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Model realization and model reduction for quantum systems

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U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

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Outline & Acknowledgements

- **Model realization and system identification**
  - Estimating unknown Hamiltonian parameters

Quantum Hamiltonian identification from measurement time traces
Jun Zhang, MS
arXiv:1401.5780, *Phys. Rev. Lett.* 113, 080401 (2014)

Jun Zhang
Shanghai Jiao Tong University

- **Model reduction**
  - Reducing simulation cost for certain many-body quantum systems

On model reduction for quantum dynamics: symmetries and invariant subspaces
Akshat Kumar, MS
arXiv:1406.7069

Akshat Kumar
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System Identification

Identify system from input-output behavior

e.g. Process tomography: identify process (CP-map, unitary) at a particular time

Alternative: **identify generator (*e.g.* Hamiltonian) of system**

Advantages: physical constraints (locality, connectivity) more naturally introduced intro generator of dynamics

e.g. n qubits: arbitrary unitary parameterized by 4^n-1 real parameters, arbitrary 2-local Hamiltonian parameterized by 3n+9n(n-1)/2 real parameters

System Identification

How powerful are time traces?

Additional considerations

- Measurements could be restricted
- May have partial information about system

Assumptions:

1. system is finite dimensional
2. Hamiltonian dynamics (closed system)

See also:

Cole, J. H., Schirmer, S. G., Greentree, A. D., Wellard, C. J., Oi, D. K. L., & Hollenberg, L. C. L. *Phys. Rev. A*, *71*, 062312 (2005)

Burgarth, D., Maruyama, K., & Nori, F. *Phys. Rev. A*, *79*, 020305 (2009)

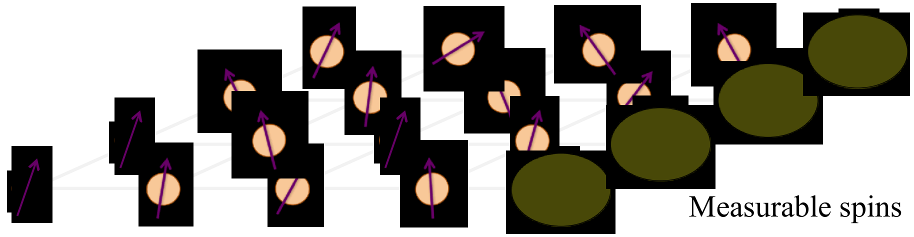
Burgarth, D., & Maruyama, K. *New J. Phys.*, *11*, 103019 (2009)

Di Franco, C., Paternostro, M., & Kim, M. *Phys. Rev. Lett.*, *102*, 187203

Burgarth, D., & Yuasa, K. *Phys. Rev. Lett.*, *108*, 080502 (2012)

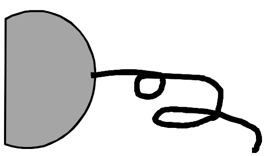
Granade, C. E., Ferrie, C., Wiebe, N., & Cory, D. G. *New J. Phys.*, *14*(10), 103013 (2012)

An example



Parametric Hamiltonian

$$H(\theta_1, \theta_2, \ldots, \theta_M)$$



$$\langle \sigma_z^1(t_0) \rangle, \langle \sigma_z^1(t_1) \rangle, \ldots, \langle \sigma_z^1(t_n) \rangle$$

Time trace of some accessible observable

Can we identify the parameters in the Hamiltonian from just this?

The setup

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Choose an orthogonal operator basis for the linear operator space (e.g. generalized Paulis)

$$[iX_j, iX_k] = \sum_{l=1}^{N^2-1} C_{jkl}(iX_l), \quad j,k = 1,\cdots,N^2-1,$$

Hamiltonian can be expanded in this basis

$$H = \sum_{m=1}^{M} a_m(\theta) X_m$$

Goal: to identify a_m

Leads to a linear, autonomous equation for state x(t)

$$\frac{\mathrm{d}}{\mathrm{d}t} x_k = \sum_{l=1}^{N^2-1} \left(\sum_{m=1}^{M} C_{mkl} a_m \right) x_l$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x}, \quad x_k(t) = \langle \psi(t)| X_k |\psi(t)\rangle$$

$$|\psi\rangle \in \mathbb{C}^N$$
$$\dim H = N \times N$$
$$\mathbf{x} \in \mathbb{R}^{(N^2-1)}$$
$$\dim A = (N^2-1) \times (N^2-1)$$

The setup

Similarly, each directly measured observable can be expanded in the same basis. Resulting in a AC system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

