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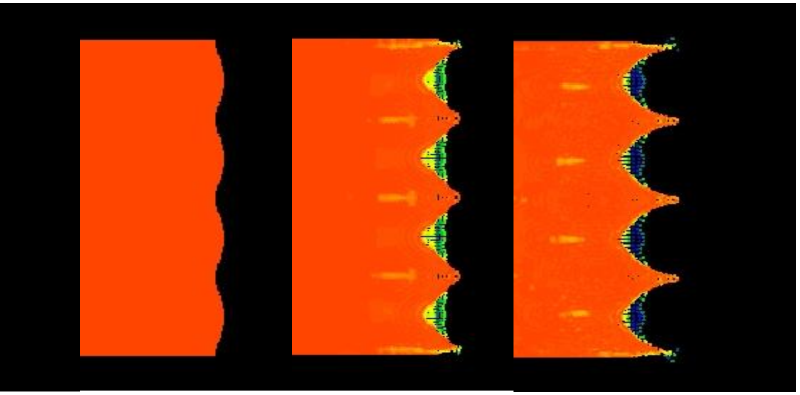
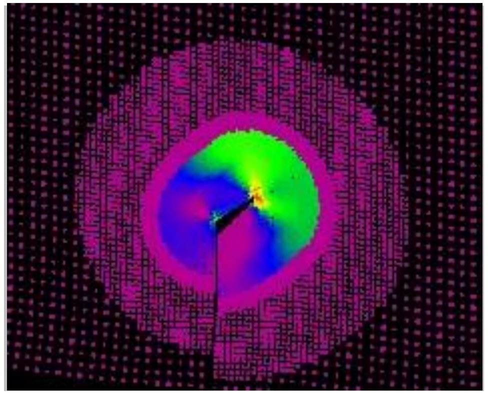
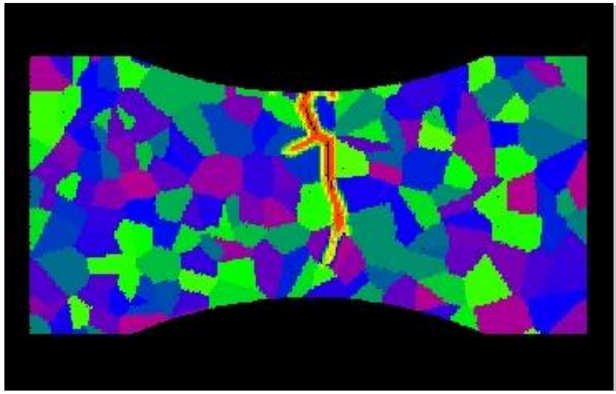

Nonlocal Waves in Continuous Media

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Outline

- Peridynamics background
- Static waves: Weird is good.
- Linear waves
  - Dispersion
  - Attenuation
- Nonlinear waves
  - Solitons
  - Shocks

Peridynamics:* What it is

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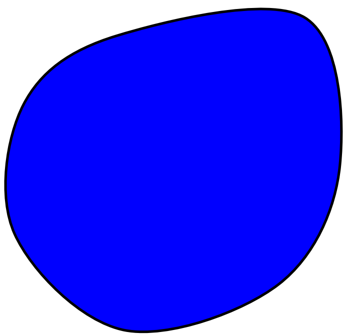
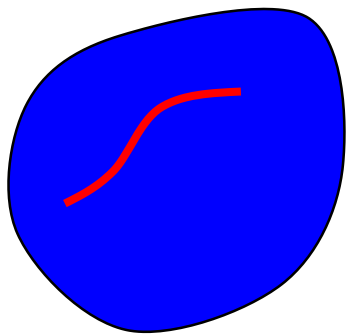
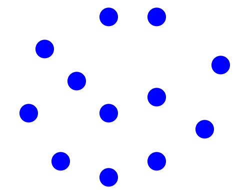
- It's an extension of continuum mechanics to media with cracks and long-range forces.
- It unifies the mechanics of continuous and discontinuous media within a single, consistent set of equations.



Continuous body



Continuous body with a defect



Discrete particles

- Our goals
  - Nucleate cracks and seamlessly transition to growth.
  - Model complex fracture patterns.
  - Communicate across length scales.

* Peri (near) + dyn (force)

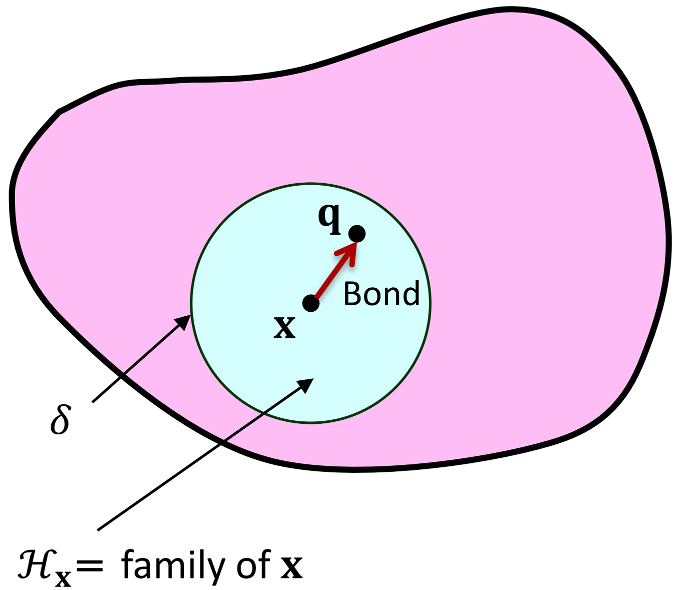
Peridynamics concepts: Horizon and family

- Any point $\mathbf{x}$ interacts directly with other points within a distance δ called the "horizon."
- The material within a distance δ of $\mathbf{x}$ is called the "family" of $\mathbf{x}$, $\mathcal{H}_{\mathbf{x}}$.

Peridynamic equilibrium equation

$$\int_{\mathcal{H}_{\mathbf{x}}} \mathbf{f}(\mathbf{q},\mathbf{x})\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}) = 0$$

$\mathbf{f}$ = bond force density



- The peridynamic field equations don't use spatial derivatives
  - so they are compatible with cracks.

General references

- SS, Journal of the Mechanics and Physics of Solids (2000)
- SS and R. Lehoucq, Advances in Applied Mechanics (2010)

Simplification: Bond-based peridynamics

- General peridynamic equation of motion:

$$\rho(x)\ddot{u}(x,t) = \int_{\mathcal{H}} f(q,x,t)\, dq + b(x,t).$$

- Bond-based: Each bond responds independently of the others, replace

$$f(q,x,t) = f(\eta,\xi), \qquad \xi = q - x, \qquad \eta = u(q) - u(x).$$

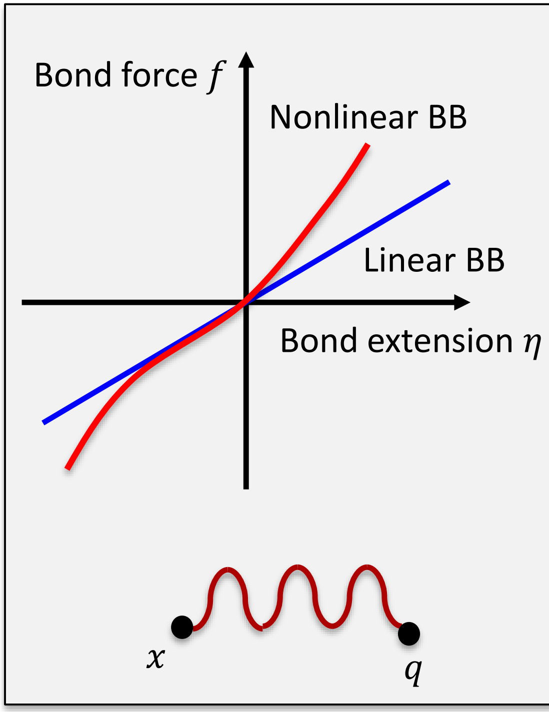
- Microelastic: Each bond is an elastic spring.

$$f(\eta,\xi) = \frac{\partial w}{\partial \eta}(\eta,\xi).$$

- Linear microelastic material:

