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Some Nonlinear Reconstruction Algorithms for Electrical Impedance Tomography

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1 Introduction

An impedance camera [Henderson and Webster, 1978; Dines and Lytle, 1981] – or what is now more commonly called electrical impedance tomography – attempts to image the electrical impedance (or just the conductivity) distribution inside a body using electrical measurements on its boundary. The method has been used successfully in both biomedical [Brown, 1983; Barber and Brown, 1986; J. C. Newell, D. G. Gisser, and D. Isaacson, 1988; Webster, 1990] and geophysical applications [Wexler, Fry, and Neuman, 1985; Daily, Lin, and Buscheck, 1987], but the analysis of optimal reconstruction algorithms is still progressing [Murai and Kagawa, 1985; Wexler, Fry, and Neuman, 1985; Kohn and Vogelius, 1987; Yorkey and Webster, 1987; Yorkey, Webster, and Tompkins, 1987; Berryman and Kohn, 1990; Kohn and McKenney, 1990; Santosa and Vogelius, 1990; Yorkey, 1990]. The most common application is monitoring the influx or efflux of a highly conducting fluid (such as brine in a porous rock or blood in the human body) through the volume being imaged. For biomedical applications, this method does not have the resolution of radiological methods, but it is comparatively safe and inexpensive and therefore provides a valuable alternative when continuous monitoring of a patient or process is desired.

The following discussion is intended first to summarize the physics of electrical impedance tomography, then to provide a few details of the data analysis and forward modeling requirements, and finally to outline some of the reconstruction algorithms that have proven to be most useful in practice. Pointers to the literature are provided throughout this brief narrative and the reader is encouraged to explore the references for more complete discussions of the various issues raised here.

2 Electrical Impedance Tomography

First, we review some facts about this problem that play an important role in the analysis that follows. Recall that the electrical power dissipated into heat is [Jackson, 1962]

$$P = \int \mathbf{J}(\vec{x}) \cdot \mathbf{E}(\vec{x})\, d^3x, \tag{1}$$

where the current $\mathbf{J}$ and electric field $\mathbf{E}$ are related to the electrical potential Φ by

$$\mathbf{J}(\vec{x}) = \sigma(\vec{x})\mathbf{E}(\vec{x}), \tag{2}$$

$$\mathbf{E}(\vec{x}) = -\nabla\Phi(\vec{x}), \tag{3}$$

and the current distribution is also divergence free

$$\nabla \cdot \mathbf{J}(\vec{x}) = 0 \tag{4}$$

away from all current sources. The remaining quantity appearing in the constitutive equation (2) is the isotropic conductivity σ whose values are to be determined as a function of the position $\vec{x}$. Substituting (2) and (3) into (4) gives Poisson's equation

$$\nabla \cdot (\sigma \nabla \Phi) = 0. \tag{5}$$

Substituting (3) into (1) and using (4), we have

$$P = -\int \mathbf{J} \cdot \nabla\Phi \, d^3x = -\int \nabla \cdot (\Phi\mathbf{J}) \, d^3x. \tag{6}$$

Then, the divergence theorem shows that

$$P = -\int \Phi\mathbf{J} \cdot \hat{n} \, dS, \tag{7}$$

where $\hat{n}$ is a unit outward normal vector and dS is the infinitesimal surface area on the boundary. If current is injected through metallic electrodes, the potential takes a constant value Φ_k on the kth electrode of surface area S_k. If there are K electrodes, then (7) becomes

$$P = \sum_{k=1}^{K} \Phi_k I_k, \tag{8}$$

where

$$I_k = -\int_{S_k} \mathbf{J} \cdot \hat{n} \, dS \tag{9}$$

is the total current injected ($I_k > 0$) or withdrawn ($I_k < 0$) at the kth electrode. Since these are the only sources and sinks, we also have the sumrule

$$\sum_{k=1}^{K} I_k = 0. \tag{10}$$

If there are only two injection electrodes, then (8) reduces to

$$P = (\Phi_1 - \Phi_2) I_1 = \Delta\Phi I, \tag{11}$$

so the power is the product of the measured potential difference $\Delta\Phi$ across the injection electrodes and the injected current I.

Reciprocity

Now suppose that two experiments have been performed on the same body. Let Φ and $\mathbf{J}_\Phi = -\sigma\nabla\Phi$ be the potential and current distribution for one experiment and Ψ and $\mathbf{J}_\Psi = -\sigma\nabla\Psi$ be those for the other experiment. Then, using the divergence theorem we have

$$\begin{aligned} \int \sigma\nabla\Phi \cdot \nabla\Psi \, d^3x &= \int \Psi\nabla \cdot \mathbf{J}_\Phi \, d^3x - \int \Psi\mathbf{J}_\Phi \cdot \hat{n} \, dS \\ &= \int \Phi\nabla \cdot \mathbf{J}_\Psi \, d^3x - \int \Phi\mathbf{J}_\Psi \cdot \hat{n} \, dS. \end{aligned} \tag{12}$$

Since both current distributions are divergence free inside the body, we conclude that the two boundary integrals satisfy

$$\int \Psi\mathbf{J}_\Phi \cdot \hat{n} \, dS = \int \Phi\mathbf{J}_\Psi \cdot \hat{n} \, dS, \tag{13}$$

which is a general statement of reciprocity for Poisson's equation.