But this may be too complex a description. *E.g.*

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{bmatrix}$$

Filtration to find minimal description

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$$O_i = \sum_j o_j^{(i)} X_j \qquad \mathcal{M} = \{X_{\nu_1}, X_{\nu_2}, \ldots, X_{\nu_p}\}$$

Directly measured set

$$H = \sum_{m=1}^{M} a_m(\theta) X_m \qquad \Delta = \{X_m\}_{m=1}^{M}$$

Hamiltonian set

Filtration recursively constructed as:

$$G_0 = \mathcal{M}, \text{ and}$$

$$G_i = [G_{i-1}, \Delta] \, \cup \, G_{i-1}$$

where

$$[G_{i-1}, \Delta] \equiv \{X_j : \mathrm{tr}\,(X_j^\dagger [g, h]) \neq 0, \text{ where } g \in G_{i-1}, h \in \Delta\}$$

Finite algebra => procedure terminates, resulting in filtration $\bar{G}$

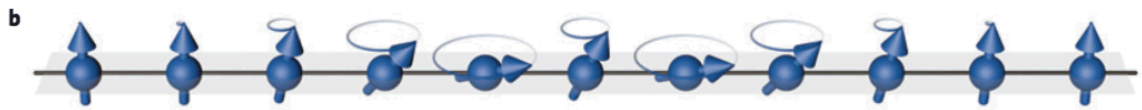
Accessible set

Results in minimal description

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_a = \tilde{\mathbf{A}}\mathbf{x}_a \qquad \mathbf{y}(t) = \tilde{\mathbf{C}}\mathbf{x}_a(t)$$

Example

XX spin chain



Fukuhara et al. Nature Physics, **9** 235 (2013)

$$H = \sum_{k=1}^{n} \frac{\omega_k}{2} \sigma_z^k + \sum_{k=1}^{n-1} \delta_k \left(\sigma_+^k \sigma_-^{k+1} + \sigma_-^k \sigma_+^{k+1} \right)$$

$$\mathcal{M} = \sigma_x^1$$ Measure end spin

$$\bar{G} = \{2^{-n/2}\sigma_x^1, \quad 2^{-n/2}\sigma_y^1\} \cup \{2^{-n/2}\sigma_z^1 \cdots \sigma_z^{k-1}\sigma_x^k, \quad 2^{-n/2}\sigma_z^1 \cdots \sigma_z^{k-1}\sigma_y^k\}_{k=2}^{n}$$

Filtration (generalized Pauli basis)

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & \omega_1 & 0 & -\delta_1 & & & & \\ -\omega_1 & 0 & \delta_1 & 0 & 0 & & & \\ 0 & -\delta_1 & 0 & \omega_2 & 0 & \ddots & & \\ \delta_1 & 0 & -\omega_2 & 0 & \ddots & \ddots & 0 & \\ & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -\delta_{n-1} \\ & & \ddots & \ddots & \ddots & 0 & \delta_{n-1} & 0 \\ & & & 0 & 0 & -\delta_{n-1} & 0 & \omega_n \\ & & & & \delta_{n-1} & 0 & -\omega_n & 0 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = [1, 0, \ldots, 0]$$

Filtered system matrix is *2n x 2n* (as opposed to *2^n x 2^n*)

$$\mathbf{x}_a = [\bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n]^{\mathsf{T}}$$

$$\bar{x}_1 = \langle \sigma_x^1 \rangle, \bar{y}_1 = \langle \sigma_y^1 \rangle$$

$$\bar{x}_k \equiv \langle \sigma_z^1 \cdots \sigma_z^{k-1} \sigma_x^k \rangle,$$

$$\bar{y}_k \equiv \langle \sigma_z^1 \cdots \sigma_z^{k-1} \sigma_y^k \rangle, \quad k \geq 2$$

Sampling and discretization

$$\mathbf{x}_a(j+1) = \tilde{\mathbf{A}}_d \mathbf{x}_a(j) \qquad \tilde{\mathbf{A}}_d = e^{\tilde{\mathbf{A}}\Delta t}$$

$$\mathbf{y}(j) = \tilde{\mathbf{C}} \mathbf{x}_a(j)$$

Note: $\mathbf{x}_a(j) \equiv \mathbf{x}_a(j\Delta t)$

$\mathbf{y}(j) \equiv \mathbf{y}(j\Delta t)$

Explicit solution

$$\mathbf{y}(j) = \tilde{\mathbf{C}} \tilde{\mathbf{A}}_d^j \mathbf{x}_a(0)$$

Goal:

Use $\{\mathbf{y}(j)\}_{j=0}^{J}$ to estimate $\{a_m\}_{m=1}^{M}$

Strategy:

1. Find the minimal linear model that generates the collected data
2. Back out the unknown parameters from this model

Eigenstate realization algorithm

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[Juang, J. N., & Pappa, R. S. *Journal of Guidance, Control, and Dynamics*, *8*, 620 (1985)]

Step 1: Form Hankel matrix from data

$$\mathbf{H}_{rs}(k) = \begin{bmatrix} \mathbf{y}(k) & \mathbf{y}(k+1) & \cdots & \mathbf{y}(k+(s-1)) \\ \mathbf{y}(1+k) & \mathbf{y}(1+k+1) & \cdots & \mathbf{y}(1+k+(s-1)) \\ \vdots & \vdots & & \vdots \\ \mathbf{y}(r-1+k) & \mathbf{y}(r-1+k+1) & \cdots & \mathbf{y}(r-1+k+(s-1)) \end{bmatrix}$$

Step 2: Take SVD of Hankel matrix at k=0

$$\mathbf{H}_{rs}(0) = P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} Q^{\mathsf{T}} = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^{\mathsf{T}} \\ Q_2^{\mathsf{T}} \end{bmatrix}$$

$$O \approx P_1 \Sigma^{-\frac{1}{2}}$$
$$C \approx \Sigma^{-\frac{1}{2}} Q_1^{\mathsf{T}}$$

Step 3: Form realizations of linear model from SVD components

$$\hat{\mathbf{A}}_d = \Sigma^{-\frac{1}{2}} P_1^{\mathsf{T}} \mathbf{H}_{rs}(1) Q_1 \Sigma^{-\frac{1}{2}}$$

$$\hat{\mathbf{C}} = e_1^{\mathsf{T}} P_1 \Sigma^{\frac{1}{2}}$$

$$\hat{\mathbf{x}}(0) \equiv \Sigma^{\frac{1}{2}} Q_1^{\mathsf{T}} e_1$$

$$e_1^{\mathsf{T}} = [1, 0, \cdots, 0]$$

Eigenstate realization algorithm

Step 3: Form realizations of linear model from SVD components

$$\hat{\mathbf{A}}_d = \Sigma^{-\frac{1}{2}} P_1^{\mathsf{T}} \mathbf{H}_{rs}(1) Q_1 \Sigma^{-\frac{1}{2}}$$

$$\hat{\mathbf{C}} = e_1^{\mathsf{T}} P_1 \Sigma^{\frac{1}{2}}$$

$$\hat{\mathbf{x}}(0) \equiv \Sigma^{\frac{1}{2}} Q_1^{\mathsf{T}} e_1$$

The triple $(\hat{\mathbf{A}}_d, \hat{\mathbf{C}}, \hat{\mathbf{x}}(0))$ is a **realization** of the triple $(\tilde{\mathbf{A}}_d, \tilde{\mathbf{C}}, \mathbf{x}_a(0))$

Certain quantities are model realization invariants,

e.g. the Markov parameters $\mathbf{C}\mathbf{A}^j\mathbf{x}(0)$ $\forall j$ for any LTI model $(\mathbf{A}, \mathbf{C}, \mathbf{x}(0))$

Therefore,

$$\mathbf{y}(j) = \tilde{\mathbf{C}}\tilde{\mathbf{A}}_d^j \mathbf{x}_a(0) = \hat{\mathbf{C}}\hat{\mathbf{A}}_d^j \hat{\mathbf{x}}(0), \quad \text{for all } j \geq 0,$$

Realization to parameter estimation

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$$\mathbf{y}(j) = \tilde{\mathbf{C}}\tilde{\mathbf{A}}_d^j \mathbf{x}_a(0) = \hat{\mathbf{C}}\hat{\mathbf{A}}_d^j \hat{\mathbf{x}}(0), \quad \text{for all } j \geq 0,$$

Define $\hat{\mathbf{A}} = \log \hat{\mathbf{A}}_d / \Delta t$

$$\tilde{\mathbf{C}}\tilde{\mathbf{A}}^j \mathbf{x}_a(0) = \hat{\mathbf{C}}\hat{\mathbf{A}}^j \hat{\mathbf{x}}(0), \quad \text{for all } j \geq 0$$

Determined by the data

Polynomial equation in unknown parameters

Solving these equations yields estimates of parameters

Notes:
1. Parameter estimates can be non-unique (gauge freedom/symmetries)
2. Δt must be satisfy Nyquist relation to one-over-fastest-frequency in system