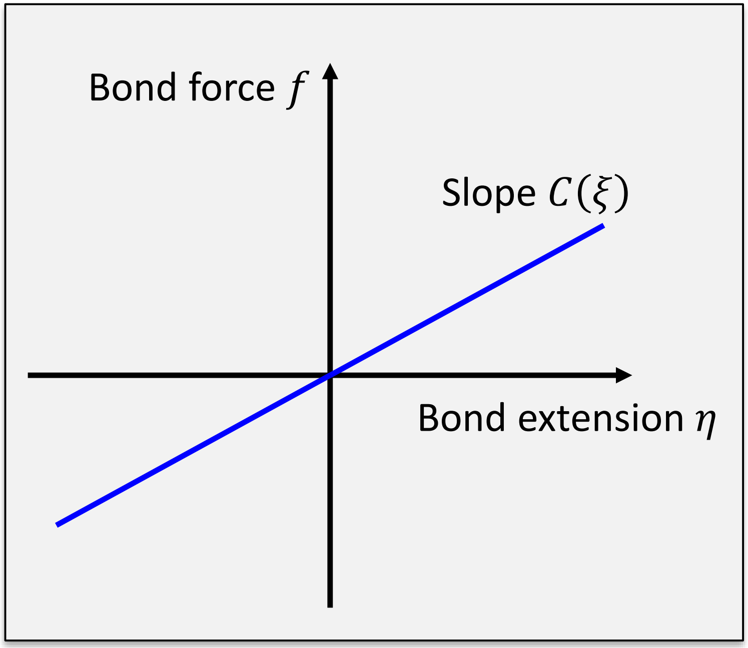
$$f(\eta,\xi) = C(\xi)\eta$$

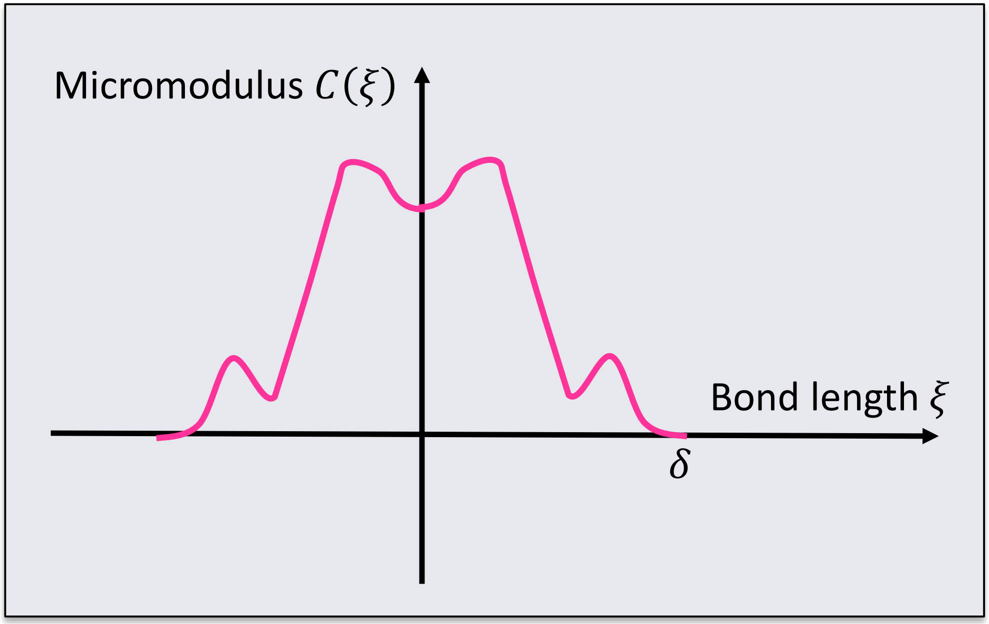
where C=micromodulus. Similar to Kunin's nonlocal theory (1983).



Micromodulus can depend on bond length

Physics requires $C(-\xi) = C(\xi)$.





Fourier transform and convolution

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- Fourier transform and inverse:

$$\bar{v}(k) = \mathcal{F}\{v\}(k) = \int_{-\infty}^{\infty} v(x) e^{-ikx}\, dx$$

$$v(x) = \mathcal{F}^{-1}\{\bar{v}\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{v}(k) e^{ikx}\, dk.$$

- Convolution:

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} g(x-p)h(p)\, dp\right\} = \bar{g}(k)\bar{h}(k).$$

Static waves

- Equilibrium equation:

$$\int_{-\delta}^{\delta} C(\xi)(u(x+\xi) - u(x))\, d\xi + b(x) = 0.$$

- Take FT:

$$(\bar{C}(k) - P)\bar{u}(k) + \bar{b}(k) = 0, \qquad P = \bar{C}(0).$$

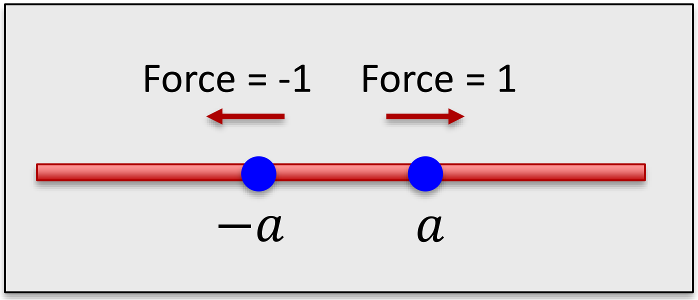
- Example: Two oppositely directed point loads.

$$b(x) = \Delta(x-a) - \Delta(x+a), \qquad a = 0.1$$

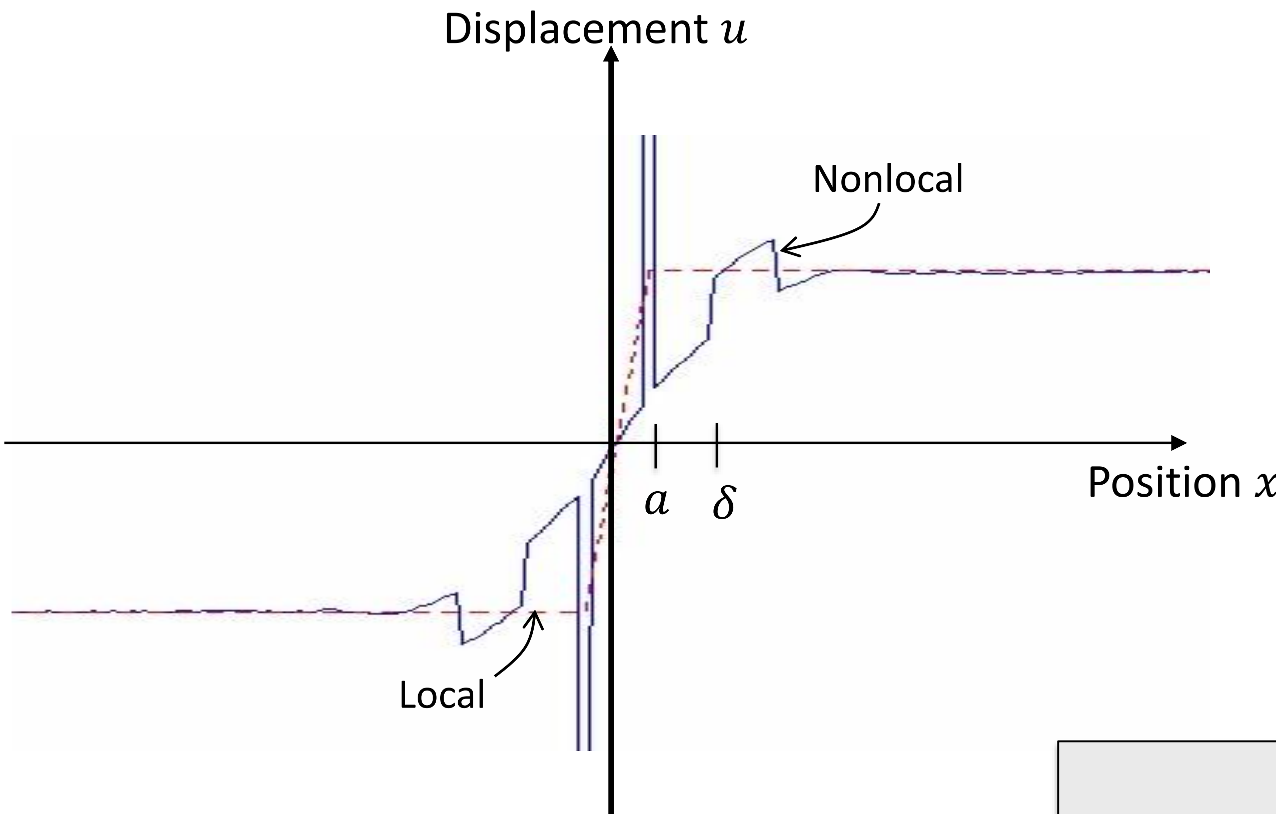
where Δ is the Dirac delta function.

