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Helmholtz decomposition in the nonlocal framework

Well-posedness analysis and applications

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Abstract We provide a Helmholtz decomposition in the nonlocal setting.
Include keywords, PACS and mathematical subject classification numbers as needed.

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Mathematics Subject Classification (2000) MSC code1 · MSC code2 · more

1 Introduction and motivation

Important applications in diffusion, elasticity, fracture propagation have benefitted from the introduction of nonlocal models. Phenomena, materials, and behaviors that are discontinuous in nature have been ideal candidates for the introduction of this framework which allows solutions with no smoothness, or even continuity properties. This advantage is counterbalanced by the fact that the theory of nonlocal calculus is still being developed.

In [3] the authors introduce a nonlocal framework with divergence, gradient, and curl versions of nonlocal operators for which they identify duality relationships via L^2 inner product topology. Integration by parts, nonlocal Poincaré inequality

significance

The classical Helmholtz decomposition [] states that for any field ...

Its numerous applications rely on duality properties, as well ...

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Emphasize that we obtain a Helmholtz decomposition for each interaction kernel α; very useful in applications when one may want to choose different gradients, curls, operators

The paper is organized as follows: in the next section we introduce the nonlocal operators for which we will obtain a nonlocal version of the Helmholtz decomposition.

2 Nonlocal vector calculus

In the analysis of the Helmholtz decomposition we use the nonlocal vector calculus (NLVC) introduced in [3] and [4] and applied to nonlocal diffusion in [2]. This theory is the nonlocal counterpart of the classical calculus for differential operators and allows one to study nonlocal diffusion problems in a very similar way as we study partial differential equations (PDEs) thanks to the formulation of nonlocal equations in a variational setting. In this work we do not consider diffusion only, but we utilize additional nonlocal operators, e.g. the nonlocal curl introduced in [3], to mimic the local Helmholtz decomposition. The basic concepts of the NLVC and the results relevant to this paper are reported below.

The NLVC is based on a new concept of nonlocal fluxes between two (possibly disjoint) domains; the derivation of a nonlocal flux strictly follows the local definition and it is based on a nonlocal Gauss theorem (the interested reader may find the complete analysis in [3]). We define the following nonlocal divergence operators acting on zero-th order, first order, and second order tensors, respectively:

$$
\begin{aligned}
\big(\mathcal{D}_{\boldsymbol{\alpha},0}\psi\big)(\boldsymbol{x}) &:= \int_{\mathbb{R}^n} \left(\psi(\boldsymbol{x},\boldsymbol{y}) + \psi(\boldsymbol{y},\boldsymbol{x})\right) \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y}, \\
\big(\mathcal{D}_{\boldsymbol{\alpha},1}\mathbf{v}\big)(\boldsymbol{x}) &:= \int_{\mathbb{R}^n} \left(\mathbf{v}(\boldsymbol{x},\boldsymbol{y}) + \mathbf{v}(\boldsymbol{y},\boldsymbol{x})\right) \cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y}, \\
\big(\mathcal{D}_{\boldsymbol{\alpha},2}\boldsymbol{\Psi}\big)(\boldsymbol{x}) &:= \int_{\mathbb{R}^n} \left(\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{\Psi}(\boldsymbol{y},\boldsymbol{x})\right) \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y},
\end{aligned} \tag{1}
$$

where $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a scalar function, $\mathbf{v} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector function, $\boldsymbol{\Psi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a matrix function, and $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an antisymmetric function, i.e. $\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{\alpha}(\boldsymbol{y},\boldsymbol{x})$.

The corresponding nonlocal gradient operators, i.e. the negative adjoint operators, $-\mathcal{D}_{\boldsymbol{\alpha}i}^* := -\big(\mathcal{D}_{\boldsymbol{\alpha}i}\big)^*$, acting on ith order tensors are defined as [3]

$$
\begin{aligned}
\left(\mathcal{G}_{\boldsymbol{\alpha},0}\mathbf{v}\right)(\boldsymbol{x},\boldsymbol{y}) &:= -\big(\mathcal{D}_{\boldsymbol{\alpha},0}^*\mathbf{v}\big)(\boldsymbol{x},\boldsymbol{y}) = -\left(\mathbf{v}(\boldsymbol{y}) - \mathbf{v}(\boldsymbol{x})\right) \cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}), \\
\left(\mathcal{G}_{\boldsymbol{\alpha},1}u\right)(\boldsymbol{x},\boldsymbol{y}) &:= -\big(\mathcal{D}_{\boldsymbol{\alpha},1}^*u\big)(\boldsymbol{x},\boldsymbol{y}) = -\left(u(\boldsymbol{y}) - u(\boldsymbol{x})\right) \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}), \\
\left(\mathcal{G}_{\boldsymbol{\alpha},2}\mathbf{v}\right)(\boldsymbol{x},\boldsymbol{y}) &:= -\big(\mathcal{D}_{\boldsymbol{\alpha},2}^*\mathbf{v}\big)(\boldsymbol{x},\boldsymbol{y}) = \left(\mathbf{v}(\boldsymbol{y}) - \mathbf{v}(\boldsymbol{x})\right) \otimes \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}).
\end{aligned} \tag{2}
$$