The practical implications of (13) may be seen by again supposing that the current is injected or withdrawn at metallic electrodes so the potential is constant (to a good approximation) on the k-th electrode of surface area S_k. If the current is injected/withdrawn at electrodes q and q' for Φ and at electrodes r and r' for Ψ, then

$$I_q^\Phi = -I_{q'}^\Phi = -\int_{S_q} \mathbf{J}_\Phi \cdot \hat{n}\, dS, \tag{14}$$

$$I_r^\Psi = -I_{r'}^\Psi = -\int_{S_r} \mathbf{J}_\Psi \cdot \hat{n}\, dS, \tag{15}$$

and (13) reduces to

$$(\Psi_q - \Psi_{q'})I_q^\Phi = (\Phi_r - \Phi_{r'})I_r^\Psi. \tag{16}$$

From (16) it follows that the transfer impedance satisfies

$$z_{qq',rr'} = \frac{\Psi_q - \Psi_{q'}}{I_r^\Psi} = \frac{\Phi_r - \Phi_{r'}}{I_q^\Phi}. \tag{17}$$

Equation (17) is probably the most common statement of the principle of reciprocity.

The reader should verify that this property of solutions of Poisson's equation is a direct consequence of the self-adjoint property of the linear operator (5).

Boundary data

The data for electrical impedance tomography have most often been gathered by injecting a measured current between two electrodes while simultaneously measuring the voltage differences between pairs of other electrodes placed around the boundary of the body being imaged. This process is then repeated, injecting current between all possible (generally adjacent) pairs of electrodes, and recording the set of voltage differences for each injection pair i. This data set has normally not included the voltage difference across the injection electrodes, because these voltages cannot be measured as reliably. A substantial contact impedance develops at the interface between the body and the injection electrodes when large currents are present. This problem can be reduced by using large electrodes or small currents. We assume that voltage differences (and therefore the powers dissipated) across the injection electrodes are known, but it is not necessary that they be known to high accuracy.

Feasibility constraints

Dirichlet's principle states that, given a conductivity distribution $\sigma(\vec{x})$ and a potential distribution $\Phi(\vec{x})$, the power dissipation p_i realized for a specified surface potential configuration is the one that minimizes the integral $\int \sigma |\nabla\Phi|^2\, d^3x$ so that

$$p_i(\sigma) = \int \sigma(\vec{x}) |\nabla\Phi_i^*(\vec{x})|^2\, d^3x = \min_{\Phi_i} \int \sigma(\vec{x}) |\nabla\Phi_i(\vec{x})|^2\, d^3x. \tag{18}$$

The trial potential field for the ith injection pair is $\Phi_i(\vec{x})$, while the particular potential field that actually minimizes the the power is $\Phi_i^*(\vec{x})$, and this one also satisfies Poisson's equation $\nabla\cdot(\sigma\nabla\Phi_i^*) = 0$ within the body. Furthermore, if the effective power dissipation associated with the trial potential $\Phi_i(\vec{x})$ is defined as

$$\bar{p}_i^{(\Phi_i)}(\sigma) \equiv \int \sigma(\vec{x})|\nabla\Phi_i(\vec{x})|^2\, d^3x, \tag{19}$$

then the measured powers P_i must satisfy

$$P_i = p_i(\sigma^*) \leq \bar{p}_i^{(\Phi_i)}(\sigma^*), \tag{20}$$

if $\sigma^*(\vec{x})$ is the true conductivity distribution. Note that if we vary the trial power dissipation (19) with respect to the trial potential, we find

$$2\int \sigma\nabla\Phi\cdot\nabla\delta\Phi\, d^3x = -2\int \nabla\cdot(\sigma\nabla\Phi)\delta\Phi\, d^3x = 0 \tag{21}$$

at a stationary point. We integrated once by parts to obtain (21); the surface integral vanishes since $\delta\Phi$ is constrained to vanish on the boundary. Since the volume variation $\delta\Phi$ is arbitrary, its coefficient inside the integral must vanish, so Poisson's equation is recovered, as expected.

To see that an analogy can be developed between seismic traveltime tomography [Lytle and Dines, 1980; Berryman, 1989; 1990] and electrical impedance tomography, consider the following set of correspondences:

$$s(\vec{x}) \to \sigma(\vec{x}),$$

$$t_i(s) \to p_i(\sigma),$$

$$\tau_i^P(s) \to \bar{p}_i^{(\Phi_i)}(\sigma),$$

$$T_i \to P_i,$$

$$dl_i^P \to |\nabla\Phi_i(\vec{x})|^2\, d^3x,$$

$$dl_i^{P^*} \to |\nabla\Phi_i^*(\vec{x})|^2\, d^3x.$$

Then the analysis of convex functionals and feasibility sets previously presented for seismic traveltime tomography [Berryman, 1991] carries over directly to the electrical impedance tomography problem formulated this way.

The feasibility constraints for electrical impedance tomography now take the form

$$\mathrm{K}\hat{\sigma} \geq \mathbf{p}, \tag{22}$$

where $\hat{\sigma}^T = (\sigma_1, \ldots, \sigma_n)$, $\mathbf{p}^T = (P_1, \ldots, P_m)$, and the E-squared matrix is given by

$$K_{ij} = \int_{\mathrm{cell}_j} |\nabla\Phi_i|^2\, d^3x, \tag{23}$$

so called because the electric field is given by $-\nabla\Phi$, and therefore (23) is the integral of the square of the electric field in cell j for current injection experiment i. Least-squares methods may be applied to this problem in much the same fashion as in traveltime tomography [Kallman and Berryman, 1992].

One other feature of the electrical impedance tomography problem not shared by the seismic tomography problem should be mentioned to emphasize the richness of this subject. So far we have discussed only Dirichlet's principle (18). In fact, there are two distinct variational principles for the conductivity problem: Dirichlet's principle and its dual, known as Thomson's principle. The second variational principle takes the form

$$P_i \leq \int \sigma^{-1}(\vec{x})|\mathbf{J}_i|^2(\vec{x})\, d^3x, \tag{24}$$

where $\mathbf{J}_i(\vec{x})$ is a trial current distribution vector for the ith current injection configuration that satisfies the continuity equation $\nabla \cdot \mathbf{J}_i = 0$, and has the correct current distribution on the boundary. The trial current distribution $\mathbf{J}_i(\vec{x})$ and the gradient of the trial potential $\nabla\Phi_i(\vec{x})$ are generally unrelated except that, when the minimum of both variational functionals is attained, then $\mathbf{J}_i^*(\vec{x}) = -\sigma\nabla\Phi_i^*(\vec{x})$. Of course, this condition is a restatement of the constitutive equation wherein the current equals the conductivity times the electric field.

