Peridynamic Multiscale Finite Element Method

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SAND2015-9137C

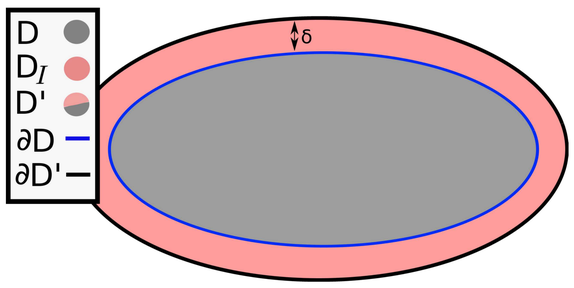
Objective

- Leverage next-generation computing platforms to include microscale information in continuum scale mechancis simulations.

Key Components

- Peridynamic Theory:
  - Unified mathematical framework from continuum scale to the scale of molecular dynamics,
  - No spatial derivatives: naturally models material failure,
  - Nonlocal model.
- Multiscale Finite Elements:
  - Multiscale basis functions computed on independent subdomains,
  - Finer resolution on subdomains → improved solution, same global degrees of freedom,
  - No communication between subdomain solves → suited for NGP systems.

Weak Linear Peridynamic Model



$$(u,v)_\rho = \int_D \rho u v \, dx, \quad a(u,v) = \int_D \int_D C(x,q)(u(q)-u(x))(v(q)-v(x)) \, dq \, dx,$$
$$V = \{v : D' \to \mathbb{R}^d \ : \ a(v,v) < \infty, v|_{D_I} = 0\}.$$

$$\text{find } u \in H^2(0,T;V) \ : \ \rho(x) u_{tt}(x,t) = a(u,v) + b(x,t), \quad \forall v \in V \text{ and a.e. } t \in [0,T].$$

u – displacement field, b – ext force density,
ρ – density, C – micromodulus function.

Nonlocal Multiscale Finite Element Method

- $\{\phi_{i,j}\}_{i,j}$ - piecewise linear Lagrangian FE basis,
- $\phi_{i,j}$ - basis function j on element T_i.
- Multiscale basis function $\Phi_{i,j} = \phi_{i,j} + q_{i,j}$ given by

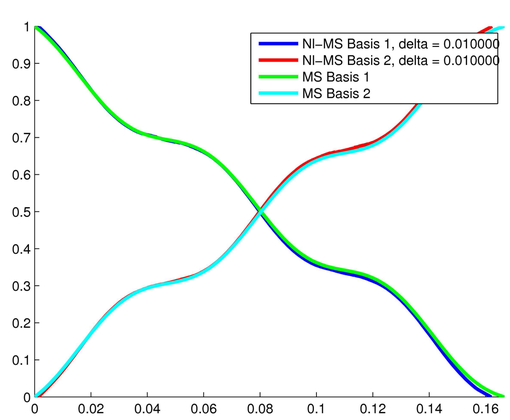
$$\text{find } q_{i,j} \in V \text{ s.t. } a_i(q_{i,j}, v) = -a_i(\phi_{i,j}, v)_{T_i} \quad \forall v \in V.$$

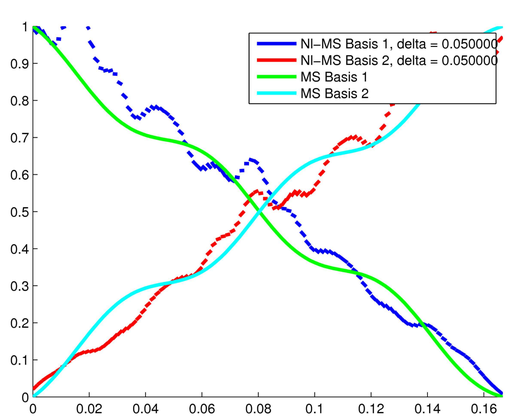
- Solve global problem with multiscale basis.
- Note: resolution of global mesh $\{T_i\}_i$ and local meshes for multiscale basis function computations both determine accuracy. MS basis function computations independent, performed concurrently.

Mixed Locality Multiscale FEM

- Use MSFEM framework to couple nonlocal and local mechanics models.
- MSFEM basis functions computed using nonlocal model.
- Global solution computed with local model.

Local & Nonlocal MS Basis Functions





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Analysis: Ambulant Galerkin Framework

- Based on global orthogonal decomposition of Hilbert space operator.

V a Hilbert space, $\mathcal{B} \in \mathcal{L}(V, V')$ coercive, $f \in V'$, abstract model problem,

$$\text{find } u \in V \ \text{ s.t. } \ \mathcal{B}u - f \in V^o.$$

$V^H \subset V$ finite dimensional with basis $\{\phi_i\}_{i=1}^{N_H}$.
$I^H : V \to V_H$ an orthogonal projection, $V^r = Ker(I^H)$. Reconstruction $R : V^H \to V$ defined by $R(\phi) = \phi + Q(\phi)$, where Q solves,

$$\text{given } \phi \in V^H, \ \text{ find } Q(\phi) \in V^r \ \text{ s.t. } \ \mathcal{B}(\phi + Q(\phi)) - f \in (V^r)^o.$$

Ambulant space $V^A = \text{span}\{R(\phi_i)\}_{i=1}^{N_H}$.