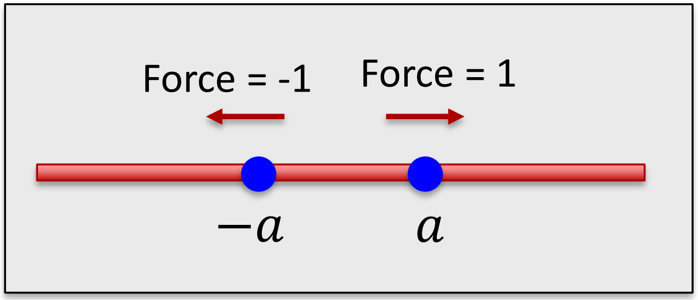
- Solution is

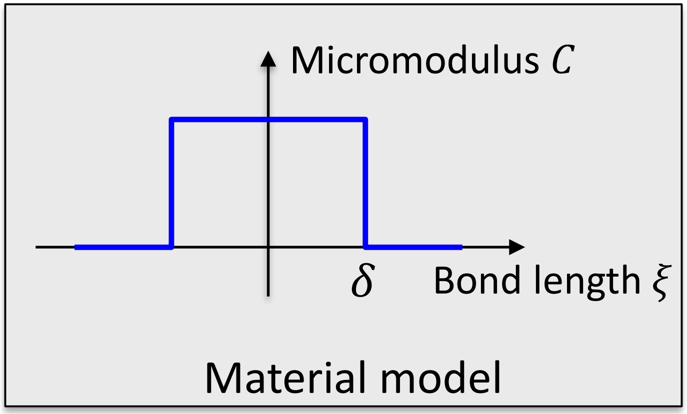
$$u(x) = \mathcal{F}^{-1}\left\{\frac{\bar{b}}{P - \bar{C}}\right\}.$$



Nonlocality creates strange features

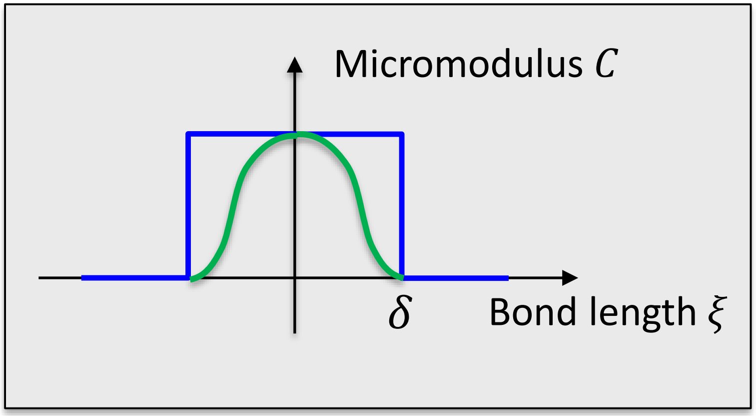






Observations about nonlocal static waves

- The displacement field has the same smoothness as the applied loads.
  - In the last example, this was a delta function.
  - But u gets smoother the farther away from the loading points you get.
- Smoother choices of the micromodulus function $C(\xi)$ would result in a smoother remote displacement field.
  - But there would still be delta functions in the displacement field.



SS, M. Zimmermann, R. Abeyaratne. "Deformation of a peridynamic bar." Journal of Elasticity 73 (2003): 173-190.

Dynamic linear waves: Dispersion

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- Equation of motion with $b = 0$:

$$\rho \ddot{u}(x,t) = \int_{-\infty}^{\infty} C(\xi)(u(x+\xi,t) - u(x,t))\, d\xi$$

- Assume a wave in an infinite homogeneous bar of the form

$$u(x,t) = e^{i(kx-\omega t)}.$$

- Then

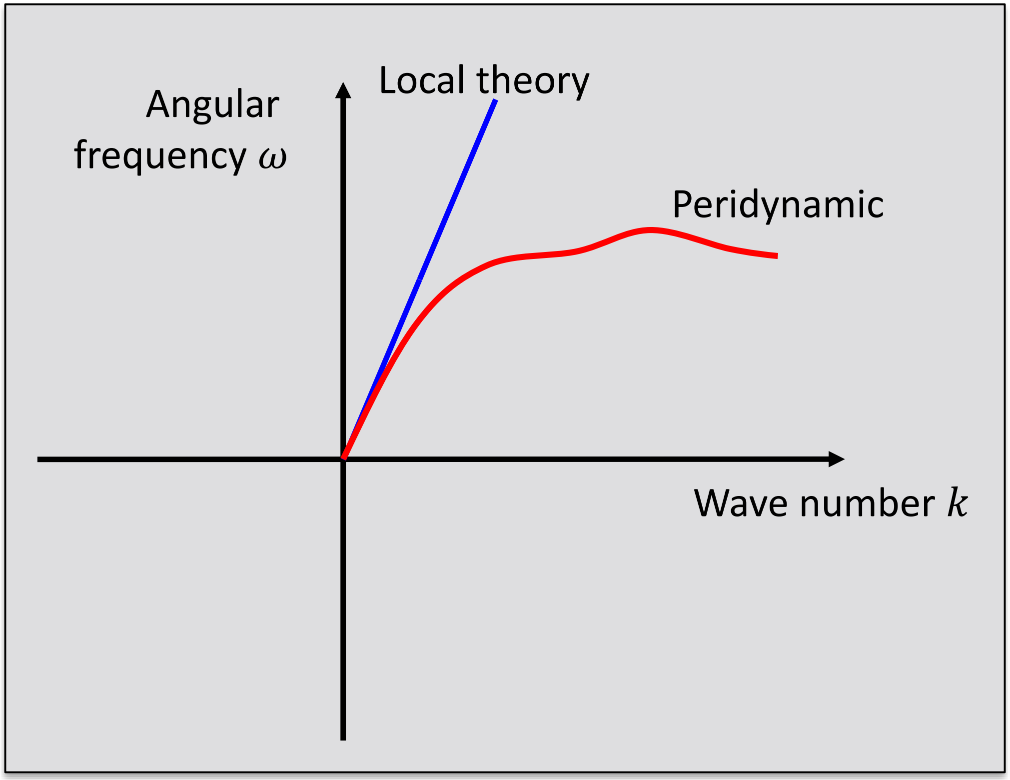
$$-\rho\omega^2 = \int_{-\infty}^{\infty} C(\xi)\left(e^{ik\xi} - 1\right)\, d\xi.$$

- Therefore the dispersion relation is

$$\omega(k) = \sqrt{\frac{P - \bar{C}(k)}{\rho}}, \qquad P = \bar{C}(0).$$

Dispersion curve

- PD coincides with the local theory for long wavelengths (small k).
- Phase velocity ω/k is not constant in PD.
- Group velocity $d\omega/dk$ can be nonpositive.

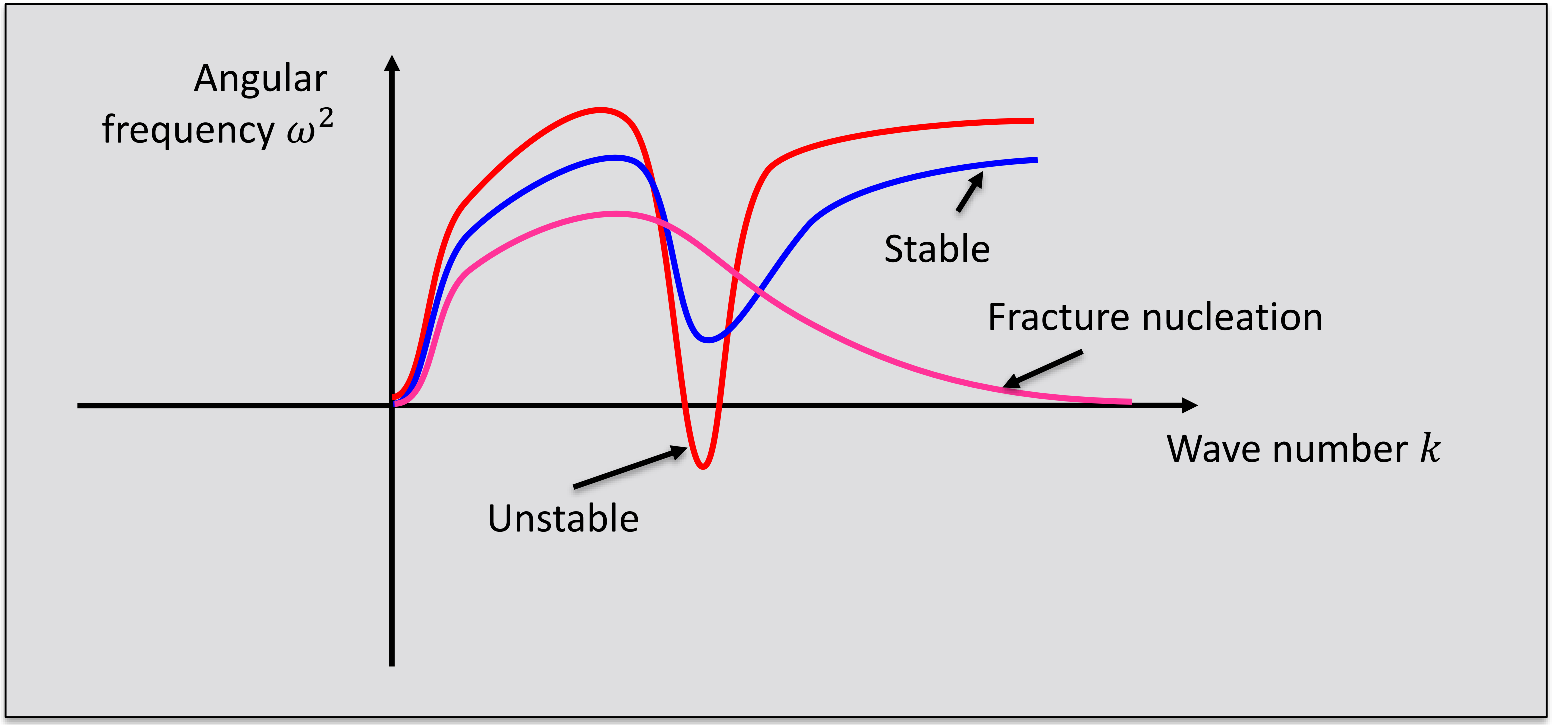


<u>Wave dispersion in (undamped) peridynamics</u>

- SS, *JMPS* (2000).
- Seleson, Parks, Gunzburger & Lehoucq, *Multiscale Modeling & Simulation* (2009).
- Weckner & SS, *Multiscale Computational Engineering* (2011).
- Gu, Zhang, Huang & Yv, *Engineering Fracture Mechanics* (2016).
- Butt, Timothy, & Meschke *Computational Mechanics* (2017) .

