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SAND2014-15071PE

Ordinary peridynamic materials based on nonlinear bond-strain measures

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17th U.S. National Congress on Theoretical and Applied Mechanics
Michigan State University, 15-20 June 2014

Institute for Soldier Nanotechnologies

U.S. DEPARTMENT OF ENERGY

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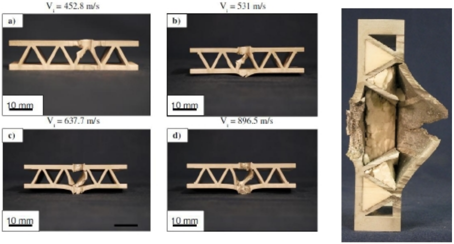
Computational framework for simulating challenging problems in mechanics.

Computational objectives

- robustness
- convergence
- avoid instabilities
- represent fracture
- scalability



Ballistic impact of aluminum sandwich structures at various impact velocities. Courtesy of Wadley, 2010.

Edge on impact of brittle materials (Umberger, Love 2011).

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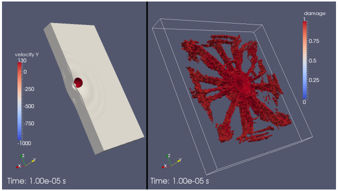
Classical damage models:



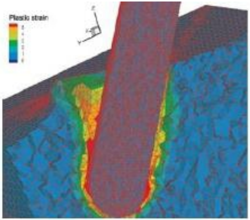
Hautefeuille, Seagraves, Talamini and Tupek 2011.

Successful for ductile failure

Mesh dependency and convergence issues

Difficulties with recontact

Adaptive remeshing:



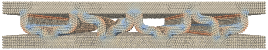
Mauch, et al. 2006.

Not robust

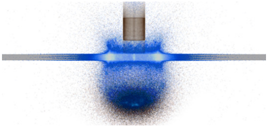
Difficult to parallelize

Reduced accuracy

Particle methods:



Tupek 2011.



Ortiz 2011.

Can't represent discrete cracks

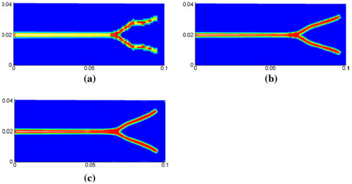
Instabilities are common

The peridynamic alternative (Silling 2000)

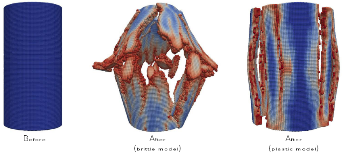
Integral formulation: $\nabla \cdot \boldsymbol{\sigma} \quad \Rightarrow \quad \int_{\mathcal{H}} \underline{\mathbf{T}} \langle \boldsymbol{\xi} \rangle - \underline{\mathbf{T}}' \langle -\boldsymbol{\xi} \rangle \; d\boldsymbol{\xi}$

"In peridynamics, cracks are part of the solution, not part of the problem"
- F. Bobaru

Convergence for 2D crack branching:
(Ha, Bobaru 2010)



Brittle and Ductile Failure in 3D:
(Parks 2012)



- Discontinuities (e.g. cracks) are automatically supported
- Nonlocal theory includes length-scale, δ
- Naturally discretized as a particle method
- Rooted in a rigorous theoretical framework
- Implementation similar to molecular dynamics $\rightarrow$ scalable

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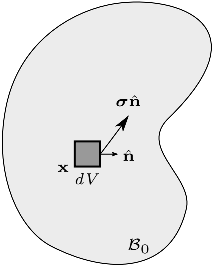
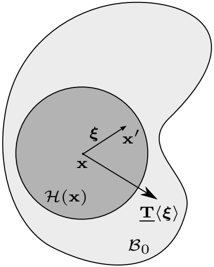
- Integral formulation of continuum mechanics originally proposed by Stewart Silling in 2000.
- Stress is replaced with long-range forces, $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$.
- Bond $\boldsymbol{\xi} := \mathbf{x}' - \mathbf{x}$.



Classical control volume, normal $\hat{\mathbf{n}}$ and traction vector $\boldsymbol{\sigma}\hat{\mathbf{n}}$.

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma}$$



Peridynamic bond $\boldsymbol{\xi}$ and bond-force $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$.

$$\rho\ddot{\mathbf{u}} = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle - \underline{\mathbf{T}}'\langle-\boldsymbol{\xi}\rangle \; d\boldsymbol{\xi}$$

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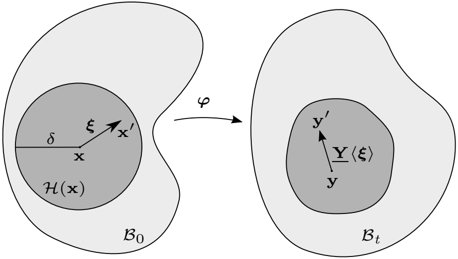
The *family* of bonds at $\mathbf{x}$ is given by

$$\mathcal{H}(\mathbf{x}) = \left\{ \boldsymbol{\xi} \in \mathbb{R}^3 \mid (\boldsymbol{\xi} + \mathbf{x}) \in \mathcal{B},\ |\boldsymbol{\xi}| < \delta \right\}.$$



Schematic representation of a body $\mathcal{B}_0$ and the family $\mathcal{H}(\mathbf{x})$ mapped by φ.

