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Algebraic Linearity Preserving Flux Limiting for Systems of PDEs

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Motivation

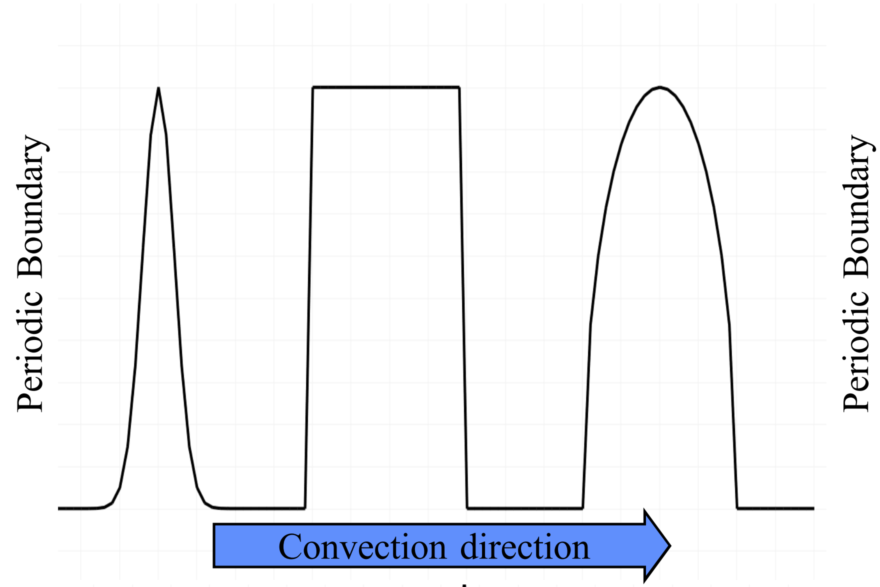
$$\frac{\partial u}{\partial t} + \nabla \cdot F(u) = 0$$

- Many physically relevant problems
  - Fluids, Drift-diffusion, Species transport
- Stability of solution is challenging
- Galerkin does poorly for shocks and steep gradients

Shu-Osher Problem

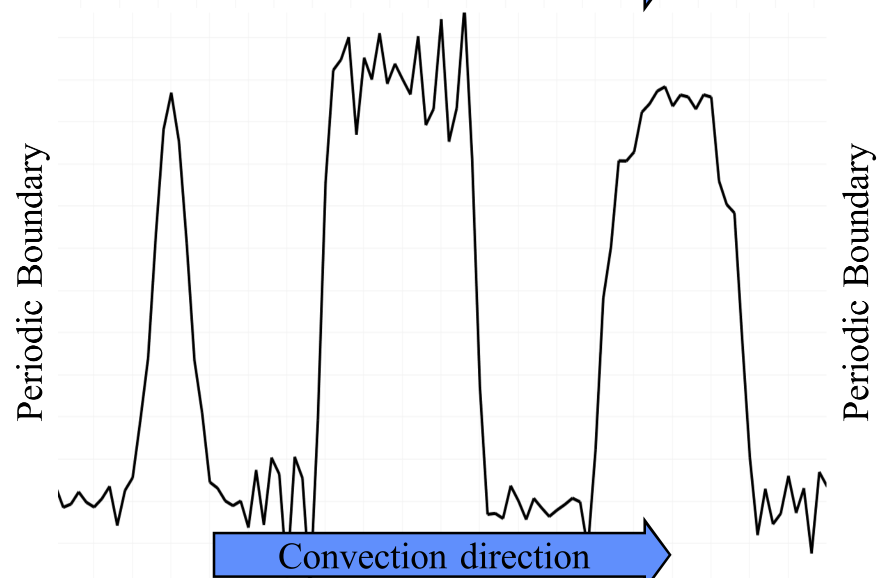
1D Transient Problem:

- smooth regions
- shock-like gradients
- sharp peak
- periodic solution



Galerkin solution (1 period):

- Crank-Nicolson
- clearly unstable
- square wave error disrupts hump



A High-Order and Low-Order Discretization

Galerkin semi-discretized problem:

$$M \frac{du}{dt} + Ku = 0$$

Define a “low-order” diffusive problem algebraically:

$$M_L \frac{du}{dt} + \tilde{K}u = 0 \qquad \text{where} \qquad \tilde{K} = K - D$$

Diffusion operator ‘D’ defined algebraically:

$$d_{ij} = \max(k_{ij}, 0, k_{ji}) \quad \text{and} \quad d_{ii} = -\sum_{j \neq i} d_{ij}$$