The existence of dual variational principles is a general property whenever the primal variational principle is a true minimum principle. Fermat's principle is only a stationary (not a minimum) principle, and so traveltime tomography does not possess this dual property. The existence of the dual variational principles for electrical impedance tomography is important because it means there are two independent sets of feasibility constraints for the conductivity model $\sigma(\vec{x})$. These two sets of constraints also allow us (in some sense) to obtain upper and lower bounds on the region of the conductivity model space that contains the solution to the inversion problem. See Berryman and Kohn [1990] for more discussion of this point.

3 Finding Nodal Potentials

When doing forward modeling in electrical impedance tomography, we assume that the input nodal currents and admittance matrix are known and that the nodal potentials need to be determined. The admittance matrix is singular, so some trick must be introduced to solve for the potential. Two methods discussed here are: (1) the generalized inverse and (2) introduction of a reference node, allowing reduction of the matrix to one of full rank. Both methods are computationally viable. The generalized inverse has the advantage that it may be applied without restriction to any problem. The reference node approach is very simple to apply, but has a few (generally minor) restrictions on its use. Both methods are discussed in some detail.

3.1 The Admittance Matrix and the Cross-power Matrix

Iterative methods for solving the inversion problem start by solving the forward problem

$$\mathbf{Y}\hat{\Phi}_i = \hat{I}_i, \tag{25}$$

where $\mathbf{Y}$ is the admittance matrix [Mitra, 1969; Brown, 1985] of the discretized problem for some assumed conductivity distribution σ, $\hat{I}_i$ is the input current vector for the i-th injection

experiment, and $\hat{\Phi}_i$ is the vector of potential values at the nodes. For a finite element model with Q nodes and n elements, these Q-vectors take the form

$$\hat{\Phi}_i^T = (\Phi_{i,1} \quad \dots \quad \Phi_{i,Q}) \tag{26}$$

and

$$\hat{I}_i^T = (I_{i,1} \quad \dots \quad I_{i,Q}). \tag{27}$$

Assuming there are m current injection experiments in the test suite, we may form the voltage matrix

$$\mathbf{V} = (\hat{\Phi}_1 \quad \dots \quad \hat{\Phi}_m) \tag{28}$$

and the current matrix

$$\mathbf{C} = (\hat{I}_1 \quad \dots \quad \hat{I}_m). \tag{29}$$

Then, it follows directly from (25) that

$$\mathbf{V}^T \mathbf{Y} \mathbf{V} = \mathbf{V}^T \mathbf{C} \equiv \mathbf{\Pi}, \tag{30}$$

where the cross-power matrix is defined by (30) and given explicitly by

$$\mathbf{\Pi} = \begin{pmatrix} \hat{\Phi}_1^T \hat{I}_1 & \hat{\Phi}_1^T \hat{I}_2 & \hat{\Phi}_1^T \hat{I}_3 & \cdots & \hat{\Phi}_1^T \hat{I}_m \\ \hat{\Phi}_2^T \hat{I}_1 & \hat{\Phi}_2^T \hat{I}_2 & \hat{\Phi}_2^T \hat{I}_3 & \cdots & \hat{\Phi}_2^T \hat{I}_m \\ \hat{\Phi}_3^T \hat{I}_1 & \hat{\Phi}_3^T \hat{I}_2 & \hat{\Phi}_3^T \hat{I}_3 & \cdots & \hat{\Phi}_3^T \hat{I}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\Phi}_m^T \hat{I}_1 & \hat{\Phi}_m^T \hat{I}_2 & \hat{\Phi}_m^T \hat{I}_3 & \cdots & \hat{\Phi}_m^T \hat{I}_m \end{pmatrix}. \tag{31}$$

The cross-power matrix may be usefully defined and studied for any choice of current injection scheme. The matrix gets its name from the fact that the diagonal terms are the powers $P_i = \hat{\Phi}_i^T \hat{I}_i$, while the off-diagonal terms have the dimensions of power.

The cross-power matrix is very important for the inversion problem, because it contains all the data we can measure. We will discuss the cross-power matrix in more detail after analyzing the spectrum of the admittance matrix.

3.2 Impedance Matrices

The generalized inverse [Penrose, 1955a; 1955b] of the admittance matrix is also known as the impedance matrix $\mathbf{Z} = \mathbf{Y}^\dagger$, which therefore satisfies

$$\mathbf{Z}\mathbf{Y} = \mathbf{I} - \mathbf{w}\mathbf{w}^T/Q = \mathbf{Y}\mathbf{Z}, \tag{32}$$

where $\mathbf{I}$ is the $Q \times Q$ identity matrix. In terms of the impedance matrix, the potential vector is given by

$$\Phi_i = \mathbf{Z}\hat{I}_i + const \times \mathbf{w}. \tag{33}$$

Therefore, the experimental voltage matrix is given (non-uniquely) by

$$\mathbf{V} = \mathbf{ZC} + const \times \mathbf{ww}^T. \tag{34}$$

Then, the cross-power matrix is given either by (30) or by the equivalent expression

$$\mathbf{\Pi} \equiv \mathbf{V}^T\mathbf{C} = \mathbf{C}^T\mathbf{ZC}. \tag{35}$$

