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Statistical Inference In Quantum Tomography - Uses of Hypothesis Testing and Information Criteria

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CQuIC Candidacy Exam
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1. Tomography
2. Better Tomography
3. Classical Hypothesis Testing
4. Model Selection
5. Results

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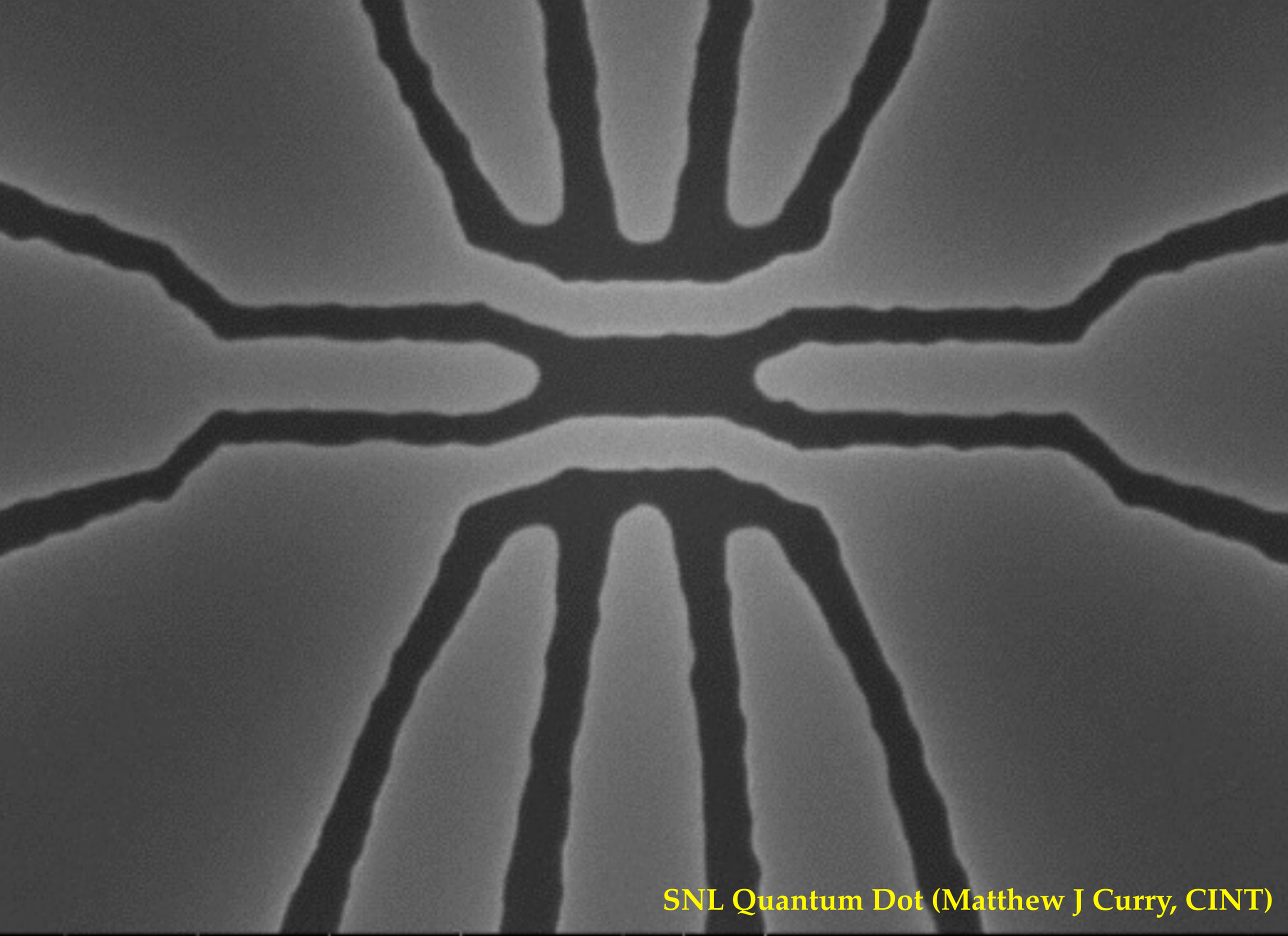
Tomography

Tomography is the characterization of systems

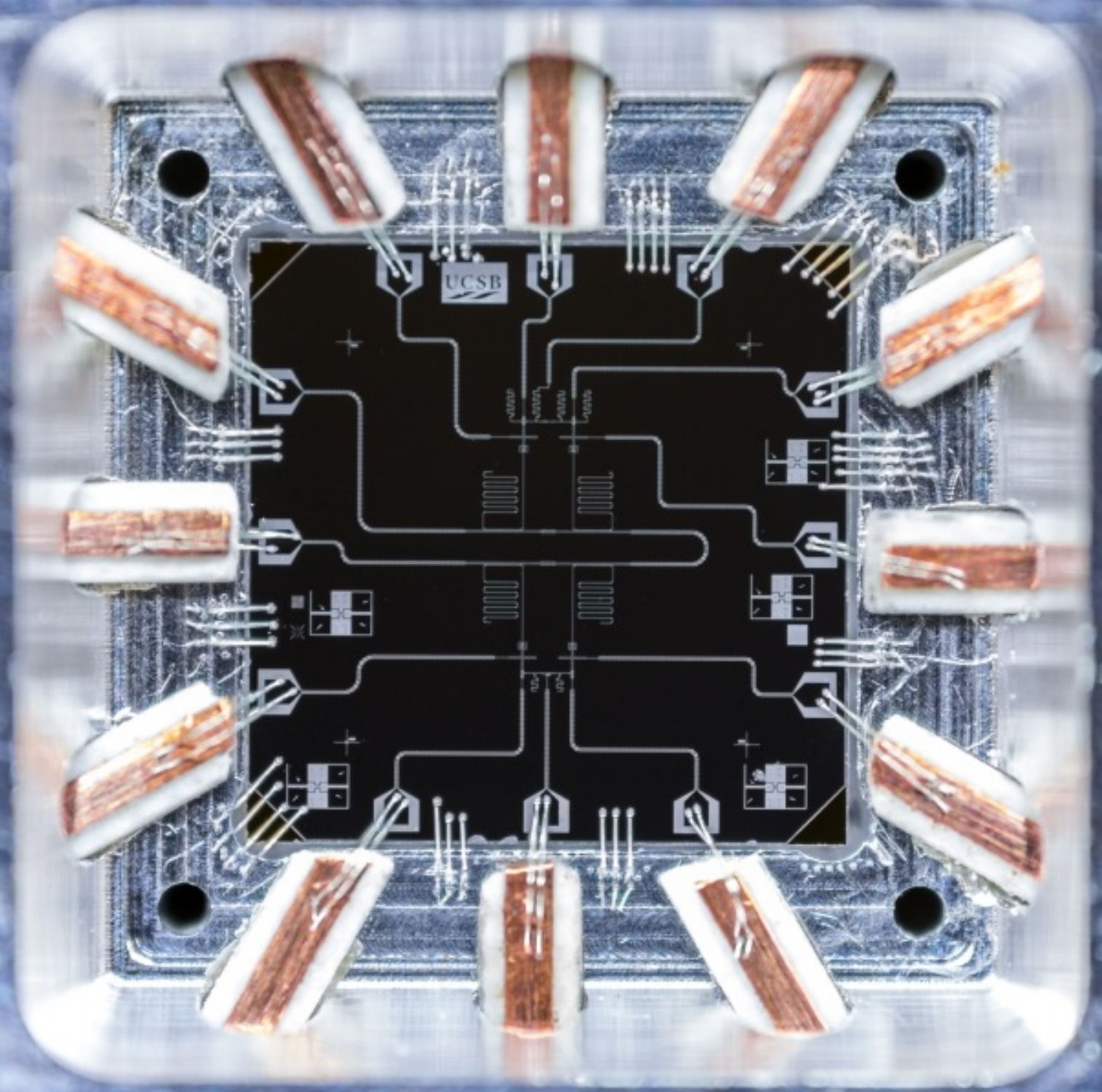
Characterization - what does this system even do?

Control - making systems do interesting things

Diagnostics - why does the system not do what I want it to?



SNL Quantum Dot (Matthew J Curry, CINT)

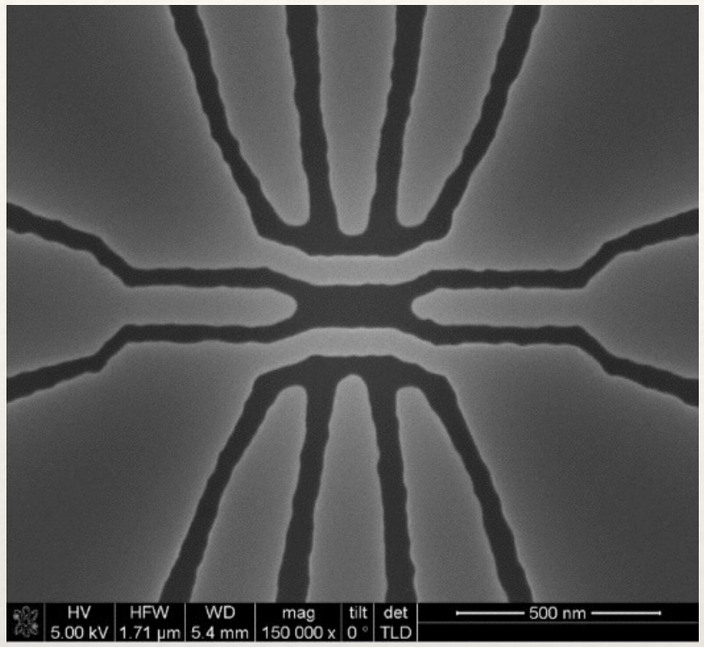


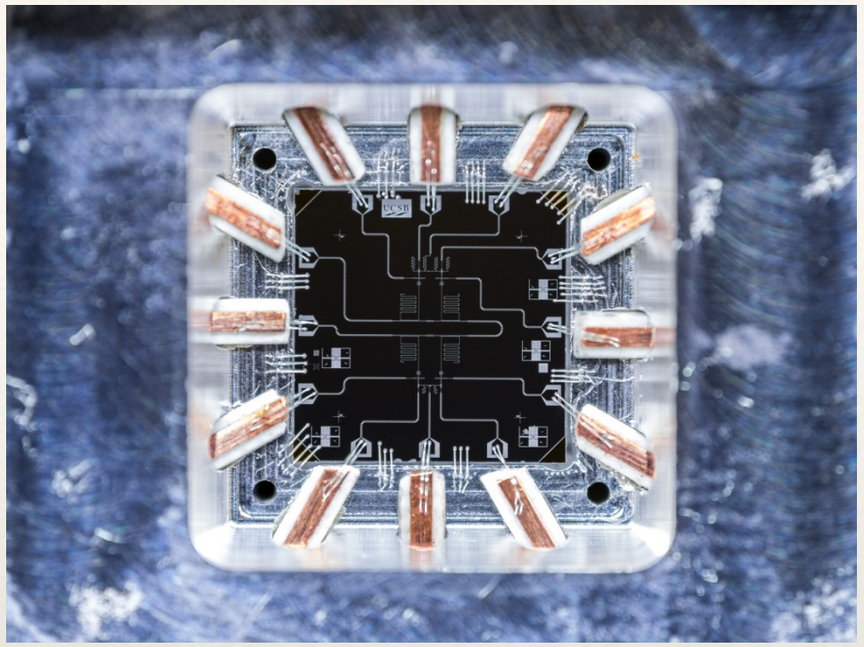
UCSB gmon Transmon Qubit (Michael Fang)