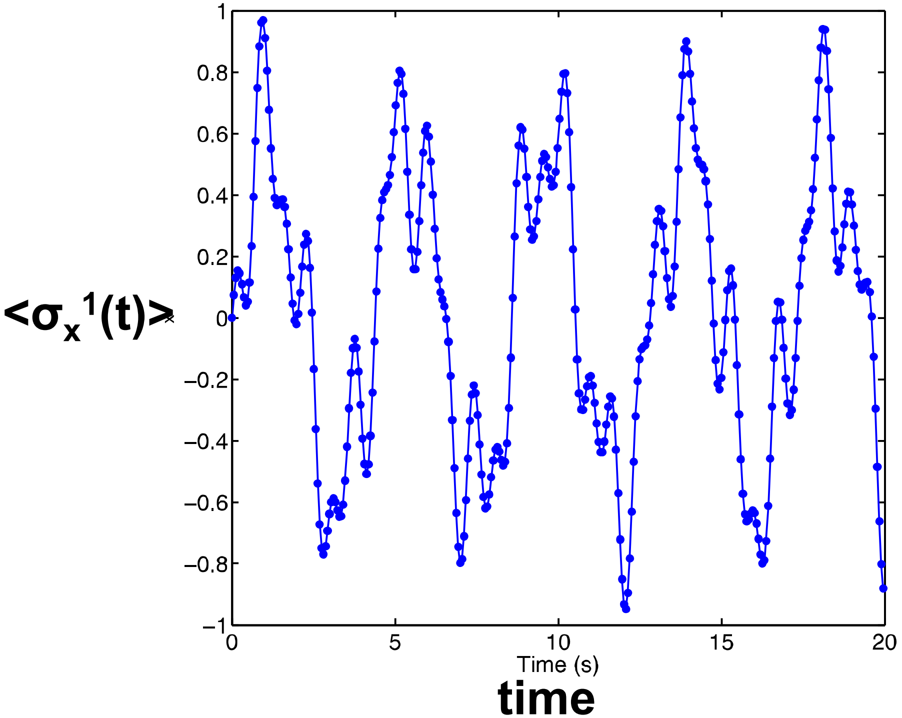
Example

XX spin chain (*n*=3 qubits)

$$H = \sum_{k=1}^{3} \frac{\omega_k}{2}\sigma_z^k + \sum_{k=1}^{2} \delta_k \left(\sigma_+^k \sigma_-^{k+1} + \sigma_-^k \sigma_+^{k+1}\right)$$

$$\omega_1 = 1.3, \omega_2 = 2.4, \omega_3 = 1.7, \delta_1 = 4.3, \delta_2 = 5.2$$

$$|\psi(0)\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}|00\rangle$$



Example

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$$\omega_1 = 1.3, \omega_2 = 2.4, \omega_3 = 1.7, \delta_1 = 4.3, \delta_2 = 5.2$$

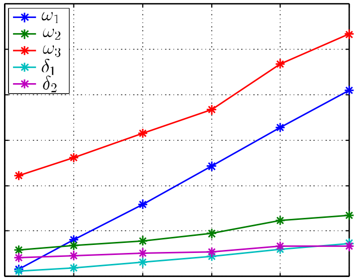
Construct Hankel matrix with *r*=100, *s*=100

$$\begin{aligned} \omega_1 &= \hat{\mathbf{C}}\hat{\mathbf{A}}\hat{\mathbf{x}}(0) = 1.3 \\ \omega_1^3 + \delta_1^2(2\omega_1 + \omega_2) &= -\hat{\mathbf{C}}\hat{\mathbf{A}}^3\hat{\mathbf{x}}(0) = 2.647 \\ \delta_1^4 (3\omega_1 + 2\omega_2) + \delta_1^2 \left(\delta_2^2 (2\omega_1 + 2\omega_2 + \omega_3) + 4\omega_1^3 + 3\omega_2\omega_1^2 + 2\omega_2^2\omega_1 + \omega_2^3 \right) + \omega_1^5 &= \hat{\mathbf{C}}\hat{\mathbf{A}}^5\hat{\mathbf{x}}(0) = 8.2942 \\ &\vdots \end{aligned}$$

Coupling parameters only occur up to even order (symmetry) => can only be determined up to sign

Summary

- System identification through model realization
- Most useful when
  - measurements are restricted
  - prior information about process is available
- Continuing work:
  - Noisy measurements
    - Use a different *model realization invariant* (transfer function)



  - Markovian open-system evolution
  - Continuous time weak measurement

Outline

- **Model realization and system identification**
  - Estimating unknown Hamiltonian parameters
- **Model reduction**
  - Reducing simulation cost for certain many-body quantum systems