To emphasize the physical character of the cross-power matrix, we rewrite it as

$$\mathbf{\Pi} = \begin{pmatrix} P_1 & \hat{\Phi}_1^T\hat{I}_2 & \hat{\Phi}_1^T\hat{I}_3 & \cdots & \hat{\Phi}_1^T\hat{I}_m \\ \hat{\Phi}_2^T\hat{I}_1 & P_2 & \hat{\Phi}_2^T\hat{I}_3 & \cdots & \hat{\Phi}_2^T\hat{I}_m \\ \hat{\Phi}_3^T\hat{I}_1 & \hat{\Phi}_3^T\hat{I}_2 & P_3 & \cdots & \hat{\Phi}_3^T\hat{I}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\Phi}_m^T\hat{I}_1 & \hat{\Phi}_m^T\hat{I}_2 & \hat{\Phi}_m^T\hat{I}_3 & \cdots & P_m \end{pmatrix}. \tag{36}$$

Reciprocity shows that the cross-power matrix is symmetric.

It follows easily from (35) that the cross-power matrix $\mathbf{\Pi}$ is closely related to the transfer-impedance matrix $\mathbf{T}$. For a nearest-neighbor excitation scheme (unit current injected and withdrawn in sequence from pairs of adjacent electrodes),

$$\mathbf{T} \equiv \begin{pmatrix} z_{12,12} & z_{12,23} & z_{12,34} & \cdots & z_{12,m1} \\ z_{23,12} & z_{23,23} & z_{23,34} & \cdots & z_{23,m1} \\ z_{34,12} & z_{34,23} & z_{34,34} & \cdots & z_{34,m1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{m1,12} & z_{m1,23} & z_{m1,34} & \cdots & z_{m1,m1} \end{pmatrix}. \tag{37}$$

In fact, except for their differing units (resistance versus power), the transfer-impedance matrix $\mathbf{T}$ is just a special case of the cross-power matrix for this particular choice of current injection scheme.

3.3 Reference Node and Matrix Reduction

The singularity of the admittance matrix is caused by the fact that any constant potential added to a potential vector does not change the resulting current vector. This undesirable feature of the admittance matrix can be eliminated by choosing a reference node and reducing the matrix by one row and column. The reference node is chosen to be at ground potential, say $\Phi_s \equiv 0$. Then, none of the products $Y_{qs}\Phi_s \equiv 0$ contributes to the current vector and we may as well take $Y_{qs} \to 0$ for $q \neq s$. The sum along row s must still give $\sum_{q'} Y_{sq'}\Phi_{q'} = I_s$. If it happens that $I_s \equiv 0$, then we may also take $Y_{sq'} \to 0$ for $q' \neq s$ and $Y_{ss} \equiv 1$. These replacements guarantee that $\Phi_s = 0$, but do not affect the physics of the computation in any other way. Since inaccessible nodes always satisfy current conservation (so $I_s \equiv 0$), we can use this trick to remove the singularity problem in general by chosing s to be any interior node, *i.e.*, any node that is never used for current injection or withdrawal. Since there is no coupling between node s and any other node (since all the off-diagonal terms now vanish), we can just eliminate row s and column s from the matrix and proceed to invert the resulting $(Q-1)\times(Q-1)$ nonsingular matrix.

The procedure just outlined can be used most efficiently if the finite element representation is chosen for convenience so the final node in the node list is always interior. Then, the choice $s = Q$ simply eliminates the last node and therefore the last row and column of the admittance matrix.

4 Nonlinear Reconstruction Algorithms

In this section, we restrict discussion to nonlinear reconstruction algorithms. Such algorithms are iterative in character. Starting from an initial guess at the conductivity model, a method is sought to update the conductivity in a way that creates a steady improvement in the agreement between the predicted and measured data. The main issues in such algorithms are stability and speed of convergence. Uniqueness of the final solution is also an issue, but the reader is referred to the section on ghosts for a thorough discussion of that topic.

4.1 Kohn-Vogelius-Wexler Schemes

The first class of algorithms to be discussed we will call the Kohn-Vogelius-Wexler (KVW) schemes. Wexler, Fry, and Neuman [1985] presented the first of these schemes. Kohn and Vogelius [1986] presented a variant of the Wexler scheme that is much easier to analyze in light of the feasibility constraints. Furthermore, Yorkey, Webster, and Tompkins [1987] showed that the results from these two schemes are virtually indistinguishable. We therefore lump the two schemes and some additional variants into the same category. We will treat the Kohn and Vogelius algorithm first.

4.1.1 Algorithm of Kohn and Vogelius

Kohn and Vogelius [1986] begin by defining an error functional

$$\varepsilon_{KV}(\sigma) \equiv \sum_{i=1}^{m} \int |\sigma(\vec{x})^{-\frac{1}{2}} \mathbf{J}_i + \sigma(\vec{x})^{\frac{1}{2}} \nabla \phi_i|^2 \, d^3x. \tag{38}$$

The constraints on the components of the integrand are that the trial current distribution for the i-th current injection experiment $\mathbf{J}_i$ must be divergence free ($\nabla \cdot \mathbf{J}_i = 0$) except at the injection and withdrawal nodes and the scalar potential field $\phi_i(\vec{x})$ must be continuous and satisfy appropriate boundary conditions. Since the full solution to the problem must satisfy $\mathbf{J}_i = -\sigma \nabla \phi_i$ for $\mathbf{J}_i$ and ϕ_i satisfying the same pair of admissibility conditions; since this equality can only be achieved at a solution, the error functional (38) will remain positive unless a solution has been attained. An iterative method that successively decreases this error functional subject to the admissibility conditions may be expected to converge to a solution.