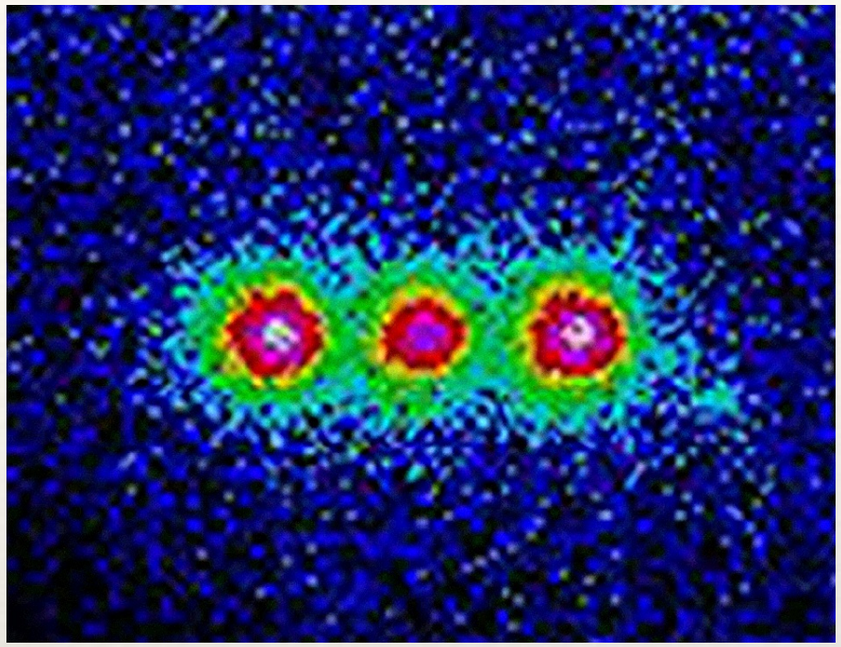
NIST Ion Trap

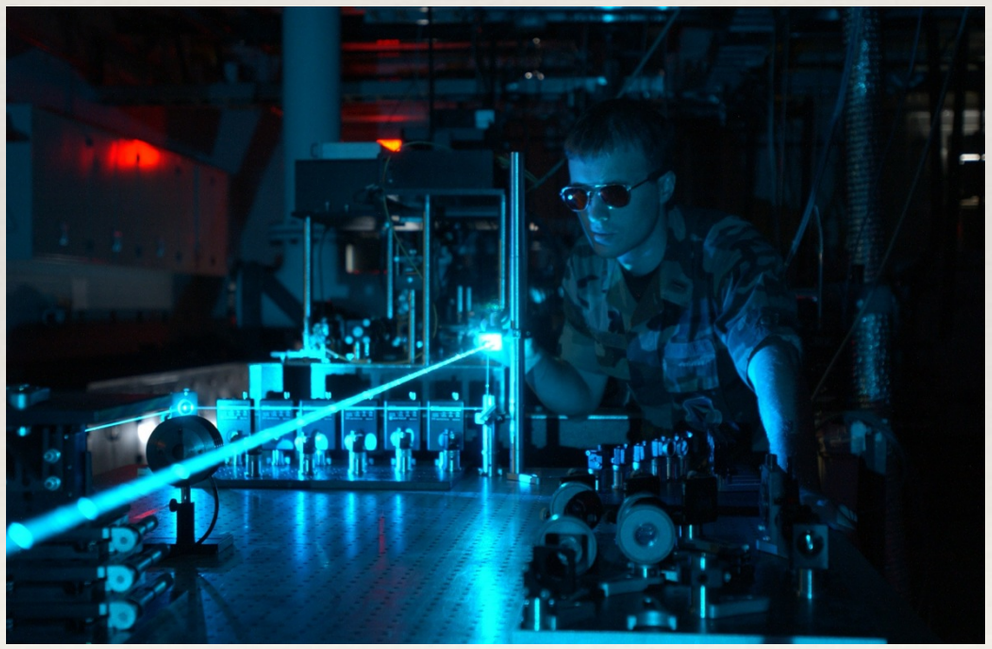
US Air Force Research Laboratory Laser

Tomography is device agnostic…



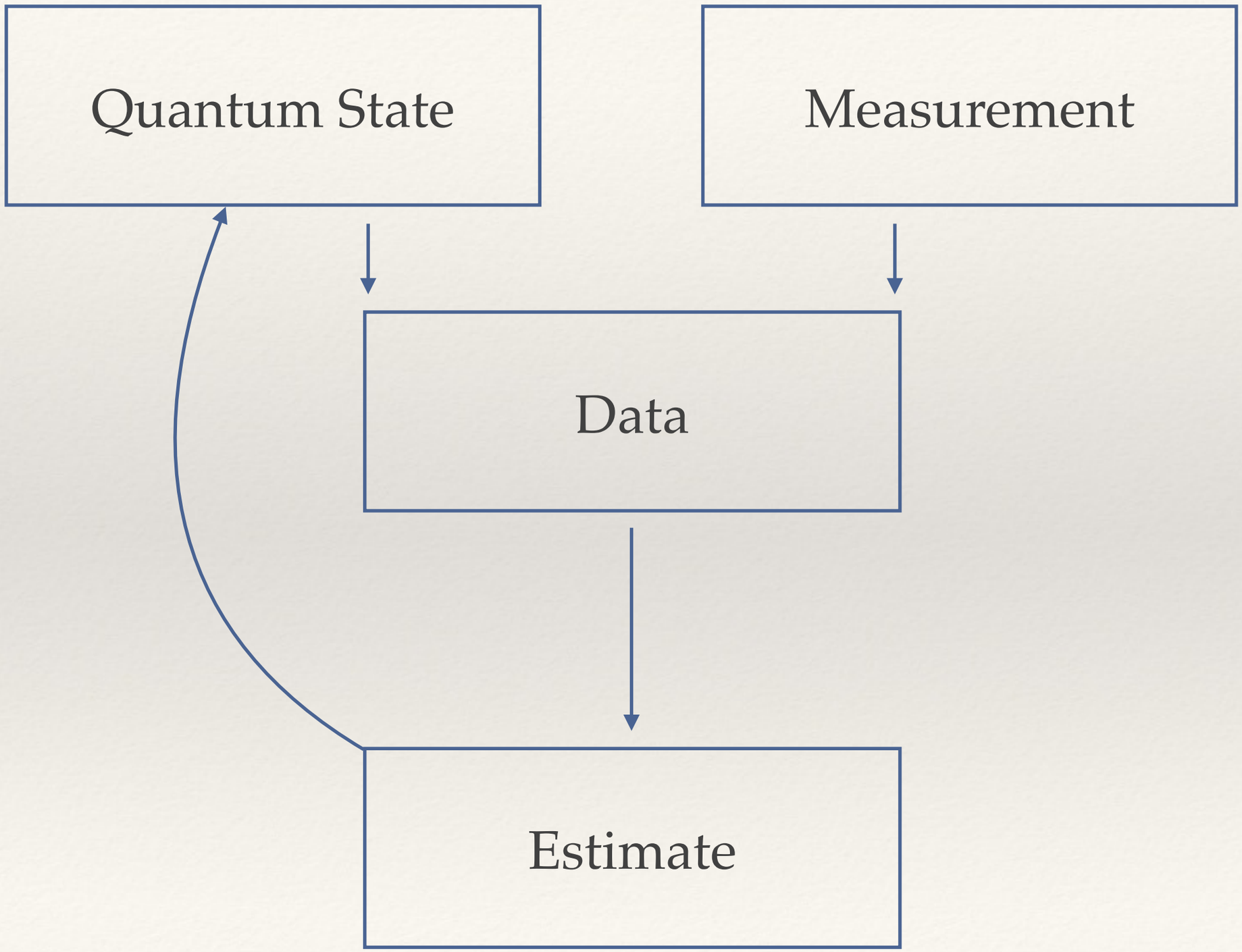




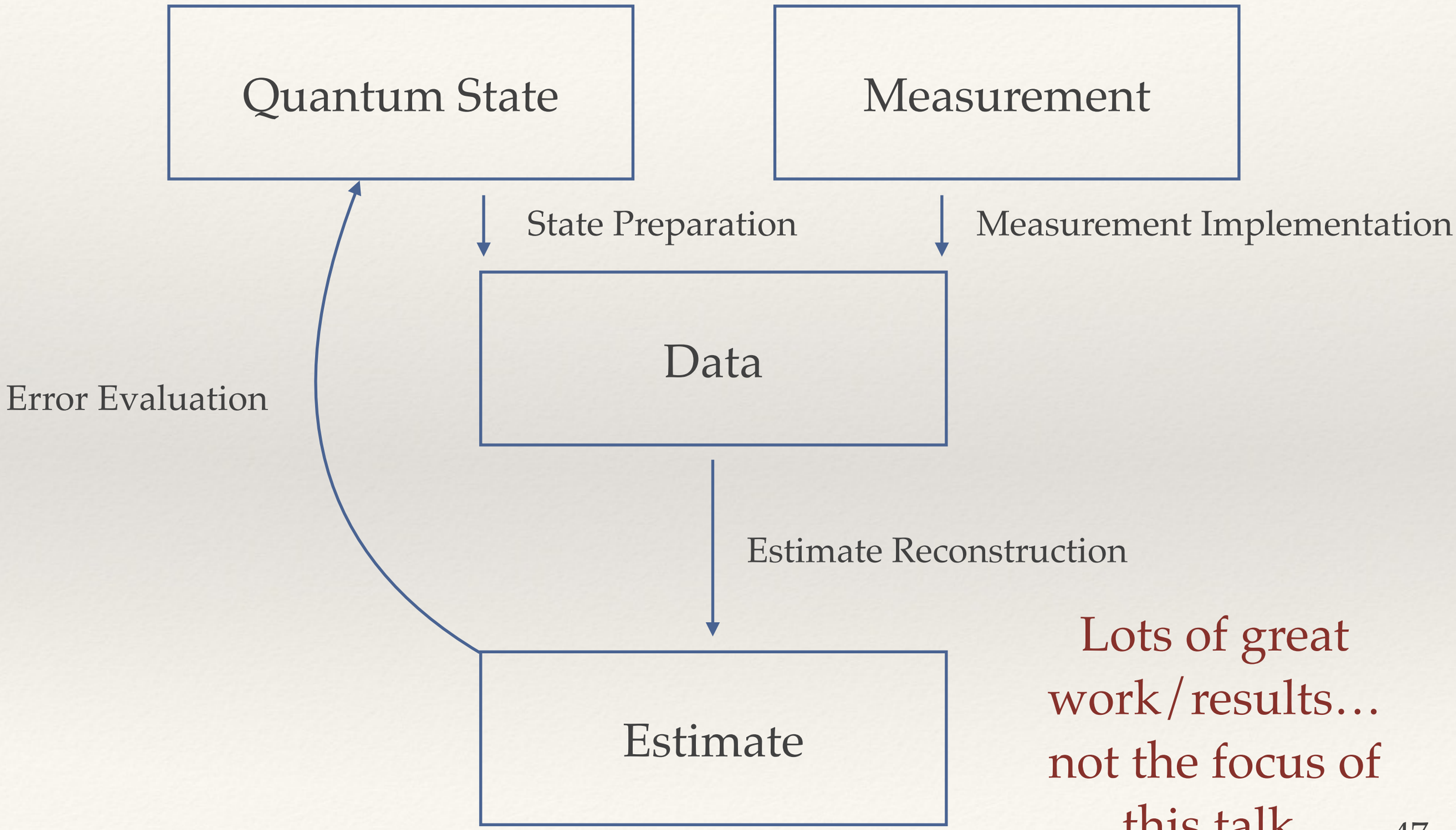


…and tells us what's going on

A Standard Tomographic Process

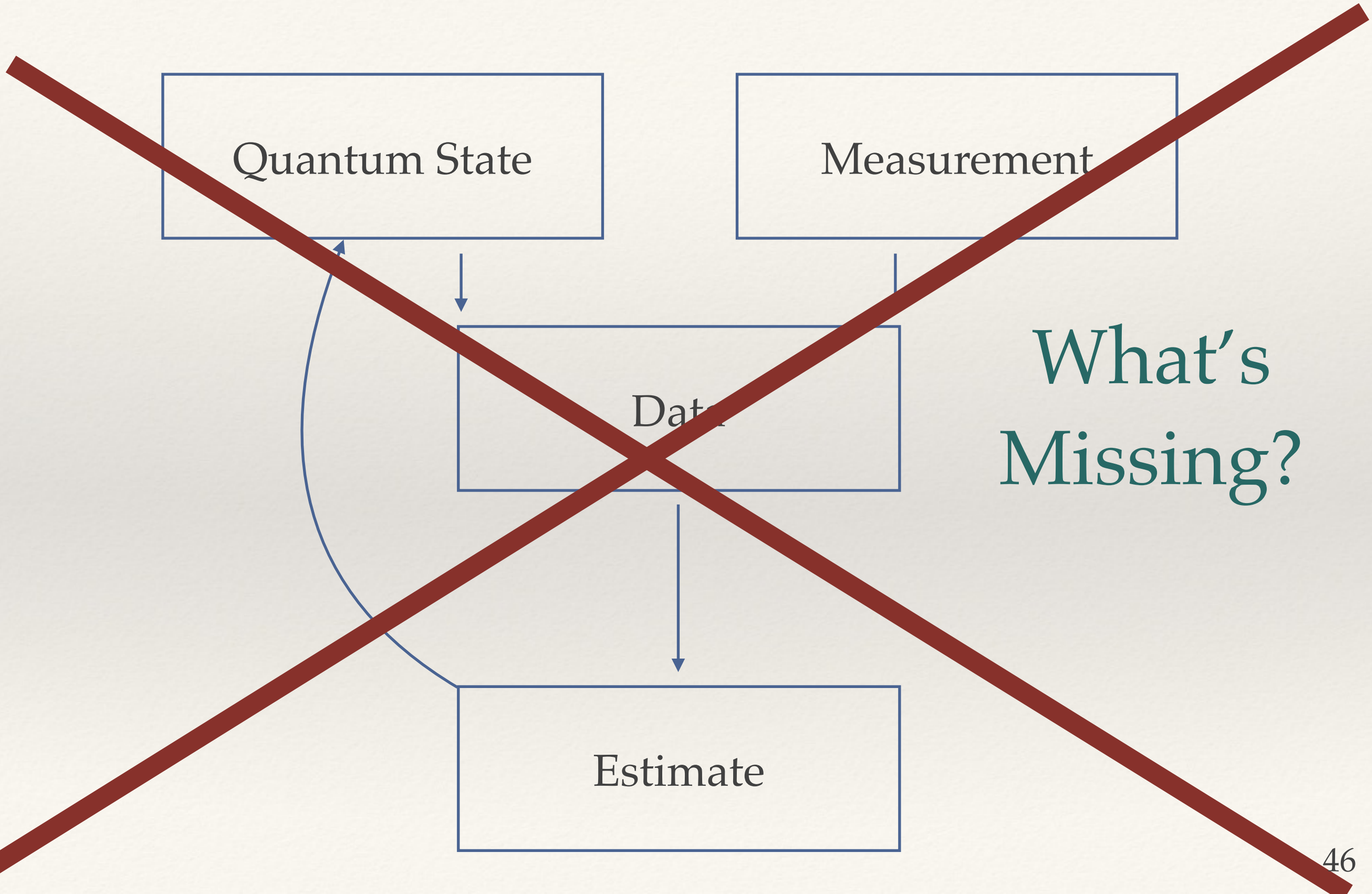


A Standard Tomographic Process



Lots of great work/results… not the focus of this talk.

A Standard Tomographic Process



What's Missing?

Better Tomography

It is of the highest importance in the art of **detection** to be able to **recognize**, out of a number of **facts**, which … [are] **vital**.

—Sherlock Holmes (The Reigate Puzzle)

Hilbert Space dimension is vital for tomography!

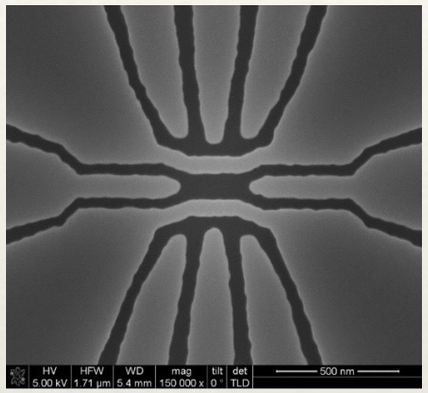
Dimension **d** analogous to number of accessible degrees of freedom

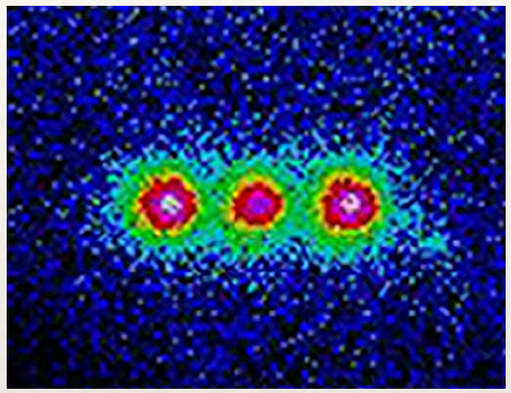
Quantum states are **d** x **d** matrices (density operator)

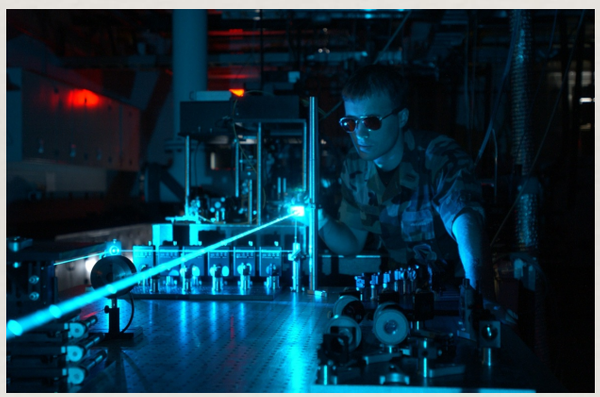
You need to know **d** to do a reconstruction!

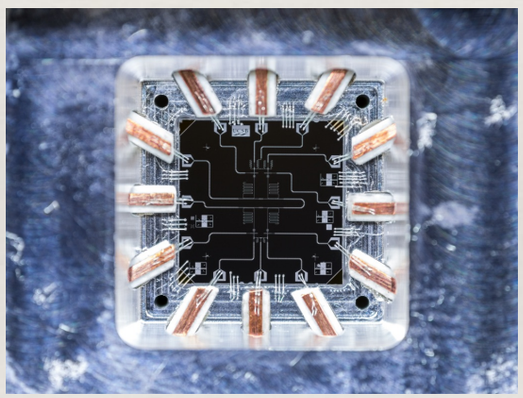
Other, more “practical” reasons?….

We want our devices to behave as qubits….









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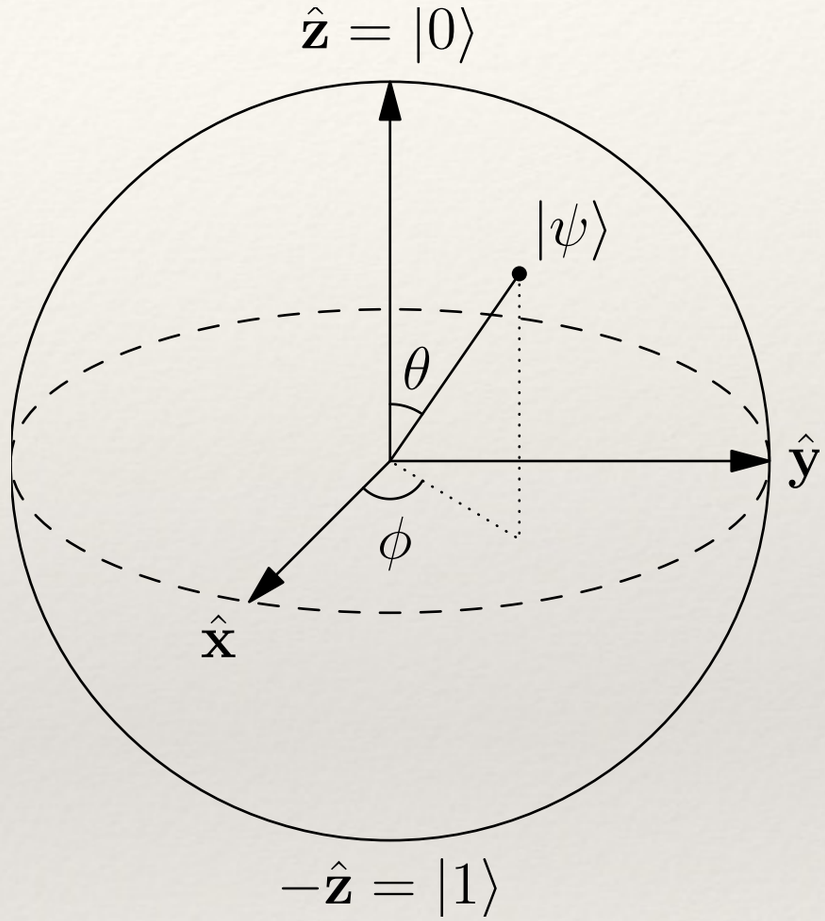
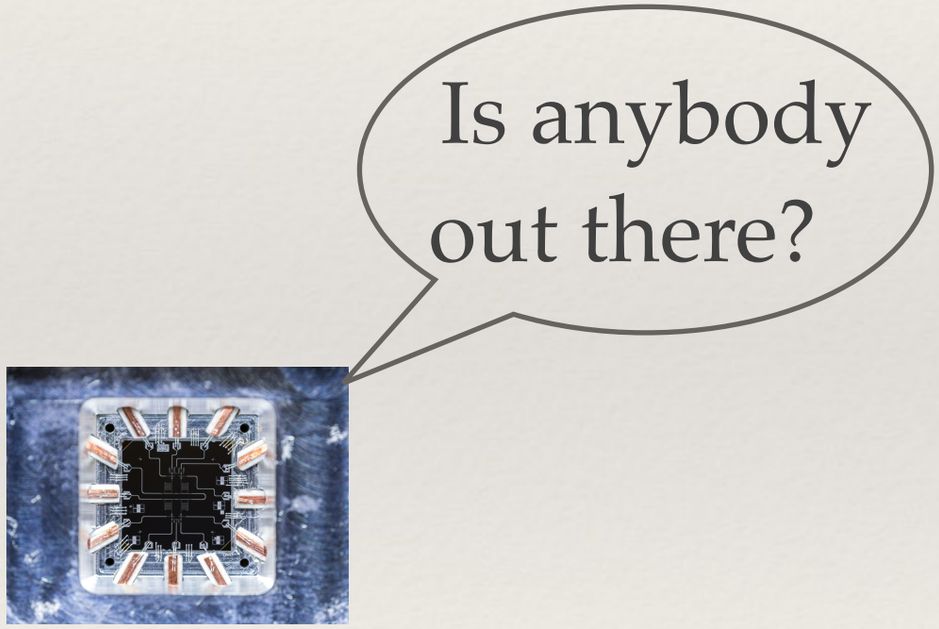


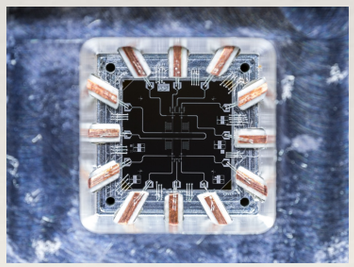
…should we not check if this is the case?

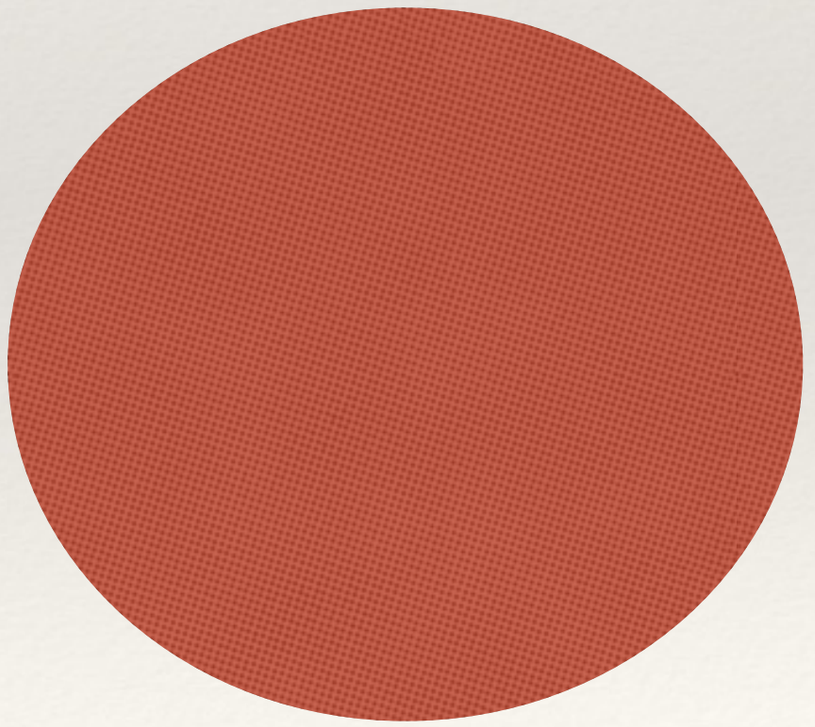
We try to isolate our devices…



…sometimes they couple to the environment, introduce non-Markovian noise…

out….



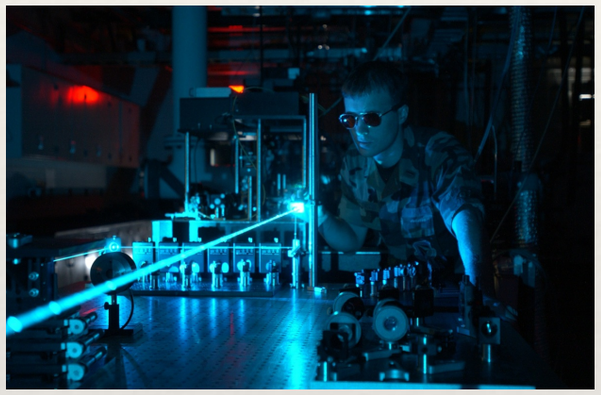


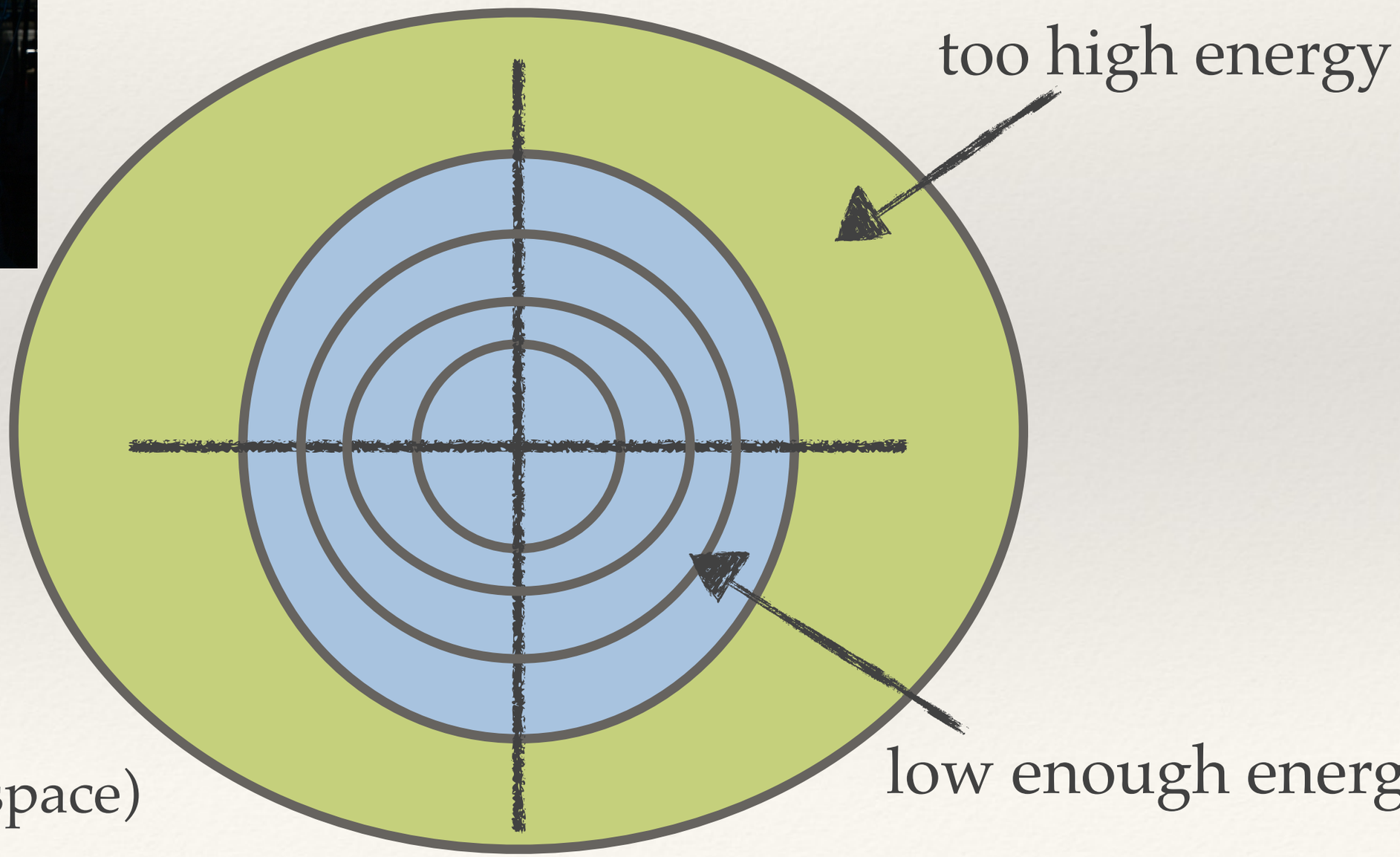
…..and back

…and our "system" is bigger than we thought!