A vector-state $\underline{\mathbf{A}}[\mathbf{x}]$ at a point $\mathbf{x} \in \mathcal{B}$ is a function

$$\underline{\mathbf{A}}[\mathbf{x}]\langle\cdot\rangle : \mathcal{H}(\mathbf{x}) \to \mathbb{R}^3.$$

Deformation vector-state: $\boxed{\underline{\mathbf{Y}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle = \mathbf{y}(\mathbf{x}+\boldsymbol{\xi}) - \mathbf{y}(\mathbf{x})}$ $\quad \forall \boldsymbol{\xi} \in \mathcal{H}(\mathbf{x}).$

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	Classical Continuum	Peridynamics
Deformation Measure	$\mathbf{F} = \nabla \mathbf{y}$	$\underline{\mathbf{Y}}$
Conjugate Force	$\mathbf{P}(\mathbf{F})$	$\underline{\mathbf{T}}(\underline{\mathbf{Y}})$
Angular Momentum	$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$	$\mathbf{0} = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \, d\boldsymbol{\xi}$
Elasticity	$\mathbf{P} = \nabla\hat{\psi}(\mathbf{F})$	$\underline{\mathbf{T}} = \partial\bar{\psi}(\underline{\mathbf{Y}})$
Kinematics	$\det(\mathbf{F}) > 0$	$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \neq \mathbf{0}$, for $\lvert\boldsymbol{\xi}\rvert \neq 0$

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Classical constitutive model: $\hat{\mathbf{P}}(\mathbf{F}) = \nabla \hat{\psi}(\mathbf{F})$

Silling, et al. 2007 proposed a nonlocal approximation to $\mathbf{F}$:

$$\boxed{\bar{\mathbf{F}}(\underline{\mathbf{Y}}) = \left(\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \left(\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle \otimes \boldsymbol{\xi} \right) d\boldsymbol{\xi} \right) \bar{\mathbf{K}}^{-1}} \qquad \bar{\mathbf{K}} = \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \; \boldsymbol{\xi} \otimes \boldsymbol{\xi} \; d\boldsymbol{\xi}$$

which is exact for affine deformations.

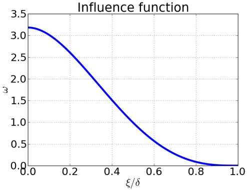
The *corresponding* constitutive model:

$$\bar{\psi}(\underline{\mathbf{Y}}) = \hat{\psi}\left(\bar{\mathbf{F}}(\underline{\mathbf{Y}})\right).$$

Bond-force follows from work conjugacy as:

$$\boxed{\underline{\mathbf{T}}\langle \boldsymbol{\xi} \rangle = \underline{\omega}\langle \boldsymbol{\xi} \rangle \, \hat{\mathbf{P}}(\bar{\mathbf{F}}) \bar{\mathbf{K}}^{-1}}$$

Influence function: $\boxed{\underline{\omega}\langle \boldsymbol{\xi} \rangle > 0}$



- Replace integrals by sums over particle volumes
- Degrees of freedom are particle displacements
- Effectively a nodally integrated particle method

For example (Foster et al. 2010):

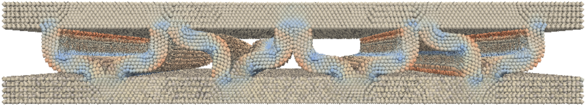
$$\bar{\mathbf{F}}_j = \sum_{i=1}^{N} V_i \, \omega\left(|\mathbf{x}_i - \mathbf{x}_j|\right) \left(\mathbf{y}_i - \mathbf{y}_j\right) \otimes \left(\mathbf{x}_i - \mathbf{x}_j\right) \bar{\mathbf{K}}_j^{-1}$$

$$\bar{\mathbf{K}}_j = \sum_{i=1}^{N} V_i \, \omega\left(|\mathbf{x}_i - \mathbf{x}_j|\right) \left(\mathbf{x}_i - \mathbf{x}_j\right) \otimes \left(\mathbf{x}_i - \mathbf{x}_j\right)$$

Semi-discrete equations of motion:

$$\rho \, \ddot{\mathbf{y}}_j = \sum_{i=1}^{N} V_i \, \omega(|\mathbf{x}_j - \mathbf{x}_i|) \left\{ \bar{\mathbf{P}}_i \bar{\mathbf{K}}_i^{-1} \left(\mathbf{x}_j - \mathbf{x}_i\right) - \bar{\mathbf{P}}_j \bar{\mathbf{K}}_j^{-1} \left(\mathbf{x}_i - \mathbf{x}_j\right) \right\}$$

C++ implementation uses standard constitutive model library



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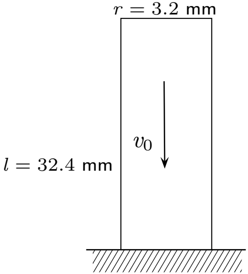
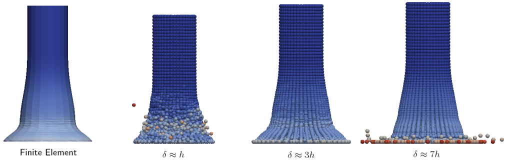


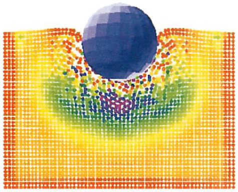
Taylor impact of a copper bar with $v_0 = 227$ m/s: fixed horizon to mesh size ratio

	l_f (mm)	r_{max} (mm)	ϵ^p_{max}
FEM, Kamoulakos 1990	21.5	7.1	2.47-3.24
FEM, Zhu and Cescotto 1995	21.3	7.1	2.47-3.24
FEM, Camacho and Ortiz 1997	21.4	7.2	2.97-3.25
OTM, Li et al. 2010	21.4	6.8	3.0
Peridynamics: coarse (left)	21.5	7.1	2.69
Peridynamics: fine (middle)	21.4	7.5	2.88
Peridynamics: finest (right)	21.4	7.4	3.29

Taylor impact revisited: instabilities



Taylor impact: increasing horizon size
(Tupek 2011)



Rigid sphere impact
(Becker, Lucas 2011)

- Instabilities have previously been observed (Littlewood 2010, Becker 2011) and can be addressed with numerical stabilization approaches.
- Nodally integrated meshless methods have energy instabilities.
- Often solved by better integration to remove zero-energy modes.
- Is this a numerical or theoretical issue?