Then, as in the local case, we define the nonlocal diffusion operator $\mathcal{L}_{\boldsymbol{\alpha}}$ as

$$
-\left(\mathcal{L}_{\boldsymbol{\alpha}}u\right)(\boldsymbol{x}) = \mathcal{D}_{\boldsymbol{\alpha},1}\left(\mathcal{D}_{\boldsymbol{\alpha},1}^*u\right)(\boldsymbol{x}) = 2\int_{\mathbb{R}^n} \left(u(\boldsymbol{x}) - u(\boldsymbol{y})\right)\big(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\big) d\boldsymbol{y}. \tag{3}
$$

Remark 1 For simplicity of notation in the sequel we will drop the numerical subscripts on the operators, unless necessary for clarity, and keep only the reference to the kernel.

Given a vector field $\mathbf{u}$, we also define the nonlocal curl operator and its corresponding adjoint as

$$\begin{aligned} \left(\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}\right)(\boldsymbol{x}) &:= \int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times \left(\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) + \mathbf{u}(\boldsymbol{y},\boldsymbol{x})\right) d\boldsymbol{y} \\ \left(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}\right)(\boldsymbol{x},\boldsymbol{y}) &:= \ \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times \left(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\right). \end{aligned} \tag{4}$$

For such operators, by substitution, we have the following result:

$$\begin{aligned} \mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}\right)(\boldsymbol{x}) &= -2\int_{\mathbb{R}^n} \boldsymbol{\alpha} \times \left[\left(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\right) \times \boldsymbol{\alpha}\right] d\boldsymbol{y} \\ &= \mathcal{D}_{\boldsymbol{\alpha},2}(\mathcal{D}^*_{\boldsymbol{\alpha},2}\mathbf{w}) - \mathcal{D}_{\boldsymbol{\alpha},0}(\mathcal{D}^*_{\boldsymbol{\alpha},0}\mathbf{w}). \end{aligned} \tag{5}$$

Note that, formally, this is the same expression as in the local calculus, i.e. for a vector field $\mathbf{r}$, $\nabla \times (\nabla \times \mathbf{r}) = \nabla(\nabla \cdot \mathbf{r}) + \nabla^2\mathbf{r}$, where the latter represents the vector Laplacian.

Remark 2 We can further simplify the expression of $\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}\right)(\boldsymbol{x})$ in (5). Plugging the definitions of (4), we have

$$\begin{aligned} &\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}\right)(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times \left[\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times (\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})) + \boldsymbol{\alpha}(\boldsymbol{y},\boldsymbol{x}) \times (\mathbf{w}(\boldsymbol{x}) - \mathbf{w}(\boldsymbol{y}))\right] d\boldsymbol{y} \\ &= -\int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times \left[(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})) \times \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) + (\mathbf{w}(\boldsymbol{x}) - \mathbf{w}(\boldsymbol{y})) \times \boldsymbol{\alpha}(\boldsymbol{y},\boldsymbol{x})\right] d\boldsymbol{y} \\ &= -2\int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times \left[(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})) \times \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\right] d\boldsymbol{y} \end{aligned}$$

Notice that for any vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have the following identify for cross product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\left(\mathbf{a}^T\mathbf{c}\right) - \mathbf{c}\left(\mathbf{a}^T\mathbf{b}\right) = \mathbf{b}\left(\mathbf{a}^T\mathbf{c}\right) - \left(\mathbf{c} \otimes \mathbf{a}\right)\mathbf{b}$$

where T denotes transpose.

So, applying it to $\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}\right)(\boldsymbol{x})$ further yields

$$\begin{aligned} &\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}\right)(\boldsymbol{x}) \\ &\quad = 2\int_{\mathbb{R}^n} \big((\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})\left(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\right) - \left(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\right)\left(\boldsymbol{\alpha}^T\boldsymbol{\alpha}\right)\big) d\boldsymbol{y}, \end{aligned} \tag{6}$$

where $\boldsymbol{\alpha}$ is short hand notation of $\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})$.

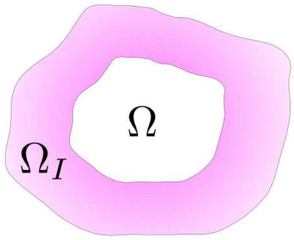

Fig. 1 Interaction domain configuration in a two-dimensional setting.

Given an open subset $\Omega \subset \mathbb{R}^n$, the corresponding *interaction domain* is defined as

$$\Omega_I := \{\boldsymbol{y} \in \mathbb{R}^n \setminus \Omega \quad \text{such that} \quad \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \neq \boldsymbol{0} \quad \text{for some } \boldsymbol{x} \in \Omega\} \qquad (7)$$

so that Ω_I consists of those points outside of Ω that interact with points in Ω, see Figure 1 for a two-dimensional configuration. For simplicity, we now focus on the divergence definition in $(1)_2$ and report important variational results. Corresponding to that divergence operator we define the action of the nonlocal *interaction operator* $\mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{\nu})\colon \mathbb{R}^n \to \mathbb{R}$ on $\boldsymbol{\nu}$ by

$$\mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{\nu})(\boldsymbol{x}) := -\int_{\Omega\cup\Omega_I} \big(\boldsymbol{\nu}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{\nu}(\boldsymbol{y},\boldsymbol{x})\big) \cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{y} \quad \text{for } \boldsymbol{x} \in \Omega_I. \qquad (8)$$