Expanding the integrand of (38) and using the divergence theorem, we find that

$$\varepsilon_{KV}(\sigma) = \sum_{i=1}^{m} \Big(\int \sigma^{-1}(\vec{x}) |\mathbf{J}_i|^2 \, d^3x + \int \sigma(\vec{x}) |\nabla \phi_i|^2 \, d^3x - 2p_i \Big), \tag{39}$$

where we used the fact that $p_i = -\int \phi_i \mathbf{J}_i \cdot \hat{n} da$ follows from the admissibility boundary conditions. The advantage of the Kohn-Vogelius approach becomes apparent when we consider that

Thomson's variational principle shows that

$$\int \sigma^{-1}(\vec{x})|\mathbf{J}_i|^2\, d^3x \geq p_i, \tag{40}$$

while Dirichlet's principle shows that

$$\int \sigma(\vec{x})|\nabla\phi_i|^2\, d^3x \geq p_i. \tag{41}$$

Thus, the absolute minimum ($\varepsilon_{KV} = 0$) of the error functional is achieved only when the integrals for both variational principles reach their minima for all m experimental configurations.

A reconstruction algorithm may be built around the observation that the error functional can be forced to decrease monotonically at each stage in an iteration scheme by noting that the values of the conductivity in the various cells producing the minimum error are determined by

$$\frac{\partial \varepsilon_{KV}}{\partial \sigma_j} = -\sigma_j^{-2} \sum_{i=1}^{m} \int_{cell_j} |\mathbf{J}_i|^2 d^3x + \sum_{i=1}^{m} \int_{cell_j} |\nabla\phi_i|^2 d^3x = 0. \tag{42}$$

It follows that the updated values of cell conductivity after an iteration step are

$$\sigma_j = \Big(\frac{\sum_{i=1}^{m} \int_j |\mathbf{J}_i|^2 d^3x}{\sum_{i=1}^{m} \int_j |\nabla\phi_i|^2 d^3x} \Big)^{\frac{1}{2}}. \tag{43}$$

The algorithm proceeds this way:

1. Given input currents, compute trial current distributions $\mathbf{J}_i$ for each injection experiment.
2. Given measured output voltages, compute trial scalar potentials ϕ_i for each injection experiment based on the latest estimate of the conductivity distribution $\sigma(\vec{x})$.
3. Update the conductivity in the j-th cell by finding the minimum of ε_{KV} according to (43).
4. Repeat this process until a convergence criterion has been met, or until some fixed number of iterations has been performed.

4.1.2 Algorithm of Wexler

Wexler et al. [1985] begin by defining the error functional

$$\varepsilon_W(\sigma) \equiv \sum_{i=1}^{m} \int |\mathbf{J}_i + \sigma(\vec{x})\nabla\phi_i|^2\, d^3x. \tag{44}$$

The admissibility conditions on $\mathbf{J}_i$ and ϕ_i are the same as for Kohn and Vogelius ($\nabla \cdot \mathbf{J}_i = 0$, ϕ_i continuous, and both satisfying appropriate boundary conditions). The main difference between this functional and that of Kohn and Vogelius is that, when the integrand is expanded, the result

$$\varepsilon_W(\sigma) = \sum_{i=1}^{m} \int (\mathbf{J}_i \cdot \mathbf{J}_i + 2\sigma(\vec{x})\nabla\phi_i \cdot \mathbf{J}_i + \sigma^2(\vec{x})|\nabla\phi_i|^2)\, d^3x \tag{45}$$

has no special physical interpretation and cannot be simplified further.

A monotonic decrease in the error functional ε_W can be forced by noting that

$$\frac{\partial \varepsilon_W}{\partial \sigma_j} = 2\sum_{i=1}^{m} \int_{cell_j} \nabla\phi_i \cdot \mathbf{J}_i \, d^3x + 2\sigma_j \sum_{i=1}^{m} \int_{cell_j} |\nabla\phi_i|^2 \, d^3x = 0. \tag{46}$$

Thus, the updated values of cell conductivity in Wexler's algorithm are given by

$$\sigma_j = -\frac{\sum_{i=1}^{m} \int_j \nabla\phi_i \cdot \mathbf{J}_i \, d^3x}{\sum_{i=1}^{m} \int_j |\nabla\phi_i|^2 \, d^3x}. \tag{47}$$

The updating rules (43) and (47) are similar but not identical.

Wexler's algorithm for reconstructing the conductivity is the same as that for Kohn and Voeglius except that in the update step the formula (43) is repaced by (47).

4.1.3 Related algorithms

A variant of the KVW schemes arises by considering the error functional

$$\varepsilon(\mathbf{a}) = \tfrac{1}{2}\sum_{i=1}^{m} \int \left| a_i^{\frac{1}{2}} \sigma^{-\frac{1}{2}}(\vec{x}) \mathbf{J}_i + a_i^{-\frac{1}{2}} \sigma^{\frac{1}{2}}(\vec{x}) \nabla\phi_i \right|^2 d^3x, \tag{48}$$

where $\mathbf{a}^T = (a_1, \ldots, a_m)$ is an m-vector of constants a_i. The absolute minimum of (48) occurs when the bracketed expression in the integrand vanishes everywhere, which requires that $\mathbf{J}_i = -\sigma\nabla\phi_i$ and $a_i = 1$ for all $i = 1 - m$. For arbitrary trial current distributions and potentials, the minimum of (48) is achieved by varying with respect to the $a_i s$. Then, we find the minimum occurs when

$$a_i^2 = \frac{\int \sigma |\nabla\phi_i|^2 \, d^3x}{\int \sigma^{-1} |\mathbf{J}_i|^2 \, d^3x}. \tag{49}$$

Substituting (49) back into (48), the error functional becomes

$$\varepsilon_S(\sigma) = \sum_{i=1}^{m} \Big[\Big(\int \sigma^{-1}(\vec{x}) |\mathbf{J}_i|^2 \, d^3x \int \sigma(\vec{x}) |\nabla\phi_i|^2 \, d^3x \Big)^{\frac{1}{2}} + \int \nabla\phi_i \cdot \mathbf{J}_i \, d^3x \Big] \geq 0. \tag{50}$$

We recognize now that the error functional ε_S makes use of the Schwartz inequality for integrals. Furthermore, like the Kohn and Vogelius functional and the Wexler functional, the absolute minimum of this error functional is achieved only when $\mathbf{J}_i = -\sigma\nabla\phi_i$ for all i.