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Outline for Section 3

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Theoretical investigation of constitutive correspondence:

$$\bar{\mathbf{F}} = \left(\int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \left(\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle \otimes \boldsymbol{\xi} \right) d\boldsymbol{\xi} \right) \bar{\mathbf{K}}^{-1}$$

$$\bar{\mathbf{K}} = \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle \;\; \boldsymbol{\xi} \otimes \boldsymbol{\xi} \;\; d\boldsymbol{\xi}$$

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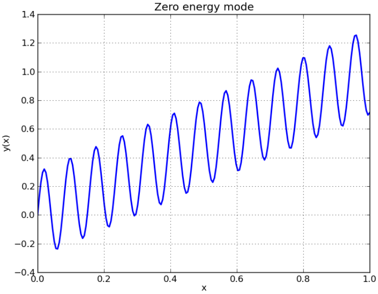
For deformation: $u(x) = a\sin(bx)$

$y(x) = x + u(x)$

$$\bar{F}(x) := \int_{\mathcal{H}} \omega(|\xi|)\,\big(y(x') - y(x)\big)\,\big(x' - x\big)\;dx'$$

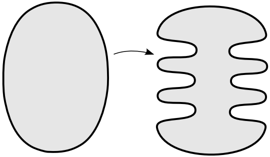
$$= 1 - \frac{a\cos(bx)}{b\bar{K}} \int_{\mathcal{H}} \frac{d}{d\xi}\left(\xi\,\omega(|\xi|)\right)\cos(b\xi)\,d\xi$$

and therefore $\lim_{b\to\infty} \bar{F}(x) = 1$



Assume a 'smooth' influence function: $\displaystyle\int_{\mathcal{H}} \left|\frac{d}{d\xi}\left(\xi\,\omega(|\xi|)\right)\right| d\xi < \infty$

- Fast oscillating unphysical deformations are undetectable
- Unphysical behavior is averaged out by the integration



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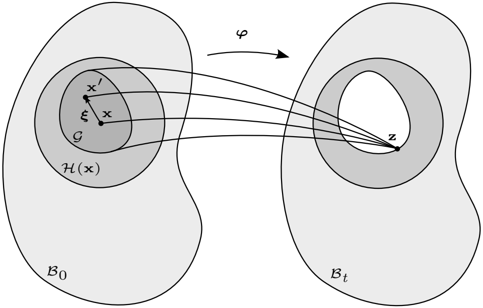
Sub-horizon material collapse: Consider a small volume of material $\mathcal{G} \subset \mathcal{B}_0$ collapsing to a single point $\mathbf{z} \in \mathcal{B}_t$. Then $\underline{\mathbf{Y}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle = \mathbf{0}$ for $\mathbf{x}$, $\mathbf{x} + \boldsymbol{\xi} \in \mathcal{G}$.



Schematic showing a region $\mathcal{G} \subset \mathcal{B}_0$ collapsing to a single point $\mathbf{z} \in \mathcal{B}_t$.

- $\det\left(\bar{\mathbf{F}}(\mathbf{x})\right) > 0$
- Violating the bond-level kinematic constraint does not imply the classical continuum kinematic constraint is violated:

$$\det\left(\bar{\mathbf{F}}\right) > 0 \not\Rightarrow \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \neq \mathbf{0}$$

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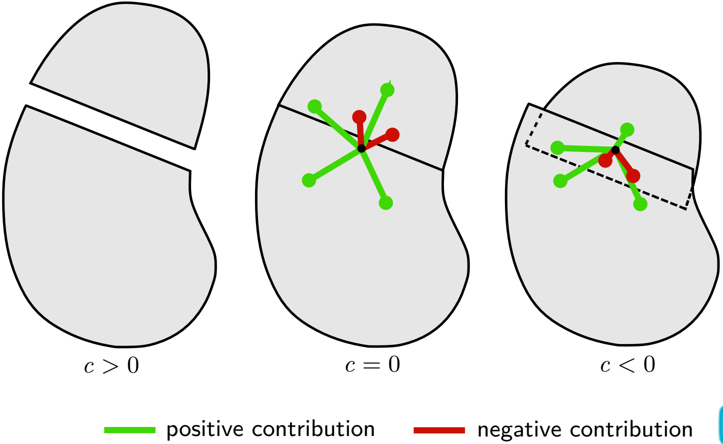
Jump discontinuities: Another example, displacement field:

$$y(x) = \begin{cases} c + x & \text{for } x > 0 \\ x & \text{for } x < 0 \end{cases}$$

for $c < 0$, may still have $\bar{F} > 0$



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Constitutive constraint for preventing matter interpenetration

The matter interpenetration condition

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \neq \mathbf{0}$$

can be enforced almost everywhere if

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \to \mathbf{0}, \ \ \forall\boldsymbol{\xi} \in \Omega \subset \mathcal{H}(\mathbf{x}) \implies \psi(\underline{\mathbf{Y}}) \to \infty,$$

where Ω is some measurable subset of $\mathcal{H}(\mathbf{x})$.
This is analogous to:

$$\det \mathbf{F} \to 0 \implies \psi(\mathbf{F}) \to \infty. \quad \text{(e.g. Ball, 1977)}$$

Alternatively, the condition:

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \to \mathbf{0} \implies \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \to -\infty.$$

does not require an elastic potential and is analogous to

$$\det \mathbf{F} \to 0 \implies P \to \infty.$$

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We want a strain measure: $\varepsilon \to -\infty$ as $|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| \to 0$.

Classical Seth-Hill strain measures:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}$$

$$\boldsymbol{\varepsilon}_{(m)} = \frac{1}{2m}\left[\mathbf{C}^m - \mathbf{1}\right]$$

$$\boldsymbol{\varepsilon}_{(0)} = \frac{1}{2}\log\left(\mathbf{C}\right)$$

For $m \leq 0$:
$\mathrm{tr}\left(\boldsymbol{\varepsilon}_{(m)}\right) \to -\infty$ as $\det\left(\mathbf{F}\right) \to 0$.

We want a strain measure: $\varepsilon \to -\infty$ as $|\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle| \to 0$.