Note the main difference between local and nonlocal: in the former case the flux out of a domain is given by a boundary integral whereas in the latter case is given by a volume integral. With $\mathcal{D}_{\boldsymbol{\alpha}}$ and $\mathcal{N}_{\boldsymbol{\alpha}}$ defined as in (1) and (8), respectively, we have the *nonlocal Gauss theorem* [3]

$$\int_{\Omega} \mathcal{D}_{\boldsymbol{\alpha}}(\boldsymbol{\nu})\, d\boldsymbol{x} = \int_{\Omega_I} \mathcal{N}_{\boldsymbol{\alpha}}(\boldsymbol{\nu})\, d\boldsymbol{x}. \qquad (9)$$

Next, let $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ denote scalar functions; then, the divergence theorem above implies the generalized *nonlocal Green's first identity* [3]

$$\int_{\Omega} v\mathcal{D}_{\boldsymbol{\alpha}}(\mathcal{D}_{\boldsymbol{\alpha}}{}^{*}u)\, d\boldsymbol{x} - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}_{\boldsymbol{\alpha}}{}^{*}v \cdot \mathcal{D}_{\boldsymbol{\alpha}}{}^{*}u\, d\boldsymbol{y}d\boldsymbol{x} = \int_{\Omega_I} v\mathcal{N}_{\boldsymbol{\alpha}}(\mathcal{D}_{\boldsymbol{\alpha}}{}^{*}u)\, d\boldsymbol{x}. \qquad (10)$$

In [3], one can find further results for the nonlocal divergence operator $\mathcal{D}_{\boldsymbol{\alpha}}$, including a nonlocal Green's second identity, as well as analogous results for nonlocal gradient and curl operators. In addition, in the same paper, further connections are made between the nonlocal operators and the corresponding local operators.

3 Nonlocal Helmholtz decomposition

3.1 Existence and uniqueness of the decomposition

The following proposition is an auxiliary result for the main decomposition theorem.

Proposition 1 *Let* $\boldsymbol{\alpha}$ *be a antisymmetric kernel,* $\boldsymbol{\alpha} \in L^2(\Omega \cup \Omega_I \times \Omega \cup \Omega_I)$ *and suppose* $\mathcal{A}_{\boldsymbol{\alpha}} : L^2(\Omega) \to L^2(\Omega)$ *is a positive or negative semidefinite symmetric linear operator such that* $ker(\mathcal{A}_{\boldsymbol{\alpha}}) \subset ker(\mathcal{C}^*_{\boldsymbol{\alpha}})$*, then the system*

$$\mathcal{A}_{\boldsymbol{\alpha}}\mathbf{w} = \mathbf{v}$$

is well-posed iff

$$\mathbf{v} = \mathcal{C}_{\boldsymbol{\alpha}}\mathbf{f},$$

for some vector field $\mathbf{f} \in L^2(\Omega)$.

Proof The necessary condition can be simply proved by the fact that []

$$\mathrm{Rng}(\mathcal{A}_{\boldsymbol{\alpha}})^{\perp} = \mathrm{Rng}(\mathcal{A}^*_{\boldsymbol{\alpha}})^{\perp} = \ker(\mathcal{A}_{\boldsymbol{\alpha}}) \subset \ker(\mathcal{C}^*_{\boldsymbol{\alpha}}) = \mathrm{Rng}(\mathcal{C}_{\boldsymbol{\alpha}})^{\perp}.$$

where Rng denotes the range of linear operator and $\perp$ signifies orthocomplementation. This immediately concludes the following

$$\mathrm{Rng}(\mathcal{A}_{\boldsymbol{\alpha}}) \supset \mathrm{Rng}(\mathcal{C}_{\boldsymbol{\alpha}}).$$

The sufficient condition is straightforward.

Theorem 1 *For each two-point vector function* $\mathbf{u}(\boldsymbol{x}, \boldsymbol{y}) \in L^2(\Omega \times \Omega)$*, there exist unique* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *and* $\mathbf{w} : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\mathbf{u}(\boldsymbol{x}, \boldsymbol{y}) = (\mathcal{G}_{\boldsymbol{\alpha}}\varphi)(\boldsymbol{x}, \boldsymbol{y}) + (\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})(\boldsymbol{x}, \boldsymbol{y}), \tag{11}$$

provided either set of volume constraints are satisfied:

- *Dirichlet (BC)* $\varphi(\boldsymbol{x}) = 0$ *on* Γ
- *Neumann (BC)* $\mathcal{N}_{\boldsymbol{\alpha}}(\mathcal{G}_{\boldsymbol{\alpha}}(\varphi)) = 0$ *on* Γ *and compatibility condition; uniqueness*

Proof We proceed by showing the existence of φ, followed by $\mathbf{w}$.