Anti-Diffusive Flux

High-order problem is sum of the low-order and "anti-diffusive flux"

$$M\dot{u} + Ku = M_L\dot{u} + \tilde{K}u + (M - M_L)\dot{u} + (K - \tilde{K})u$$

Can rewrite as a sum of numerical fluxes

$$((M - M_L)\dot{u} + (K - \tilde{K})u)_i = -\sum_{j \neq i} m_{ij}(\dot{u}_i - \dot{u}_j) - \sum_{j \neq i} d_{ij}(u_i - u_j)$$

Flux Limiting

The flux limited residual is

$$R(u) = M_L \dot{u} + K u - \bar{f}$$

where

$$\bar{f}_i = -\sum_{j \neq i} \alpha_{ij}^M m_{ij} (\dot{u}_i - \dot{u}_j) - \sum_{j \neq i} \alpha_{ij}^D d_{ij} (u_i - u_j)$$

Flux Limiting

The limiters satisfy

$$0 \leq \alpha_{ij}^M, \alpha_{ij}^D \leq 1$$

and are chosen so the LED criteria is met

$$q_i^M (u_i^{min} - u_i) \leq \sum_{j \neq i} \alpha_{ij}^M m_{ij} (\dot{u}_i - \dot{u}_j) \leq q_i^M (u_i^{max} - u_i)$$

$$q_i^D (u_i^{min} - u_i) \leq \sum_{j \neq i} \alpha_{ij}^D d_{ij} (u_i - u_j) \leq q_i^D (u_i^{max} - u_i)$$

where N_i is the set of nodes neighboring i

$$u_i^{min} = \min_{j \in N_i} (u_j), \quad u_i^{max} = \max_{j \in N_i} (u_j)$$

Flux Limiting: Algorithm

Flux limiting algorithm assures that LED bounds are met:

1. Seperate positive and negative anti-diffusive fluxes:

$$P_i^- = \sum_{j \neq i} \min(0, f_{ij}) \quad \text{and} \quad P_i^+ = \sum_{j \neq i} \max(0, f_{ij})$$

2. Build bounding quantities:

$$Q_i^- = q_i(u_i^{min} - u_i) \quad \text{and} \quad Q_i^+ = q_i(u_i^{max} - u_i)$$

3. Build limiting ratios (with symmetry):

$$R_i^- = \min(1, \frac{Q_i^-}{P_i^-}) \quad \text{and} \quad R_i^+ = \min(1, \frac{Q_i^+}{P_i^+})$$

4. Construct limiters:

$$\alpha_{ij} = \begin{cases} \min(R_i^-, R_j^+) & f_{ij} \leq 0 \\ \min(R_i^+, R_j^-) & f_{ij} \geq 0 \end{cases}$$

Linearity Preserving Flux Limiting

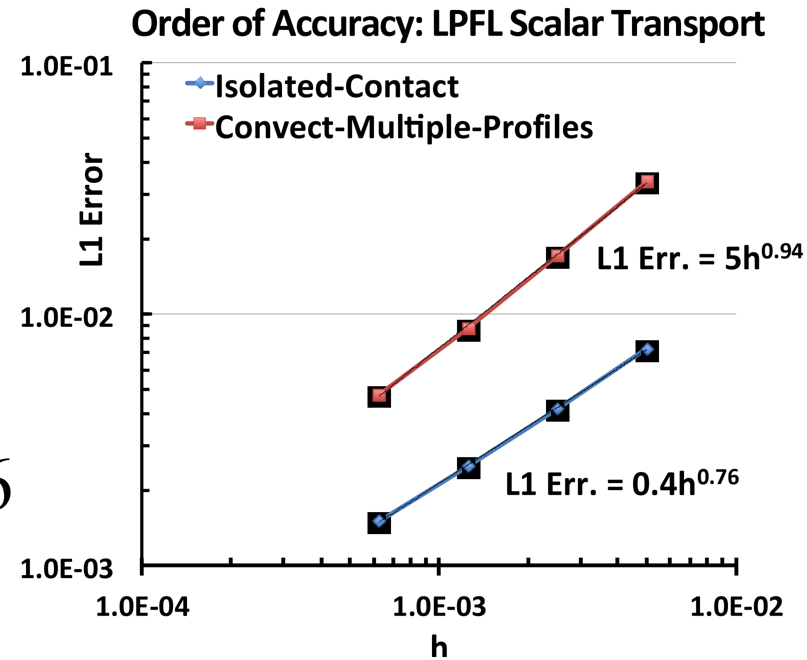
The bounds in the LED criteria are designed so that linear functions are not limited (thus the high-order method is used)