Outline

Quantum state space: exponential

- Full-scale simulation of quantum systems very difficult
- Formal state is exponentially large in the number of particles

$$\rho_1 \in \mathcal{H}_1 \qquad \dim\mathcal{H}_1 = n_1$$

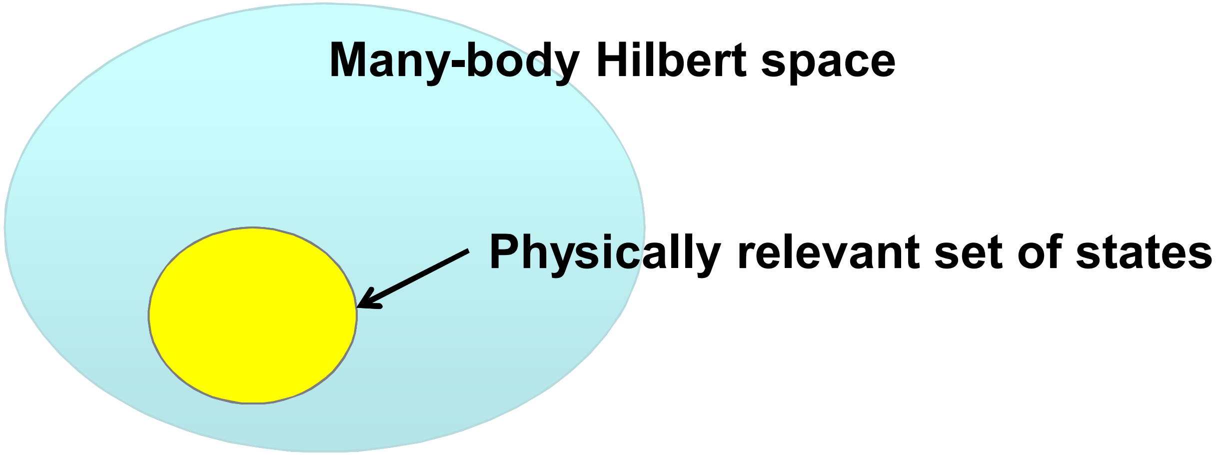
$$\rho_2 \in \mathcal{H}_2 \qquad \dim\mathcal{H}_2 = n_2$$

$$\rho_c \in \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad \dim\mathcal{H}_1 \otimes \mathcal{H}_2 = n_1 n_2 \neq n_1 + n_2$$

Outline

Quantum state space: not really exponential?

- For most practical systems, this exponential scaling is only formal

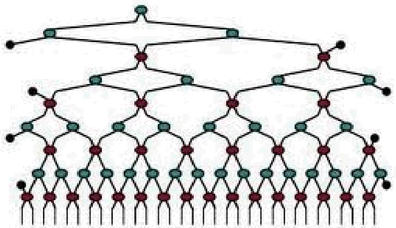


- Identifying this set of relevant states is difficult
- Dynamics within this relevant set of states is a *reduced order model* (ROM)

Identifying reduced order models

Existing techniques/results:

- **static**: DMRG, MPS, etc.



- **dynamic**: Nakajima-Zwanzig (statistical), Bloch equations, adiabatic elimination

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}x(t) = \mathbf{PAP}x(t) + \mathbf{PB}u(t) + \mathbf{PA}\mathcal{G}(t,0)\mathbf{Q}x(0) + \int_0^t \mathbf{PA}\mathcal{G}(t,s)\mathbf{QAP}x(s)ds + \int_0^t \mathbf{PA}\mathcal{G}(t,s)\mathbf{QB}u(s)ds.$$

Model reduction

Resources available / Desired Output	State snapshots	Input-output map	Dynamical model
Full state vector want to reproduce $\lvert\psi(t)\rangle$	Proper orthogonal decomposition (POD), manifold learning	?	Identify or approximate invariant subspaces
Input-output map want to reproduce *y(t)*	Empirical balanced truncation (BPOD)	Minimal model realization algorithms (*e.g.* ERA)	Balanced truncation

- In the context of continuous measurement / quantum filtering:
Mabuchi, PRA 78, 015801 (2008), Nielsen, Hopkins, Mabuchi, NJP 11, 105043 (2009)

- In the context of coherent feedback control / quantum optical networks:
Nurdin, arXiv:1308.6062, Nurdin, Gough arXiv:1309.0562

Compressible dynamics

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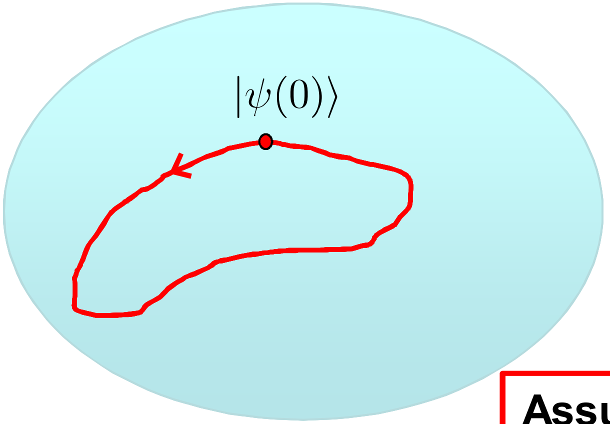
$$|\psi(t)\rangle = e^{iHt}\,|\psi(0)\rangle$$

$$H(\lambda) = \sum_i \lambda_i h_i$$



Assumption:

1. system is finite dimensional

Problem:

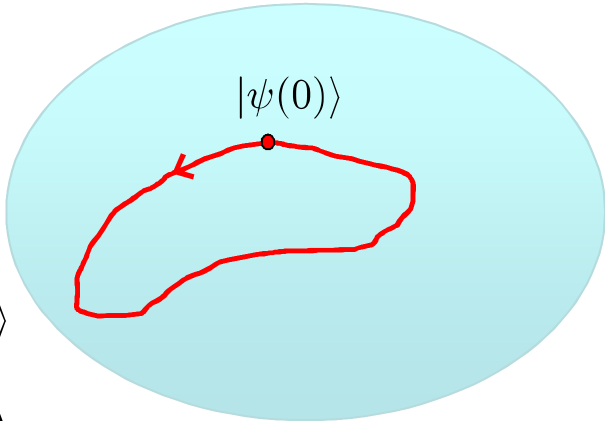
Identify subspace of Hilbert space that contains $|\psi(0)\rangle$ and is invariant under Hamiltonian for all choices of λ

e.g. quantum Ising model:

$$H = -B\sum_i \sigma_x^i - J\sum_{<i,j>} \sigma_z^i \sigma_z^j$$

Compressible dynamics

$$H(\lambda) = \sum_i \lambda_i h_i$$



Problem:

Identify subspace of Hilbert space that contains $|\psi(0)\rangle$ and is invariant under Hamiltonian for all choices of λ

Solutions:

1. Certificates
   Is this dynamics compressible?
2. Computing reduced order models
   What is the invariant subspace and moreover, what is the reduced order dynamical model?

Projective linear model reduction: columns of P are basis vectors in this invariant subspace

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P^\dagger \left|\psi(t)\right\rangle\right) = P^\dagger H P \left(P^\dagger \left|\psi(t)\right\rangle\right) \qquad \mathrm{dim} P = N \times q, \quad q \ll N$$

q x q compressed description

Certificate

$$H(\lambda) = \sum_i \lambda_i h_i \qquad H \in L(\mathcal{H}) \qquad \dim \mathcal{H} = N$$

$$\mathrm{Coeff}(H) \equiv \{h_i\}$$

Theorem: (algebraic certificate)
The Hamiltonian acting on $\mathcal{H}$ keeps invariant a non-trivial proper subspace iff the subalgebra generated by $\mathrm{Coeff}(H)$ is a proper subalgebra of $L(\mathcal{H})$.

Kumar, MS. arXiv: 1406.7069

Intuition:

$$\begin{aligned} |\psi(t)\rangle &= \exp\{i(\lambda_1 h_1 + \lambda_2 h_2)t\}\,|\psi(0)\rangle \\ &= \sum_n \frac{(it)^n}{n!}(\lambda_1 h_1 + \lambda_2 h_2)^n\,|\psi(0)\rangle \end{aligned}$$

Products of h_i generate an algebra. If the full operator algebra is not generated, there are directions in state space that are not explored

Certificate

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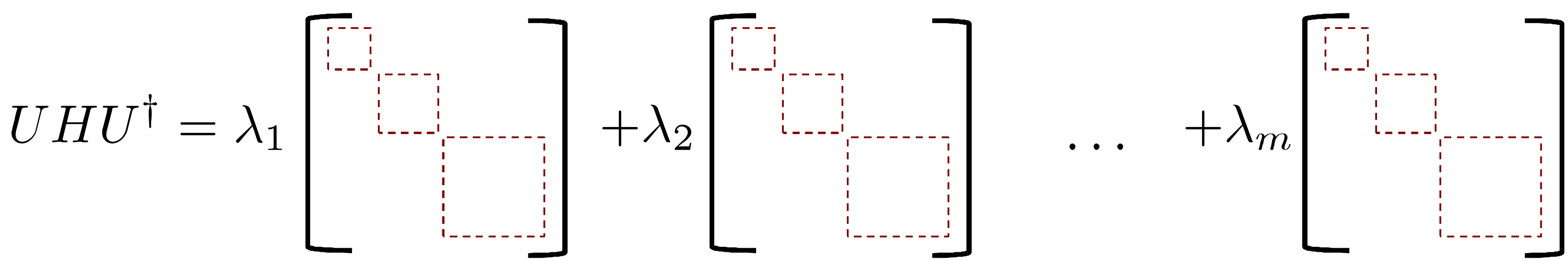
Identifies a symmetry:

Certifies the existence of a unitary matrix that simultaneously diagonalizes all h_i



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$$\dim\ \mathcal{A}(\mathrm{Coeff}(H)) < \dim\ L(\mathcal{H})?$$

To answer this: generate linear basis for $\mathcal{A}(\mathrm{Coeff}(H))$ (**the Burnside basis**) and count dimension.