A. Find potential φ.

We apply $\mathcal{D}_{\boldsymbol{\alpha}}$ to both sides of (11) and get

$$\begin{aligned}(\mathcal{D}_{\boldsymbol{\alpha}}\mathbf{u})(\boldsymbol{x}) =& \big(\mathcal{D}_{\boldsymbol{\alpha}}\left(\mathcal{G}_{\boldsymbol{\alpha}}\varphi\right)\big)(\boldsymbol{x}) + \big(\mathcal{D}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})\big)(\boldsymbol{x}) \\ =& \big(\mathcal{D}_{\boldsymbol{\alpha}}\left(\mathcal{G}_{\boldsymbol{\alpha}}\varphi\right)\big)(\boldsymbol{x}) = -\big(\mathcal{D}_{\boldsymbol{\alpha}}\left(\mathcal{D}^*_{\boldsymbol{\alpha}}\varphi\right)\big)(\boldsymbol{x}) = \big(\mathcal{L}_{\boldsymbol{\alpha}}\varphi\big)(\boldsymbol{x}),\end{aligned}$$

where $\mathcal{D}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}) = 0$. With appropriate volume constraints, φ exists and is unique.

B. Find $\mathbf{w}$.

We apply $\mathcal{C}_{\boldsymbol{\alpha}}$ to both sides of (11) and get

$$\begin{aligned}(\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u})(\boldsymbol{x}) =& \big(\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{G}_{\boldsymbol{\alpha}1}\varphi\right)\big)(\boldsymbol{x}) + \big(\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})\big)(\boldsymbol{x}) \\ =& -\big(\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{D}_{\boldsymbol{\alpha}1}^*\varphi\right)\big)(\boldsymbol{x}) + \big(\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})\big)(\boldsymbol{x}) = \big(\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})\big)(\boldsymbol{x}) \\ =& -2\int_{\mathbb{R}^n} \boldsymbol{\alpha} \times \big[\big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big) \times \boldsymbol{\alpha}\big] d\boldsymbol{y} \\ =& -2\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}) d\boldsymbol{y} + 2\int_{\mathbb{R}^n} (\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})\big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big) d\boldsymbol{y},\end{aligned}$$

where $\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{D}_{\boldsymbol{\alpha}}{}_1^*\varphi\right) = 0$.

We prove the well-posedness of the following equation

$$-2\int_{\mathbb{R}^n}\big(\mathbf{w}(\boldsymbol{y})-\mathbf{w}(\boldsymbol{x})\big)(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})d\boldsymbol{y}+2\int_{\mathbb{R}^n}(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})\big(\mathbf{w}(\boldsymbol{y})-\mathbf{w}(\boldsymbol{x})\big)d\boldsymbol{y}=(\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u})(\boldsymbol{x}), \tag{12}$$

with arbitrary $\mathbf{u}(\boldsymbol{x},\boldsymbol{y}):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ given.

In the operator form, given $\mathbf{u}$, we seek for $\mathbf{w}$ satisfying

$$\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}) = \mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}.$$

The corresponding weak solution $\mathbf{w}$ satisfies:

$$\langle\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}),\mathbf{v}\rangle_{L^2(\mathbb{R}^n)} = \langle\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u},\mathbf{v}\rangle_{L^2(\mathbb{R}^n)}, \quad \mathbf{v}\in\mathcal{V}_{\boldsymbol{\alpha}}, \tag{13}$$

where $\mathcal{V}_{\boldsymbol{\alpha}}$ is some proper function space that the solution $\mathbf{w}$ lives in.

Notice that, by the definition of adjoint, we have that

$$\langle\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}),\mathbf{v}\rangle_{L^2(\mathbb{R}^n)} = \langle\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w},\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{v}\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)},$$

and

$$\langle(\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}),\mathbf{v}\rangle_{L^2(\mathbb{R}^n)} = \langle\mathbf{u},\mathcal{C}^*_{\boldsymbol{\alpha}}(\mathbf{v})\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Hence, we can first define a bilinear operator $B(\cdot,\cdot)$ for $\mathbf{w}$

$$B(\mathbf{w},\mathbf{w}) := \langle(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}),(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}.$$

$B(\cdot,\cdot)$ is not coercive because $K_{\mathcal{C}^*_{\boldsymbol{\alpha}}} := \mathsf{kernel}\,(\mathcal{C}^*_{\boldsymbol{\alpha}})$ is not trivial. More specifically, its rank is equal to the one of $K_{\mathcal{C}^*_{\boldsymbol{\alpha}}} = \mathsf{span}\{\boldsymbol{\alpha}\}$.

Therefore, we will restrict our function space to be $\mathcal{V} := L^2(\mathbb{R}^n)/K_{\mathcal{C}^*_{\boldsymbol{\alpha}}}$. Then on this function space $\mathcal{V}$, we have the coercivity of B:

$$B(\mathbf{w},\mathbf{w}) = \langle(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}),(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \geq \|\mathbf{w}\|^2_*$$

with $\|\mathbf{w}\|^2_* := B(\mathbf{w},\mathbf{w})$.

