SAND2017-1733C

# Investigating the Quantum Approximate Optimization Algorithm's Advantage over Classical Algorithms

Ojas Parekh[1], Ciarán Ryan-Anderson[1, 2, 3], Yang Jiao[1,4]

1) Center for Computing Research, Sandia National Laboratories

2) Center for Quantum Information and Control, UNM

3) Department of Physics and Astronomy, UNM

4) Tepper School of Business, Carnegie Mellon University

SQuInT - Feb 24, 2017

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000. SAND2016-11153 PE

# Mission

Study Quantum Approximate Optimization Algorithm (QAOA):

Power of quantum?

QAOA: arXiv:1411.4028

# AQC Connection

- QAOA ≅ AQC ≅ Quantum Circuits

- Adiabatic Quantum Computation (AQC)

$$H(t) = H_{simple}(1 - s(t)) + H_{solution}\, s(t)$$

where $s(t): 0 \rightarrow 1$ as $t:\ 0 \rightarrow T$

- As $T \rightarrow \infty$, AQC:

$$|G_{simple}\rangle \rightarrow |G_{solution}\rangle$$

# Discretizing AQC

$$\left|\Psi(T)\right\rangle = U(T,0)\,|\Psi(0)\rangle$$

- Time dependence:

$$U(T,0) \approx \prod_{j=0}^{N} \exp(-iH(j\Delta t)\,\Delta t)$$

- Anti-commutation:

$$e^{-i(H_1+H_2)} \approx \left(e^{-iH_1/M} e^{-iH_2/M}\right)^M$$

# QAOA Connection

- AQC:

$$U_{AQC}(T,0) \approx \prod_{j=0}^{N} \left(e^{-i\, H_1\, s_1(j\Delta t)\, \Delta t/M}\, e^{-i\, H_2\;\, s_2(j\Delta t)\, \Delta t/M}\right)^M$$

- QAOA-p:

$$U_{QAOA-p} \equiv \prod_{j=1}^{p} e^{-i\, \beta_j\, H_1}\, e^{-i\, \gamma_j\, H_2}$$

- as $p \to \infty$, $U_{QAOA-p} \to U_{AQC}$

# QAOA Connection

- AQC:

$$U_{AQC}(T,0) \approx \prod_{j=0}^{N} \left(e^{-i\, H_1\, s_1(j\Delta t)\, \Delta t/M} e^{-i\, H_2\;\, s_2(j\Delta t)\, \Delta t/M}\right)^M$$

- QAOA-p:

$$U_{QAOA-p} \equiv \prod_{j=1}^{p} e^{-i\, \beta_j\, H_1}\, e^{-i\, \gamma_j\, H_2}$$

- Let $\vec{\beta}$ and $\vec{\gamma}$ be free! *(and find "good angles"...)*

# Classically Hard Distributions

- Quantum distributions:
  - Boson Sampling
  - IQP
  - Forrelation

- Hard distribution ⇒ Solvers with quantum advantage?

- Test bed: $U_{QAOA-1} = e^{-i\beta\, B} e^{-i\gamma\, C}$

- QAOA-1 is classically hard to sample

“Quantum Supremacy”: arXiv:1602.07674

# Optimization Problems

$$C(\vec{z}) = \sum_{\alpha=1}^{m} C_\alpha(\vec{z})$$

where $C_\alpha(\vec{z}) = \begin{cases} 1 & \text{If } \vec{z} \text{ satisfies} \\ 0 & \text{else} \end{cases}$ and $\vec{z} \epsilon \{0,1\}^n$

- QAOA-1:

$$\underset{\vec{z},\gamma,\beta}{\text{Max}} \ \langle \gamma, \beta | C | \gamma, \beta \rangle$$

where $|\gamma, \beta\rangle = e^{-i\beta\, B} e^{-i\gamma\, C} |+\rangle^{\otimes n}$

and $B = \sum_{j=1}^{n} X_j$

# Trivialize C

- QAOA-1: $|\gamma, \beta\rangle = e^{-i\beta\, B} e^{-i\gamma\, C} |+\rangle^{\otimes n}$

where $C(\vec{z}) = \sum_{\alpha=1}^{m} C_\alpha(\vec{z})$

$$e^{-i\gamma\, C} = e^{-i\gamma \left[\, \delta_I + \sum \delta_i\, Z_i + \sum \delta_{ij}\, Z_i\, Z_j + \cdots \right]}$$

$$\delta_k \rightarrow \delta_k' \equiv \delta_k + \frac{2\, n_k\, \pi}{\gamma}$$