Algorithm for basis generation: repeatedly multiply out h_i and keep linearly independent results.

This generates: $\{B_1, B_2, ...B_K\}$

Constructing reduced order models

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P^{\dagger}\left|\psi(t)\right\rangle\right)=P^{\dagger}HP\left(P^{\dagger}\left|\psi(t)\right\rangle\right)\qquad \mathrm{dim}P=N\times q,\quad q\ll N$$

q x q compressed description

Two methods:

1. **Sampling from full-order model**
   1. Fix $H(\lambda)$
   2. Generate samples $\left|\psi_{\lambda}(t_1)\right\rangle, \left|\psi_{\lambda}(t_2)\right\rangle, \ldots \left|\psi_{\lambda}(t_k)\right\rangle$
   3. The columns of *P* are formed from the (orthonormalized) minimal linearly independent set of these state samples (Krylov/cyclic subspace)

   Careful: cyclic subspace ≠ invariant subspace for model

2. **Use Burnside basis generated in algebraic characterization (certificate)**
   1. Find $|\psi_1\rangle = B_1|\psi(0)\rangle, |\psi_2\rangle = B_2|\psi(0)\rangle, \ldots|\psi_K\rangle = B_K|\psi(0)\rangle$
   2. The columns of *P* are formed from the (orthonormalized) minimal linearly independent set of these states

   Generates true reduced subspace for model

Kumar, MS. arXiv: 1406.7069

Example: quench dynamics

Quantum Ising model

Fukuhara et al. Nature Physics, **9** 235 (2013)

$$H = -B\sum_i \sigma_x^i - J\sum_{<i,j>} \sigma_z^i\sigma_z^j$$

- Basic model for magnetism in crystalline material
- Competition between B and J results in phase transition behavior
- Can be emulated using cold atoms
- As a result: intense interest in dynamical phase transitions, quenching dynamics