Minimizing (50) with respect to the conductivity σ_j in the j-th cell as before, we find that the updated $\sigma_j s$ are given by

$$\sigma_j^2 = \frac{\sum_{i=1}^{m} \int_j |\mathbf{J}_i|^2 \, d^3x \int \sigma |\nabla\phi_i|^2 \, d^3x}{\sum_{i=1}^{m} \int_j |\nabla\phi_i|^2 \, d^3x \int \sigma^{-1} |\mathbf{J}_i|^2 \, d^3x}. \tag{51}$$

The major difference between (51) and the updates of Kohn and Vogelius or Wexler is that σ_j is only defined implicitly by (51), since the conductivity appears on the right hand side of the equation inside the integrals. Thus, (51) is an implicit formula for the updated conductivities. In contrast, the updates (43) and (47) are actually independent of the conductivity and therefore provide explicit formulas.

4.2 Yorkey's Output Least-squares Method

The simplest output least-squares method for electrical impedance tomography takes the optimization function to be

$$f(\hat{\sigma}) \equiv \tfrac{1}{2} \sum_{i,i'}^{m} W_{ii'} (\Pi_{ii'} - \Pi_{ii'}^{(d)})^2, \tag{52}$$

where the weights $W_{ii'}$ are generally chosen to be either one or zero — depending on whether the data are available or not. For example, in the common nearest-neighbor excitation scheme, those terms involving injection electrodes have generally not been available so $W_{ii} = W_{ii+1} = W_{ii-1} = 0$, etc. Equation (52) is the functional used by Yorkey and Webster [1987] and Yorkey, Webster, and Tompkins [1987].

The minimum of (52) is found when

$$\frac{\partial f}{\partial \sigma_j} = \sum_{i,i'=1}^{m} W_{ii'} \frac{\partial \Pi_{ii'}}{\partial \sigma_j} (\Pi_{ii'} - \Pi_{ii'}^{(d)}) = 0, \tag{53}$$

for all elements $1 \leq j \leq n$. To make use of this equation in updating the conductivity, consider the Taylor expansion of $\Pi_{ii'}$ given by

$$\Pi_{ii'}(\hat{\sigma}^{(0)} + \delta\hat{\sigma}) \simeq \Pi_{ii'}(\hat{\sigma}^{(0)}) + \sum_{k=1}^{n} \Pi_{ii',k} \delta\sigma_k, \tag{54}$$

where we have introduced the notation

$$\Pi_{ii',k} \equiv \frac{\partial \Pi_{ii'}}{\partial \sigma_k}. \tag{55}$$

Retaining only the first order terms in $\delta\sigma_k$, (53) may then be rearranged approximately as

$$\sum_{i,i'=1}^{m} \sum_{k=1}^{n} W_{ii'} \Pi_{ii',j} \Pi_{ii',k} \delta\sigma_k = \sum_{i,i'=1}^{m} W_{ii'} \Pi_{ii',j} (\Pi_{ii'}^{(d)} - \Pi_{ii'}), \tag{56}$$

where $\mathbf{\Pi}$ is evaluated at $\hat{\sigma}^{(0)}$. Equations (56) are approximate normal equations for the least-squares minimization problem associated with (52).

To solve for the model corrections $\delta\hat{\sigma}$, we must first evaluate the Jacobian derivative $\Pi_{ii',j}$. Recalling that

$$\mathbf{Y}\hat{\Phi}_i = \hat{I}_i \tag{57}$$

and noting that the input current vector is not a function of the model conductivities, we find that

$$\frac{\partial \mathbf{Y}}{\partial \sigma_j} \hat{\Phi}_i + \mathbf{Y} \frac{\partial \hat{\Phi}_i}{\partial \sigma_j} = 0. \tag{58}$$

Multiplying on the left by $\hat{\Phi}_{i'}^T$ gives

$$\Pi_{ii',j} = -\hat{\Phi}_{i'}^T \mathbf{k}^{(j)} \hat{\Phi}_i, \tag{59}$$

where we have used the fact that the stiffness matrix $\mathbf{k}^{(j)} = \partial \mathbf{Y}/\partial \sigma_j$. Equation (59) has the important advantage that it provides a simple means of evaluating the Jacobian without requiring any numerical derivatives. Nevertheless, the formula is not trivial to evaluate since it depends implicitly on $\hat{\sigma}$ through $\hat{\Phi}_i$, where $\hat{\Phi}_i$ solves (57) for some $\hat{\sigma}$ and $\hat{I}_i$.

An important practical detail for implementation of inversion algorithms should be stressed now, in light of (59). Reciprocity guarantees both that $\Pi_{ii'} = \Pi_{i'i}$ and also that $\Pi_{ii',j} = \Pi_{i'i,j}$, so we might think it should be possible to avoid doing some of the computations from (57) for the nodal potentials. However, the potential vector $\hat{\Phi}_i$ must be computed for every electrode excitation vector $\hat{I}_i$ to take advantage of formula (59). Thus, complete sets of reciprocal data are required by the output least-squares inversion method, if it is to be implemented as a Newton-Raphson iteration scheme. Since it is always good experimental practice to check how well reciprocity is satisfied for any excitation scheme (in order to evaluate the linearity and repeatability of the experiment), this requirement really costs us nothing.