Need a method to describe dynamics on everything

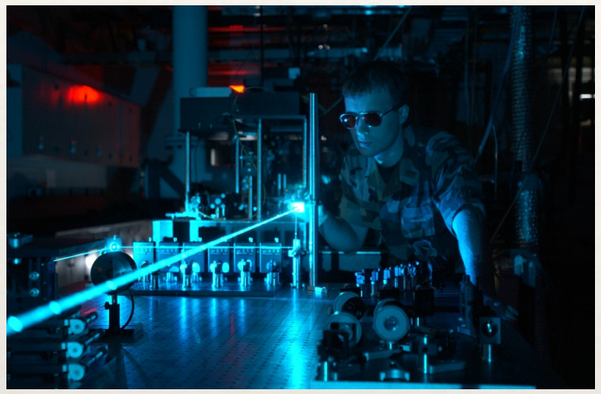
Physical intuition tells us some systems have small dimension…

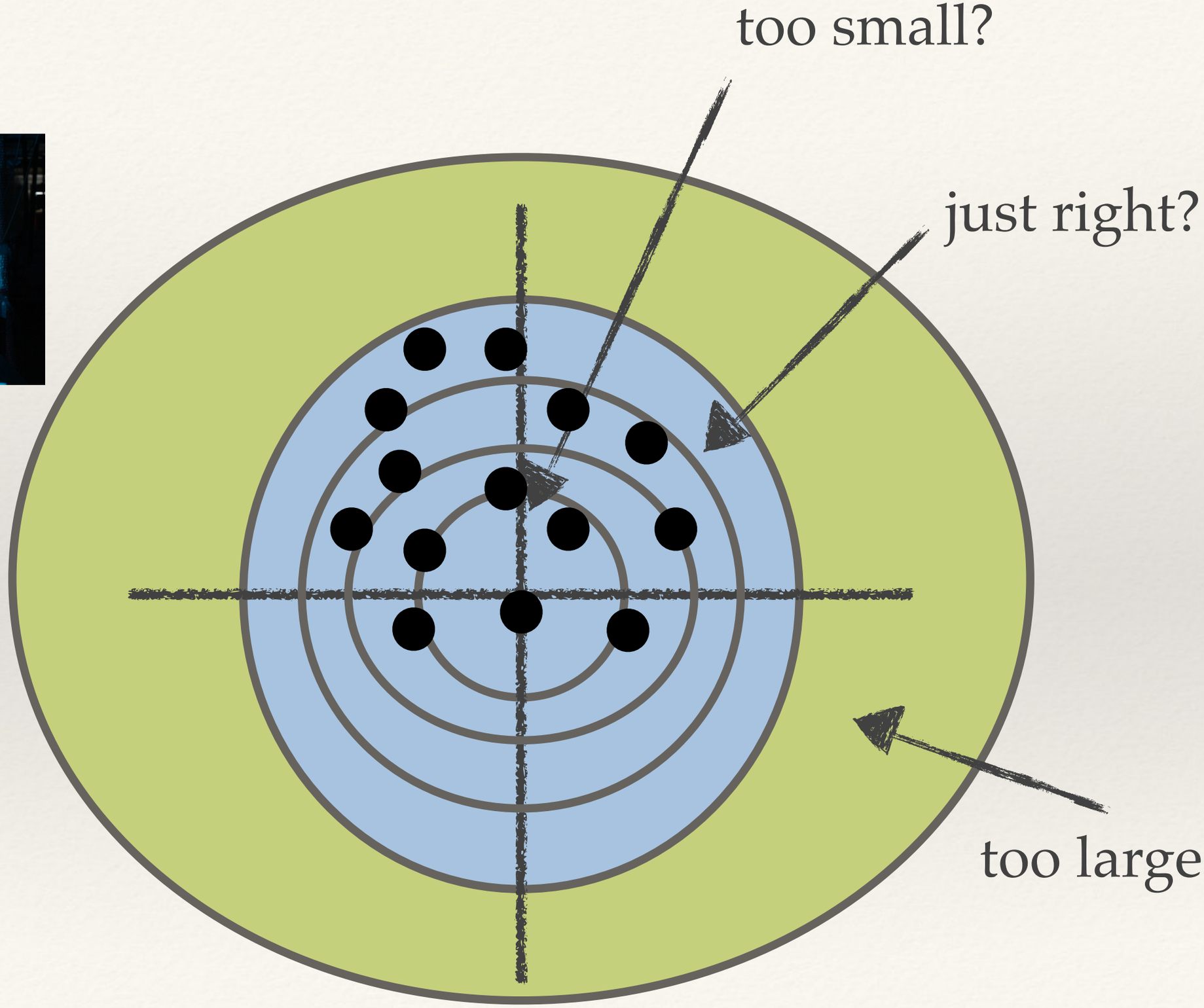




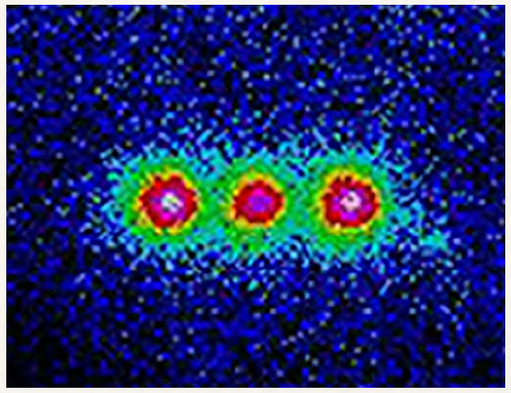
(optical phase space)

…but how small is too large?

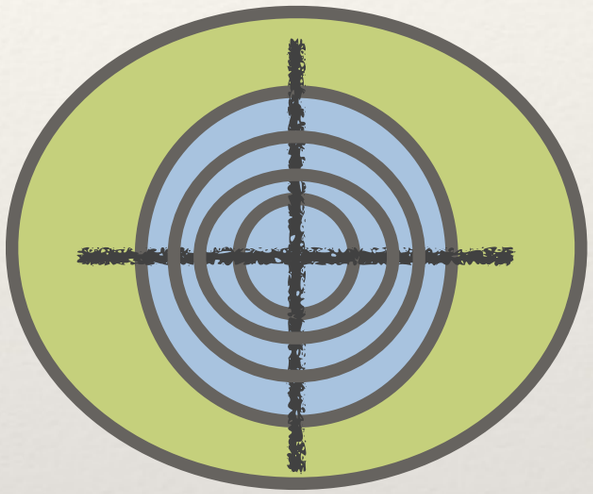




Why Dimension Matters



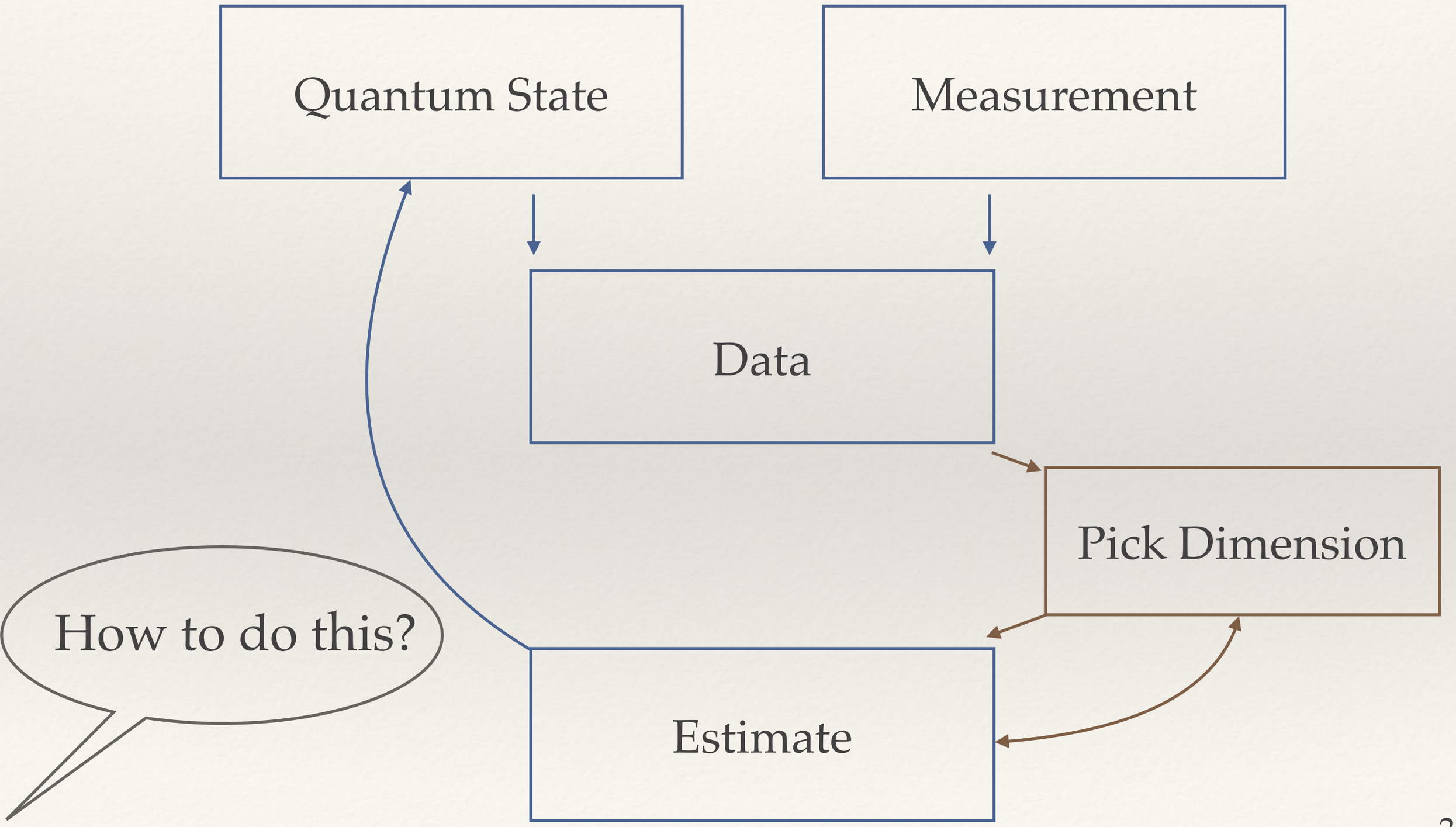
Implementation-independent qubits



Do not know where to truncate

Coupling expands system size

A Better Tomographic Process



2	3	4
5	?	6
7	8	9

Too many choices!

2

3

4

5

6

7

8

9

We need a tool to help choose…

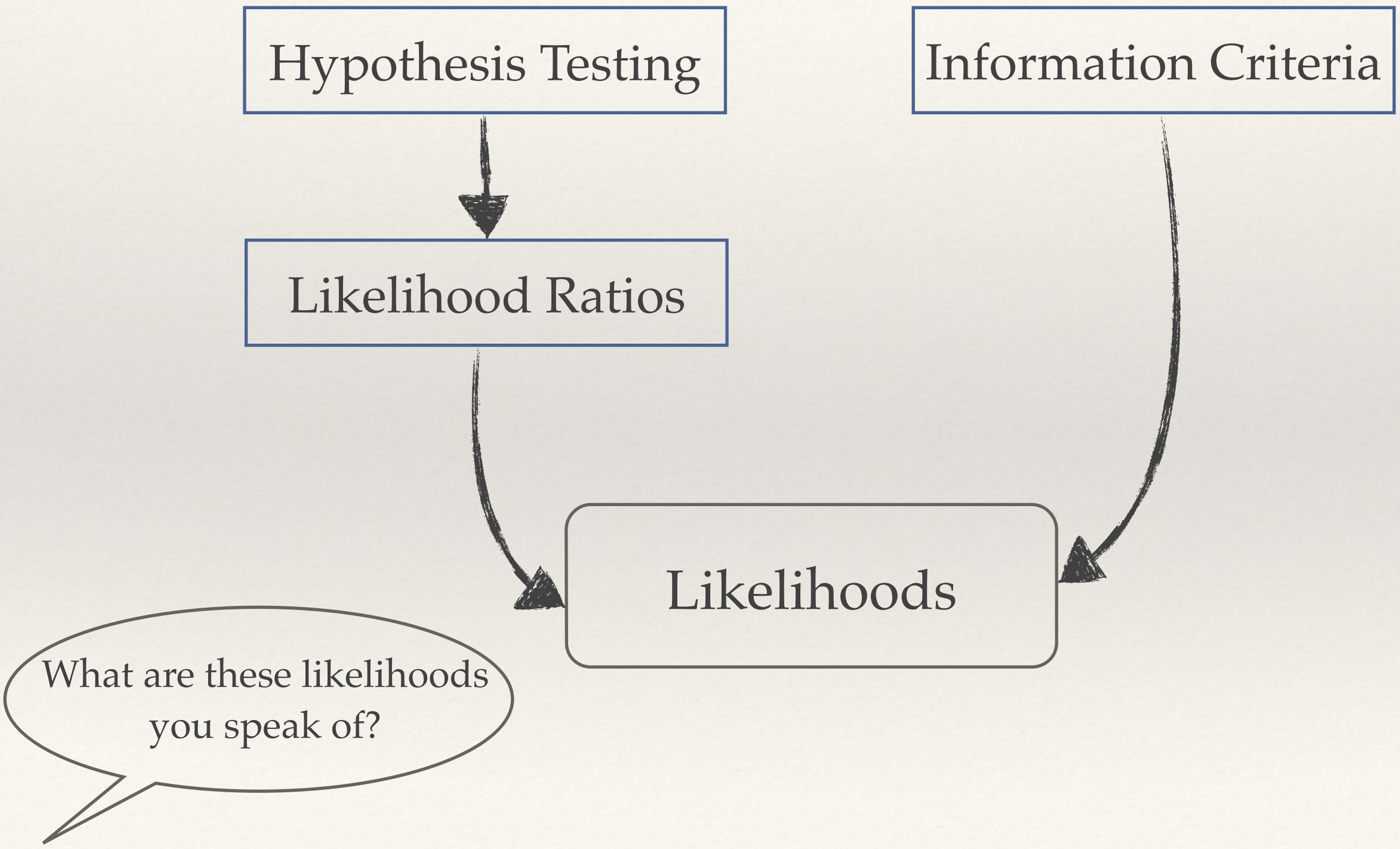
Classical Statistical Inference







Inference can use hypothesis tests and/or information criteria



Likelihoods are Data Driven

A coin with bias **b**
comes up heads **n** times
out of **N** throws.

What is the probability p this happens?

$$p(n) = \binom{N}{n} b^n (1-b)^{N-n}$$

Likelihoods are Data Driven

A coin comes up heads
n times out of **N** throws.

Can we say anything about the bias **b**?

What inferences do the data support?

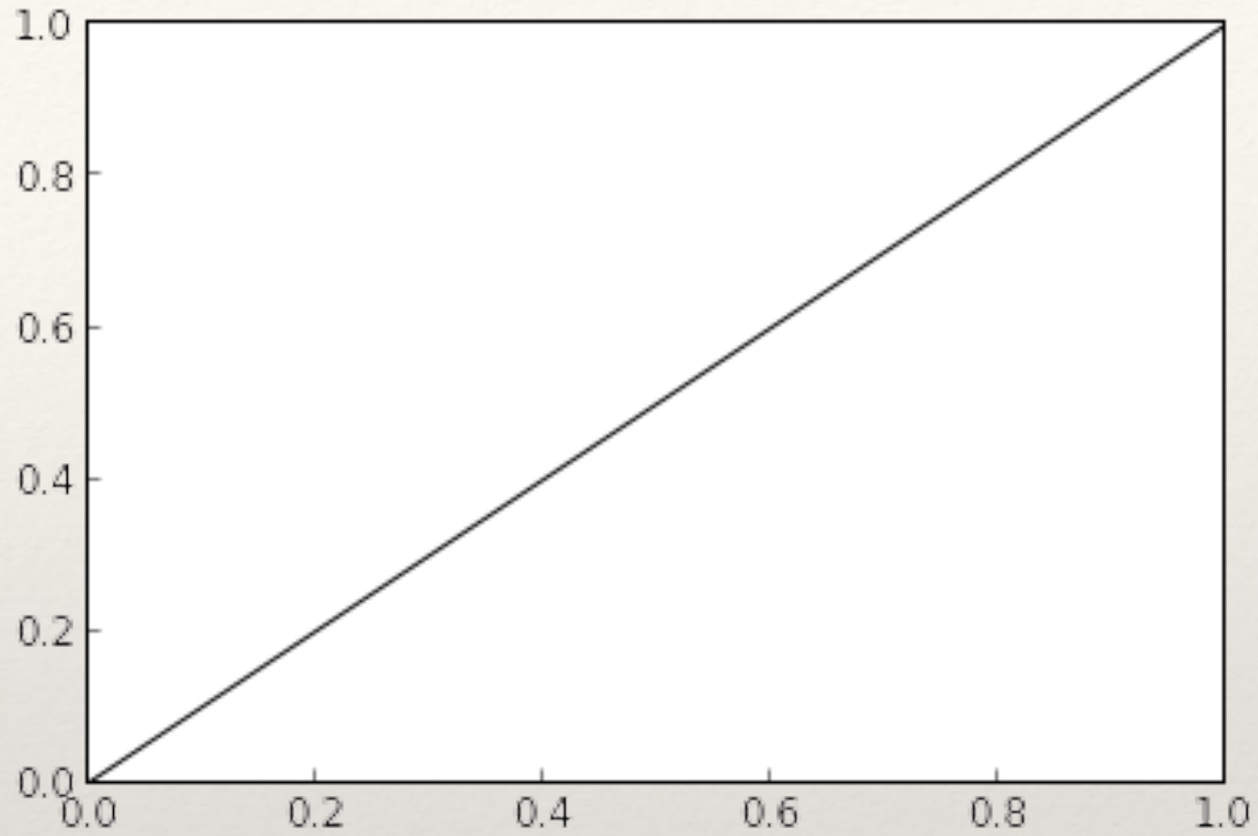
Likelihoods are Data Driven

A coin comes up heads
n times out of **N** throws.