Classical Seth-Hill strain measures:

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

$$\boldsymbol{\varepsilon}_{(m)} = \frac{1}{2m} \left[\mathbf{C}^m - \mathbf{1} \right]$$

$$\boldsymbol{\varepsilon}_{(0)} = \frac{1}{2} \log \left(\mathbf{C} \right)$$

For $m \leq 0$:
tr $\left(\boldsymbol{\varepsilon}_{(m)} \right) \to -\infty$ as det $\left(\mathbf{F} \right) \to 0$.

Nonlinear bond-strain measures:

$$\underline{c}\langle \boldsymbol{\xi} \rangle = \frac{\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle \cdot \underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle}{\boldsymbol{\xi} \cdot \boldsymbol{\xi}}$$

$$\boxed{\underline{\varepsilon}_{(m)}\langle \boldsymbol{\xi} \rangle = \frac{1}{2m} \left[\underline{c}\langle \boldsymbol{\xi} \rangle^m - 1 \right]}$$

$$\underline{\varepsilon}_{(0)}\langle \boldsymbol{\xi} \rangle = \frac{1}{2} \log(\underline{c}\langle \boldsymbol{\xi} \rangle)$$

For $m \leq 0$:
$\underline{\varepsilon}_{(m)}\langle \boldsymbol{\xi} \rangle \to -\infty$ as $|\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle| \to 0$.

[1] M. Tupek, R. Radovitkzy. *An extended constitutive correspondence formulation of peridynamics based on nonlinear bond-strain measures*, submitted to JMPS.

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Notation: $\text{tr}_\omega(\underline{a}) = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\, \underline{a}\langle\boldsymbol{\xi}\rangle\ d\boldsymbol{\xi}$. Normalize weight: $\text{tr}_\omega(\underline{1}) = 3$.

Use bond-strain measure $\underline{\varepsilon} = \underline{\varepsilon}_{(m)}$, for any m.

Elastic energy: $\bar{\psi}(\underline{\varepsilon}) = \frac{1}{2}\bar{\kappa}\,\text{tr}_\omega(\underline{\varepsilon})^2 + \bar{\mu}\,\text{tr}_\omega\big(\underline{\varepsilon}_{dev}^2\big) = \frac{1}{2}\bar{\lambda}\,\text{tr}_\omega(\underline{\varepsilon})^2 + \bar{\mu}\,\text{tr}_\omega\big(\underline{\varepsilon}^2\big)$

$$\underline{\varepsilon}_{dev}\langle\boldsymbol{\xi}\rangle = \underline{\varepsilon}\langle\boldsymbol{\xi}\rangle - \tfrac{1}{3}\text{tr}_\omega(\underline{\varepsilon})$$

Bond-force: $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \big[\bar{\lambda}\,\text{tr}_\omega(\underline{\varepsilon}) + 2\,\bar{\mu}\,\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle\big]\,\underline{\omega}\langle\boldsymbol{\xi}\rangle\,\underline{c}\langle\boldsymbol{\xi}\rangle^{m-1}\,|\boldsymbol{\xi}|^{-2}\,\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$

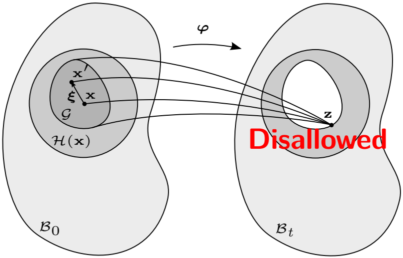
Bond-based peridynamics (Silling 2000): $\bar{\lambda} = 0$

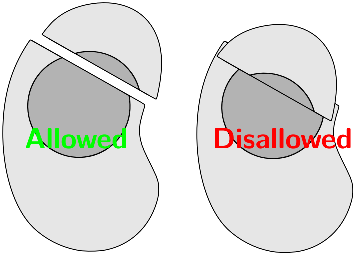
Linear isotropic (Silling et al. 2007): $m = \frac{1}{2}$

For uniform, small strains: $\boxed{\bar{\kappa} = \kappa, \quad \bar{\mu} = \frac{5}{2}\mu, \quad \bar{\lambda} = \lambda - \mu}$

Property: $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \to -\infty$ as $|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| \to 0$ for $m \leq 0$

- No zero-energy modes or material collapse allowed
- Matter interpenetration is prevented for strain measures with $m < -\frac{1}{4}$
- Crack-like opening jump discontinuities are still valid solutions!

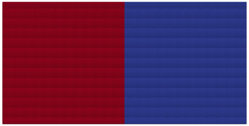




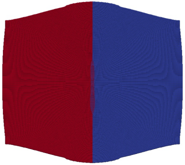
a) opening b) interpenetrating

Bond-strain measure with $m = \frac{1}{2}$ (linear model)

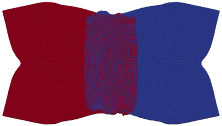
$$v_x(\mathbf{x}, t=0) = \begin{cases} -1200 \text{ m/s} & \text{for } x > 0 \\ 1200 \text{ m/s} & \text{for } x < 0 \end{cases}$$



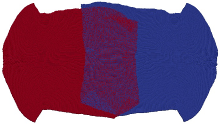
$t = 0\ \mu s$



$t = 50\ \mu s$



$t = 100\ \mu s$

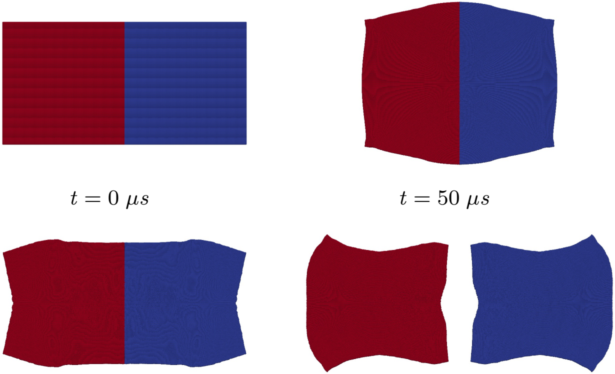


$t = 150\ \mu s$

2D peridynamic bar: length $l = 2$ height $h = 1$, $\delta = 0.05$ m, $\bar{\lambda} = 10$ GPa, $\bar{\mu} = 10$ GPa, $\rho = 1180$ kg/m^3.