Yorkey's output least-squares method works very well on synthetic data with low noise content. Essentially perfect reconstructions can be easily obtained if the noise level is 0.1% or less. However, if boundary voltage data with noise/signal ratio of 1.0% or more and the conductivity contrasts present are large (20% or more), the iterative method tends to diverge after a few iterations. Since the repeatability tests of reciprocity measurements has been shown to have accuracy of about 1.0% for our field data, modifications in the method are therefore required to improve stability of the reconstruction algorithm.

4.3 Modifications Based on Feasibility Constraints

Contamination of the measurements due to the presence of contact impedance is the reason usually given for avoiding the use of voltages involving the injection and withdrawal electrodes in the data set for inversion. Such terms are eliminated in a nearest-neighbor excitation scheme in two-dimensions by setting the tridiagonal matrix elements to zero, *e.g.*, $W_{ii} = W_{ii+1} = W_{ii-1} = 0$. For all other excitation patterns that exclude injection electrodes, the diagonal terms of the matrix $\mathbf{W}$ always vanish, while other elements may or may not vanish depending on the particular pattern used.

To make use of feasibility constraints, we should have a measurement or an estimate of the power dissipation for each excitation pattern in our scheme. Furthermore, it is preferable to have as many of these power measurements as possible; having at least as many constraints as unknowns in the conductivity model is desirable. The inaccuracy of the power measurements may appear to be an issue, however feasibility constraints based on comparatively inaccurate power measurements can still be used quite effectively. The constraints are used not to compute the *direction* of the least-squares model correction step, but rather to choose the *magnitude* of an underrelaxation parameter used as a multiplier to reduce the length of that model correction step [Berryman and Kohn, 1990]. Thus, the constraints are used only to prevent the iteration scheme from producing a sequence of models that wanders rapidly away from the feasibility boundary (whose location is known approximately from values of the power measurements). Since the data are assumed to depend linearly on any measurement errors and the location of the feasibility boundary depends linearly on the data, our estimates of this boundary are also linearly dependent on the data error. Thus, inaccuracies in the power measurements may be seen to have a very small effect on the outcome of the inversion routine at each iteration step.

The feasibility constraints have been implemented to modify the correction step in Yorkey's algorithm. Our experience has been that boundary voltage data with noise/signal ratios of 1.0% for conductivity models containing as much as 100% contrast produce stable/convergent reconstructions, whereas the same code without the constraints produced rapidly diverging reconstructions.

4.4 Least-squares for Power Measurements

Extraordinary measures may be taken to produce accurate estimates of the power dissipated in each current injection scheme. For example, the experiment may be repeated several times at successively smaller current injection levels; then the linear transfer resistance across the injection electrodes may be deduced by extrapolation to zero current level.

Assuming that accurate power measurements become available, we may consider a somewhat different least-squares approach. Now the *only* data are the power measurements. Let the data vector $\mathbf{p}$ be the measured power dissipation, an m-vector such that $\mathbf{p}^T = (p_1, \ldots, p_m)$, where p_i is the power dissipation of the i-th current injection configuration. The model conductivity n-vector $\hat{\sigma}^T = (\sigma_1, \ldots, \sigma_n)$ with σ_j being the conductivity of the j-th cell satisfies

$$\mathrm{K}\hat{\sigma} = \mathbf{p}, \tag{60}$$

where $\mathbf{K}$ is an $m \times n$ matrix whose matrix elements K_{ij} are determined by the square magnitude of the electric field of the i-th current injection pattern through the j-th cell of the model (*i.e.* $K_{ij} = \int_{cell_j} |\nabla \Phi_i|^2 d^3x$). This matrix is just the E-square matrix defined previously in (23). In general, we do not have good *a priori* approximations to the square magnitude of the electric field if the medium is very inhomogeneous. However, for the present application, we will assume that the E-square integral matrix has been fixed with a known set of (what are possibly trial) electric field values.

Once a set of trial E^2s and corresponding values of $K_{ij} = \int_{cell_j} |\nabla \Phi_i|^2 d^3x$ are known, we try to solve (60) for $\hat{\sigma}$. The resulting model vector $\hat{\sigma}$ may be underdetermined if $m < n$ or overdetermined if $m > n$. Finding a "solution" to (60) requires the use of a generalized inverse which implies a least-squares estimate of the model conductivity vector. A general objective function to be minimized might have the form suggested by Herman [1980]

$$G_\mu(\hat{\sigma}) = (\mathbf{p} - \mathrm{K}\hat{\sigma})^T \mathbf{W}_1 (\mathbf{p} - \mathrm{K}\hat{\sigma}) + \mu (\hat{\sigma} - \hat{\sigma}_b)^T \mathbf{W}_2 (\hat{\sigma} - \hat{\sigma}_b), \tag{61}$$

where $\mathbf{W}_1$ and $\mathbf{W}_2$ are (respectively) $m \times m$ and $n \times n$ real, symmetric weight matrices, μ is some scalar (called the damping parameter) determined by the relative importance of the second term compared to the first, and $\hat{\sigma}_b$ is some special background value of the model conductivity vector to which the final result should be close.