Lemmas 1 & 2

Model problem is equivalent to,

$$(1) \text{ find } u \in V^A \ \text{ s.t. } \ \mathcal{B}u - f \in V^o, \quad (2) \text{ find } u \in V^A \ \text{ s.t. } \ \mathcal{B}u - f \in (V^A)^o.$$

Approximate Reconstruction

$\{V^a\}_a$, $a \in (0, \infty)$ family of Hilberty spaces s.t.
- For each a, $V^H \subset V^a \subset V$.
- For each $v \in V$ $\exists$ a sequence $\{v^a \in V^a\}$ satisfying,

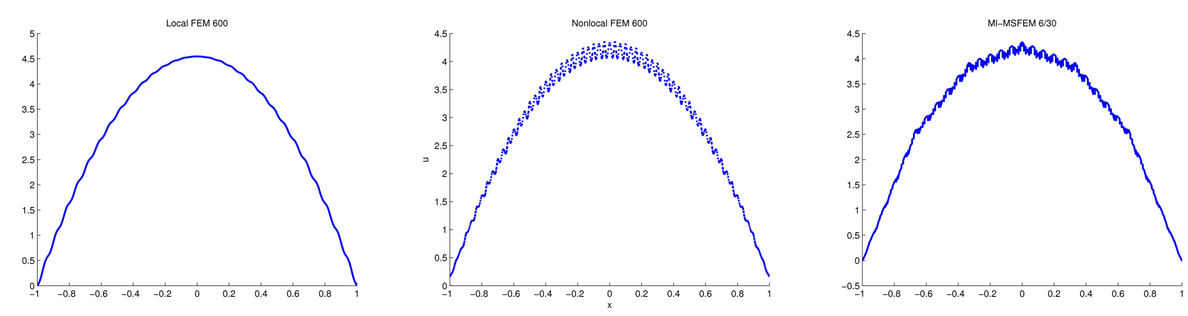
$$\lim_{a \to 0} \|v - v^a\|_V = 0.$$

For each a, $V^{r,a} = Ker(I^H|_{V^a})$. Reconstruction defined by

$$\text{given } \phi \in V^H, \ \text{ find } Q^a(\phi) \in V^{r,a} \ \text{ s.t. } \ \mathcal{B}(\phi + Q^a(\phi)) - f \in (V^{r,a})^o.$$

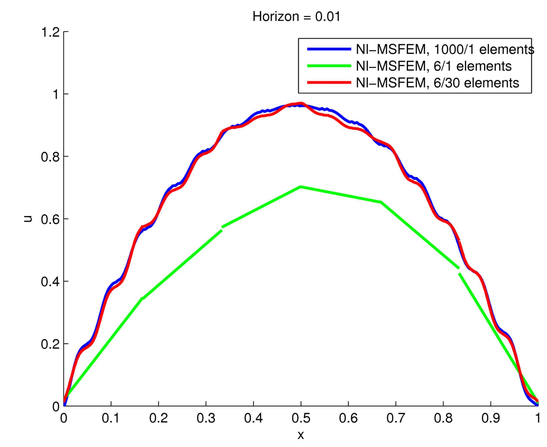
- Particular choices of V^H, V^a give multiscale FEM.

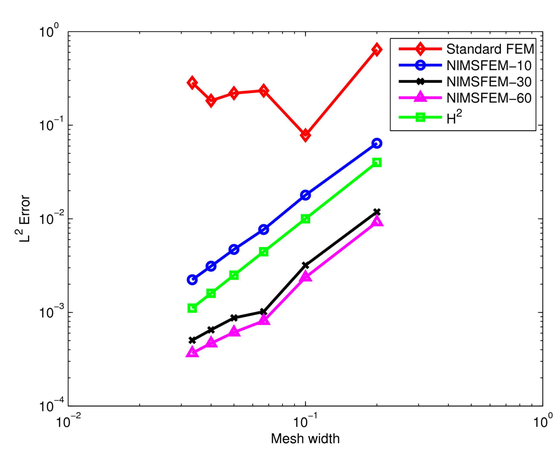
Local, Nonlocal, and Mixed-Locality Comparison



NIMSFEM vs FEM 1D Simulation & Convergence

$$\alpha(x) = \frac{2}{4 - \frac{3}{5}\pi \cos(30\pi x)}, \ C(x,q) = \frac{\alpha(x) + \alpha(q)}{2}, \ b(x) = 1.$$





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