What is probability of the data,
given a fixed choice of bias?

$$\mathcal{L}(b) = p(n|b) = \binom{N}{n} b^n (1-b)^{N-n}$$

A Single Flip…



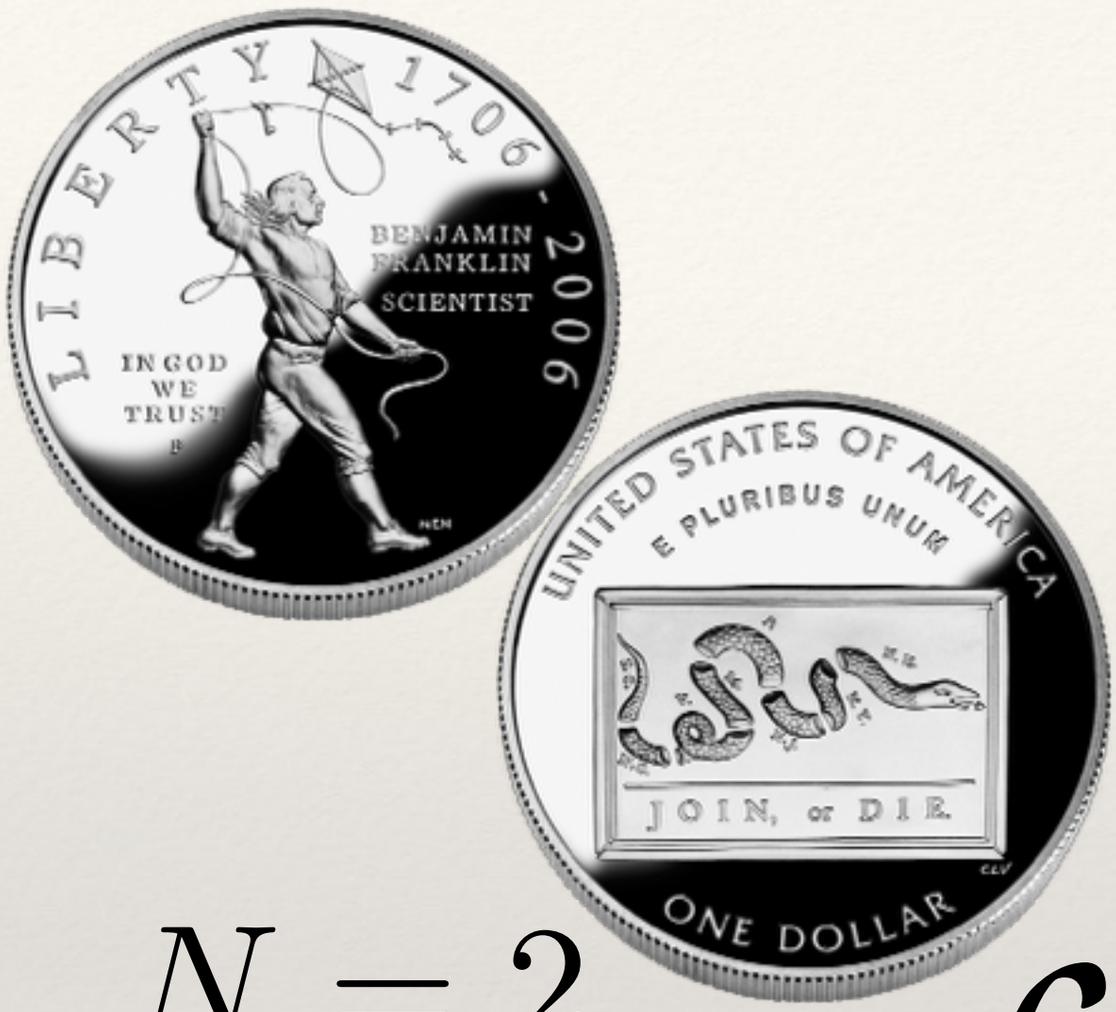
$$N = 1$$

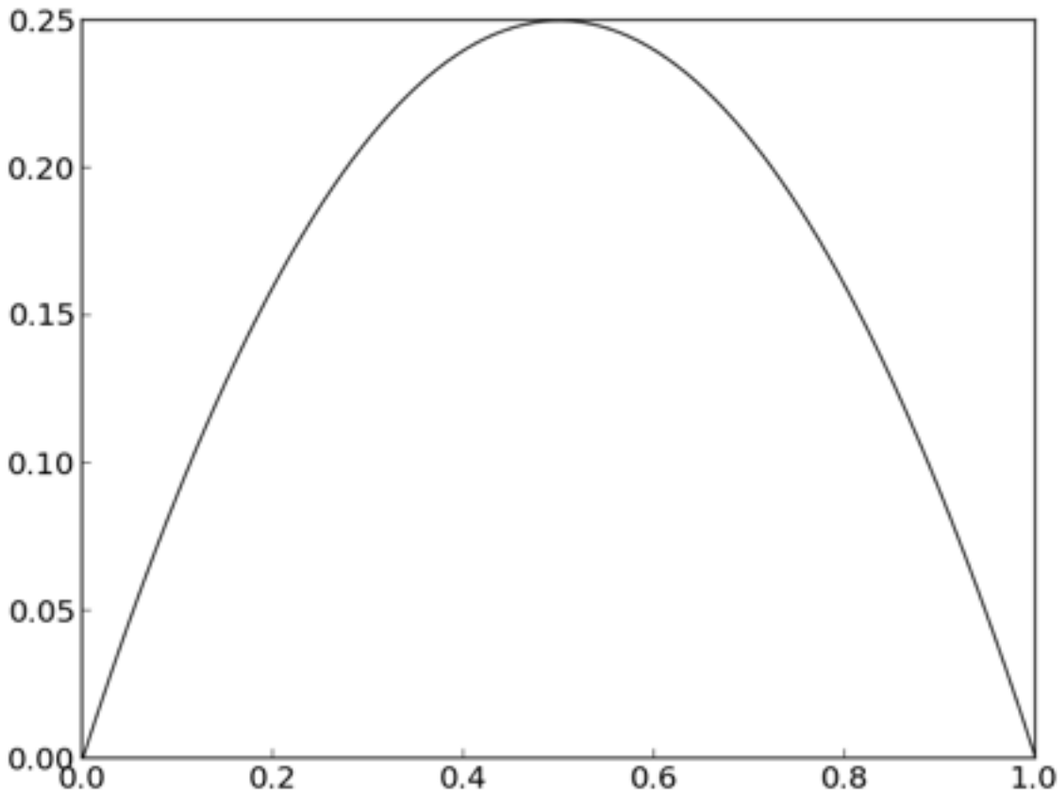
$$n = 1$$

$$\mathcal{L}(b) \sim b$$

The likelihood for the bias being 0 is 0.

Flip Once More…





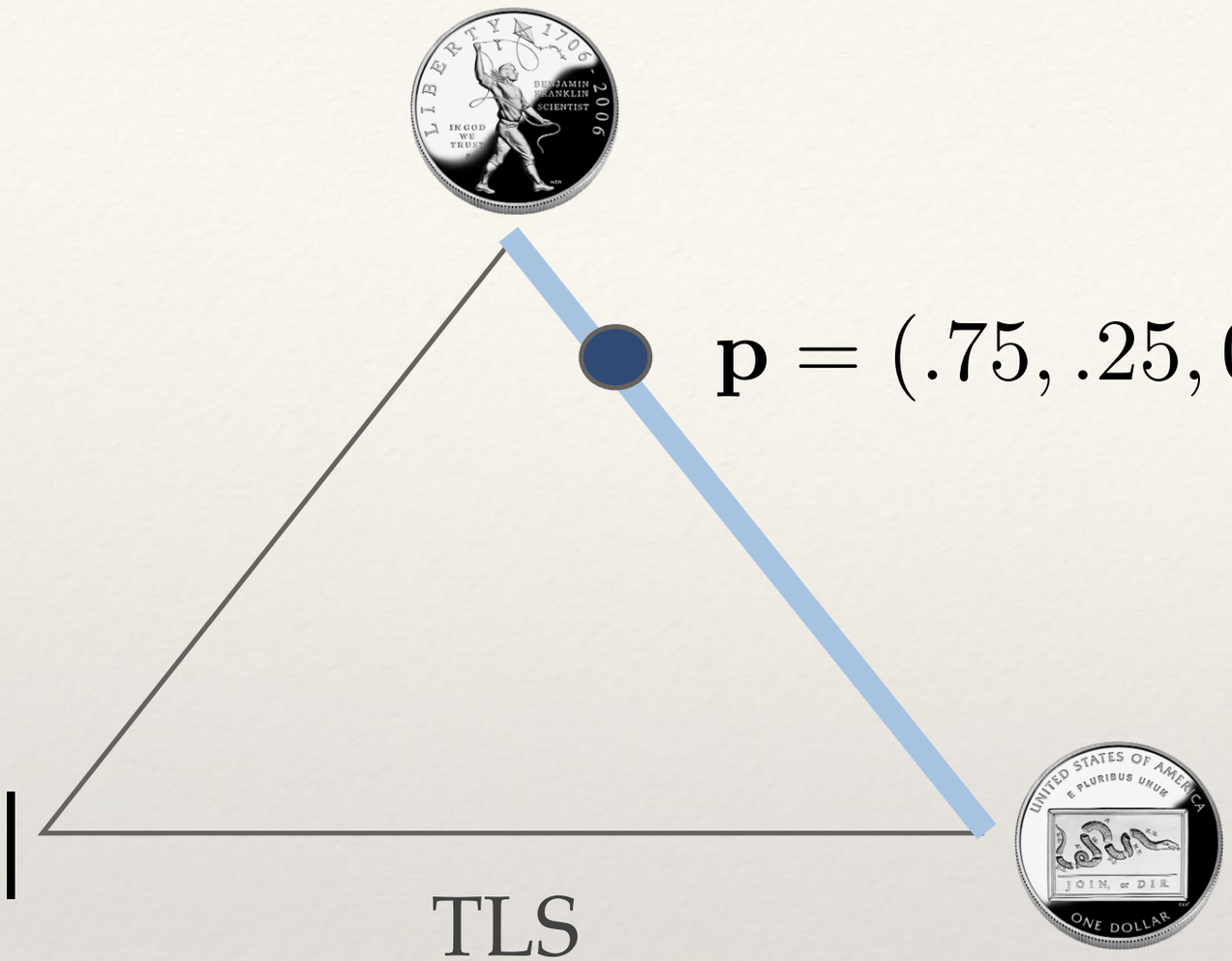
$$N = 2$$

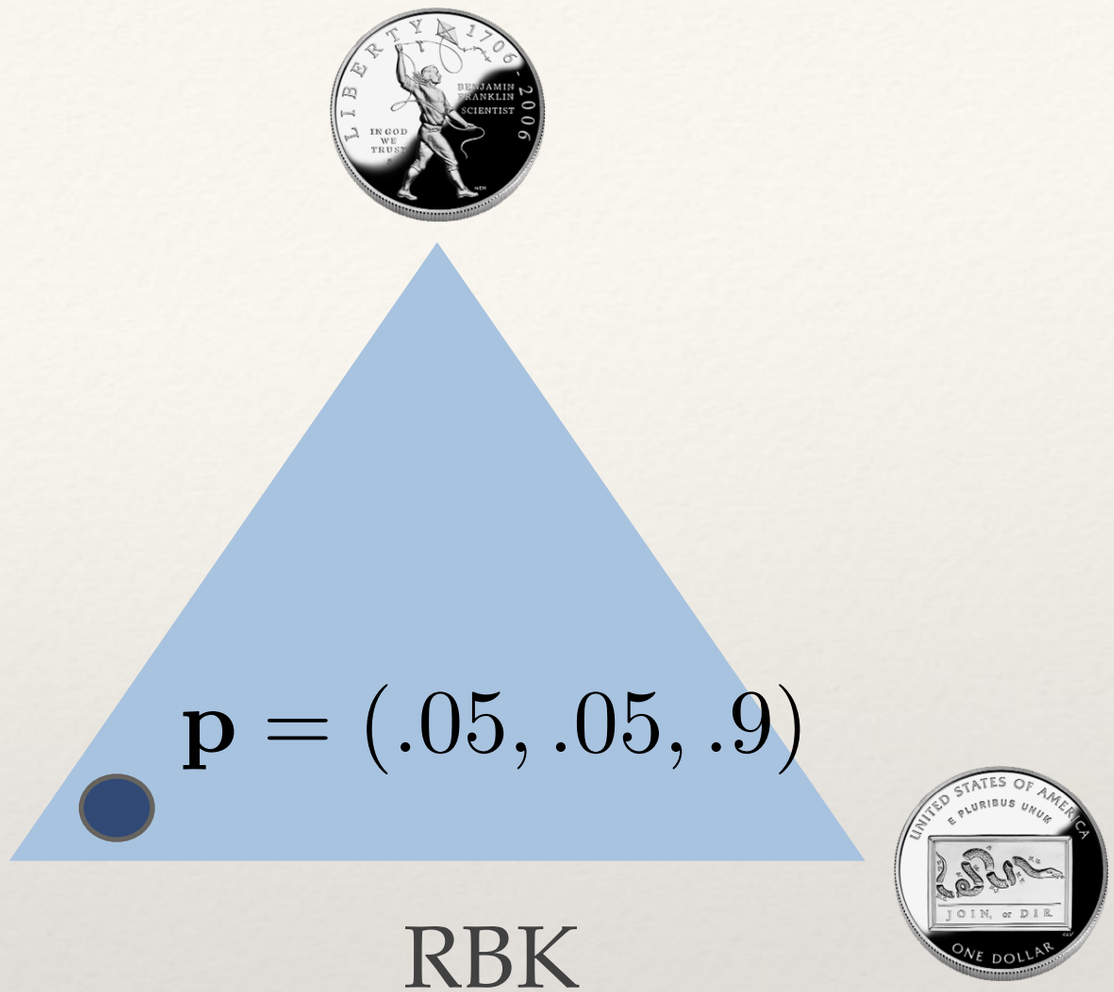
$$n = 1$$

$$\mathcal{L}(b) = b(1-b)$$

The likelihood for the bias being 1 is 0.

Likelihoods of Models for Coins



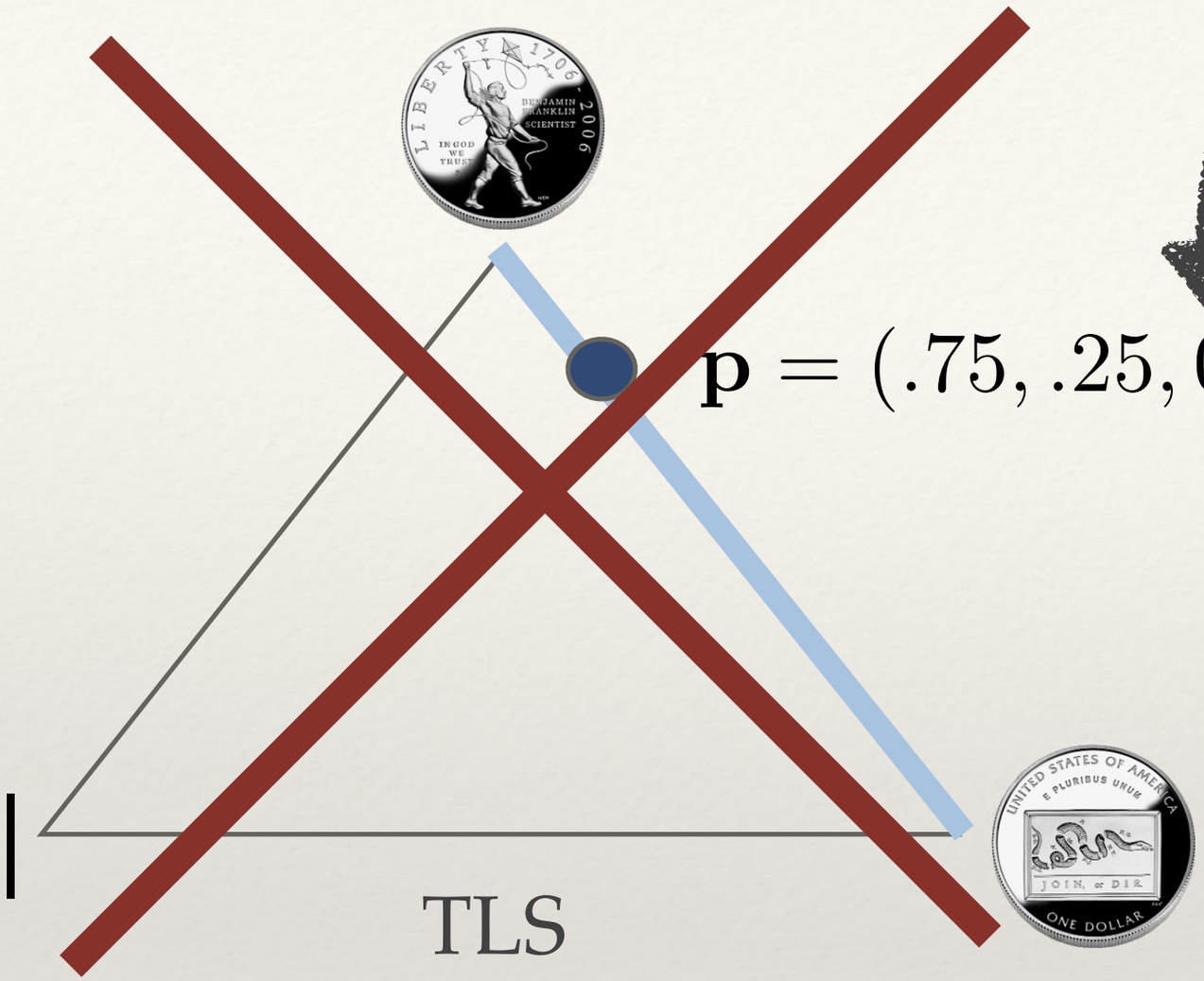


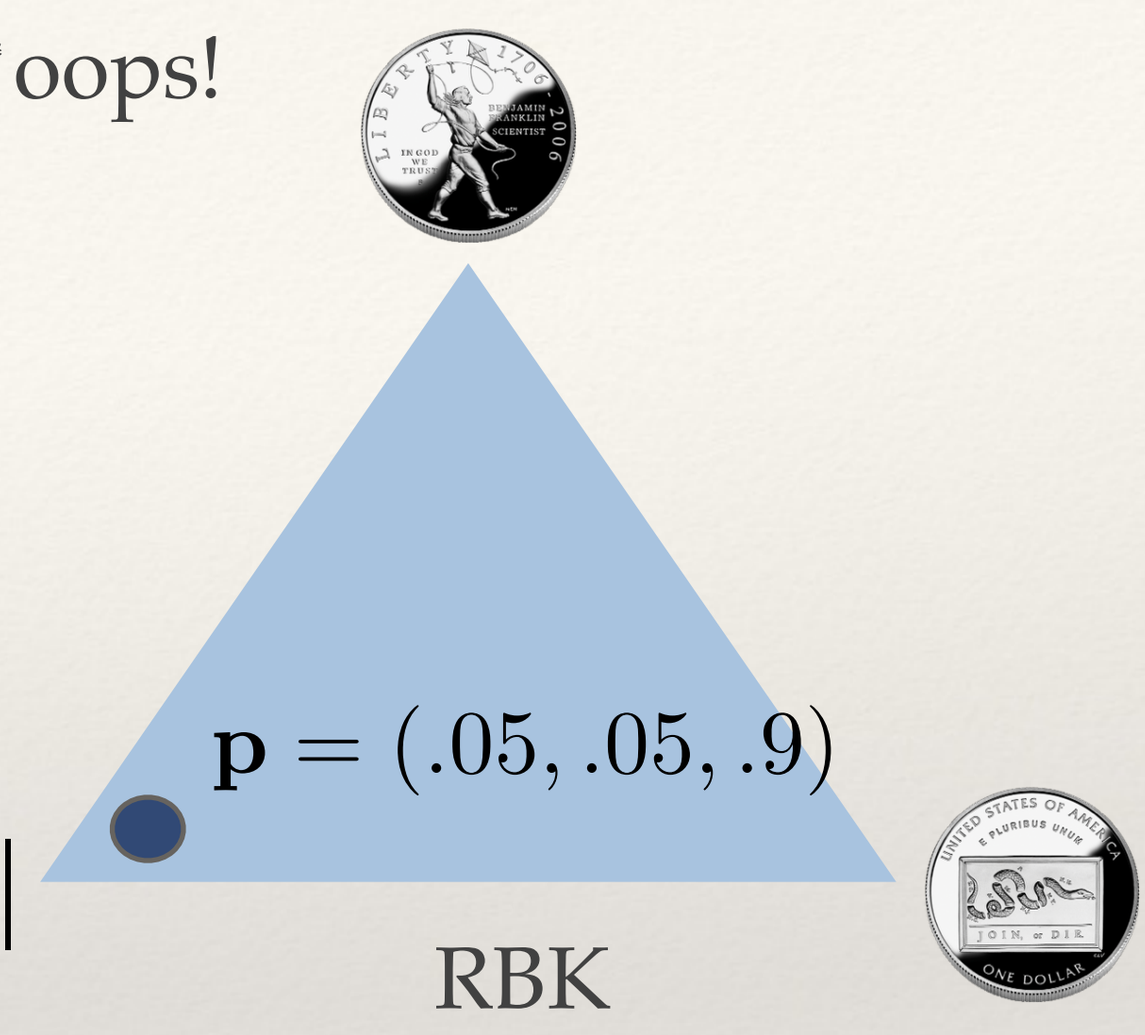
$$\mathbf{p} = (\text{heads}, \text{tails}, \text{edge})$$

Model = description of possible outcomes

Model = probability *simplex*

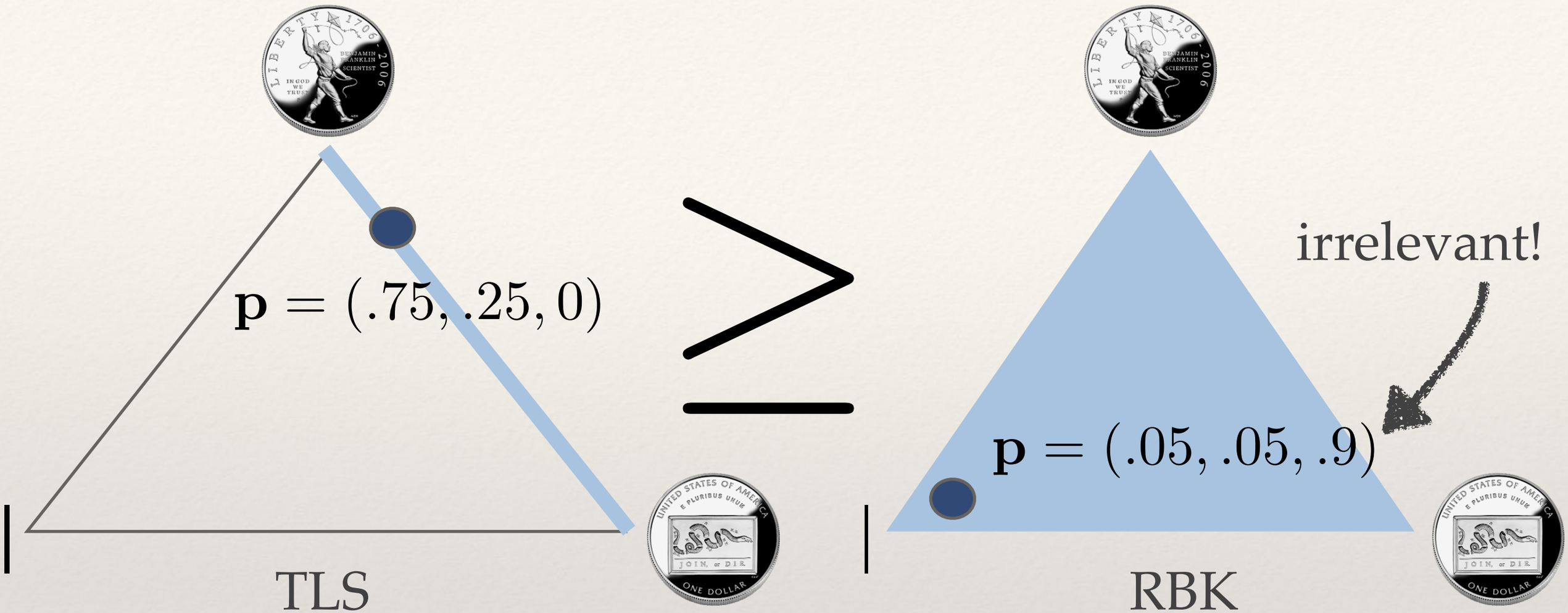
Likelihoods of Models for Coins





If **any edge shows up**, TLS is wrong - assigned 0 probability to that outcome!