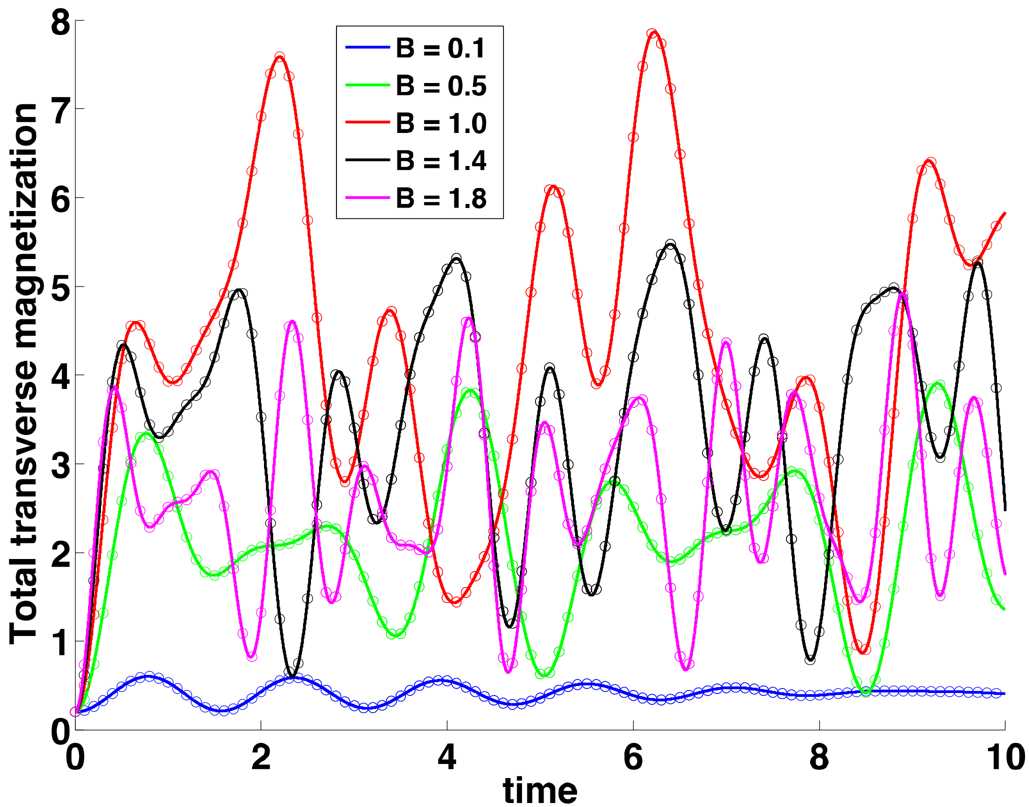
<u>Quenching dynamics:</u>

1. Prepare ground state of $H^0 = -B^0\sum_i \sigma_x^i - J\sum_{<i,j>} \sigma_z^i\sigma_z^j$
1. Rapidly change B and evolve system under $H^1 = -B^1\sum_i \sigma_x^i - J\sum_{<i,j>} \sigma_z^i\sigma_z^j$
1. The resulting dynamics is very informative; *e.g.* contains information about static phases of system

Example

Simulation of quench dynamics in quantum Ising model
(Circles: full model, lines: reduced order model)

Quenches to different parameters are indicated by different colors



N=8 qubits

Full order model:
2^8-1 = 255 complex numbers
Reduced order model:
23 complex numbers

Expense of computing ROM

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P^{\dagger}\left|\psi(t)\right\rangle\right) = P^{\dagger}HP\left(P^{\dagger}\left|\psi(t)\right\rangle\right) \qquad H(\lambda) = \sum_i \lambda_i h_i$$

- Certificate:
  - Multiply *d x d* matrices entering the Hamiltonian
  - Compute linear dependency of the results
- Construction of model reduction matrix *P*
  - *Time sampling:*
    - Need samples from full order system
    - Compute linear dependency of samples
  - *Burnside basis construction*
    - Multiply *d x d* matrices entering the Hamiltonian
    - Compute linear dependency of the results
    - Multiply Burnside basis elements by $|\psi(0)\rangle$

Rational computations, but exponential complexity in general

However, complexity simplifies greatly in special cases (*e.g. Pure Pauli spin models).* See Kumar, MS. arXiv: 1406.7069

Certificate
Special case: Pauli Hamiltonian

$$H(\lambda) = \sum_i \lambda_i \sigma_i \qquad H \in L(\mathcal{H}) \qquad \mathcal{H} = \mathbb{C}^{2n} \qquad \dim \mathcal{H} = 2^n$$

$$e.g. \quad \sigma_i : \sigma_x^{(1)} \otimes \mathbf{1} \otimes \ldots \otimes \sigma_y^{(n)}$$

Theorem: (Pauli algebraic certificate)
Any Pauli Hamiltonian acting on *n* qubits with fewer than *2n* terms has a non-trivial proper invariant subspace.

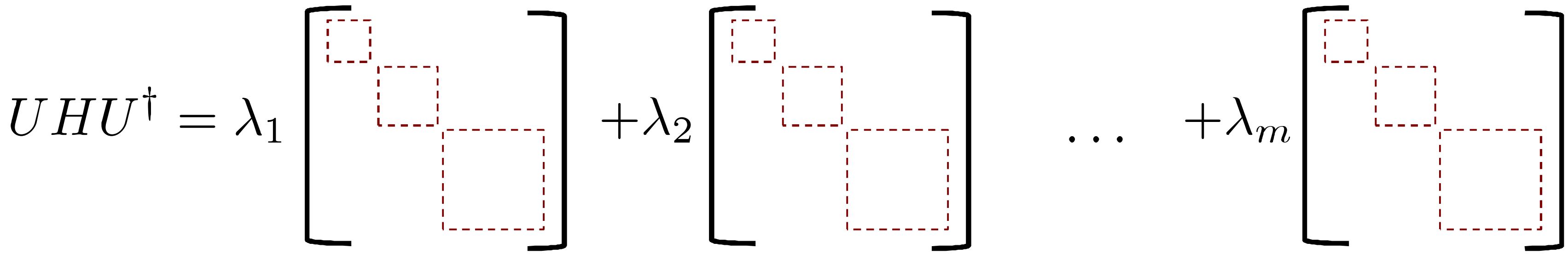
Kumar, MS. arXiv: 1406.7069

Note: a sufficient condition

e.g. Random quantum (transverse field) Ising model with open boundary conditions

Continuing work

- Approximations to invariant subspaces? “True” model reduction



Methods for *approximate* simultaneous block diagonalization

- Reduce computational complexity of ROM computation for special cases

- Algebraic approach to observability-based model reduction (*i.e.* only care to reproduce some observables)

Thanks!