^T B_2 (\tilde{x} - Rx_c)$
 if additive correction **then**
  $\gamma = 1$
 else if multiplicative correction **then**
  $\gamma = 0$
 else if combined correction **then**
  $\gamma = \frac{f_{\text{hi}}(x_p) - f_{\text{lo}}(x_p)\beta(x_p)}{f_{\text{lo}}(x_p) + \alpha(x_p) - f_{\text{lo}}(x_p)\beta(x_p)}$
 end if
 return $\hat{f}(\tilde{x}) = \gamma \left(f_{\text{lo}}(\tilde{x}) + \alpha(\tilde{x})\right) + (1 - \gamma) f_{\text{lo}}(\tilde{x}) \beta(\tilde{x})$
end procedure

Results

1D Diffusion Problem:

High-Fidelity:

$$\frac{d}{dx}\left(a\frac{du}{dx}\right) = -f, \quad x \in (0,1)$$
$$u(0) = u(1) = 0$$
$$a = 2 + \cos(2\pi x) + 0.4\sin(6\pi x)$$

Low-Fidelity:

$$\frac{d}{dx}\left(a\frac{du}{dx}\right) = -f, \quad x \in (0,1)$$
$$u(0) = u(1) = 0$$
$$a = 2 + \cos(2\pi x)$$

Optimization Problem:

$$\underset{f}{\text{minimize}} \quad \int (u(f) - u^*)^2 dx$$
$$\text{subject to} \quad c(f, u(f)) = 0$$
$$u^* = \sin^2(2\pi x)$$

Discretization:
- 2nd-order finite differences
- Fine grid: $N_1 = 100$
- Coarse grid: $N_2 = 50$

Prolongation / Restriction:
- 2nd-degree Lagrange interpolation
- $R = cP^T$

Number of function evaluations:

1 fidelity and 1 level

	Fine
High-Fidelity	229

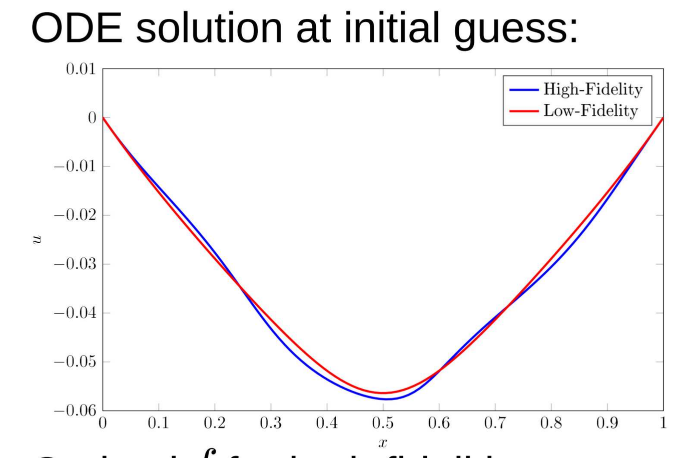
1 fidelity and 2 levels

	Fine	Coarse
High-Fidelity	112	146

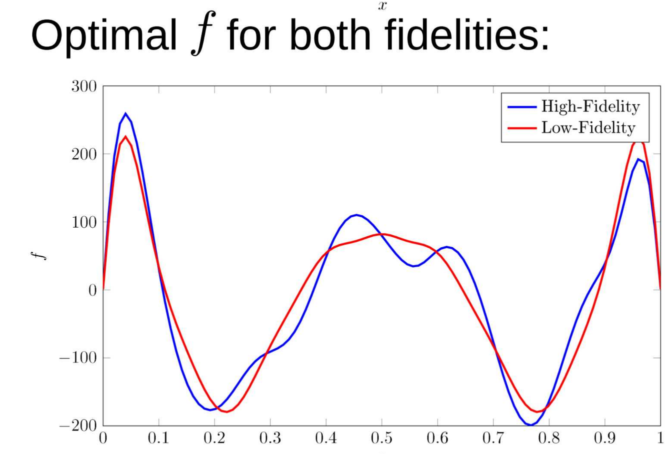
2 fidelities and 2 levels

	Fine	Coarse
High-Fidelity	65	82
Low-Fidelity	26	29

ODE solution at initial guess:



Optimal f for both fidelities:



Transonic Airfoil Problem: (Calculations performed using SU2)