Likelihoods of Models for Coins



If **no edge shows up**, RBK can do no better than TLS
- has extra outcome to account for

Can we quantify who has a better model?

We compare models with loglikelihood ratio statistics (LLRS).

Compare models = compare *maximum* likelihoods

Whose model assigns higher probability to data we saw?

Quantify *weight of evidence* for one model or another.

$$\lambda = -2 \log \left(\frac{\max\limits_{\mathbf{p} \in TLS} \mathcal{L}(\mathbf{p})}{\max\limits_{\mathbf{p} \in RBK} \mathcal{L}(\mathbf{p})} \right)$$

We compare models with loglikelihood ratio statistics (LLRS).

Quantify *weight of evidence* for one model or another.

$$\lambda = -2\log\left(\frac{\max\limits_{\mathbf{p}\in TLS}\mathcal{L}(\mathbf{p})}{\max\limits_{\mathbf{p}\in RBK}\mathcal{L}(\mathbf{p})}\right)$$

Any edge: $\lambda = -2\log(0) = \infty$ (Go with RBK)

No edge: $\lambda = -2\log(1) = 0$ (Go with TLS)

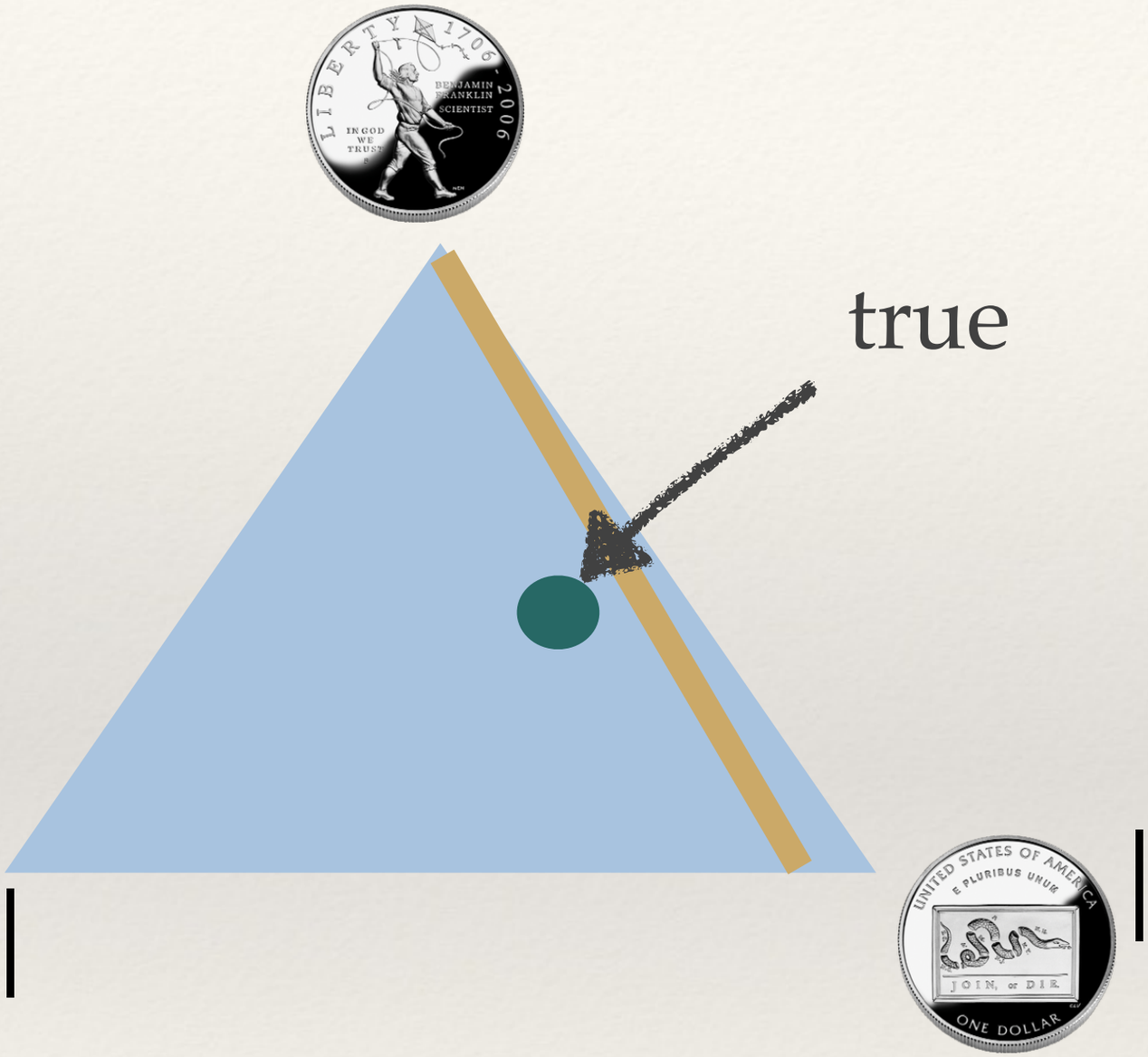
General Framework/Important Results

$$\lambda = -2 \log \left(\frac{\max\limits_{\mathbf{p} \in 1} \mathcal{L}(\mathbf{p})}{\max\limits_{\mathbf{p} \in 2} \mathcal{L}(\mathbf{p})} \right)$$

Quantify *weight of evidence* for one model or another.

LLRS λ is a random variable;
in some cases, its distribution $p(\lambda)$
can be computed.

Two General Cases….

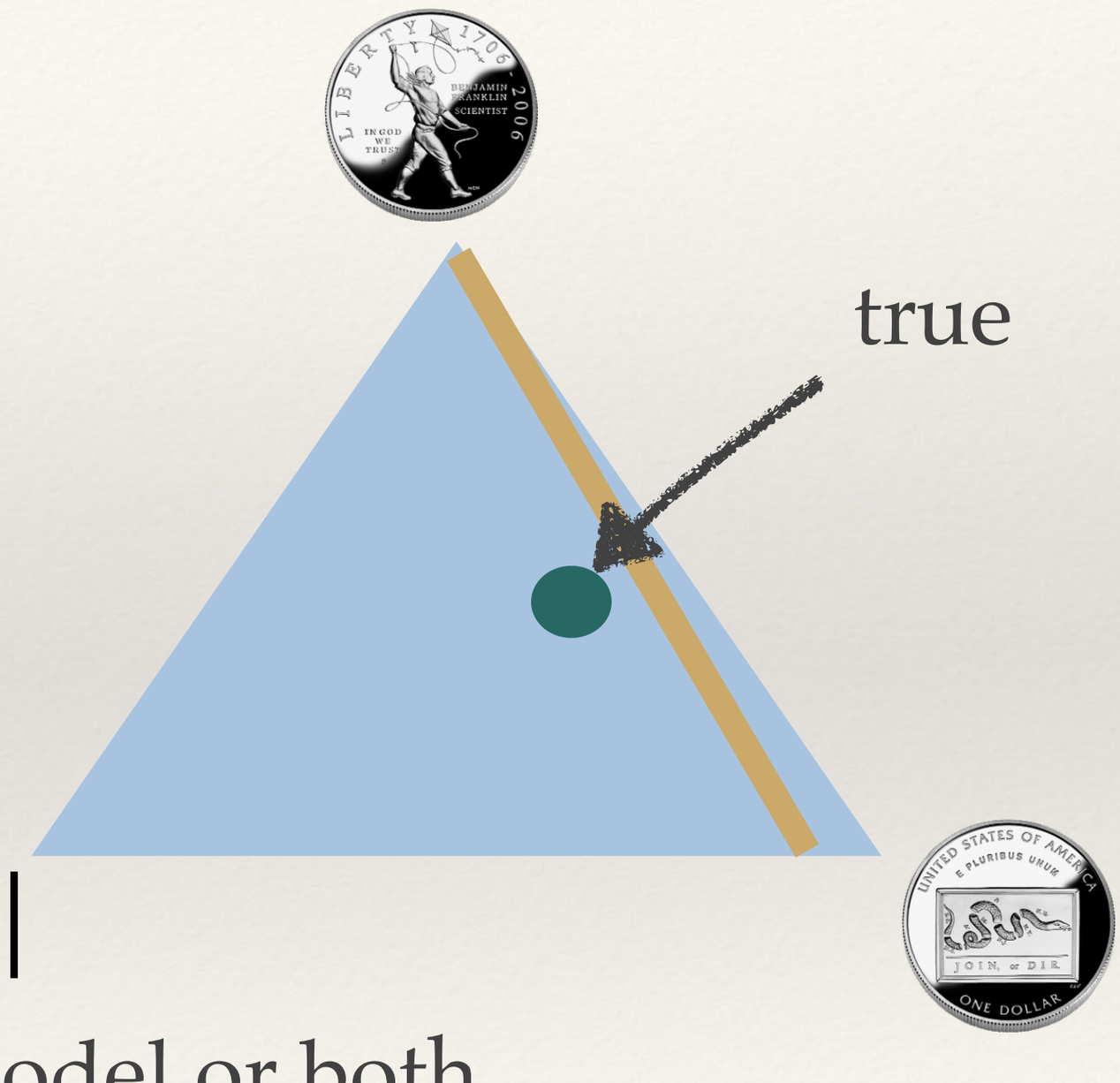


Where is the truth?

Case 1

Model Mismatch:

$$\lambda \propto N$$

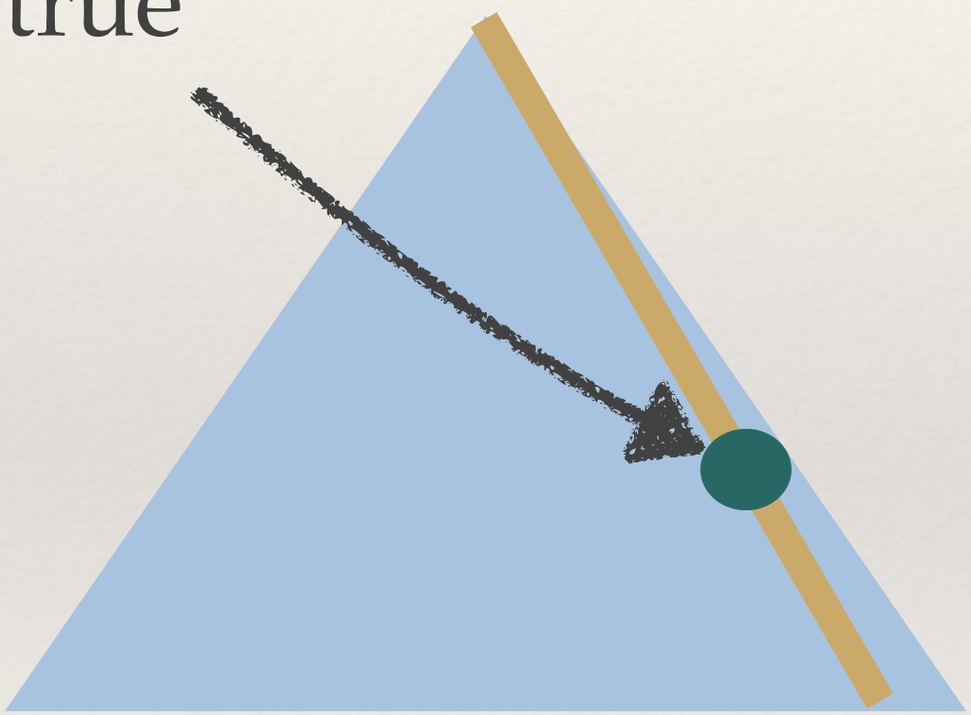


When truth outside one model or both
LLRS grows with sample size

Case 2

The Wilks theorem: $\lambda \sim \chi^2_{k_2-k_1}$

true

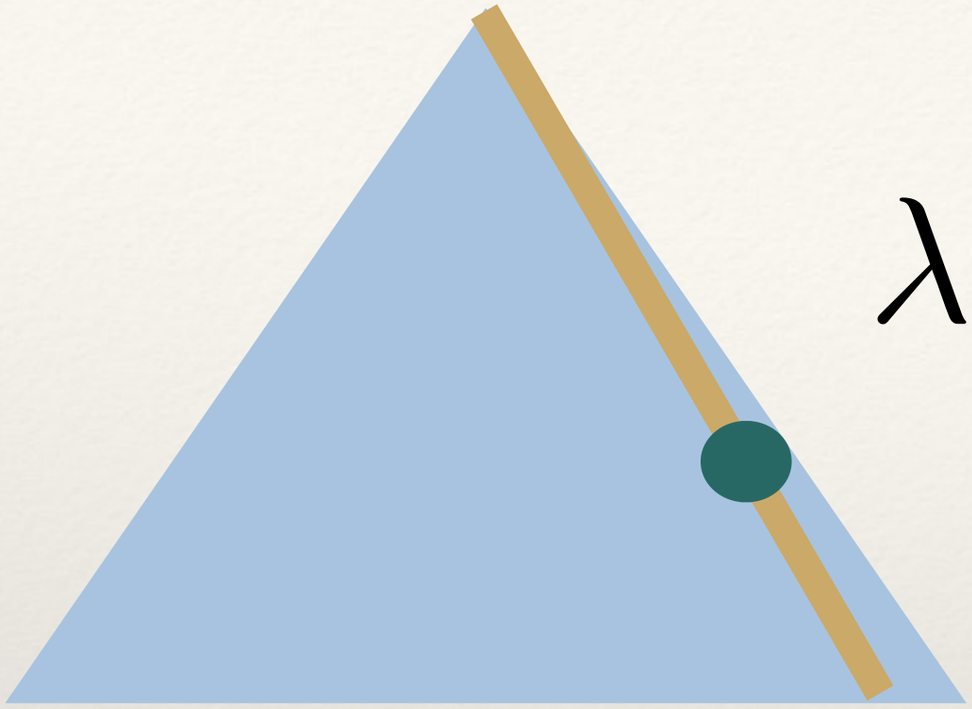


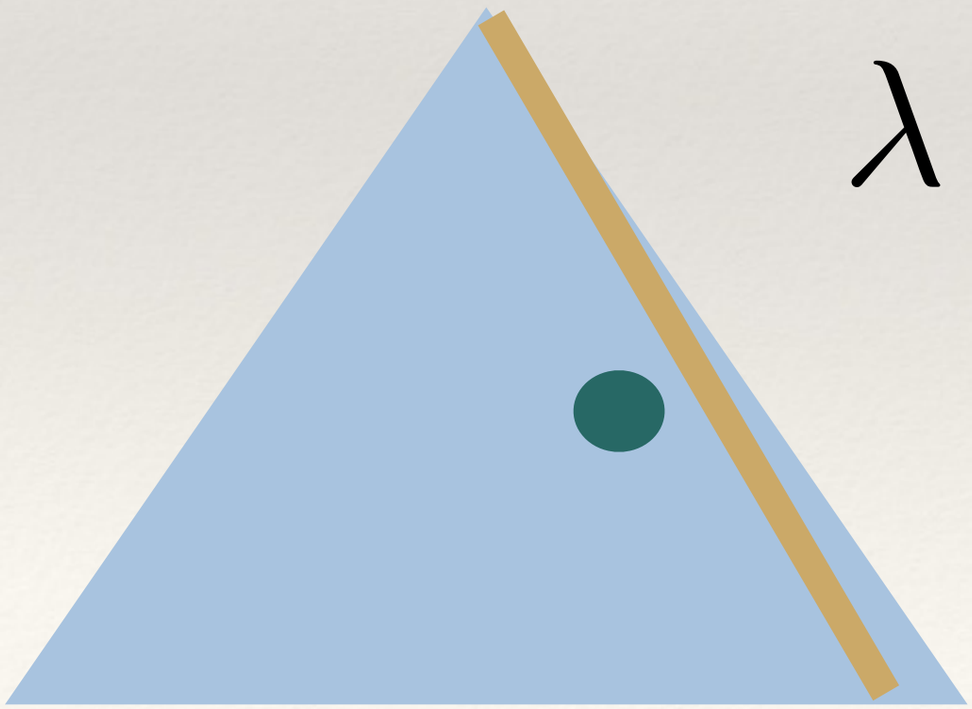
$$\langle \lambda \rangle = k_2 - k_1$$
$$(\Delta\lambda)^2 = 2(k_2 - k_1)$$

k_j is number of parameters in model j

When truth inside smaller model
LLRS has chi-squared distribution

Behavior of LLRS





To Sum Up....