- All $C \rightarrow$ some trivial $C'$

# Max E3Lin with Bounded Degree

- $C_\alpha = \frac{1}{2}(1 \pm Z_a\, Z_b\, Z_b)$ where $C(\vec{z}) = \sum_{\alpha=1}^{m} C_\alpha(\vec{z})$

- Original best known approximation:
  - Classical: $\left(\frac{1}{2} + \frac{const.}{D}\right) m$ QAOA-1: $\left(\frac{1}{2} + \frac{1}{22\, D^{3/4}}\right) m$

- Afterwards:
  - Classical: $\left(\frac{1}{2} + \frac{const.}{D^{1/2}}\right) m$ QAOA-1: $\left(\frac{1}{2} + \frac{1.}{101\, D^{1/2} \ln D}\right) m$

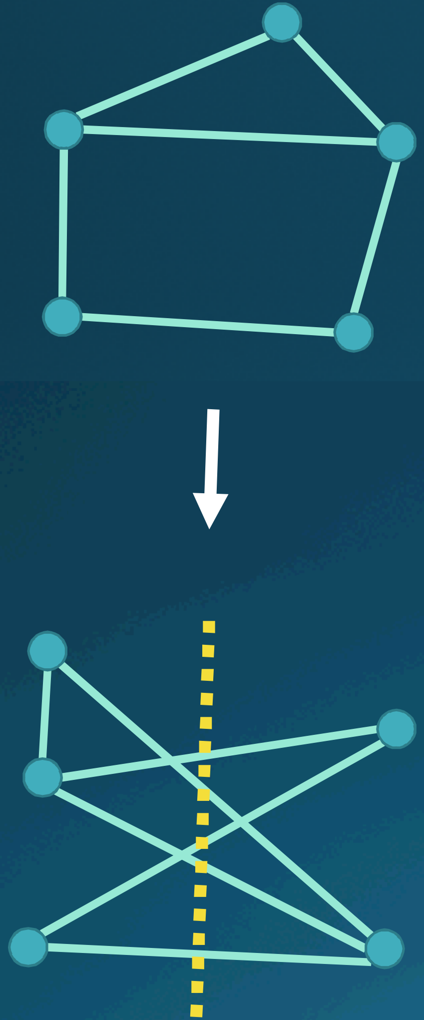
Max E3Lin: arXiv:1412.6062

# Simpler QAOA-1

$$C = \sum_{\alpha=1}^{m} C_\alpha = \delta_I + \sum_i \delta_i Z_i + \sum_{i<j} \delta_{i,j} Z_i Z_j + \cdots$$

- Boolean functions → Sum of $Z$s
- Two-body QAOA-1 ⇒ Hard distribution

# Max-Cut



$$C_\alpha = C_{\langle i,j \rangle} = \frac{1}{2}\left(I - Z_i Z_j\right)$$

$$|00\rangle, |11\rangle \rightarrow 0 \qquad |01\rangle, |10\rangle \rightarrow 1$$

$$|\gamma, \beta\rangle = e^{-i\beta\, B} e^{-i\gamma\, C} |+\rangle^{\otimes n}$$

$$C = \sum_{\langle i,j \rangle \in E} C_{\langle i,j \rangle}$$

# QAOA-1 and Max-Cut

- Given $C_{\langle i,j\rangle} = \frac{1}{2}\left(I - Z_i Z_j\right)$ what does QAOA-1 do?