Dispersion curve and stability

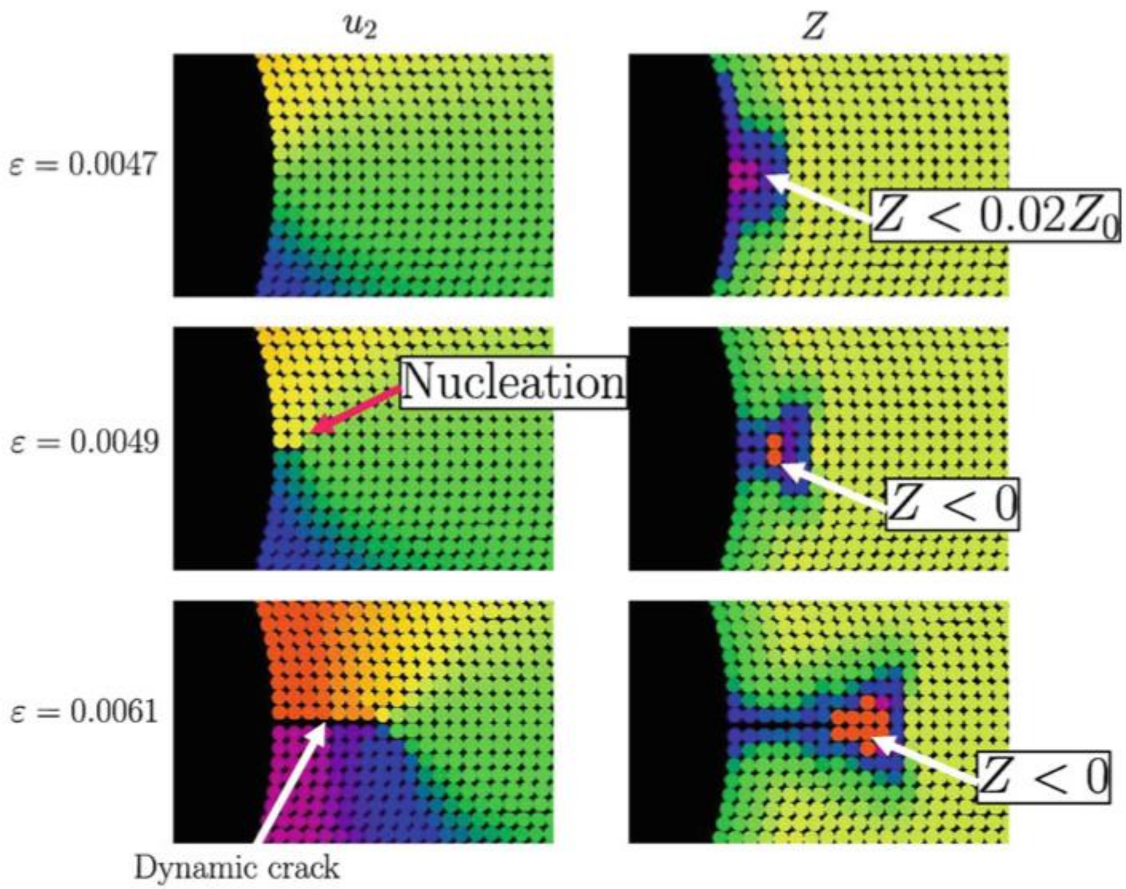
- Nonpositive $\omega(k)$ means waves can grow unboundedly over time.
- "Imaginary wave speed" since $\mathrm{V} = \omega(k)/k$.
- If $\omega(\infty) \to 0$, a small discontinuity can grow: fracture nucleation*.



- *Silling, S. A., Weckner, O., Askari, E., & Bobaru, F. (2010) International Journal of Fracture, 162, 219-227.
- Lipton, R. (2014) Journal of Elasticity, 117, 21-50.

Material instability can be a good thing

- There is a small elastically unstable region surrounding the tip of a growing crack.

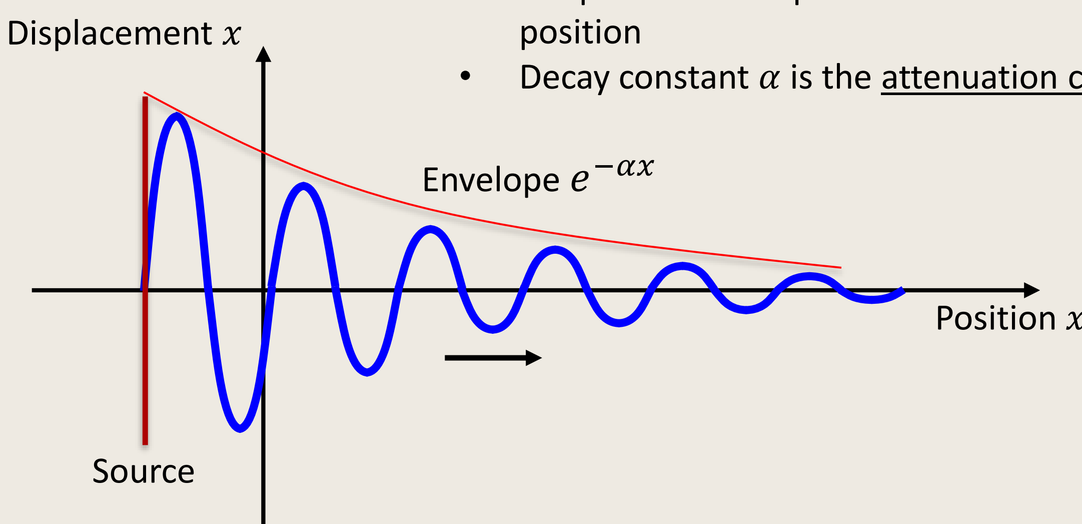


- *Silling, S. A., Weckner, O., Askari, E., & Bobaru, F. (2010) International Journal of Fracture, 162, 219-227.
- Lipton, R. (2014) Journal of Elasticity, 117, 21-50.

Wave attenuation limits the usefulness of ultrasonic imaging

- Long wavelengths have low attenuation.
  - Can see deeply but with low resolution.
- Short wavelengths have higher attenuation.
  - How resolution but small depth.

Attenuated steady wave:
- Amplitude is independent of time at any fixed position
- Decay constant α is the <u>attenuation coefficient</u>.



Microviscoelastic material

- General peridynamic equation of motion:

$$\rho(x)\ddot{u}(x,t) = \int_{\mathcal{H}} f(q,x,t)\, dq + b(x,t).$$

- Bond-based linear microviscoelastic material:

$$f(\eta,\dot{\eta},\xi) = C(\xi)\eta + D(\xi)\dot{\eta}$$

where C=micromodulus, D=damping modulus.

- Requirement for linear momentum conservation:

$$C(-\xi) = C(\xi) \qquad D(-\xi) = D(\xi).$$

- C and D can have different horizons (cutoff distances).
- Second law of thermodynamics implies

$$D(\xi) \geq 0.$$

Viscoelasticity in peridynamics

- Weckner & Mohamed, *Applied Mathematics and Computation* (2013).
- Mitchell, SAND2011-8064 (2011).
- Madenci & S. Oterkus. *Engineering Fracture Mechanics* (2017).
- Nadimi, Miscovic & McLennan, *Journal of Petroleum Science and Engineering* (2016).

Transformed equation of motion

- Transformed equation of motion leads to the following condition on $\omega(k)$:

$$\omega^2(k) + 2ir(k)\omega(k) - \omega_0^2(k) = 0,$$

$$r(k) := \frac{Q - \bar{D}(k)}{2\rho}, \qquad \omega_0(k) := \sqrt{\frac{P - \bar{C}(k)}{\rho}}.$$

- r and ω_0^2 depend only on the material properties.
- For an undamped wave, $D \equiv 0 \implies \omega(k) = \pm\omega_0(k)$.
- Otherwise, $\omega(k)$ is in general complex (for real k).