Bond-strain measure with $m = 0$ (logarithmic model)

$$v_x(\mathbf{x}, t=0) = \begin{cases} -1200 \text{ m/s} & \text{for } x > 0 \\ 1200 \text{ m/s} & \text{for } x < 0 \end{cases}$$



$t = 0\ \mu s$ $t = 50\ \mu s$

$t = 100\ \mu s$ $t = 150\ \mu s$

2D peridynamic bar: length $l = 2$ height $h = 1$, $\delta = 0.05$ m,
$\bar{\lambda} = 10$ GPa, $\bar{\mu} = 10$ GPa, $\rho = 1180$ kg/m^3.

Preventing instabilities: Summary

- Numerical simulations of peridynamics exhibit instabilities: zero-energy modes, matter interpenetration
- Not just a numerical issue, but due to poor kinematic assumptions
- Proposed formulating constitutive response in terms of nonlinear bond-strain measures: $\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle$
- Introduced nonlinear isotropic elastic peridynamic solids
- Numerical examples demonstrate that the new theory fixes previous limitations

Normalize weight: $\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \; d\boldsymbol{\xi} = 1.$

Elastic energy: $$\bar{\psi}(\underline{\varepsilon}) = \frac{9}{2}\bar{\kappa}\left(\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\,\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle \; d\boldsymbol{\xi}\right)^2 + 3\bar{\mu}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\,\underline{\varepsilon}^2_{dev}\langle\boldsymbol{\xi}\rangle \; d\boldsymbol{\xi}$$

Interpret integrals as first and second moments of a random variable $\underline{\varepsilon}$:

$$\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\,\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle \; d\boldsymbol{\xi} = \mathsf{E}[\underline{\varepsilon}]$$

and

$$\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\,\underline{\varepsilon}^2_{dev}\langle\boldsymbol{\xi}\rangle \; d\boldsymbol{\xi} = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\left(\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle - \mathsf{E}[\underline{\varepsilon}]\right)^2 \; d\boldsymbol{\xi} = \mathsf{Var}(\underline{\varepsilon})$$

so

$$\psi(\underline{\varepsilon}) = k_1\mathsf{E}[\underline{\varepsilon}]^2 + k_2\mathsf{Var}(\underline{\varepsilon})\,,$$

where $k_1 = \frac{9}{2}\hat{\kappa}$ and $k_2 = 3\hat{\mu}$.

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In other words

First moment:

$$\mathsf{E}_\omega[\underline{\varepsilon}] \propto I_1(\boldsymbol{\varepsilon})$$

and

Second central moment:

$$\mathsf{Var}_\omega(\underline{\varepsilon}) \propto J_2(\boldsymbol{\varepsilon})$$

Quiz: What is the third central moment of $\underline{\varepsilon}$?

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Relation between moments and invariants

In other words

First moment:

$$\mathsf{E}_\omega[\underline{\varepsilon}] \propto I_1(\boldsymbol{\varepsilon})$$

and

Second central moment:

$$\mathsf{Var}_\omega(\underline{\varepsilon}) \propto J_2(\boldsymbol{\varepsilon})$$

Quiz: What is the third central moment of $\underline{\varepsilon}$?

$$\mathsf{skewness}(\underline{\varepsilon}) \propto \frac{J_3(\boldsymbol{\varepsilon})}{\sqrt{J_2(\boldsymbol{\varepsilon})^3}}$$

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For a generic peridynamic scalar-state $\underline{a}$ with

$$\underline{a}\langle\boldsymbol{\xi}\rangle \in \mathbb{R}$$

its invariants $(I_1,\ J_n)$ with respect to a normalized influence function $\underline{\omega}$ with

$$\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \ d\boldsymbol{\xi} = 1$$

can be defined as:

$$I_1 := \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle\, \underline{a}\langle\boldsymbol{\xi}\rangle \ d\boldsymbol{\xi}$$

and for $n > 1$:

$$J_n := \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left(\underline{a}\langle\boldsymbol{\xi}\rangle - I_1\right)^n \ d\boldsymbol{\xi}.$$

Maximum bond-stretch criterion for the *bond-based* theory:

Review:

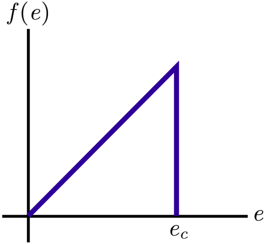
- Bond-based peridynamics is the special case:

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \mathbf{f}\left(e(\boldsymbol{\xi})\right)$$

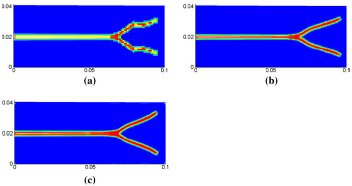
- $e(\boldsymbol{\xi}) = |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|$
- Based on maximum bond stretch criteria:

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \mathbf{0} \text{ when } e \geq e_c$$

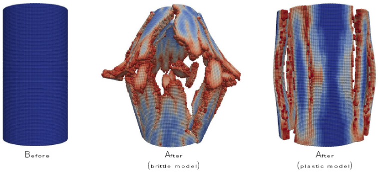
- Bond-based formulation restricts $\nu = 0.25$



Convergence for 2D crack branching Ha, Bobaru 2010:



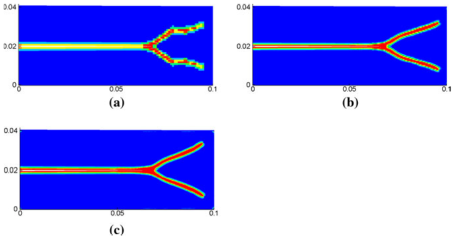
Extension to the general theory are more recent, Parks 2012:



Brittle failure modeling in peridynamics

Bond-severing criterion works well for bond-based version of the theory.