Therefore, the weak PDE (13) becomes: find $\mathbf{w}\in\mathcal{V}_{\boldsymbol{\alpha}}$, such that

$$\langle(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}),(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{v})\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = \langle\mathbf{u},(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{v})\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}, \quad \forall\mathbf{v}\in\mathcal{V}_{\boldsymbol{\alpha}}. \tag{14}$$

Because for an arbitrary given $u\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$, $\langle\mathbf{u},(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{v})\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}$ defines a bounded linear functional with respect to any $\mathbf{v}\in\mathcal{V}_{\boldsymbol{\alpha}}$. Therefore, apply the Lax-Milgram theorem, we have the existence and uniqueness of the solution to (14). If the instead of $\mathcal{V}_{\boldsymbol{\alpha}}$, we search our solution in the space of $L^2(\mathbb{R}^n)$, then we have the solution $\widetilde{\mathbf{w}} = \mathbf{w}+\mathbf{z}$ with $z\in K_{\mathcal{C}^*_{\boldsymbol{\alpha}}}$.

3.2 Convergence to the local limit

In this section we study the limit of the operators involved in the decomposition as the extent of nonlocal interactions vanishes. To this end, we consider kernels with support on a ball or radius ε, called horizon or interaction radius, and we study the convergence behavior as $\varepsilon \to 0$. We assume that γ satisfies the following properties for all $\boldsymbol{x} \in \Omega$.

$$\begin{aligned} &\gamma(\boldsymbol{x},\boldsymbol{y}) = 0 \qquad \boldsymbol{y} \in B_\varepsilon(\boldsymbol{x}) \\ &\gamma(\boldsymbol{x},\boldsymbol{y}) \geq \gamma_0 \quad \boldsymbol{y} \in B_{\varepsilon/2}(\boldsymbol{x}). \end{aligned} \tag{15}$$

For simplicity and clearness in the exposition, we consider an integrable constant kernel $\gamma = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$, such that

$$\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) = \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|}\mathcal{X}(\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x})). \tag{16}$$

We study the limiting behavior of the operators in **A.** and **B.**, $\mathcal{L}_{\boldsymbol{\alpha}}$ and $\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}_{\boldsymbol{\alpha}}{}^*)$ respectively. While the limit of the nonlocal Laplacian $\mathcal{L}_{\boldsymbol{\alpha}}$ has been widely studied [], no results have been proved on the limit of $\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}_{\boldsymbol{\alpha}}{}^*)$; thus, we proceed by analyzing the integral in (6). First, we note that the first term is exactly the Laplacian operator, for which we already know that

$$\mathcal{L}_{\boldsymbol{\alpha}}\mathbf{w}(\boldsymbol{x}) = \Delta\mathbf{w}(\boldsymbol{x}) + \mathcal{O}(\varepsilon^2).$$

More specifically, the first component of the vector Laplacian (the other two are obtained in the same way) is given by

$$-\mathcal{L}_{\boldsymbol{\alpha}} w_1(\boldsymbol{x}) = -\frac{4\pi}{15}\varepsilon^5(w_{1,x_1x_1} + w_{1,x_2x_2} + w_{1,x_3x_3}) + \text{HOT}.$$

Thus, we analyze the second term in (6), i.e. $\int_{\mathbb{R}^n} \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x}))\, d\boldsymbol{y}$. We expand the integrand: for any vector $\mathbf{v} \in \mathbb{R}^3$, we have

$$\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}\mathbf{v} = \begin{bmatrix} \alpha_1^2 v_1^2 + \alpha_1\alpha_2 v_2 + \alpha_1\alpha_3 v_1 v_3 \\ \alpha_2\alpha_1 v_1 + \alpha_2^2 v_2^2 + \alpha_2\alpha_3 v_2 v_3 \\ \alpha_3\alpha_1 v_1 + \alpha_3\alpha_2 v_2 + \alpha_3^2 v_1 v_3^2 \end{bmatrix}.$$

Due to symmetry, we only study the first component; we have

$$2\int_{\mathbb{R}^n} \left[\alpha_1^2(w_1(\boldsymbol{y}) - w_1(\boldsymbol{x})) + \alpha_1\alpha_2(w_2(\boldsymbol{y}) - w_2(\boldsymbol{x})) + \alpha_1\alpha_3(w_2(\boldsymbol{y}) - w_2(\boldsymbol{x}))\right] d\boldsymbol{y} = I + II + III.$$

We analyze each term separately.

$$
\begin{aligned}
I &= 2\int\limits_{\mathbb{R}^n} \frac{(y_1-x_1)^2}{|\boldsymbol{y}-\boldsymbol{x}|^2}(w_1(\boldsymbol{y})-w_1(\boldsymbol{x}))d\boldsymbol{y} \\
&= 2\int\limits_{\mathbb{R}^n} \frac{h_1^2}{|\mathbf{h}|^2}(w_1(\boldsymbol{x}+\mathbf{h})-w_1(\boldsymbol{x}))d\mathbf{h} \\
&= 2\int\limits_{\mathbb{R}^n} \frac{h_1^2}{|\mathbf{h}|^2}(\nabla w_1(\boldsymbol{x})\cdot\mathbf{h}+\frac{1}{2}\mathbf{h}^T\nabla^2 w_1(\boldsymbol{x})\mathbf{h}+\text{HOT})d\mathbf{h}
\end{aligned}
$$