Attenuated steady waves

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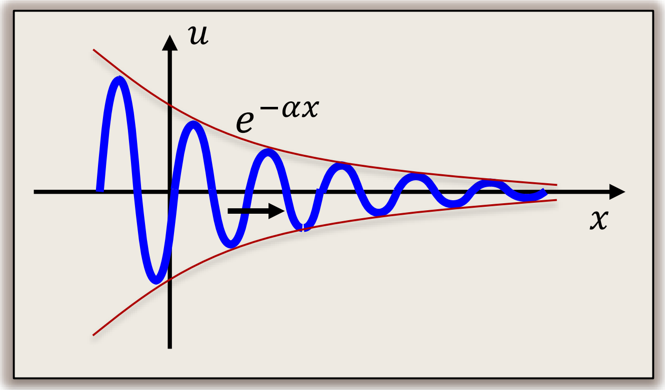
- Seek an attenuated wave solution of the form

$$u(x,t) = e^{-\alpha(k_0)x} e^{i(k_0 x - \omega(k_0)t)}, \qquad k_0, \omega \text{ real}$$

where $\alpha(k_0)$ is the *attenuation coefficient* (real).

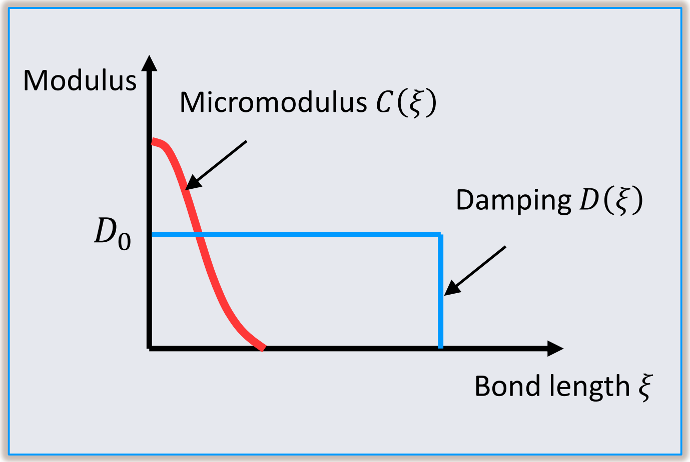


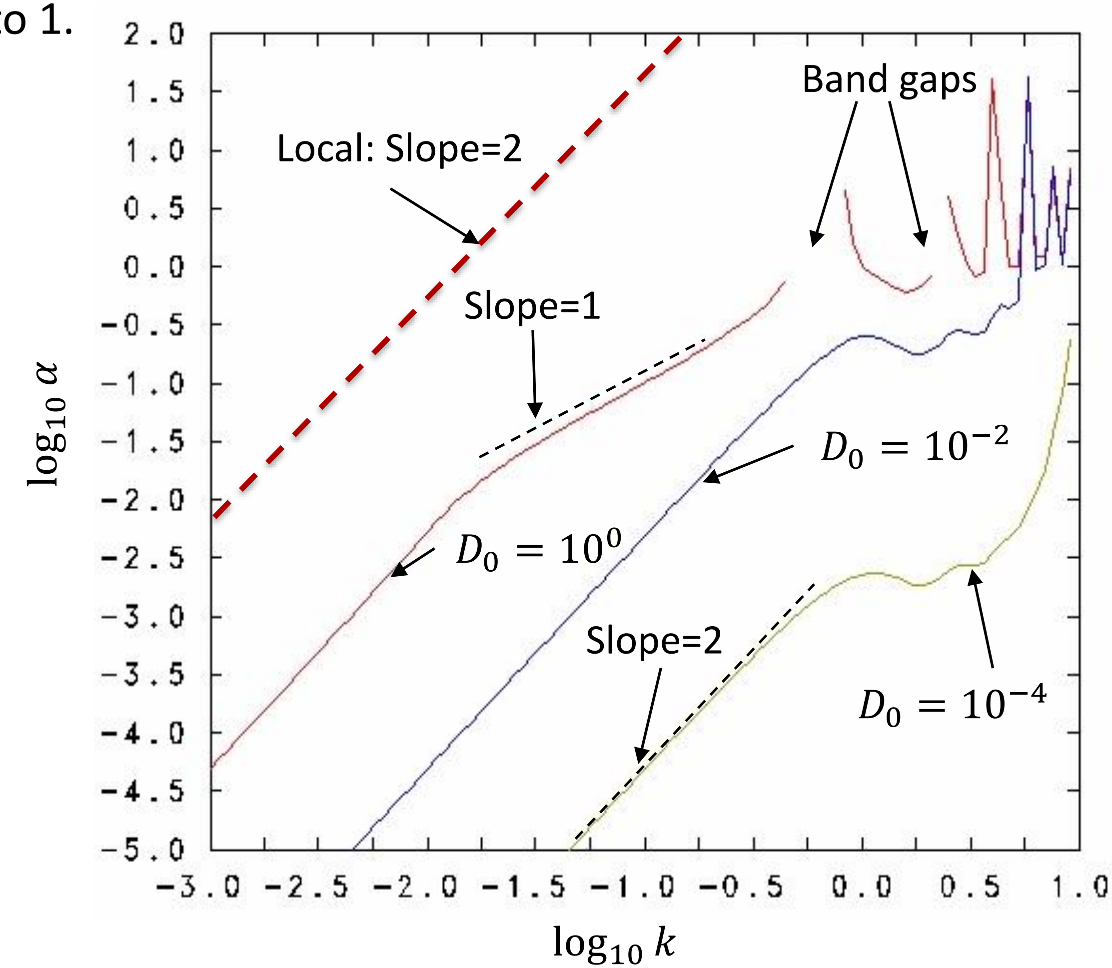
- Condition on α turns out to be

$$\text{Im}\left\{-ir(k+i\alpha) + \sqrt{\omega_0^2(k+i\alpha) - r^2(k+i\alpha)}\right\}$$

Example of a nonlocal attenuation curve

- Solve numerically the equation for $\alpha(k)$.
- Interesting features:
  - Transition in exponent from 2 to 1.
  - Band gaps.

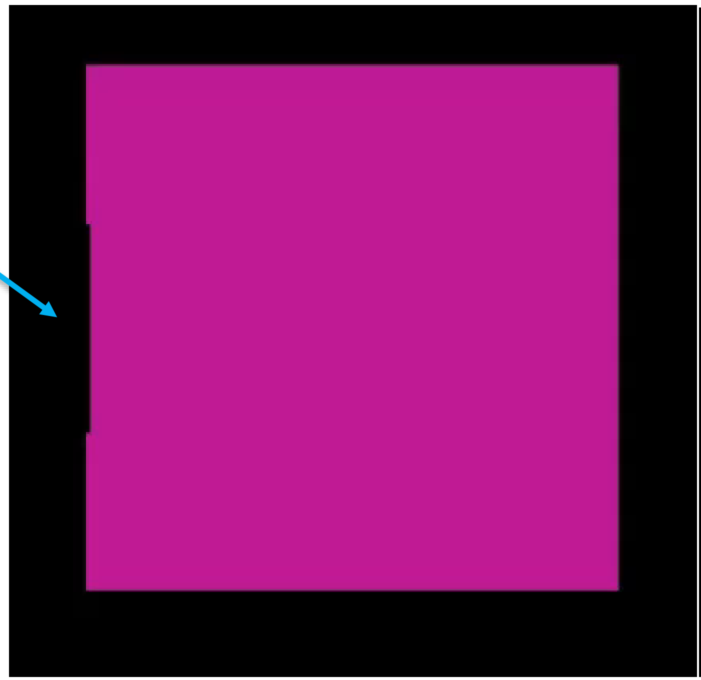




Linear vs. nonlinear waves: Example

VIDEOS

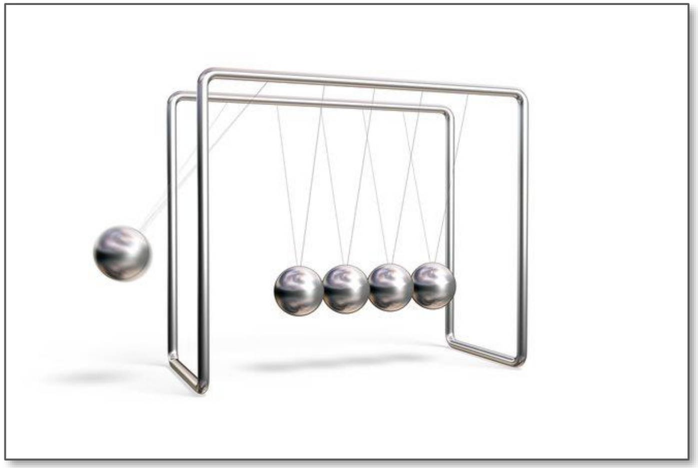
Step function for displacement at boundary