2D crack branching: (Ha, Bobaru 2010)



Proposed extension of existing bond-failure criteria

- Based on G_c: consistent with the Griffith criterion
- Extension of (Silling 2005) and (Foster 2011)
- Bond failure function, $s_c(\xi)$, sets the maximum energy per bond

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Brittle failure model for nonlinear peridynamic solids

Elastic free energy density: $\psi(\mathbf{x}) = \frac{1}{2}\bar{\lambda}\left(\mathrm{tr}_\omega\left(\underline{\phi}\,\underline{\varepsilon}\right)\right)^2 + \bar{\mu}\,\mathrm{tr}_\omega\left(\underline{\phi}\,\underline{\varepsilon}^2\right)$

Bond integrity: $\underline{\phi}\langle\boldsymbol{\xi}\rangle = \{1 \text{ for intact bonds}, \quad 0 \text{ for severed bonds}\}$

There are multiple ways to define the bond-energy:

Average bond-energy

$$\underline{s}_{ave}\langle\boldsymbol{\xi}\rangle := \tfrac{1}{2}\bar{\lambda}\mathrm{tr}_\omega\left(\underline{\phi}\,\underline{\varepsilon}\right) + \bar{\mu}\,\underline{\phi}\,\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle$$

Instantaneous bond-energy

$$\underline{s}_{inst}\langle\boldsymbol{\xi}\rangle := \bar{\lambda}\mathrm{tr}_\omega\left(\underline{\phi}\,\underline{\varepsilon}\right) + \bar{\mu}\,\underline{\phi}\,\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle$$

Foster, et al. 2011 proposed the work done on the bond:

$$\underline{s}_{work}\langle\boldsymbol{\xi}\rangle(t) := \int_0^t \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\dot{\mathbf{Y}}}\langle\boldsymbol{\xi}\rangle\, dt.$$

We use the **instantaneous bond-energy**: $\boxed{\underline{s}_{inst}\langle\boldsymbol{\xi}\rangle := \psi_{,\underline{\phi}}\langle\boldsymbol{\xi}\rangle}$

- The instantaneous bond-energy is the energy dissipated when severing a single bond (all other bonds held fixed).
- In general: $\underline{s}_{inst} \neq \underline{s}_{work}$.

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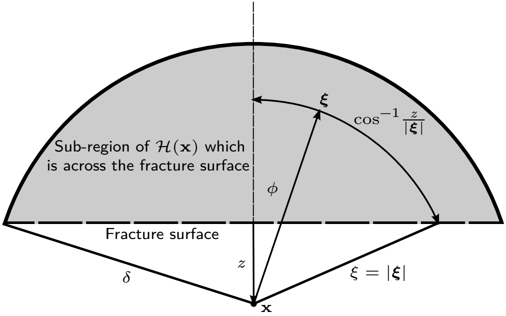
Energy dissipated by breaking bonds across a fracture surface:

$$\int_0^\delta \int_0^{2\pi} \int_z^\delta \int_0^{\cos^{-1}(z/\xi)} s_c(\xi)\, \xi^2 \sin(\phi)\, d\phi\, d\xi\, d\theta\, dz = \frac{1}{2} G_c$$

Silling et al. 2005: $s_c(\xi) = \alpha \underline{\omega}\langle \boldsymbol{\xi} \rangle$ Foster et al. 2011: $s_c(\xi) = s_c$



Schematic for the domain of integration (modified from Silling 2005).

Bond failure criterion

- Assume the failure energy decreases for distant material points:

$$s_c(|\boldsymbol{\xi}|) = \beta\, \underline{\omega}\langle\boldsymbol{\xi}\rangle\, |\boldsymbol{\xi}|^{-2}$$

- This choice tends to result in sharper crack features.
- β is calibrated to match the cohesive energy, G_c

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Bond-force: $\underline{\mathbf{T}}_{(m)}\langle\boldsymbol{\xi}\rangle = \underline{\phi}\langle\boldsymbol{\xi}\rangle \left[\bar{\lambda}\,\mathrm{tr}_{\omega}\!\left(\underline{\phi\,\varepsilon}\right) + 2\,\bar{\mu}\,\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle\right] \underline{\omega}\langle\boldsymbol{\xi}\rangle\, \underline{c}\,\langle\boldsymbol{\xi}\rangle^{m-1}\,|\boldsymbol{\xi}|^{-2}\,\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$

$$\underline{\omega}\langle\boldsymbol{\xi}\rangle = \frac{10}{\pi\delta^6}(3|\boldsymbol{\xi}| + \delta)(\delta - |\boldsymbol{\xi}|)^3$$

Bond failure criterion: $\bar{\lambda}\,\underline{\varepsilon}\,\langle\boldsymbol{\xi}\rangle\,\mathrm{tr}_{\omega}\!\left(\underline{\phi\,\varepsilon}\right) + \bar{\mu}\,\underline{\varepsilon}\,\langle\boldsymbol{\xi}\rangle\,\underline{\varepsilon}\,\langle\boldsymbol{\xi}\rangle \geq \frac{\pi}{16}\,\frac{\delta\,G_c}{|\boldsymbol{\xi}|^2}, \quad \underline{\varepsilon}\langle\boldsymbol{\xi}\rangle \geq 0$

This failure criterion satisfies the laws of thermodynamics.

Free-surface detection parameter: $D(\mathbf{x}) = 1 - \frac{1}{3}\int_{\mathcal{H}(\mathbf{x})} \underline{\omega}\langle\boldsymbol{\xi}\rangle\;\underline{\phi}\langle\boldsymbol{\xi}\rangle\; d\boldsymbol{\xi}.$

Summary: key ingredients for this failure model are $s_c(\xi)$ and $\underline{s}_{inst}\,\langle\boldsymbol{\xi}\rangle$.