If $\mathbf{\Pi}$ is the cross-power matrix for any complete set of measurements, the least-squares functional for power measurements satisfies

$$(\mathbf{p} - \mathrm{K}\hat{\sigma})^T \mathbf{W} (\mathbf{p} - \mathrm{K}\hat{\sigma}) = \tfrac{1}{2} \sum_{ii';ll'} (\Pi_{ii'} - \Pi^{(d)}_{ii'}) \mathcal{W}_{ii';ll'} (\Pi_{ll'} - \Pi^{(d)}_{ll'}).$$

For a complete data set, both the estimated and measured powers may be found in terms of a definite set of coefficients $\{a_{k;i}\}$ related to the input current by

$$\hat{I}_{\mathbf{a}_k} = \mathbf{C}\mathbf{a}_k, \tag{62}$$

where

$$\mathbf{a}_k^T = (a_{k;1}, a_{k;2}, \ldots, a_{k;n}), \tag{63}$$

and (from the admittance equation $\mathbf{YV} = \mathbf{C}$) to the potential by

$$\hat{\Phi}_{\mathbf{a}_k} = \mathbf{V}\mathbf{a}_k, \tag{64}$$

so that

$$\mathbf{a}_k^T \mathbf{V}^T \mathbf{C} \mathbf{a}_k = \mathbf{a}_k^T \mathbf{\Pi}^{(d)} \mathbf{a}_k = P_k. \tag{65}$$

Thus, the measured power is given explicitly by

$$P_k = \sum_{ii'} a_{k;i'} \Pi_{i'i}^{(d)} a_{k;i} \tag{66}$$

and the predicted power is similarly found to be given by

$$(\mathbf{K}\hat{\sigma})_k = \sum_{ii'} a_{k;i'} \Pi_{i'i} a_{k;i}. \tag{67}$$

Then, $\mathcal{W}_{ii';ll'}$ is

$$\mathcal{W}_{ii';ll'} = \sum_{k=1}^{m} a_{k;i} a_{k;i'} w_{kk} a_{k;l} a_{k;l'}, \tag{68}$$

where w_{kk} are the elements of the diagonal weight matrix $\mathbf{W}$. It is not hard to show that $\mathcal{W}_{ii';ll'}$ satisfies (??).

Thus, we see that least-squares for the power measurements is another special case of the general output least-squares method based on the functional (??). Since solution of the normal equations for the least-squares problem often requires inversion of a singular or poorly conditioned normal matrix, the second term in (61) is used to regularize the problem. Kallman and Berryman [1992] have shown that the resulting algorithms for electrical impedance tomography are completely analogous to the ones that have been shown to be successful for stablizing seismic traveltime tomography algorithms [Berryman, 1990]. These algorithms have been implemented and tested on both synthetic and real data. The results show that the feasibility constraints do stabilize the reconstructions as indicated by the analysis.

5 Discussion

We have concentrated on nonlinear inversion algorithms based on the output least-squares approach. Among other nonlinear reconstruction methods that deserve mention here are those of Wexler, Fry, and Neuman [1985], Kohn and Vogelius [1987], Kohn and McKenney [1990], and Santosa and Vogelius [1990]. Spatial constraints prevent a detailed analysis of all these methods, so we limit discussion to the method of Kohn and Vogelius.

Beginning with the constitutive relation

$$\mathbf{J}_i = -\sigma(\vec{x}) \nabla \Phi_i \tag{69}$$

which must be satisfied by the true solution for every experimental current injection pattern i, Kohn and Vogelius [1987] consider the error functional

$$\varepsilon_{KV}(\sigma) \equiv \tfrac{1}{2}\sum_{i=1}^{m}\int |\sigma^{-\frac{1}{2}}(\vec{x})\mathbf{J}_i + \sigma^{\frac{1}{2}}(\vec{x})\nabla\Phi_i|^2\, d^3x. \tag{70}$$

The constraints on the components of the integrand are that the trial current distribution for the i-th current injection experiment $\mathbf{J}_i$ must be divergence free ($\nabla \cdot \mathbf{J}_i = 0$) except at the injection and withdrawal nodes while the scalar potential $\Phi_i(\vec{x})$ must be continuous and satisfy appropriate boundary conditions. The error functional will be positive unless a solution of the inversion problem has been found, in which case the functional vanishes identically. To see the significance of this limiting value, we may expand the integrand of (70) and use the divergence theorem to show that

$$\varepsilon_{KV}(\sigma) = \tfrac{1}{2}\sum_{i=1}^{m}\Big(\int \sigma^{-1}(\vec{x})|\mathbf{J}_i|^2\, d^3x + \int \sigma(\vec{x})|\nabla\Phi_i|^2\, d^3x - 2p_i\Big) \geq 0, \tag{71}$$

where the admissibility boundary conditions guarantee that $p_i = -\int \Phi_i \mathbf{J}_i \cdot \hat{n} da$. In essence, Kohn and Vogelius [1987] minimize the average of the power estimates using both the Dirichlet and Thomson variational integrals. Generally, both power integrals do not simultaneously satisfy the feasibility constraints. However, (71) shows that the average of the two integrals is bounded below by the true power. Furthermore, by minimizing the sum with respect to the conductivity value in all the cells, the two integrals take the same value at the minimum (the reader can easily verify this) and therefore at the end of each iteration step Kohn and Vogelius [1987] have found a conductivity lying in the dual feasibility region for the data.

In the language of either linear or nonlinear programming [Fiacco and McCormick, 1990], functional minimization algorithms are either interior or exterior methods depending on whether the iterates remain inside or outside the feasible set. Methods producing iterates always lying on the feasibility boundary are also interior methods. Mixed methods may have iterates that alternate between the interior and exterior of the feasible set. The main point of contrast between the method of Kohn and Vogelius and the output least-squares algorithms discussed earlier is that, whereas the least-squares methods may be shown to be exterior methods [Berryman, 1991], algorithms such as Kohn and Vogelius [1987] are interior methods (or they would be if all the power data were always available). An analysis of the feasibility constraints shows that both types of algorithms attempt to converge to a solution on the feasibility boundary, but the output least-squares methods converge from outside the feasible set, while these other methods generally converge from inside the feasible set. There are significant advantages to both techniques, notably the exterior methods may achieve convergence quickly but can be subject to instabilities induced by data errors if an insufficient number of constraints is available while the interior methods are insensitive to data error and therefore very stable but are often slowly converging.

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