- Previously:

$$\langle C \rangle = \sum_{\langle i,j\rangle \in E} \left\langle C_{\langle i,j\rangle} \right\rangle =$$

$$\sum_{\langle i,j\rangle \in E} \langle + |^{\otimes n} e^{i\gamma\, C} e^{i\beta \sum X_k} \; C_{\langle i,j\rangle} \; e^{-i\beta \sum X_k} e^{-i\gamma\, C} | + \rangle^{\otimes n}$$

# QAOA-1 and Max-Cut

- Given $C_{\langle i,j\rangle} = \frac{1}{2}\left(I - Z_i Z_j\right)$ what does QAOA-1 do?

- Previously:

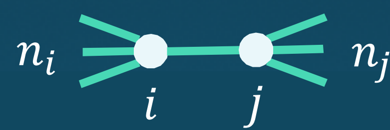
$$\langle C\rangle = \sum_{\langle i,j\rangle \in E} \left\langle C_{\langle i,j\rangle}\right\rangle =$$

$$\sum_{\langle i,j\rangle \in E} \langle +|^{\otimes n} e^{i\gamma\, C} e^{i\beta(X_i + X_j)}\ C_{\langle i,j\rangle}\ e^{-i\beta(X_i+X_j)} e^{-i\gamma\, C} |+\rangle^{\otimes n}$$

# QAOA-1 and Max-Cut

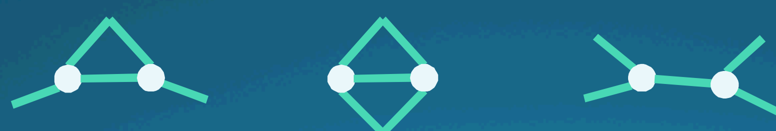
$$\left\langle C_{\langle i,j\rangle} \right\rangle = \langle + |^{\otimes n} e^{i\gamma\, C} e^{i\beta\,(X_i + X_j)}\; C_{\langle i,j\rangle}\; e^{-i\beta(X_i+X_j)} e^{-i\gamma\, C} | + \rangle^{\otimes n}$$

$$\left\langle C_{\langle i,j\rangle} \right\rangle \rightarrow$$




$$\left\langle \gamma, \beta \middle| C_{\langle i,j\rangle} \middle| \gamma, \beta \right\rangle = \sum_{g} w_g \left\langle C_{g \cong \langle i,j\rangle} \right\rangle$$

*3-regular:*

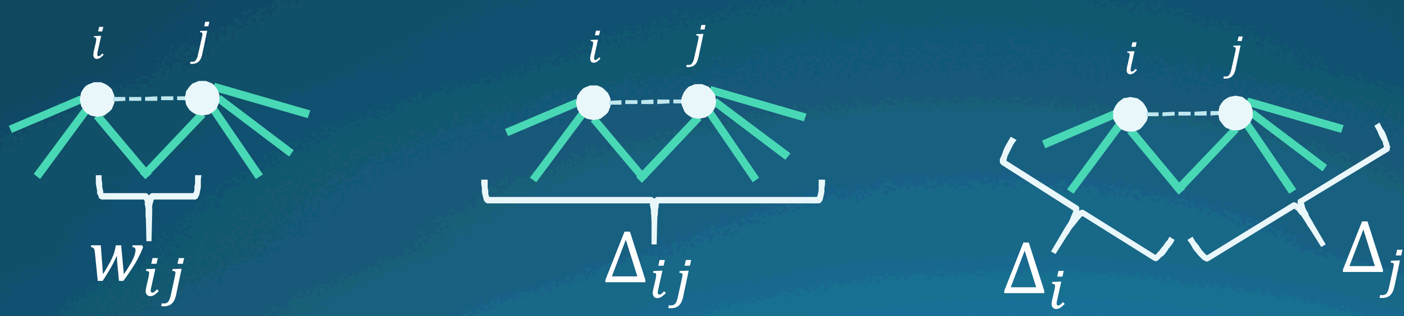


# Closed Form

- But what is QAOA-1 doing with Max-Cut?

$$\left\langle C_{\langle i,j\rangle}\right\rangle =$$

$$\frac{1}{2}\Big[1 + \sin(4\beta)\sin(4\gamma)\left\{\cos^{|\Delta_i|-1}(2\gamma) + \cos^{|\Delta_j|-1}(2\gamma)\right\} + \sin^2(2\beta)\cos^{|\Delta_{ij}|-2|w_{ij}|}(2\gamma)\left\{1 - \cos^{|w_{ij}|}(4\gamma)\right\}\Big]$$