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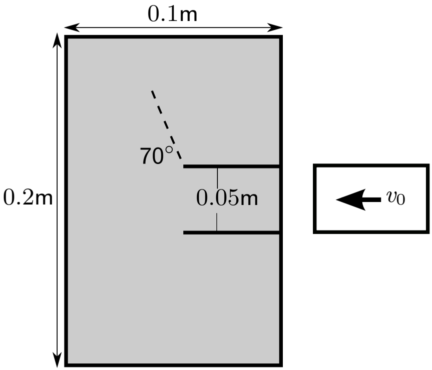

- Based on experiments by Kalthoff, Winkler 1987.
- $\rho = 8000$ kg/m^3, $E = 190$ GPa, $\nu = 0.3$ and $G_c = 22,000$ J/m^2.
- Impact velocity: $v_0 = 16$ m/s.
- Experimental crack propagation angle of 70°.

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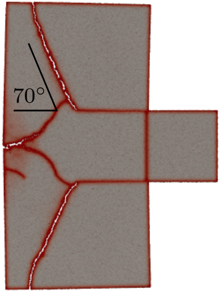
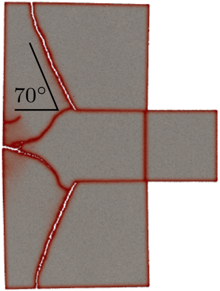
Kalthoff test simulation results

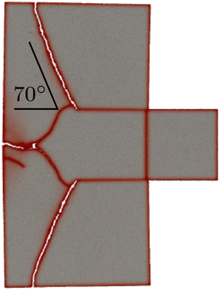
$h = 1.2$ mm
$\delta = 5.0$ mm



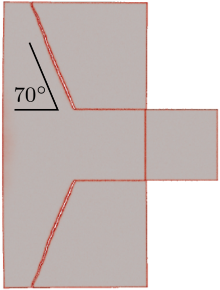
$h = 1.0$ mm
$\delta = 5.0$ mm



$h = 0.8$ mm
$\delta = 5.0$ mm

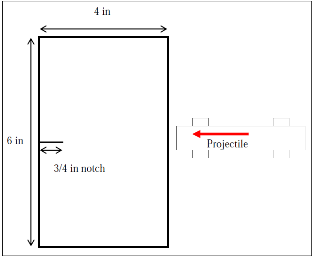


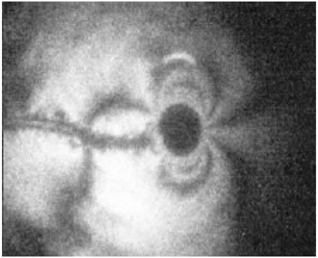
$h = 0.4$ mm
$\delta = 1.8$ mm



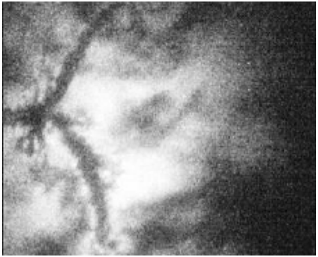
Validation: edge on impact of PMMA

- Mode-I crack propagation for low impact velocities
- Crack branching at higher velocities
- $\rho = 1180$ kg/m^3, $E = 3.5$ GPa, $\nu = 0.35$, $\boxed{G_c = 400 \text{ J/m}^2}$





$v_0 = 50.5$ m/s

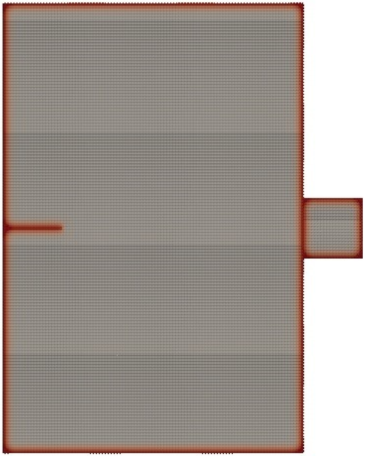


$v_0 = 65.9$ m/s

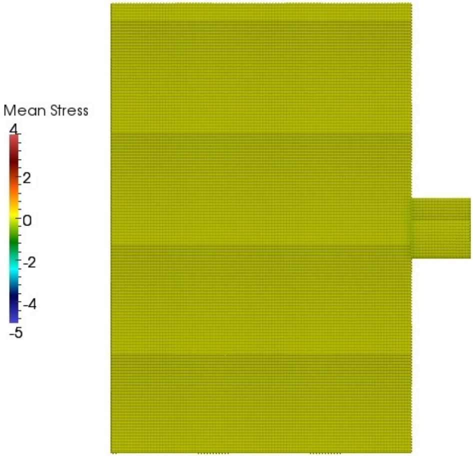
Coherent gradient sensing: experimental results of Umberger and Love, 2010.

Edge-on impact simulations: 61,600 particles, $\delta = 0.0055$, $v_0 = 15$ m/s

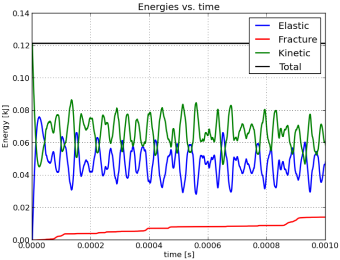
Free-surface parameter

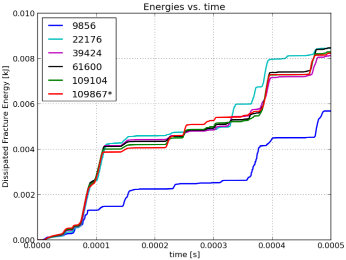


Mean stress (MPa)



Sensitivity to particle density and evaluating the fracture energy

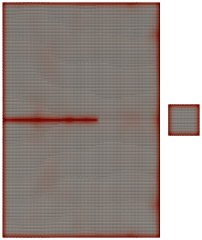




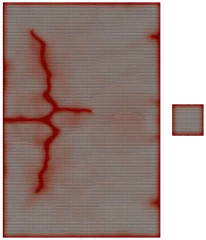
Dissipated energy over time for varying particle densities, with $v_0 = 15$ m/s.