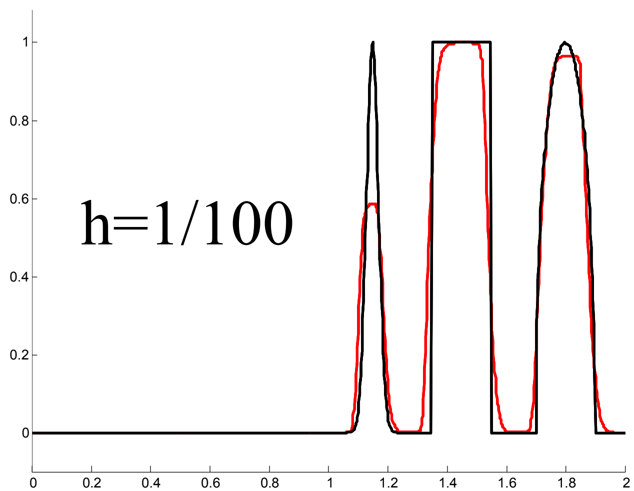
$$
\begin{aligned}
u_i - u_j &= \nabla u_i \cdot (x_i - x_j) \\
&= \frac{1}{m_i} \sum_{k \neq i} \int \phi_i \nabla \phi_k (u_k - u_i) \cdot (x_i - x_j) \\
&\leq \left(\frac{1}{m_i} \sum_{k \neq i} \left| (x_i - x_j) \cdot \int \phi_i \nabla \phi_k \right| \right) (u_i^{max} - u_i)
\end{aligned}
$$

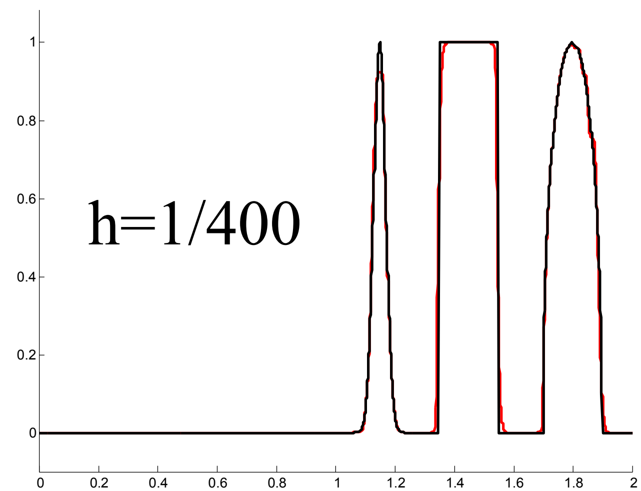
$$
q_i = \sum_{j \neq i} d_{ij} \left(\frac{1}{m_i} \sum_{k \neq i} \left| (x_i - x_j) \cdot \int \phi_i \nabla \phi_k \right| \right)
$$

Results for a Scalar PDE

- Two different problems
  1. Isolated Contact
  2. Multiple Profiles
- Spatial convergence rate: 0.76







Coupled Linear PDEs

Use scalar dissipation to construct 'D'

1. Compute maximum e.v. of $\mathbf{K_{ij}}$ (power method)
2. Construct diffusive operator by

$$\mathbf{D}_{ij} = \lambda_{max}(\mathbf{K}_{ij})\mathbf{I} \ \text{ and } \ \mathbf{D}_{ii} = -\sum_{j \neq i} \mathbf{D}_{ij}$$

1. Limiters applied to each coupled entry

$$\sum_{j \neq i} \alpha_{ij} \mathbf{D}_{ij} (U_i - U_j)$$

2. Use synchronized LPFL (D and M-M_L are diagonal)

$$\alpha_{ij} = \min(\alpha_{ij}^{u1}, \alpha_{ij}^{u2}, \ldots)$$

A Convected Wave

In primitive variables (two coupled PDEs):

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} a & c \\ c & a \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} = 0$$

In characteristic variables (two decoupled PDEs):

$$\frac{\partial}{\partial t}\begin{bmatrix} q \\ r \end{bmatrix} + \begin{bmatrix} a+c & \\ & a-c \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} q \\ r \end{bmatrix} = 0$$