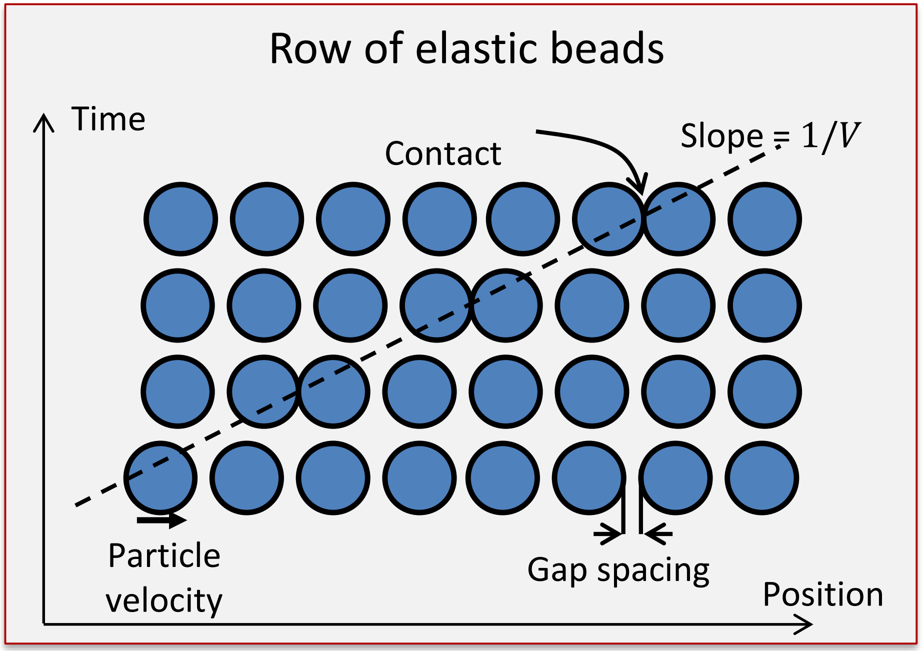


Linear

Nonlinear

What are solitary waves?





A solitary wave is a nonlinear wave that moves

- without dispersion
- without changing shape
- without dissipation
- without changing the state of the material it passes through.

Peridynamic soliton model

- Peridynamic 1D equation of motion:

$$\rho u_{tt}(x,t) = \int_{-\delta}^{\delta} f(u(x+\xi,t) - u(x,t), \xi)\, d\xi + b(x,t)$$

where u is the displacement, f is the bond force density, δ is the horizon, $b \equiv 0$ is the body force, and ξ is the bond.

- Material model:

$$f(\eta,\xi) = F(s)\mathrm{sgn}\,(\xi), \qquad s = \frac{\eta}{\xi}, \qquad 0 < |\xi| \leq \delta$$

$$\eta = u(x+\xi) - u(x).$$

where F is a function and s is the bond strain.

Nonlinear bond force

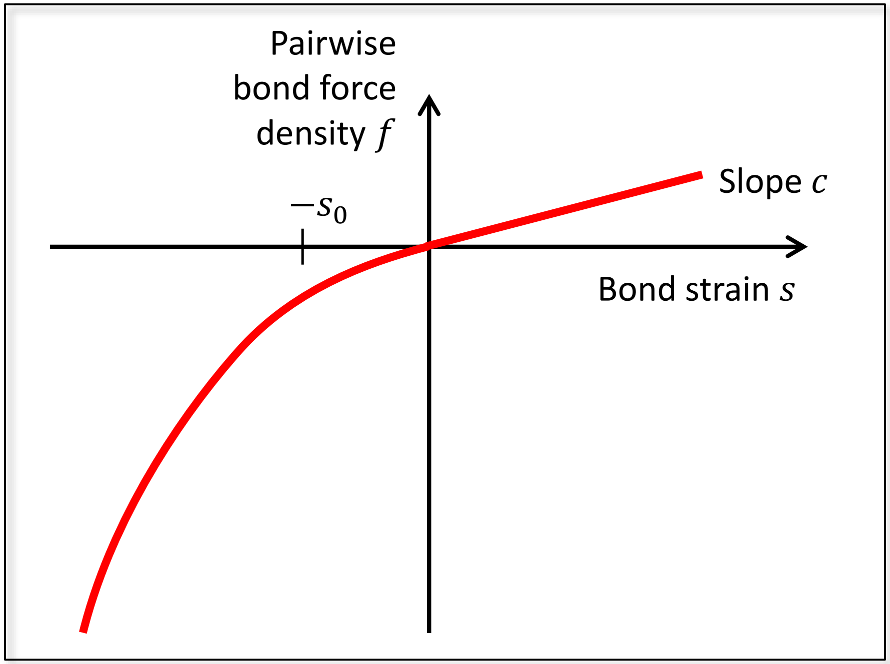
- Consider a material model that stiffens in compression (similar to Fermi-Pasta-Ulam lattice model):

$$F(s) = \begin{cases} c(1 - s/s_0)s & \text{if } s < 0, \\ cs & \text{otherwise} \end{cases}$$

where c and s_0 are positive constants.

- c is calibrated to the Young's modulus E:

$$c = \frac{2E}{\delta^2}.$$



Steady wave assumption

- We seek steady-wave solutions of the form

$$u(x,t) = U(z), \qquad z = x - Vt$$

where V is the wave velocity (to be determined).

- The equation of motion becomes

$$\rho V^2 U''(z) = \int_{-\delta}^{\delta} f(\eta(z,\xi),\xi)\, d\xi$$

- Taylor expansion

$$U(z+\xi) = U(z) + U'(z)\xi + \frac{U''(z)\xi^2}{2} + \frac{U'''(z)\xi^3}{6} + \frac{U''''(z)\xi^4}{24} + O(\delta^5).$$

Our ODE resembles the KdV equation

- The Taylor expansion leads to

$$\left(\frac{\rho V^2}{E} - 1\right)\epsilon' = \left[-\frac{2\epsilon\epsilon'}{s_0}\right] + \left[\frac{\epsilon'''}{24} - \frac{\epsilon'\epsilon''}{6s_0} - \frac{\epsilon\epsilon'''}{12s_0}\right]\delta^2.$$

where the *local strain* field ϵ is defined by

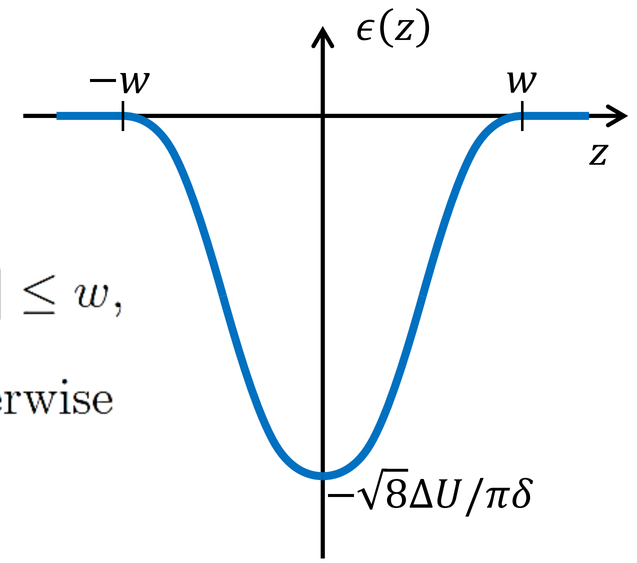
$$\epsilon(z) = U'(z).$$

- Compare KdV equation:

$$\phi_t + \alpha\phi_{xxx} + \beta\phi\phi_x = 0.$$

- Some terms are the same, others different.

Exact solution to the 3rd order nonlinear ODE



- An exact solution to the ODE is

$$\epsilon(z) = \begin{cases} -\frac{\sqrt{8}\Delta U}{\pi\delta}\cos^2\left(\frac{\sqrt{2}\ z}{\delta}\right) & \text{if } |z| \le w, \\ 0, & \text{otherwise} \end{cases}$$