It can be shown that, due to symmetry, the first derivative term has no contribution, in fact, the integral is 0. We analyze the term with the Hessian:

$$
\begin{aligned}
&2\int\limits_{\mathbb{R}^n} \frac{h_1^2}{|\mathbf{h}|^2}\,\frac{1}{2}\mathbf{h}^T\nabla^2 w_1(\boldsymbol{x})\mathbf{h}\,d\mathbf{h} \\
&=\int\limits_{\mathbb{R}^n} \frac{h_1^2}{|\mathbf{h}|^2}(h_1^2 w_{1,x_1x_1}+2h_1h_2w_{1,x_1x_2}+2h_1h_3w_{1,x_1x_3} \\
&\qquad + h_2^2 w_{1,x_2x_2}+2h_2h_3w_{1,x_2x_3}+h_3^2w_{1,x_3x_3})\,d\mathbf{h} \\
&= A+B+C+D+E+F
\end{aligned}
$$

We treat each term separately. It can be shown that $B=C=E=0$; furthermore

$$
A=\frac{4\pi}{25}\varepsilon^5 w_{1,x_1x_1},\quad D=\frac{4\pi}{75}\varepsilon^5 w_{1,x_2x_2},\quad F=\frac{4\pi}{75}\varepsilon^5 w_{1,x_3x_3}.
$$

Applying a similar procedure to II and III, we have the following:

$$
\begin{aligned}
&I+II+III \\
&=\frac{4\pi}{25}\varepsilon^5 w_{1,x_1x_1}+\frac{4\pi}{75}\varepsilon^5 w_{1,x_2x_2}+\frac{4\pi}{75}\varepsilon^5 w_{1,x_3x_3}+\frac{8\pi}{75}\varepsilon^5 w_{2,x_1x_y}+\frac{8\pi}{75}\varepsilon^5 w_{3,x_1x_3} \\
&=\frac{4\pi}{75}\varepsilon^5\Delta w_1+\frac{8\pi}{75}\varepsilon^5\nabla(\nabla\cdot\mathbf{w}).
\end{aligned}
$$

In summary, the first component of $\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}_{\boldsymbol{\alpha}}{}^*\mathbf{w})(\boldsymbol{x})$ is given by

$$
\begin{aligned}
\big(\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}_{\boldsymbol{\alpha}}{}^*\mathbf{w})\big)_1 &= \frac{8\pi}{75}\varepsilon^5\Big[\big(\nabla(\nabla\cdot\mathbf{w})-\Delta w_1\big)-\Delta w_1\Big]+\text{HOT} \\
&= \frac{8\pi}{75}\varepsilon^5\Big[\nabla\times(\nabla\times\mathbf{w})-\Delta w_1\Big]+\text{HOT}
\end{aligned}
$$

Note that this is not consistent with the first component of the local operator $\nabla\times(\nabla\times\mathbf{w})(\boldsymbol{x})$ which reads

$$
\big(\nabla\times(\nabla\times\mathbf{w})\big)_1=\nabla(\nabla\cdot\mathbf{w})-\Delta w_1,
$$

i.e. $\kappa\,\mathcal{C}_{\boldsymbol{\alpha}}(\mathcal{C}_{\boldsymbol{\alpha}}{}^*\mathbf{w})\to\nabla\times(\nabla\times\mathbf{w})-\Delta\mathbf{w}$, where $\kappa=\frac{75}{8\pi\varepsilon^5}$.

3.3 Geometrical aspects

3.4 Connection to Du&Mengesha

In [6], the well-posedness of the linearized peridynamics equilibrium system

$$-\int_{B_\delta(\boldsymbol{x})\cap\Omega} \mathbb{C}(\boldsymbol{y}-\boldsymbol{x})\,(\mathbf{w}(\boldsymbol{y})-\mathbf{w}(\boldsymbol{x}))d\boldsymbol{y} = \mathbf{b}(\boldsymbol{x}), \tag{17}$$

is established for $\delta > 0$, where $\mathbf{w}(\boldsymbol{x})$ denotes a displacement field, and where $\mathbf{b}(\boldsymbol{x})$ is a given loading force density function and $\mathbb{C}(\boldsymbol{\xi})$ is the micromodulus tensor defined by

$$\mathbb{C}(\boldsymbol{\xi}) = 2\frac{\rho(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}^2|}\boldsymbol{\xi}\otimes\boldsymbol{\xi} + 2F_0(|\boldsymbol{\xi}|)\mathbb{I}. \tag{18}$$

The functions ρ and F_0 are given radial functions and their properties determine the well-posedness of (17). We mention the works of [6] and others for well-posedness theory in the case where $F_0 \equiv 0$, although [7] argues that the condition $F_0 \equiv 0$ is too restrictive for equations of motion for bond-based materials. Moreover, equation (17) defines a non-local boundary value problem with Drichlet-type volumetric boundary conditions and approaches the Navier equations of elasticity with Poisson ration $1/4$ as $\delta \to 0$. Well-posedness is studied in the function space

$$\mathcal{S}(\Omega) = \left\{ \mathbf{w} \in L^2(\Omega:\mathbb{R}^d) \;:\; \int_\Omega\int_\Omega \rho(|\boldsymbol{y}-\boldsymbol{x}|)\left|\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}(\mathbf{w}(\boldsymbol{y})-\mathbf{w}(\boldsymbol{x}))\right|^2 d\boldsymbol{y}d\boldsymbol{x} < \infty \right\}. \tag{19}$$

See [3] and references therein, particularly for Sobolev scale stuff.