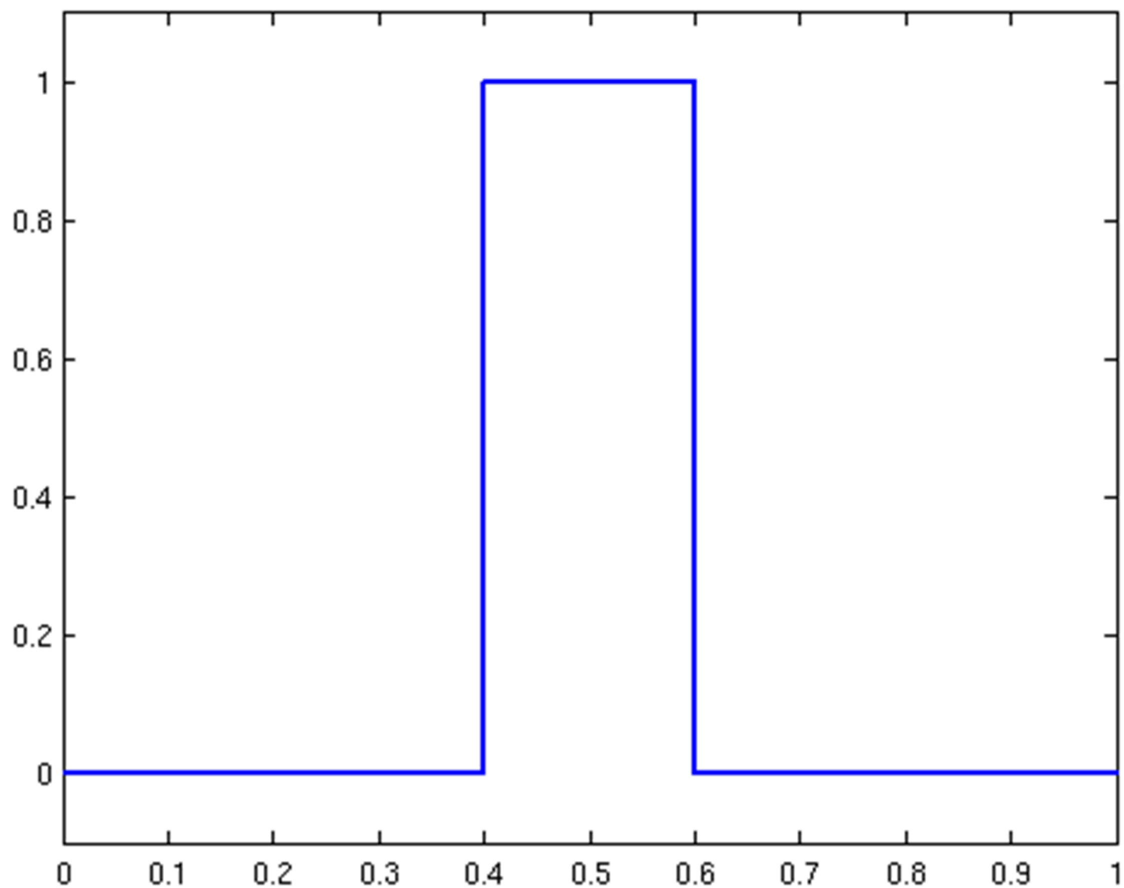
where

$$\begin{bmatrix} u \\ v \end{bmatrix} = 0.5 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix}$$

A Convected Wave

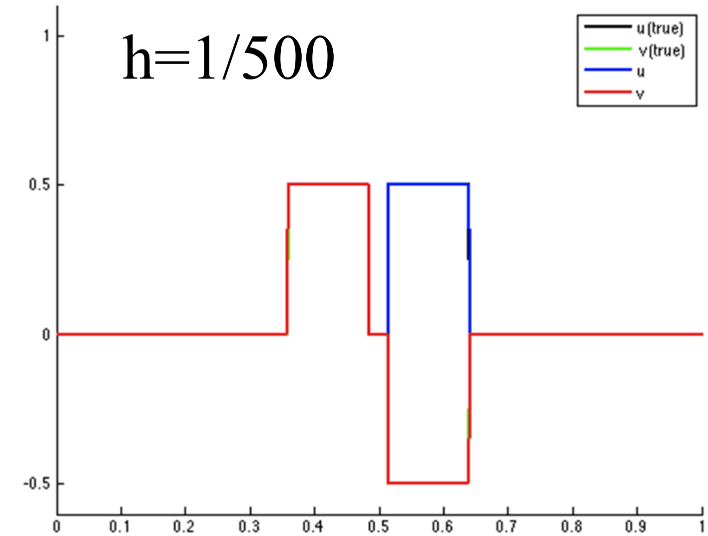
Two speeds seen in plot of ‘u’:

1. a+c (moving to the right)
2. a-c (moving to the left)

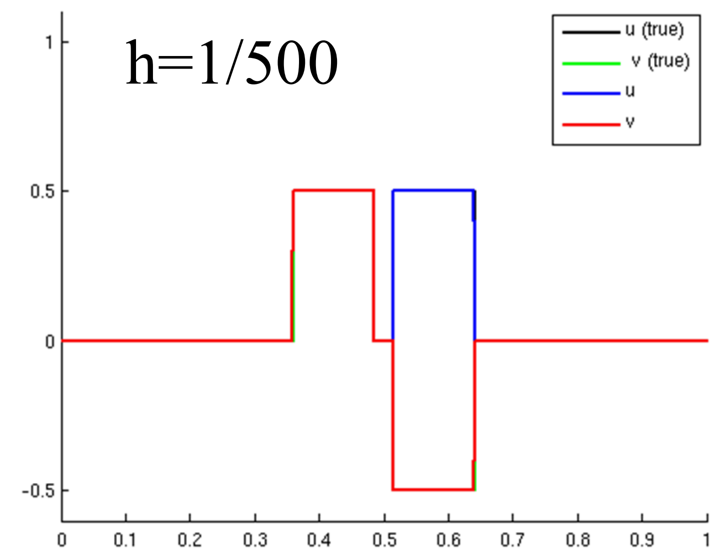


A Convected Wave: Primitive Variables

Limiting primitive variables with coupled limiting and scalar dissipation:

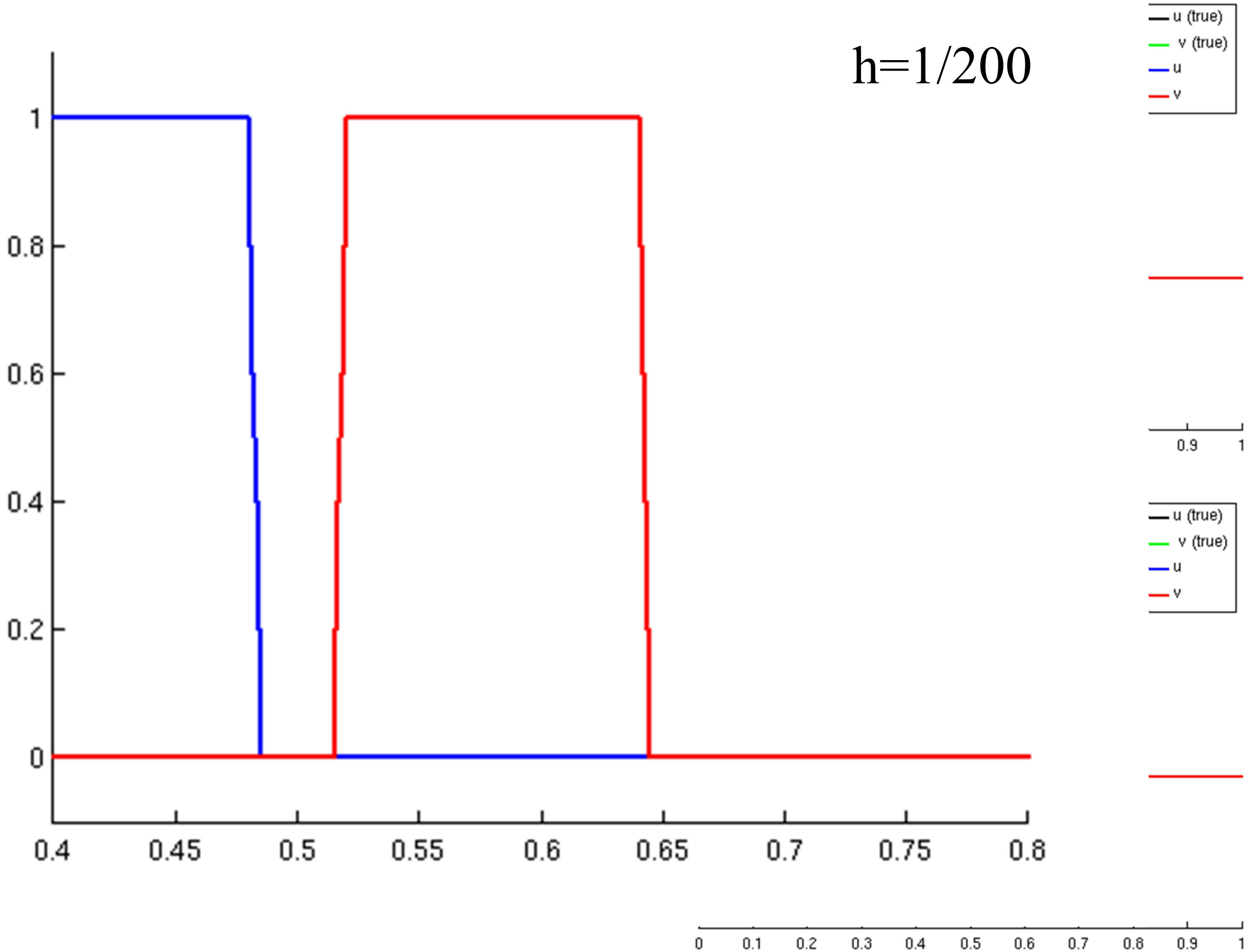


Limiting characteristic variables with algebraic linearity preserving FCT:



A Convected Wave: Characteristic Variables

Lim
witl
scal

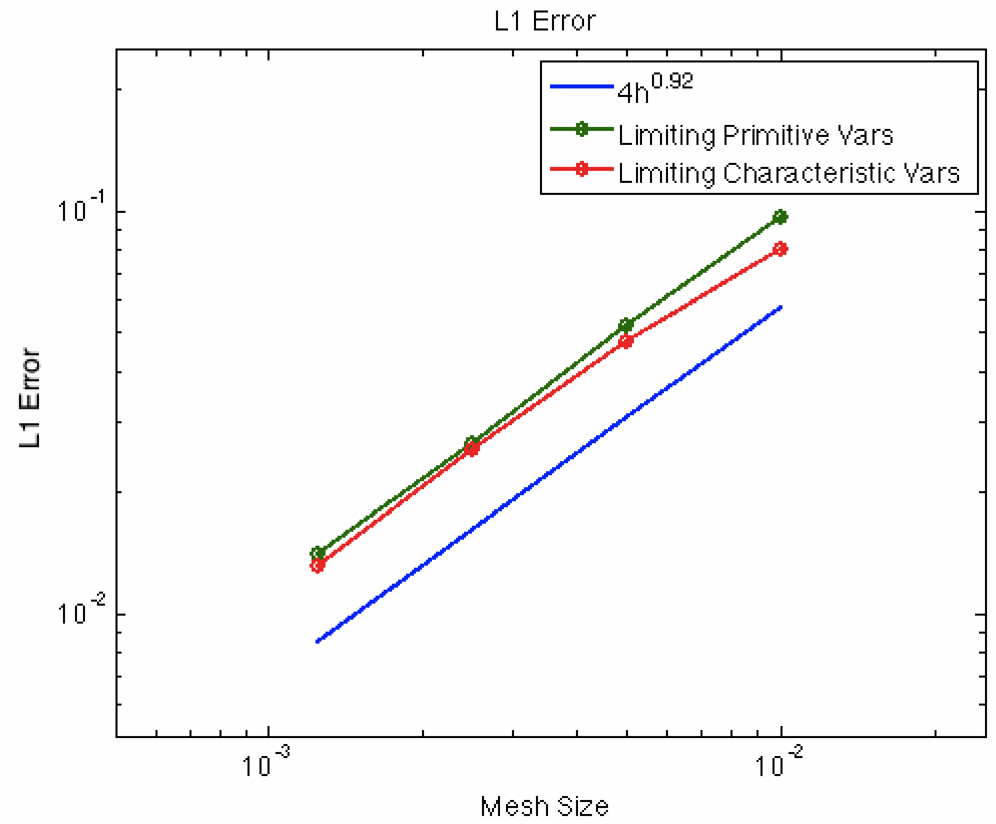


Lim
vari
line

A Convected Wave: Convergence

Convergence rate of 0.9

- Limiting characteristics slightly more accurate
- Over/undershoots characteristics with coupled limiting
- Caveat: Only a linear problem



Summary

- Presented an algebraic framework for flux corrected transport
- Discussed a linearity preserving limiter
- Showed results for a Shu-Osher problem
- Talked about a coupled wave problem:
  - Presented an algebraic diffusion operator based on scalar dissipation
  - Results comparing limiting of characteristic equations and coupled equations