where the pulse half-width is

$$w = \frac{\pi\delta}{\sqrt{8}}$$

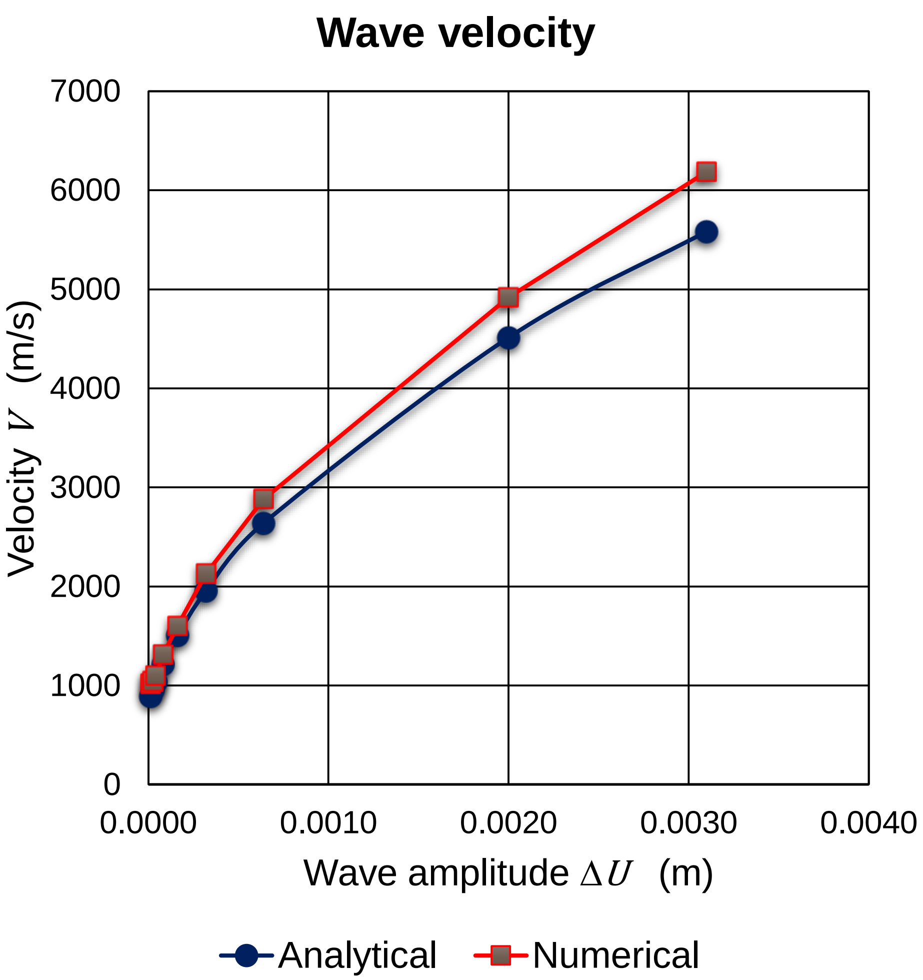
and ΔU is the total displacement change through the pulse.

- The wave velocity depends on the total displacement:

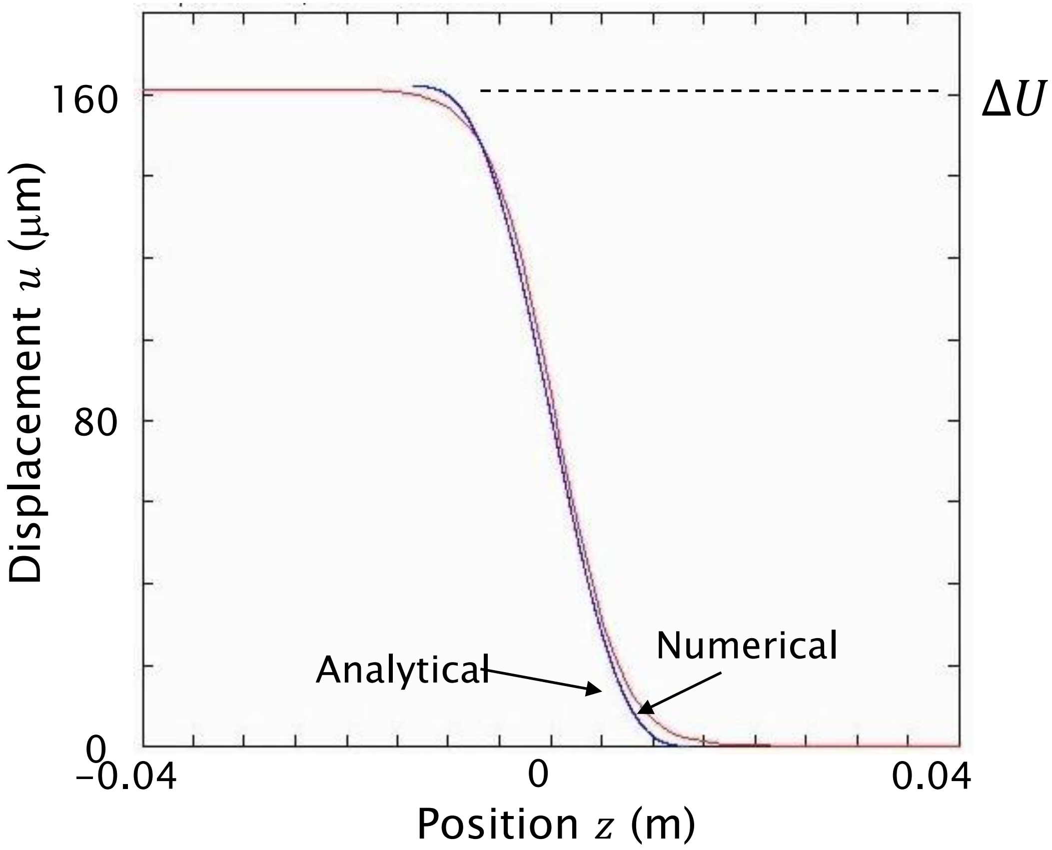
$$V = \pm\sqrt{\frac{2E}{3\rho}\left(1 \pm \frac{\sqrt{8}\Delta U}{\pi\delta s_0}\right)}.$$

Solitary wave velocity (ODE vs Emu)

$$V = \pm\sqrt{\frac{2E}{3\rho}\left(1 + \frac{\sqrt{8}\Delta U}{\pi \delta s_0}\right)}$$



Wave shape (ODE vs. Emu)



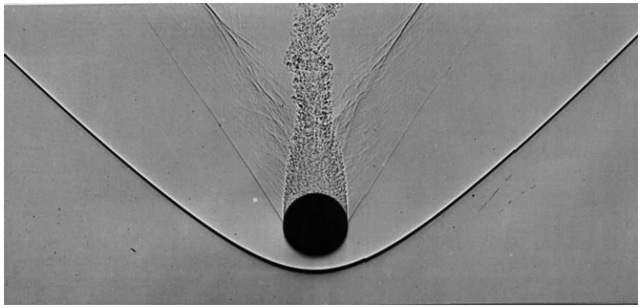
Collisions between solitary waves

VIDEOS

- SS, Solitary waves in a peridynamic elastic solid, JMPS 96 (2016) 121-132.
- Pego, Robert L., and Truong-Son Van. "Existence of solitary waves in one dimensional peridynamics." arXiv preprint arXiv:1802.00516 (2018).

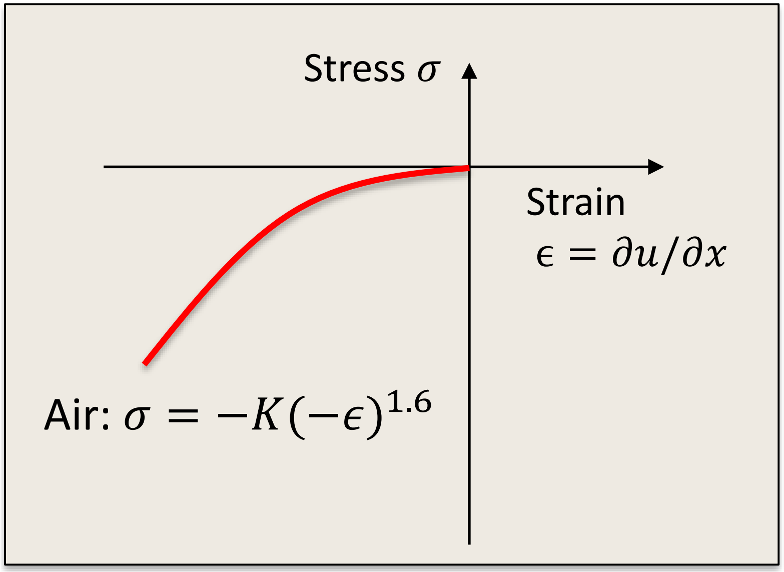
Shock waves

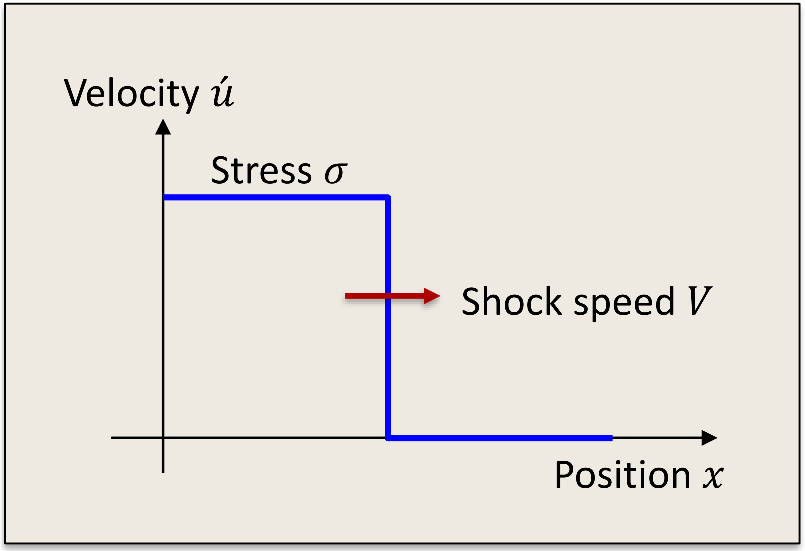
- A shock is a wave that carries a large jump in velocity.
  - (A solitary wave carries a large jump in displacement.)
- Important applications:
  - Supersonic gas flow
  - Impact and detonation waves