Need tool for dimension decisions

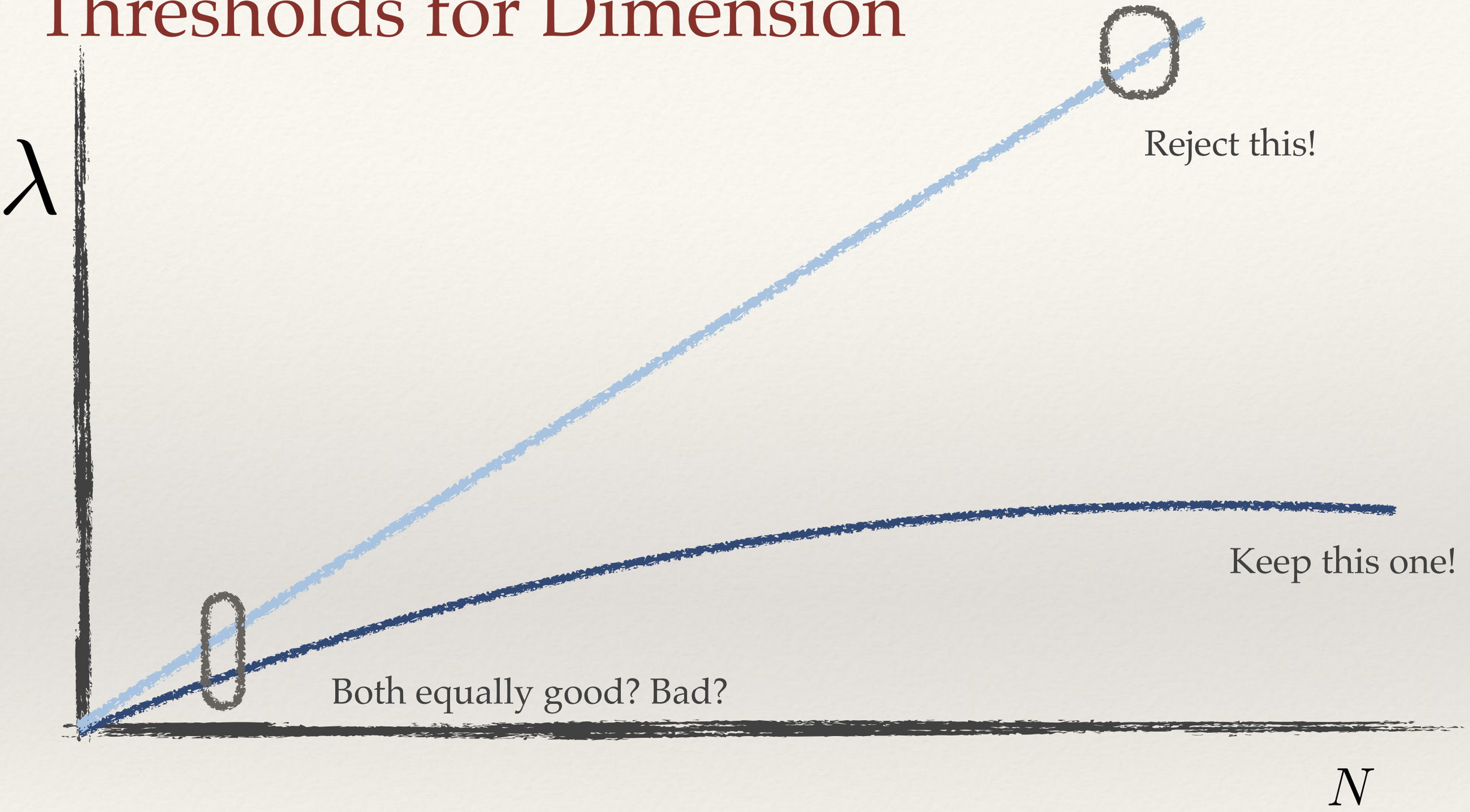
Hypothesis testing provides a framework via LLRS λ

Hypotheses: state is **d** or **D** dimensional

Project Goal: Devise, Use, and Evaluate a rule based on LLRS

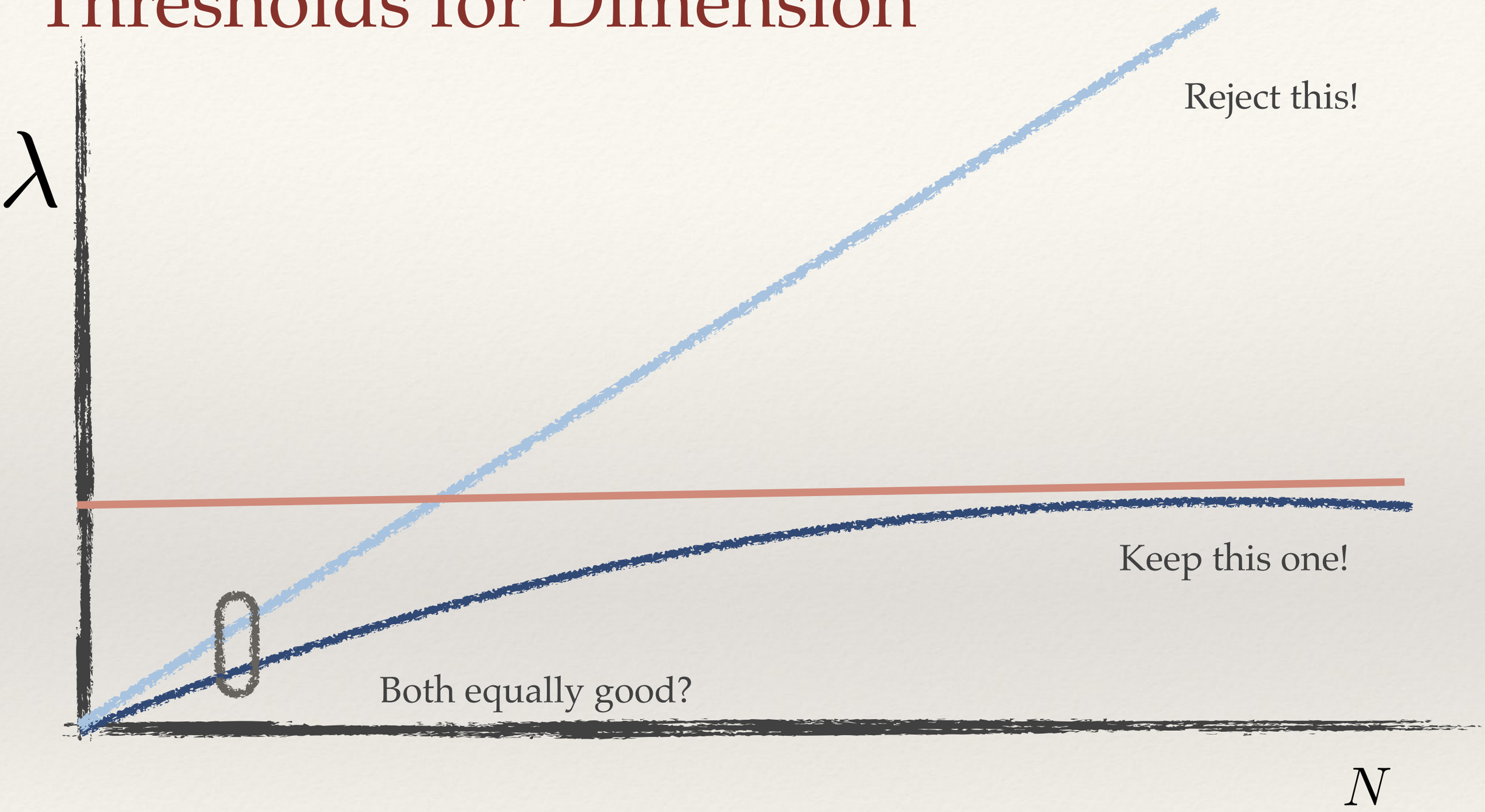
Ideas and Results

Using Hypothesis Testing to Establish Thresholds for Dimension



Can we use pattern matching (“just plot it!”) and achieve the same result?

Using Hypothesis Testing to Establish Thresholds for Dimension



Can we compute some threshold value of LLRS and compare data against it?

Testing the Rule

Pick a quantum system ⟶ optical modes

Pick some (fiducial) quantum state ρ_{true}

Simulate measurements ⟶ coherent state projection $|\alpha\rangle\langle\alpha|$

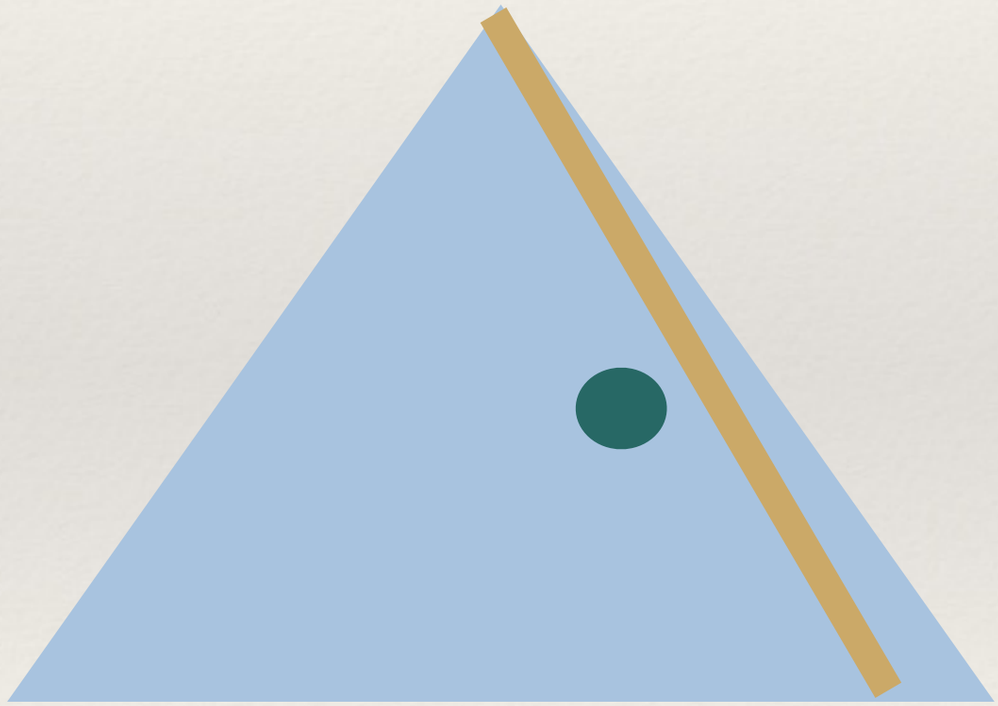
Estimate (do MLE) in many dimensions

Compute LLRS λ

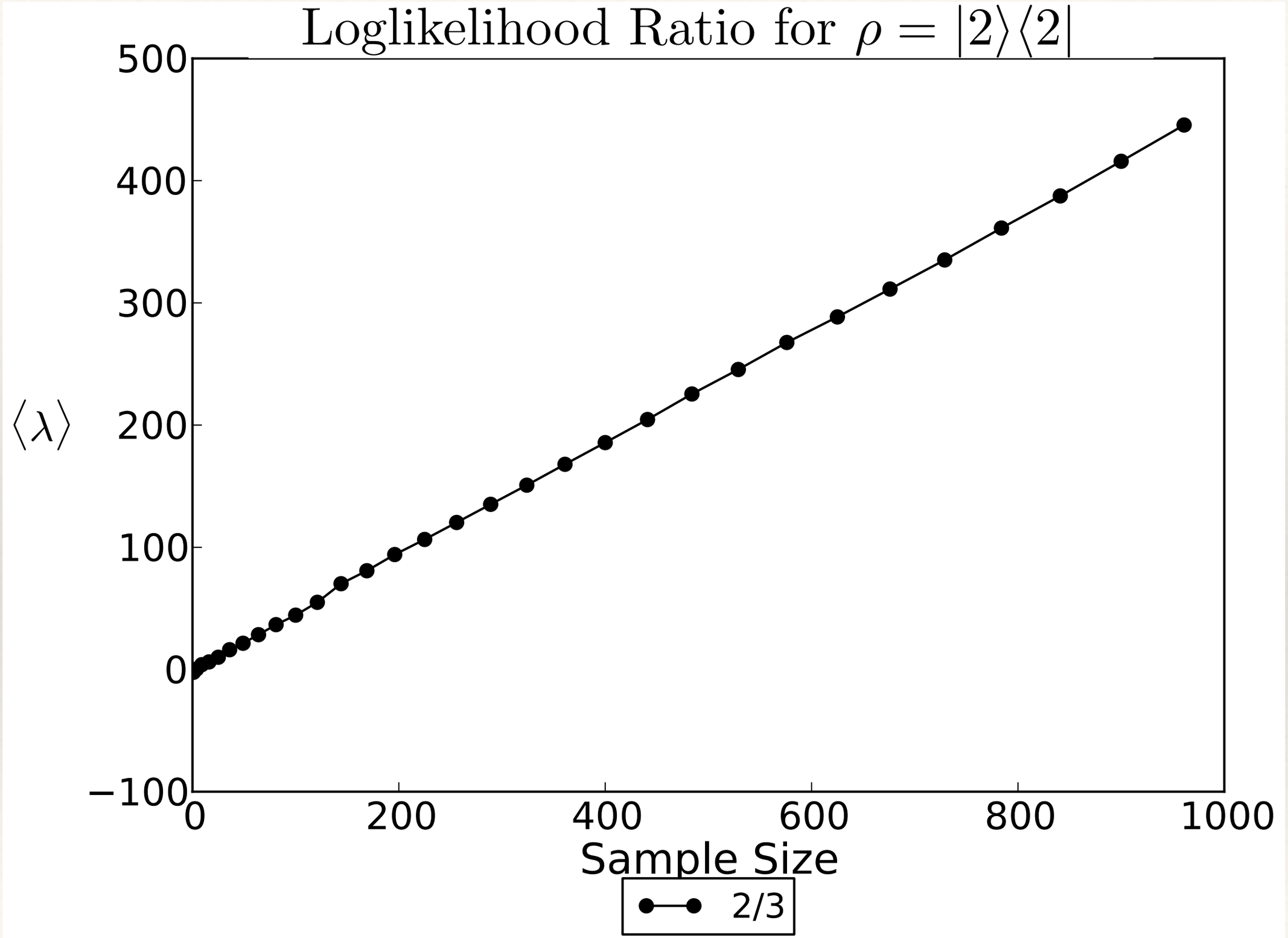
Examine relationship between
LLRS and sample size

Case 1:

Larger model contains true state; smaller model does not

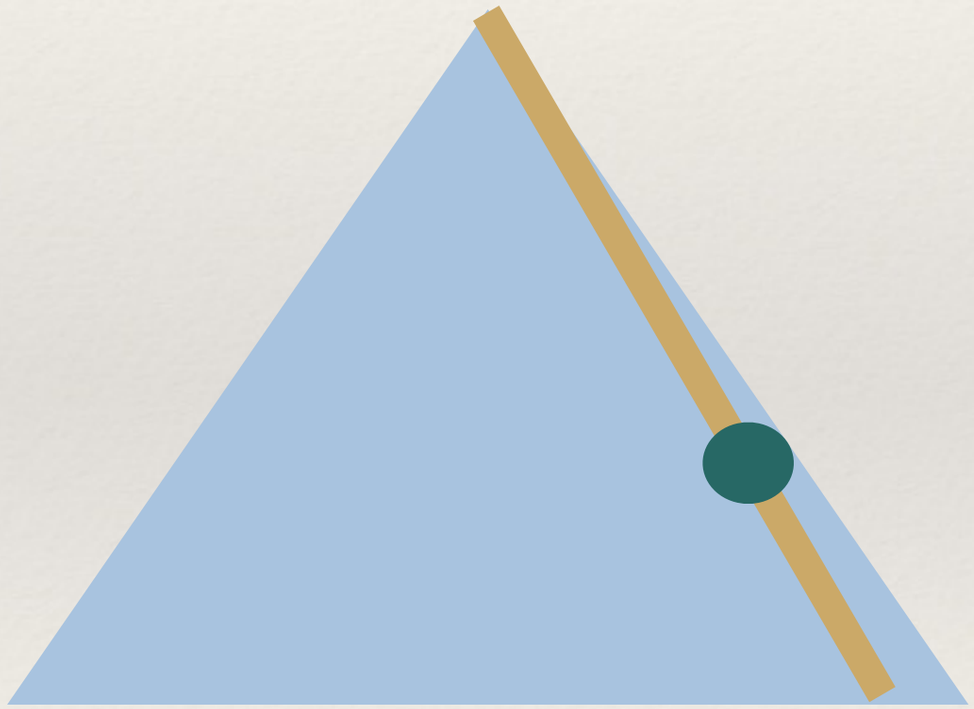


Linear Growth Observed

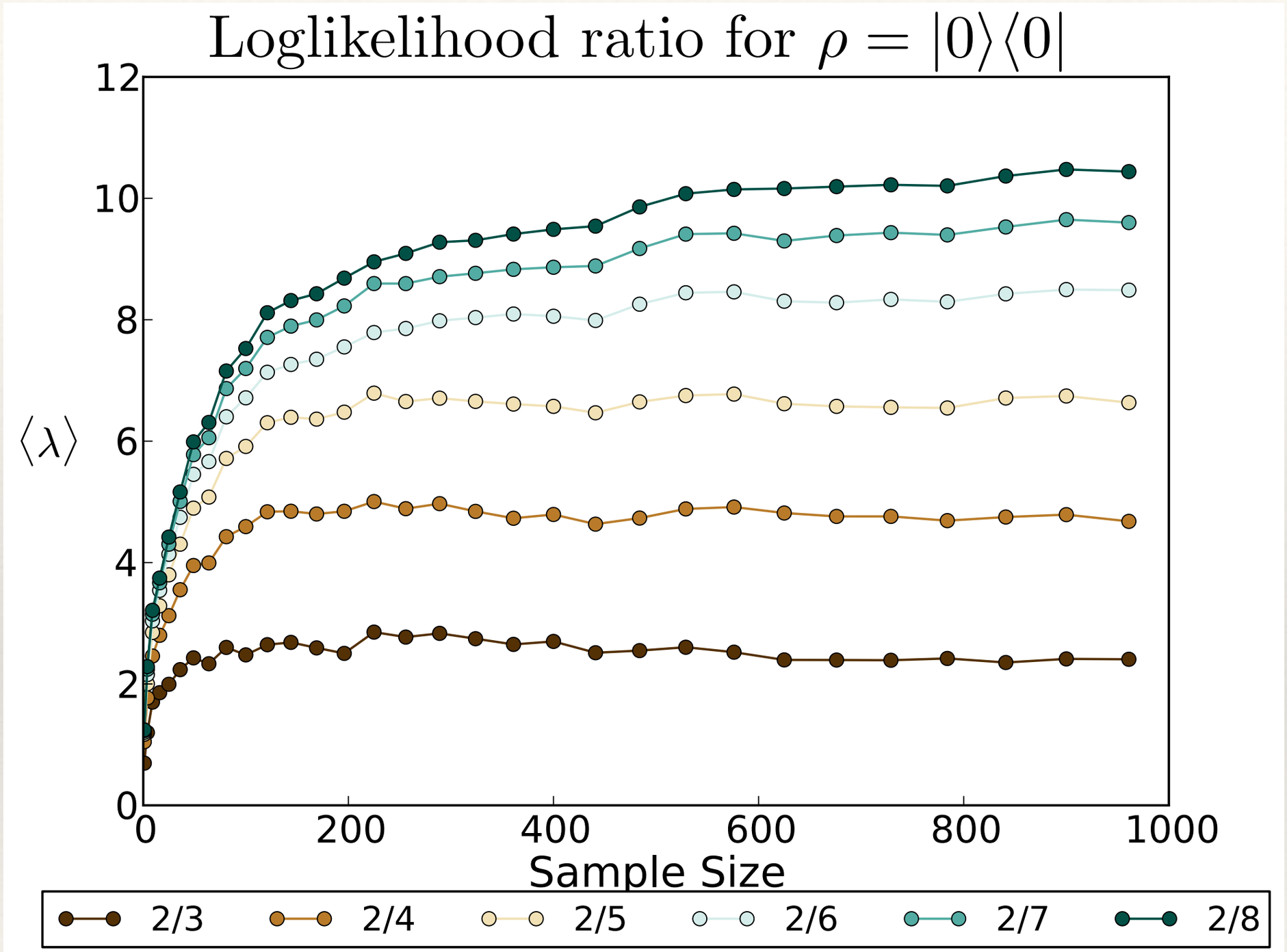


Case 2:

Smaller model contains true state

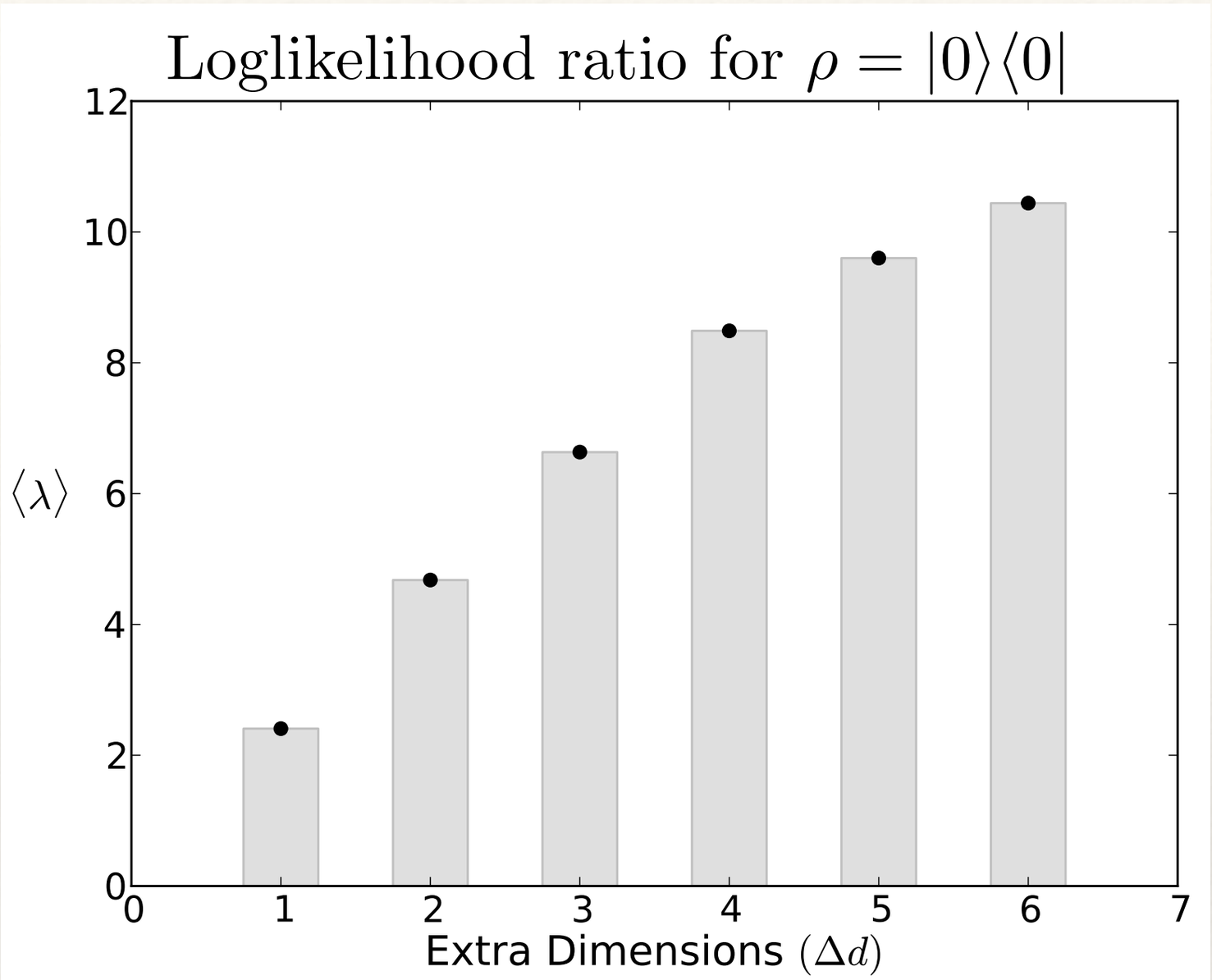


Asymptotic Convergence?



$$n/m = -2\log\left(\frac{\max\limits_{\mathbf{p}\in n}\mathcal{L}(\mathbf{p})}{\max\limits_{\mathbf{p}\in m}\mathcal{L}(\mathbf{p})}\right)$$

Does the Wilks theorem work?