number of discrete particles	crack length	dissipated energy	cohesive energy
9,856	4 mm	2.5 J	**620 J/m^2**
22,176	12 mm	4.6 J	**400 J/m^2**
39,424	11 mm	4.4 J	**420 J/m^2**
61,600	11 mm	4.4 J	**410 J/m^2**
109,104	10 mm	4.2 J	**420 J/m^2**
109,867 (unstructured)	10 mm	4.1 J	**410 J/m^2**

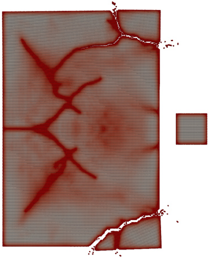
Edge-on impact simulations: varying impact velocity
61,600 particles, $\delta = 0.0055$ at $t = 0.001$



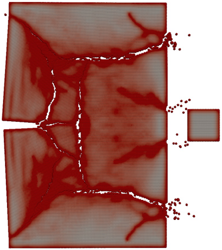
$v_0 = 18$ m/s



$v_0 = 25$ m/s



$v_0 = 35$ m/s



$v_0 = 50$ m/s

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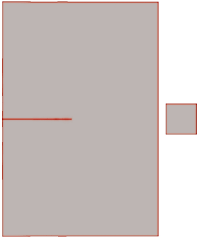
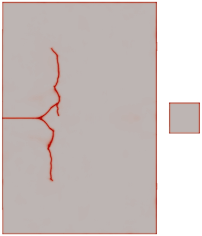
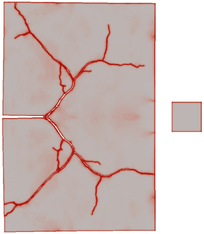
Effect of changing horizon to $\delta = 0.0015$ m



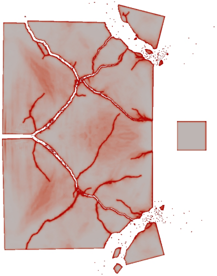
$v_0 = 15$ m/s



$v_0 = 25$ m/s



$v_0 = 35$ m/s



$v_0 = 50$ m/s

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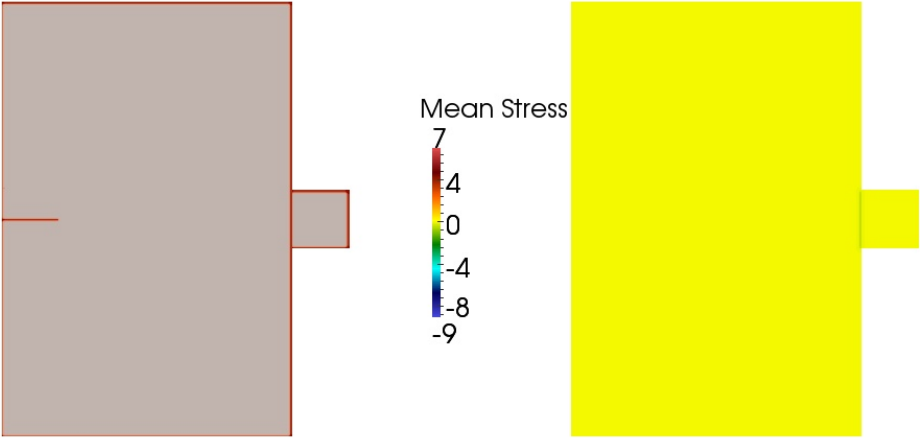
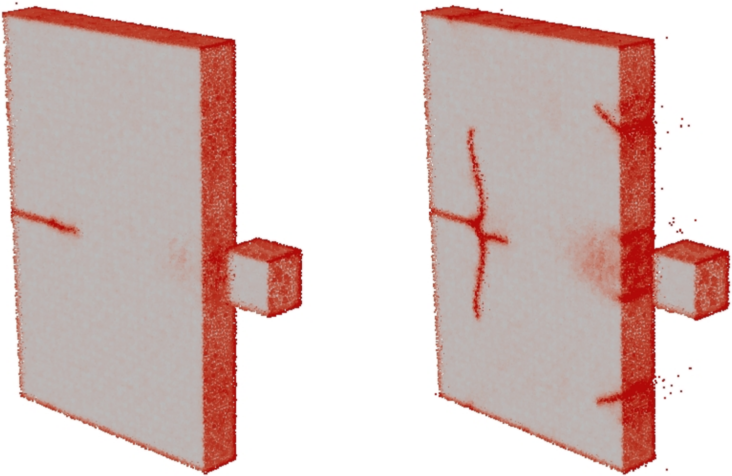
Animation of crack propagation and branching: $v_0 = 35$ m/s, $\delta = 0.0015$ m

Free-surface parameter **Mean stress** (MPa)



Theory and discretization extends directly to 3D



$v_0 = 30$ m/s $\qquad$ $v_0 = 50$ m/s

Nonlocal material damage at $t = 0.001$ s for edge-on impact in 3D. A discretization with 573,346 particles and $\delta = 0.004$ m is shown sliced through the thickness of the plate.

Summary of contributions

Extensions to the peridynamic theory

- Identified a cause of instabilities and matter interpenetration.
- Proposed a constitutive modeling framework based on new nonlinear bond-strain measures.
- Demonstrated that opening (crack-like) discontinuities are allowed, while matter interpenetration is prevented.

Introduced a probabilistic interpretation of isotropic ordinary peridynamics

- Showed that moments of strain-states are equivalent to strain invariants.
- Proposed a peridynamic analog for invariants of a scalar-state.

State-based peridynamic failure modeling

- Introduced a new bond-failure criterion for brittle fracture:
  - Captures realistic crack patterns, including branching and coalescence.
  - Dissipates a pre-specified fracture energy.