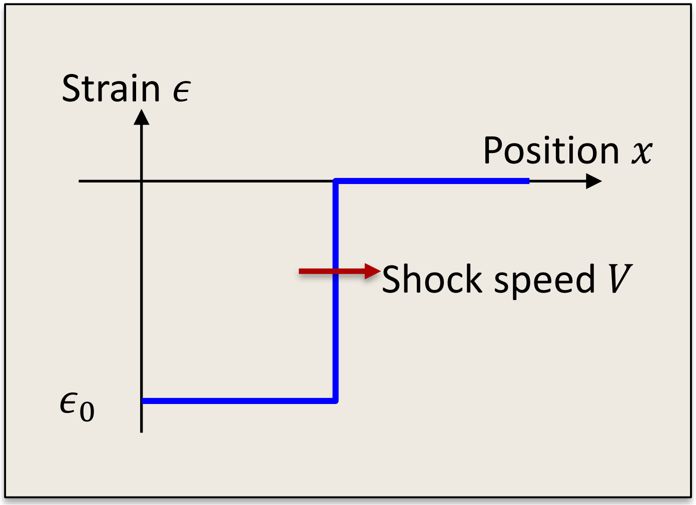
Shock in air

Image: Milton Van Dyke, An Album of Fluid Motion (1982).





Shock waves: Elastic model predicts wave velocity but is wrong physically



- Local theory: jump conditions allow weak solutions

$$\epsilon_0 = \dot{u}/V$$

$$\sigma(\epsilon_0) = -\rho V \dot{u}.$$

- Given $\dot{u}$ we can find V and σ.
- However the resulting W fails to satisfy the energy jump (Rankine-Hugoniot) condition:

$$e = -\frac{1}{2}\dot{u}\sigma \neq W$$

where e is the internal energy density behind the shock.

- So without some way of dissipating energy, this is not an admissible solution.

Same material model as with solitons but with nonlinear bond damping

- 1D equation of motion:

$$\rho \ddot{u}(x,t) = \int_{-\delta}^{\delta} f(\eta, \dot{\eta}, \xi)\, d\xi + b(x,t)$$

- Material with nonlinear elastic and damping terms:

$$f(\eta, \dot{\eta}, \xi) = \left(F^e(s) + F^d(\dot{s})\right) \operatorname{sgn}(\xi), \qquad s = \frac{\eta}{\xi}, \qquad 0 < |\xi| \le \delta$$

where s=bond strain, ξ=bond vector.

$$F^e(s) = \begin{cases} c(1 - s/s_0)s & \text{if } s < 0, \\ cs & \text{otherwise} \end{cases},$$

$$F^d(\dot{s}) = \begin{cases} -D\dot{s}^2 & \text{if } \dot{s} < 0, \\ 0 & \text{otherwise} \end{cases}$$

New

where c and D are constants.

Dissipative material model leads to a stable shock wave

- Use the same trick again to get an ODE:

$$\left[\frac{\rho V^2}{E} - 1\right]\epsilon' = \left[-\frac{2\epsilon\epsilon'}{s_0} - 2\beta\epsilon'\epsilon''\right] + \left[\frac{\epsilon'''}{24} - \frac{\epsilon'\epsilon''}{6s_0} - \frac{\epsilon\epsilon'''}{12s_0} - \frac{\beta\epsilon'\epsilon''''}{12} - \frac{\beta\epsilon''\epsilon'''}{6}\right]\delta^2.$$

- Ansatz:

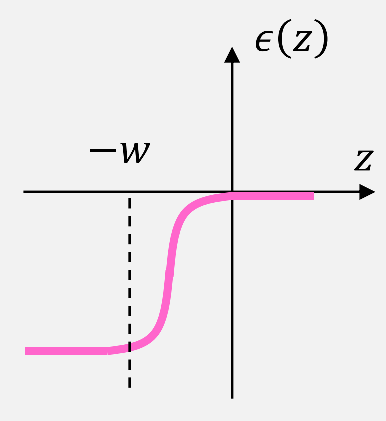
$$\epsilon(z) = \begin{cases} \frac{\epsilon_0}{2}(1-\cos kz) & \text{if } -\pi \le kz \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

- This $\epsilon(z)$ satisfies the ODE with

$$V = \sqrt{\frac{E}{2\rho}\left[L + \sqrt{L\left(L - \frac{\delta^2\rho}{3Ds_0}\right)}\right]}, \qquad L = 1 - \frac{\epsilon_0}{s_0}$$

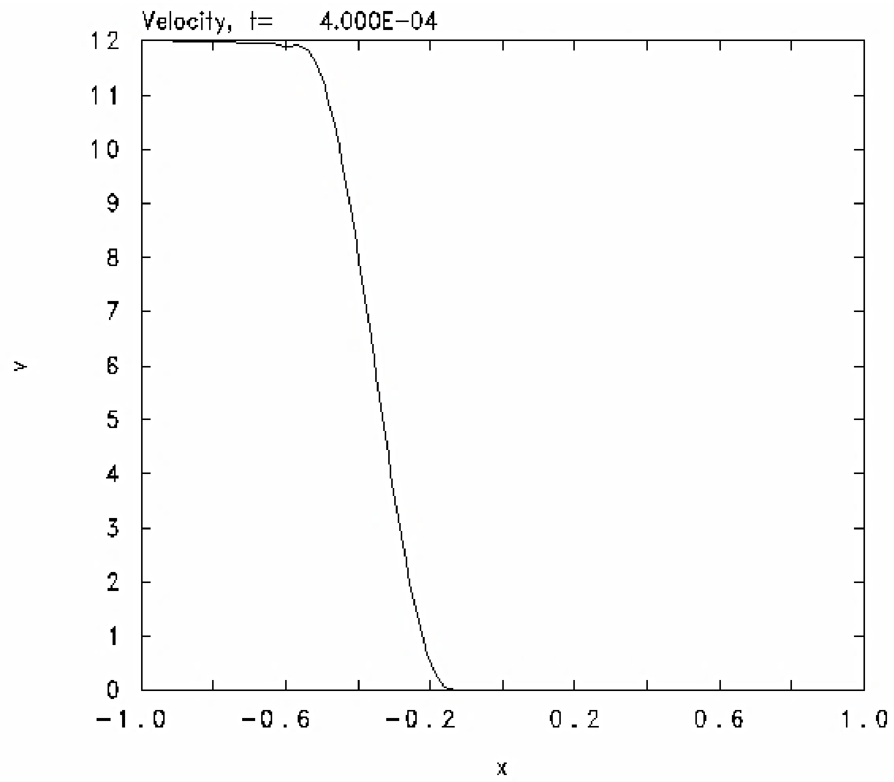
- Shock thickness is

$$w = \frac{\pi V}{2}\sqrt{\frac{D}{c}}.$$



Shock wave structure

Numerical solution also shows a stable shock wave



Nonlinear nonlocal waves: an almost completely unexplored area

- Just by choosing an appropriate material model, the basic equations without modification reproduce
  - Solitary waves
  - Phase boundaries
  - Fracture
  - Bending waves of beams and shells (O'Grady & Foster)
  - Unknown:
    - Periodic media
    - Metamaterials
    - Band gaps
    - Quantized energy levels
    - Monotonicity
    - Blow-up
    - Scattering

Extra slides

General solution for the attenuation coefficient

- Set $p = k - i\alpha$.

$$\omega^2(p) + 2ir(p + i\alpha)\omega(p) - \omega_0^2(p + i\alpha) = 0.$$

- Solve for $\omega(p)$:

$$\omega(p) = -ir(p + i\alpha) + \sqrt{\omega_0^2(p + i\alpha) - r^2(p + i\alpha)}.$$

- Our ansatz was that $\omega(p)$ is real whenever is p real.
- For any real p, find α such that

$$\mathrm{Im}\left\{-ir(p + i\alpha) + \sqrt{\omega_0^2(p + i\alpha) - r^2(p + i\alpha)}\right\} = 0.$$

- This is a nonlinear algebraic equation that can be solved numerically for $\alpha(p)$. (Examples later.)

Wave solution dissipates energy

- Rankine-Hugoniot condition:

$$e = \frac{\sigma}{2}\epsilon = \frac{E}{2}\left(\epsilon^2 - \frac{\epsilon^3}{s_0}\right)$$

- Define the dissipated energy Φ by

$$\Phi = e - W.$$

- Compute

$$\Phi = \left[\frac{E}{2}\left(\epsilon^2 - \frac{\epsilon^3}{s_0}\right)\right] - \left[\frac{E}{2}\left(\epsilon^2 - \frac{2\epsilon^3}{3s_0}\right)\right] = \frac{-E\epsilon^3}{6s_0} > 0$$

- So our shock wave obeys the dissipation inequality $\dot{\Phi} > 0$ which is consistent with the second law of thermodynamics.

Waves interact weakly (Emu)

- Larger wave overtakes a smaller wave.