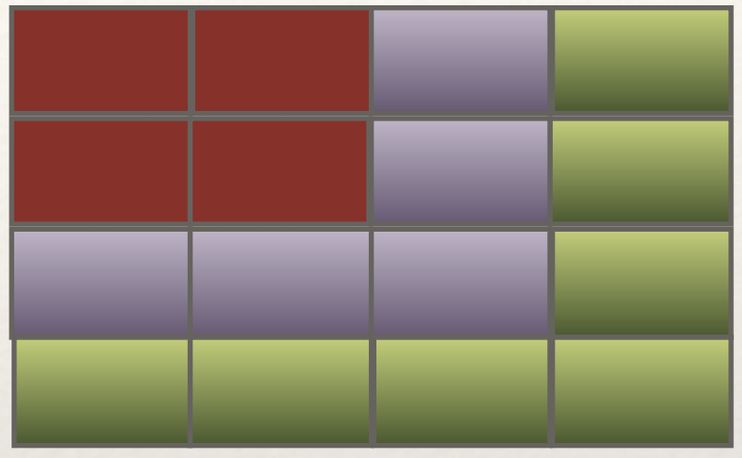


Observed: $\langle\lambda\rangle \propto (\Delta d)$

Expected: $\langle\lambda\rangle = (\Delta d)^2 + 2d_1(\Delta d)$

Mixed State Estimates

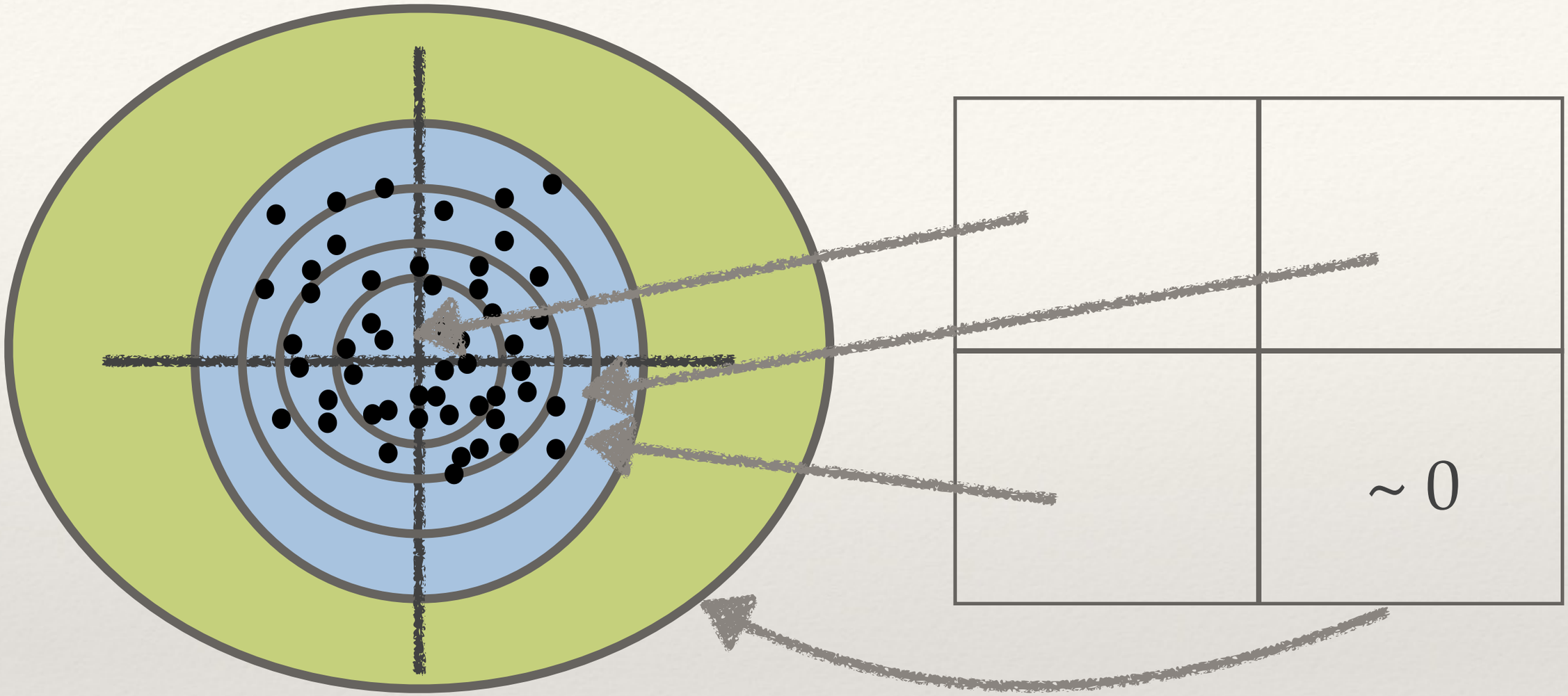
$\rho \sim$



$$k_2 - k_1 = d_2^2 - d_1^2$$

The Wilks theorem predicts values which are way too high?

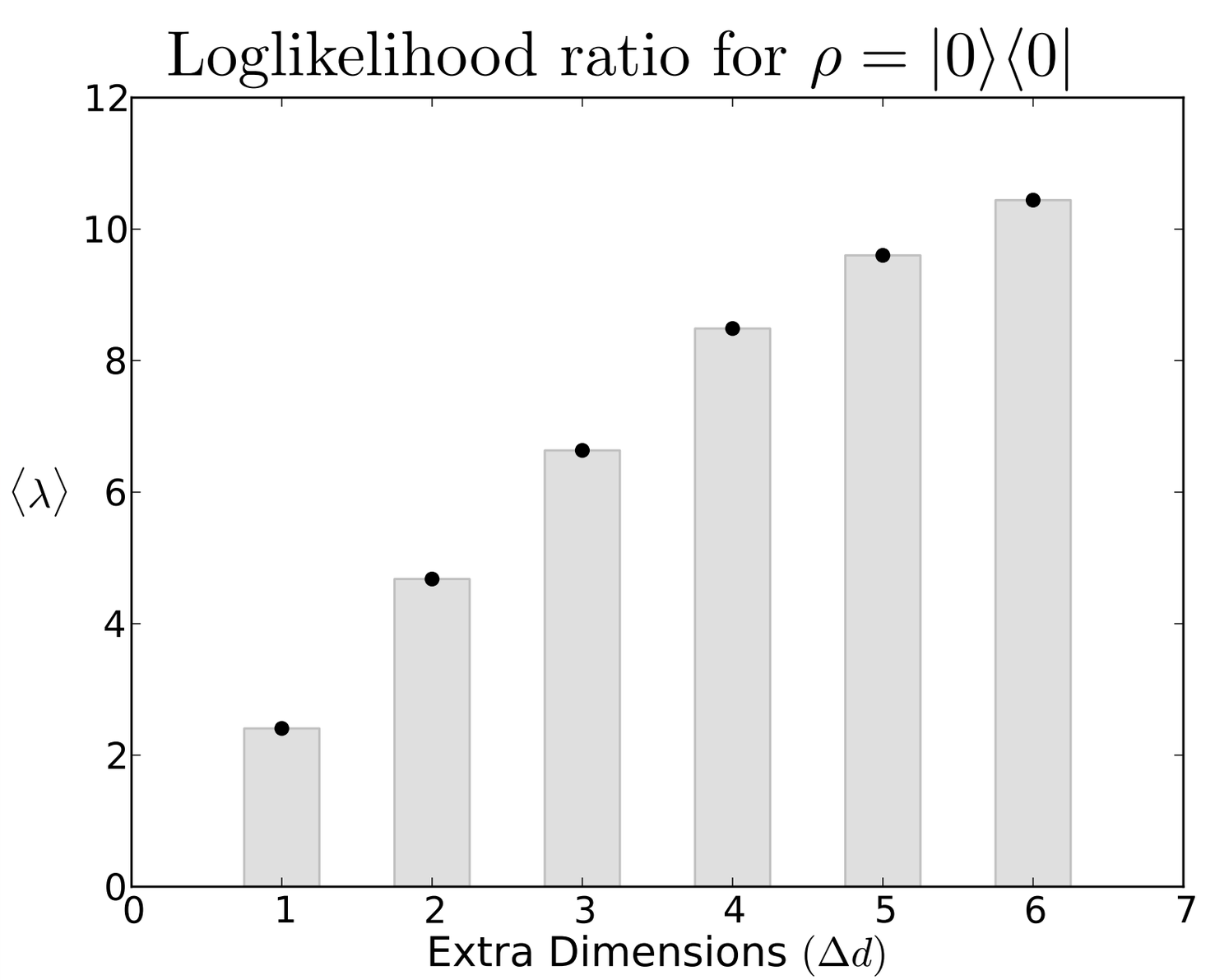
Does the Wilks theorem work?



Too high energy Fock state = contributes 0 to density matrix

Only “coherencies” with smaller model are retained?

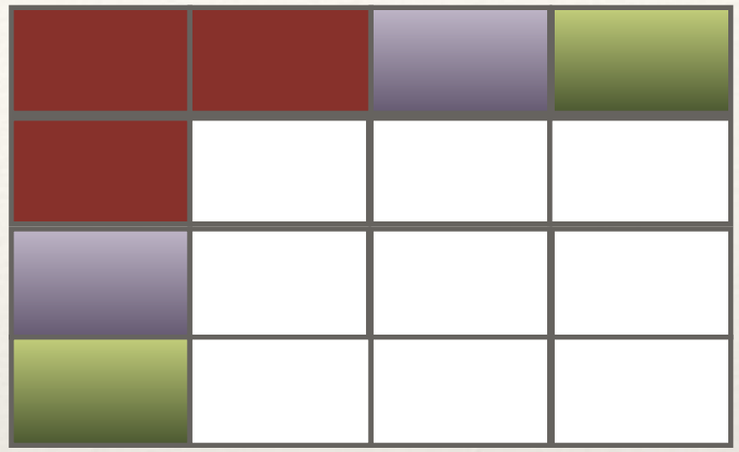
Does the Wilks theorem work?



Observed: $\langle\lambda\rangle \propto (\Delta d)$

Expected: $\langle\lambda\rangle \propto (\Delta d)$

$\rho \sim$

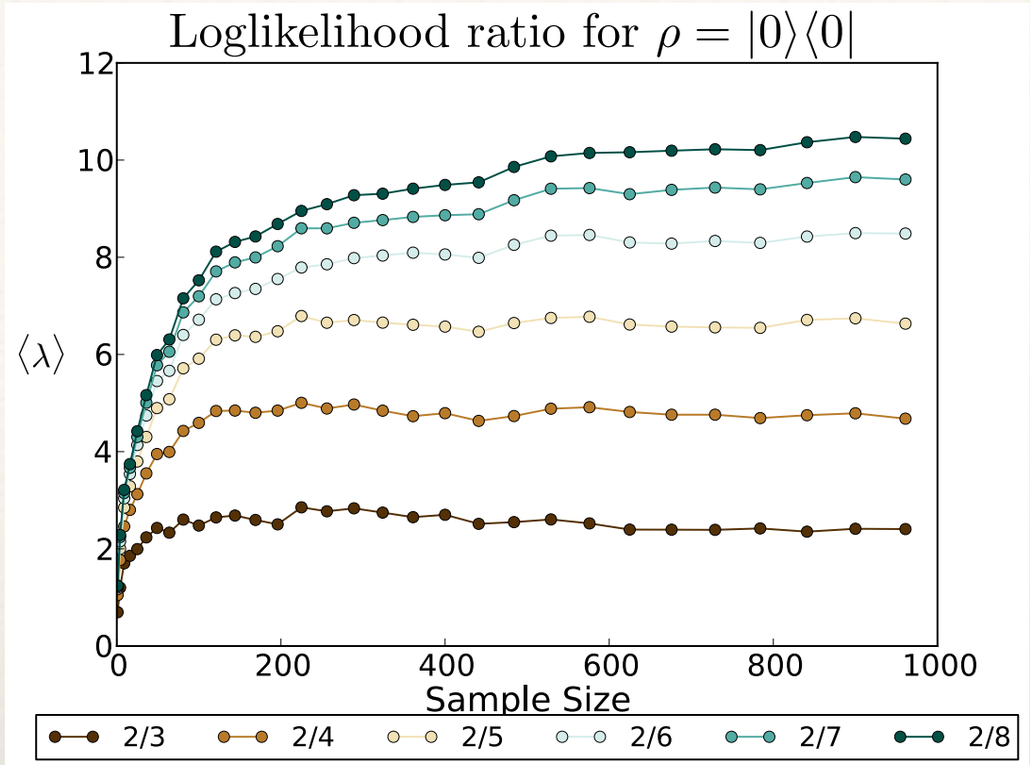


$$k_2 - k_1 = 2(d_2 - d_1)$$

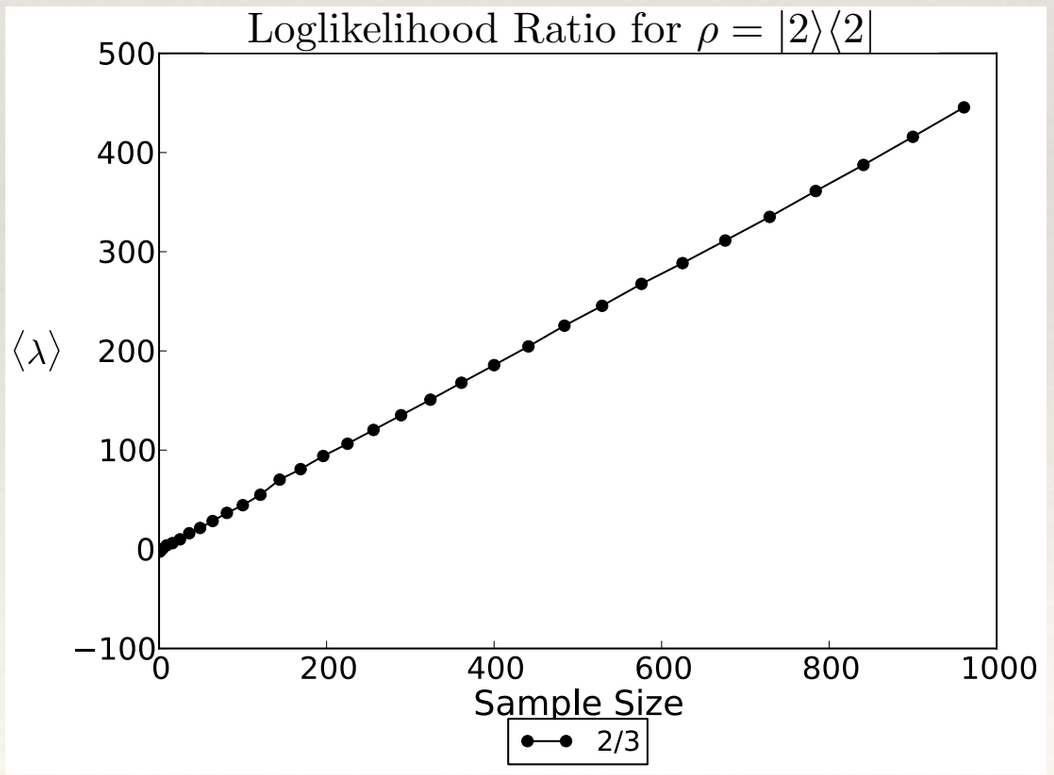
An effect of reconstruction? Of true state?

Observations

Asymptotic convergence observed, but *not* as expected from naive application of the Wilks theorem



When smaller model does not fit, we observe linear growth



The Road Ahead

- Develop a general theory for LLRS and tomography
- Use information criteria instead?
- "Real life" use?

	Right Now	Future
	Hypothesis Testing	Information Criteria
Tries to Fit	Past Data	Future Data
True Model?	Yes	No
Arbitrary Complexity?	Possible	Not Usually

I have a question!

Take Away

- Determining system dimension is a big deal.
- Practical use for diagnosing errors/ couplings
- LLRS is a way to go
- Develop a theory for LLRS & quantum tomography
- Use information criteria?