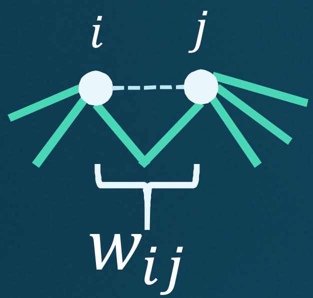
# High Level Sketch

$$\langle C_{\langle i,j\rangle}\rangle = \langle +|^n e^{i\gamma C} e^{i\frac{\pi}{4}(X_i+X_j)} \frac{1}{2}\left(1 - Z_i Z_j\right) e^{-i\frac{\pi}{4}(X_i+X_j)} e^{-i\gamma C} |+\rangle^n$$

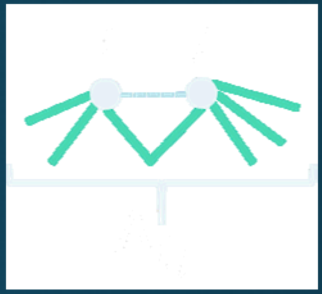
$$= \frac{1}{2} - \frac{1}{2^{n+1}} \sum_{z,a\in\{\pm 1\}^n} \langle z|\, e^{i\gamma\, C}\; Y_i Y_j\; e^{-i\gamma\, C} |a\rangle$$

$$= \frac{1}{2} + \frac{1}{2^{n+1}} \sum_{z\in\{\pm 1\}^n} z_i z_j\; e^{i\gamma\, C\left(z^{\overline{ij}}\right) - C(z)}$$

# High Level Sketch




$$= \frac{1}{2} + \frac{1}{2^{n+1}} \sum_{z \in \{\pm 1\}^n} z_i z_j \, e^{i\gamma\, \Delta_{ij}(z)}$$



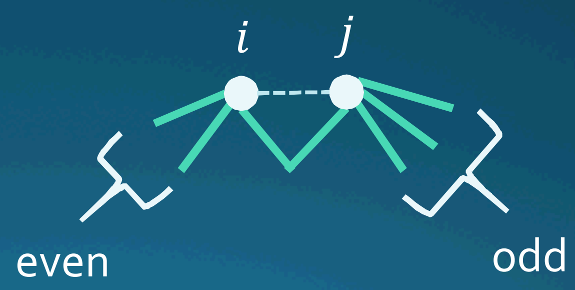
$$= \frac{1}{2} + \frac{1}{2^{n+1}} \sum_{z \in \{\pm 1\}^n} z_i z_j \sum_{\substack{k\ odd \\ 1 \le k \le |w_{ij}|}} \binom{|w_{ij}|}{k} z_i z_j \, i^{2k} \sin(\gamma)^{2k} \cos(\gamma)^{|\Delta_{ij}|-2k}$$

$$= \frac{1}{2} - \frac{1}{2} \cos(\gamma)^{|\Delta_{ij}|-2|w_{ij}|} \left[\frac{1-\cos(2\,\gamma)^{|w_{ij}|}}{2}\right]$$

# Learning from the Closed Form

$$\left\langle C_{\langle i,j\rangle}\right\rangle_{\beta={}^{\pi}/_{4}} = \frac{1}{2} - \frac{1}{2}\cos(\gamma)^{|\Delta_{ij}|-2|w_{ij}|}\left[\frac{1-\cos(2\,\gamma)^{|w_{ij}|}}{2}\right]$$

- $C_{\langle i,j\rangle}(z)$: $00, 11\ (not\ cut) \rightarrow 0; \quad 01, 10\ (cut) \rightarrow 1$
- $\left\langle C_{\langle i,j\rangle}\right\rangle \Rightarrow$ On average, how often the edge is cut
- For an edge to have an advantage:



# Future Work

- Design better classical algorithms

- Otherwise, gain insight into the quantum advantage

- Apply techniques to QAOA-1 on other problems:
  - Understand Cs with terms like $Z_i Z_j Z_k$ and so on…

- Examine QAOA-2. Extend arguments beyond p=2?

# Thanks!