Baseline airfoil: NACA 0012, $M_\infty = 0.8$, $AoA = 1^\circ$

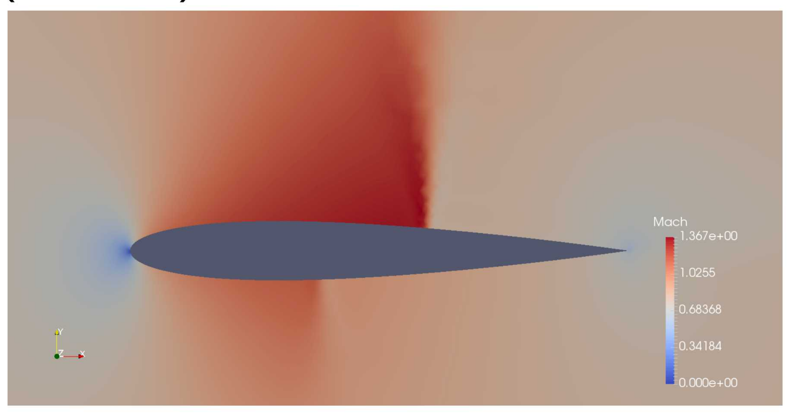
Optimization problem:
(x parameterizes 38 Hicks-Henne bump functions)

$$\underset{x}{\text{minimize}} \quad C_D(u(x))$$
$$\text{subject to} \quad c(x, u(x)) = 0$$

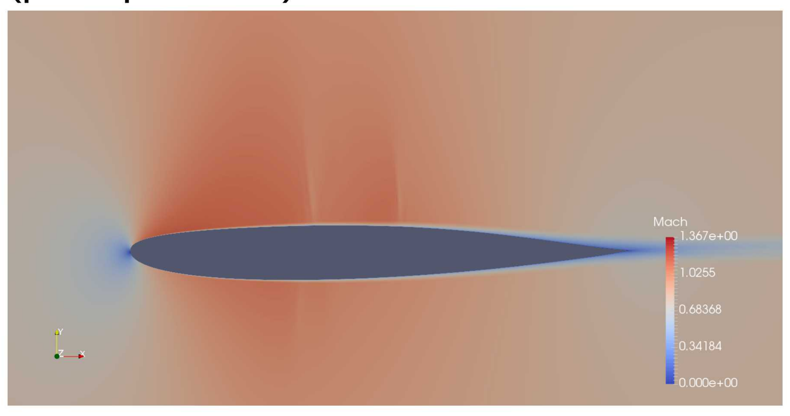
High-fidelity: RANS-SA
- Fine-grid: 319,859 cells (C-grid)
- Coarse-grid: 140,768 cells (C-grid)

Low-fidelity: Euler
- Fine-grid: 32,626 cells (unstructured grid)
- Coarse-grid: 6,131 cells (unstructured grid)

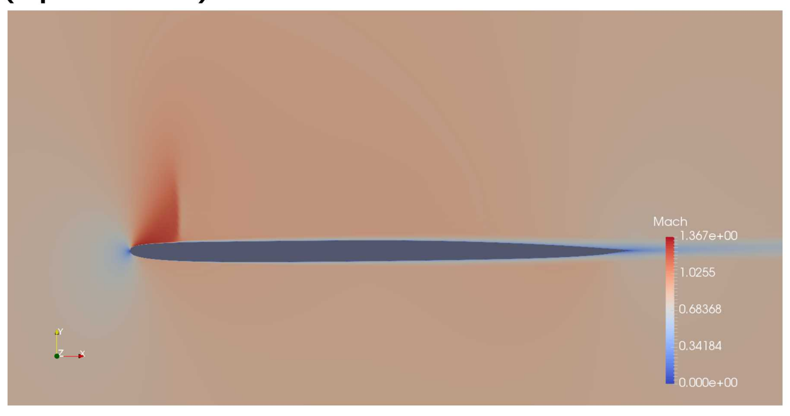
Euler, fine grid, $C_D = 0.0168$ (Baseline)



RANS-SA, coarse grid, $C_D = 0.0127$ (pre-optimized)



RANS-SA, coarse grid, $C_D = 0.0111$ (optimized)



Function evaluations

	Fine	Coarse
High-Fidelity	-	11
Low-Fidelity	23	-

Conclusions & Future Work

An up to second-order consistent multilevel-multifidelity optimization scheme was presented. This optimization scheme was applied to a 1D diffusion problem and transonic airfoil optimization. Both problems converged leveraging lower-fidelity and/or coarser discretization models.

Future work will compare trust regions to line searches and incorporate general constraints.

References:

[1] Lewis, Robert Michael, and Stephen G. Nash. "Model problems for the multigrid optimization of systems governed by differential equations." SIAM Journal on Scientific Computing 26.6 (2005): 1811-1837.

[2] Eldred, M. S., et al. "Second-order corrections for surrogate-based optimization with model hierarchies." Proceedings of the 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, Albany, NY, Aug. 2004.

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