Supplemental Material

We think a lot about Hilbert space dimension…

PHYSICAL REVIEW A **66**, 012303 (2002)

Qudit quantum-state tomography

PHYSICAL REVIEW A **69**, 042108 (2004)

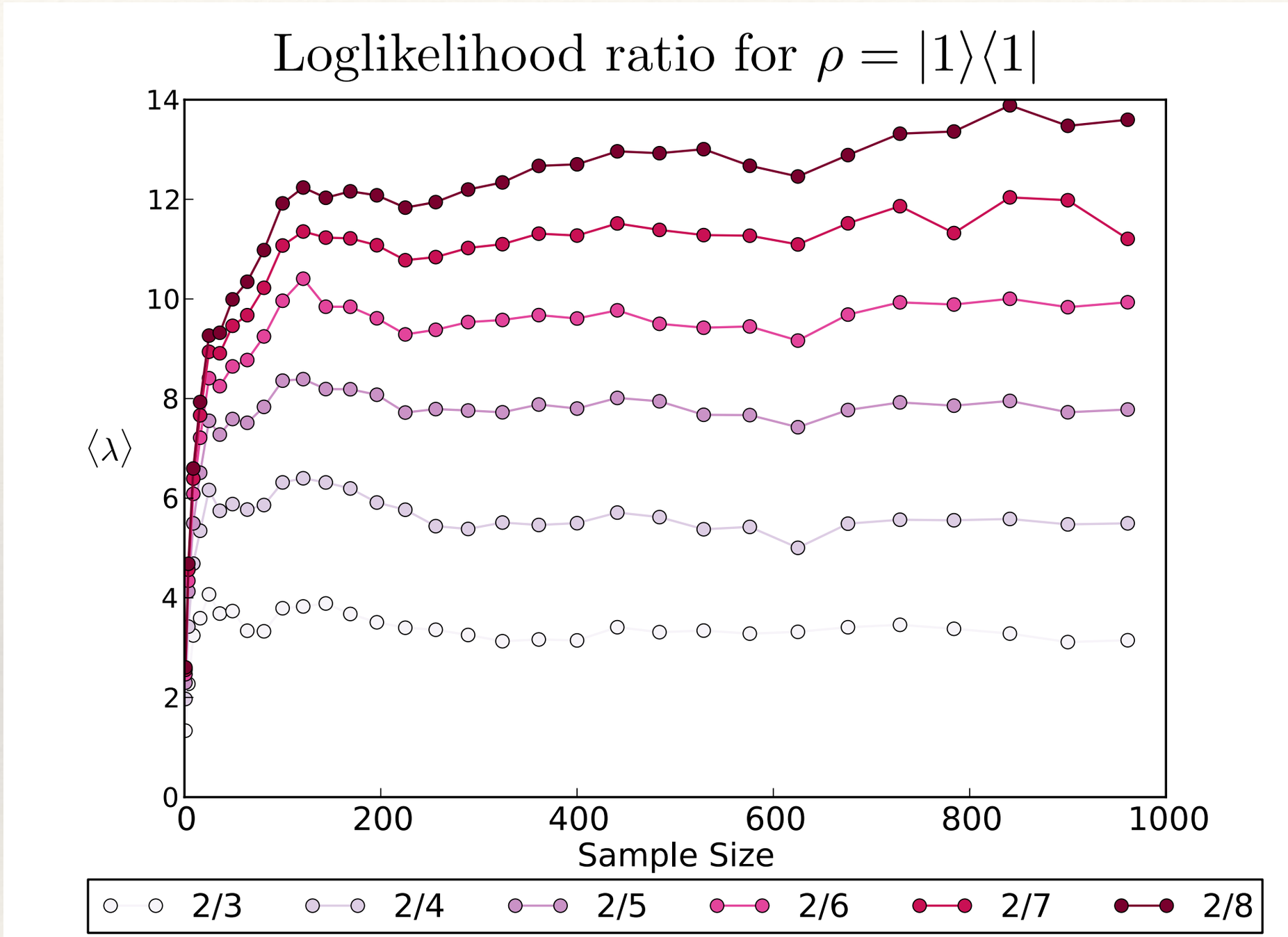
Quantum-state tomography for spin-*l* systems

PHYSICAL REVIEW A **84**, 062101 (2011)

Tomography of the quantum state of photons entangled in high dimensions

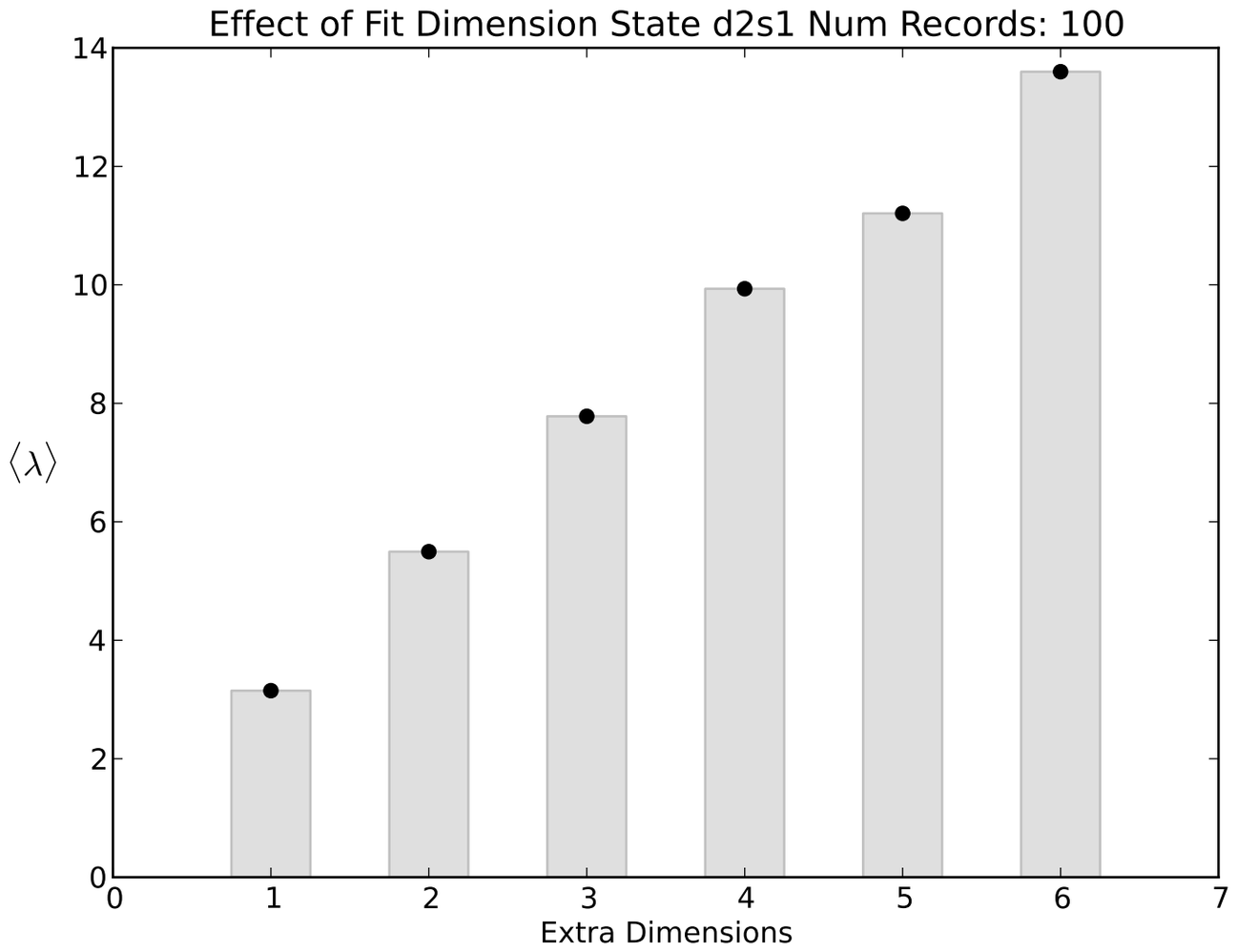
…but often assume we know what it is!

Asymptotic Convergence?

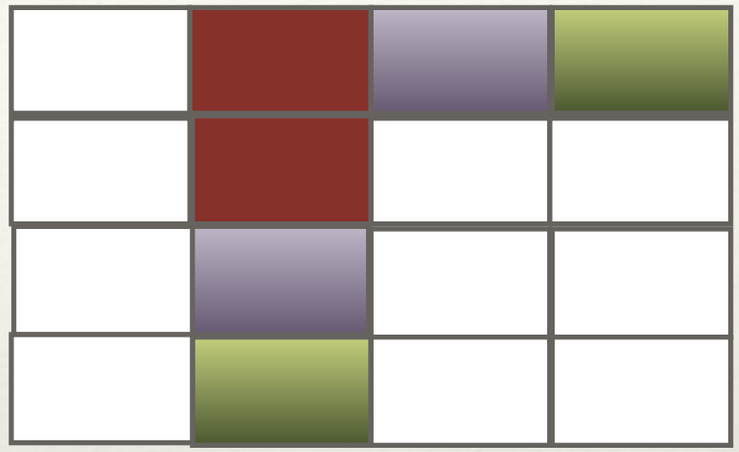


$$n/m = -2\log\left(\frac{\max\limits_{\mathbf{p}\in n}\mathcal{L}(\mathbf{p})}{\max\limits_{\mathbf{p}\in m}\mathcal{L}(\mathbf{p})}\right)$$

Does the Wilks theorem work?



$\rho \sim$



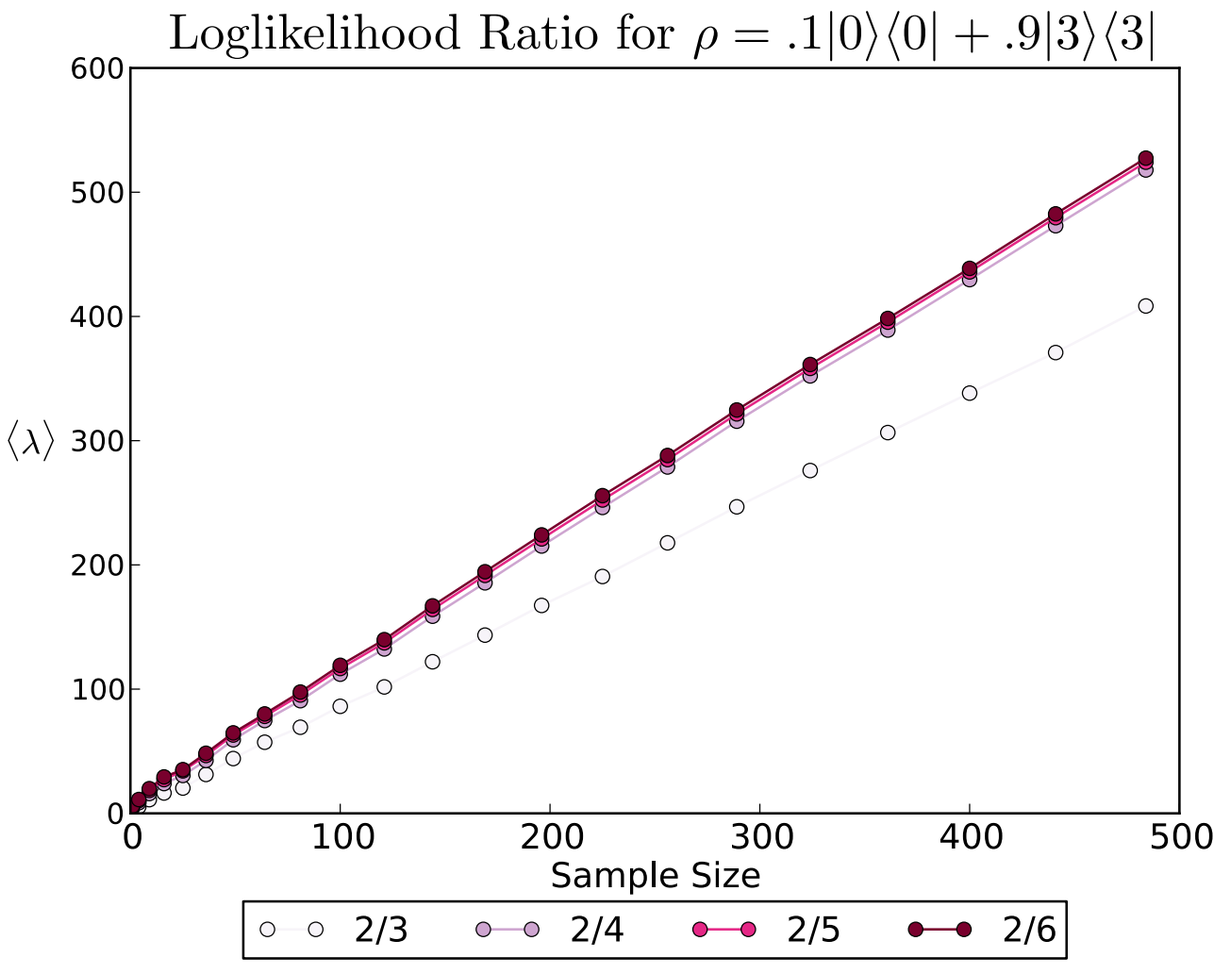
"Pure" State Estimates

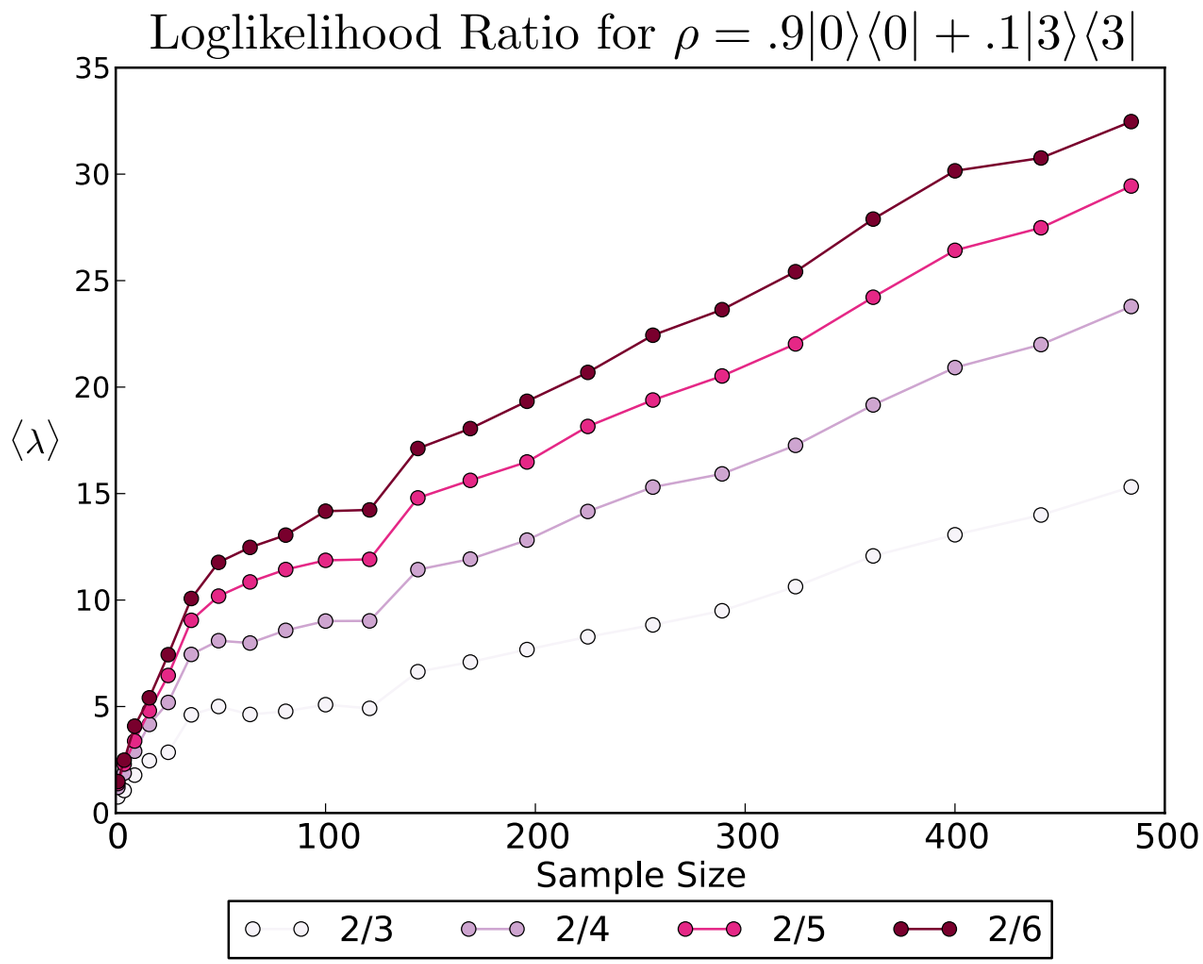
$$k_2 - k_1 = 2(d_2 - d_1)$$

Observed: $\langle\lambda\rangle \propto (\Delta d)$

Expected: $\langle\lambda\rangle \propto (\Delta d)$

An effect of reconstruction? Of true state?


Loglikelihood Ratio for $\rho = .1|0\rangle\langle 0| + .9|3\rangle\langle 3|$
$\langle\lambda\rangle$
Sample Size
0
100
200
300
400
500
600
2/3
2/4
2/5
2/6


Loglikelihood Ratio for $\rho = .9|0\rangle\langle 0| + .1|3\rangle\langle 3|$
$\langle\lambda\rangle$
Sample Size
0
5
10
15
20
25
30
35
100
200
300
400
500
2/3
2/4
2/5
2/6

Technical Aside

- Simple hypotheses = Neyman-Pearson
  - Guaranteed best statistic is likelihood ratio
- Composite hypotheses = NO Neyman-Pearson
  - A different idea of best test wrt power
  - Allows nesting Hilbert space dimensions

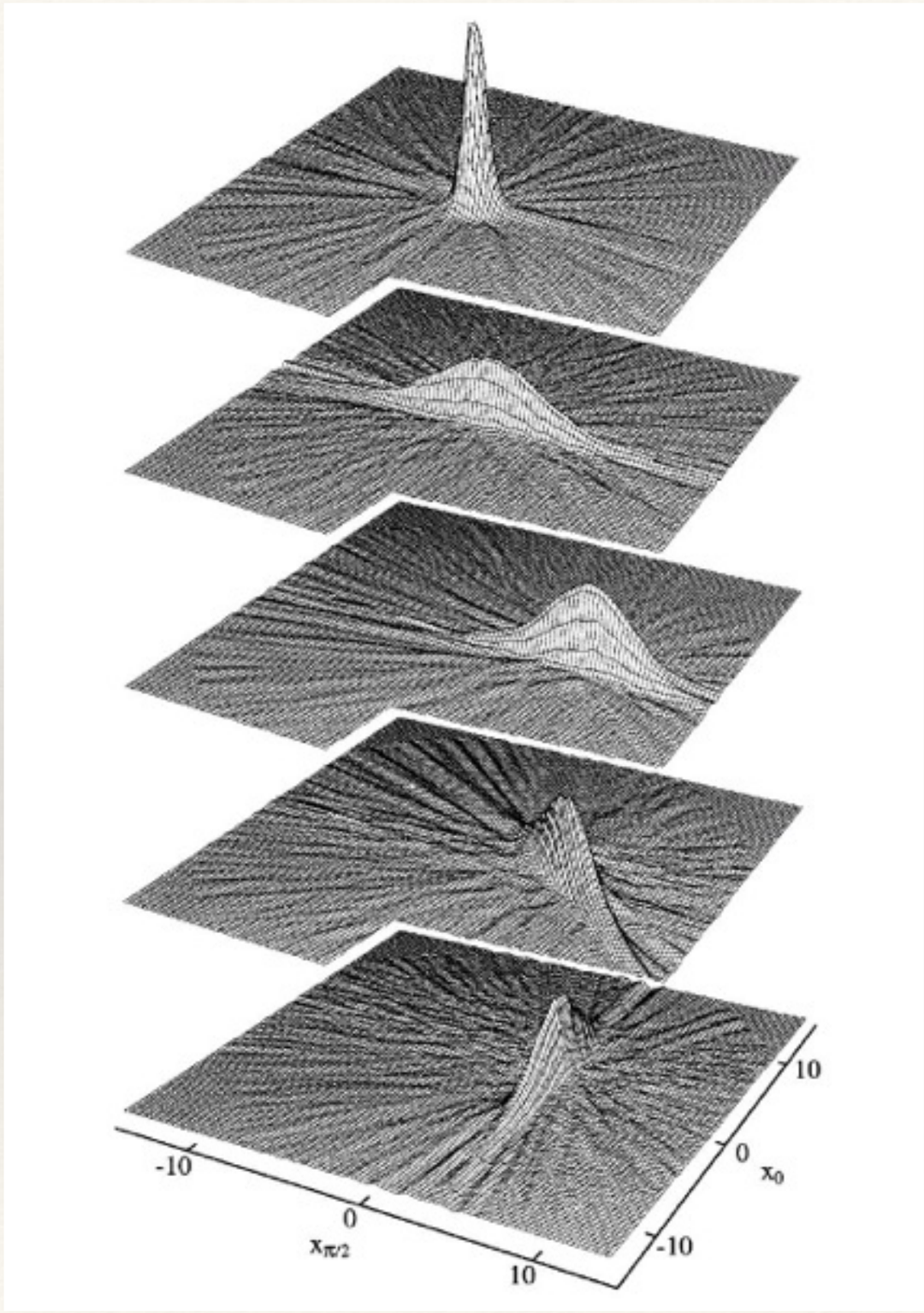
Wigner Function Reconstruction

Wigner function = representation of quantum state in phase space

Make plots to visualize states

Are the wiggles real?

Can we even tell?



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Categories of POVMs	POVM finite dimensional	POVM infinite dimensional
Finite Number of Outcomes	Pauli eigenbasis measurement	Parity measurement
Countably Infinite Outcomes		Photon number counting
Uncountably Infinite Outcomes	Haar uniform qudit projection	Heterodyne / Homodyne / Coherent state projections