We can reformulate (12) to (17) by selecting appropriate $\boldsymbol{\alpha}$. Particularly, we are interested in the prototype kernels

$$\boldsymbol{\alpha} = \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{1+\beta}},$$

where $\beta = \beta(d) > 0$. By letting $\boldsymbol{\xi} = \boldsymbol{x}-\boldsymbol{y}$, the formulations (12) and (17) agree and a simple calculation finds

$$\rho(|\boldsymbol{\xi}|) = \frac{1}{|\boldsymbol{\xi}|^\beta} = F_0(|\boldsymbol{\xi}|). \tag{20}$$

For reference, the simple calculation is included below but can be removed if necessary. Consider the tensor

$$\boldsymbol{\alpha}\otimes\boldsymbol{\alpha} - \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\mathbb{I}.$$

Letting $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{y}$, we have

$$\begin{aligned}\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} - \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}\mathbb{I} &= \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^{1+\beta}} \otimes \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^{1+\beta}} - \frac{(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{2(1+\beta)}}\mathbb{I} \\ &= \frac{1}{|\boldsymbol{\xi}|^{2(1+\beta)}}\boldsymbol{\xi} \otimes \boldsymbol{\xi} + \frac{\boldsymbol{\xi} \cdot \boldsymbol{\xi}}{|\boldsymbol{\xi}|^{2(1+\beta)}}\mathbb{I} \\ &= \frac{1}{|\boldsymbol{\xi}|^2} |\boldsymbol{\xi}|^{-\beta}\boldsymbol{\xi} \otimes \boldsymbol{\xi} + |\boldsymbol{\xi}|^{-\beta}\mathbb{I} \\ &= \frac{1}{|\boldsymbol{\xi}|^2}\rho(|\boldsymbol{\xi}|)\boldsymbol{\xi} \otimes \boldsymbol{\xi} + F_0(|\boldsymbol{\xi}|)\mathbb{I} = \frac{1}{2}\mathbb{C}(\boldsymbol{\xi}).\end{aligned}$$

This concludes the simple calculation.

The identity (20) implies that the results in [6, see Thms 4.2 & 4.5] cannot be used to obtain unique solutions to (12) for a given $\mathbf{u}$. That is, the results of Du-Mengesha apply in two scenarios. The first requires $F_0 \in L^1_{loc}(\mathbb{R}^d)$ while $|\boldsymbol{\xi}|^2\rho(\boldsymbol{\xi}) \in L^1_{loc}(\mathbb{R}^d)$. This would require $\beta < d+2$, while simultaneously having $\beta < d$. In the Sobolev scale, we take $\beta = d + 2s$, for $s \in (0, 1)$, hence this scenario is not fruitful. The second requires F_0 to have zero mean value and for $\rho \in L^1_{loc}(\mathbb{R}^d)$ and does not apply to the work presented here. Therefore, the results from Theorems 1 and 1 lift previous restrictions mentioned in [7].

Remark 3 Note that the assumption $|\boldsymbol{\xi}|^2\rho(\boldsymbol{\xi}) \in L^1_{loc}(\mathbb{R}^d)$ is a simplification of the requirement that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\displaystyle\int_{B_\epsilon(0)} |\boldsymbol{\xi}|^2\rho(\boldsymbol{\xi})d\boldsymbol{\xi}} = 0.$$

3.5 Connection to graph theory

In this section we will make connections with Hodge decompositions on graphs; see Lemma 1 in [1] which generalized the proof given in [5].

In graph theory, points are represented by vertices and two-point functions are weights on a graph. Define operators on graphs to obtain

Lemma 1 (Lemma 1 [1]) *Given a vector field on a graph* $v = (v_{ij})_{(ij)\in E}$ *with* $v(i, j) = -v(j, i)$ *and a measure* ρ *... there exists a unique*

4 Example

We will give an example of a decomposition with an application. For an integro-differential equation perform the decomposition, show how it works in finding the solution, or performing other analysis.

4.1 Matrix analysis with 3D results

In this section we aim to provide an illustration of the existence of solution $\mathbf{w}$ in Theorem 1 when $n=3$. Here we denote $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = (\alpha_1(\mathbf{x},\mathbf{y}), \alpha_2(\mathbf{x},\mathbf{y}), \alpha_3(\mathbf{x},\mathbf{y}))$, $\mathbf{w}(\mathbf{x}) = (w_1(\mathbf{x}), w_2(\mathbf{x}), w_3(\mathbf{x}))$, and $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}), v_3(\mathbf{x}))$, where $\alpha_i : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ are antisymmetric functions, $w_i : \mathbb{R}^3 \to \mathbb{R}$ and $v_i : \mathbb{R}^3 \to \mathbb{R}$ are scalar functions, $i = 1,2,3$. To show well-posedness of (12), we can rewrite its left hand side:

$$\begin{aligned}
&-2\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})d\boldsymbol{y} + 2\int_{\mathbb{R}^n} (\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})\big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)d\boldsymbol{y} \\
=&2\int_{\mathbb{R}^n} \begin{pmatrix} -(w_1(\boldsymbol{y}) - w_1(\boldsymbol{x}))(\alpha_2^2+\alpha_3^2) + (w_2(\boldsymbol{y}) - w_2(\boldsymbol{x}))\alpha_1\alpha_2 + (w_3(\boldsymbol{y}) - w_3(\boldsymbol{x}))\alpha_1\alpha_3 \\ (w_1(\boldsymbol{y}) - w_1(\boldsymbol{x}))\alpha_1\alpha_2 - (w_2(\boldsymbol{y}) - w_2(\boldsymbol{x}))(\alpha_1^2+\alpha_3^2) + (w_3(\boldsymbol{y}) - w_3(\boldsymbol{x}))\alpha_2\alpha_3 \\ (w_1(\boldsymbol{y}) - w_1(\boldsymbol{x}))\alpha_1\alpha_3 + (w_2(\boldsymbol{y}) - w_2(\boldsymbol{x}))\alpha_2\alpha_3 - (w_3(\boldsymbol{y}) - w_3(\boldsymbol{x}))(\alpha_1^2+\alpha_2^2) \end{pmatrix} d\boldsymbol{y} \\
=&-2\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)\mathbf{A}(\mathbf{x},\mathbf{y})d\boldsymbol{y}
\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} \alpha_2^2+\alpha_3^2 & -\alpha_1\alpha_2 & -\alpha_1\alpha_3 \\ -\alpha_1\alpha_2 & \alpha_1^2+\alpha_3^2 & -\alpha_2\alpha_3 \\ -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & \alpha_1^2+\alpha_2^2 \end{bmatrix}$$

is a symmetric matrix function. When apply a test function $\mathbf{v}(\mathbf{x})$ to the above function and integrate, since $\mathbf{A}$ is symmetric,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{v}^T(\mathbf{x})d\boldsymbol{y}d\boldsymbol{x} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{x}) - \mathbf{w}(\boldsymbol{y})\big)\mathbf{A}(\mathbf{y},\mathbf{x})\mathbf{v}^T(\mathbf{y})d\boldsymbol{x}d\boldsymbol{y} \\
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{v}^T(\mathbf{y})d\boldsymbol{y}d\boldsymbol{x},
\end{aligned}$$

which yields

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(-2\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})d\boldsymbol{y} + 2\int_{\mathbb{R}^n} (\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})\big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)d\boldsymbol{y}\right)\mathbf{v}^T(\mathbf{x})d\boldsymbol{x} \\
=&-2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)\mathbf{A}(\mathbf{x},\mathbf{y})\mathbf{v}^T(\mathbf{x})d\boldsymbol{y}d\boldsymbol{x} \\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)\mathbf{A}(\mathbf{x},\mathbf{y})(\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x}))^T d\boldsymbol{y}d\boldsymbol{x}.
\end{aligned}$$

That means, the bilinear operator $B(\cdot,\cdot)$ can be explicitly expressed as

$$B(\mathbf{w},\mathbf{v}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x})\big)\mathbf{A}(\mathbf{x},\mathbf{y})(\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x}))^T d\boldsymbol{y}d\boldsymbol{x}. \tag{21}$$

Note that $B(\mathbf{w},\mathbf{w}) = 0$ if and only if $\mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x}) = C(\alpha_1(\mathbf{x},\mathbf{y}), \alpha_2(\mathbf{x},\mathbf{y}), \alpha_3(\mathbf{x},\mathbf{y}))$. Therefore, $Ker(B) := \{\mathbf{w} : B(\mathbf{w},\mathbf{w}) = 0\} = \{\mathbf{w} : \mathbf{w}(\boldsymbol{y}) - \mathbf{w}(\boldsymbol{x}) = C(\alpha_1(\mathbf{x},\mathbf{y}), \alpha_2(\mathbf{x},\mathbf{y}), \alpha_3(\mathbf{x},\mathbf{y}))\}$, and we can define the cosets of $\mathbf{w}$ on the quotient space of $Ker(B)$ and associated norm:

$$|||[\mathbf{w}]|||^2_{L^2/Ker(B)} := \inf_{\tilde{\mathbf{w}}\in Ker(B)} B(\mathbf{w}+\tilde{\mathbf{w}}, \mathbf{w}+\tilde{\mathbf{w}}) = B(\mathbf{w},\mathbf{w}). \tag{22}$$

On the other hand, for a given two-point vector function $\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) = (u_1(\boldsymbol{x},\boldsymbol{y}), u_2(\boldsymbol{x},\boldsymbol{y}), u_3(\boldsymbol{x},\boldsymbol{y}))$, after applying the test function, with the antisymmetric property of $\boldsymbol{\alpha}$ the right hand side of (12) can be similarly written as

$$\begin{aligned}((\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}),\mathbf{v}) =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times (\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) + \mathbf{u}(\boldsymbol{y},\boldsymbol{x}))\cdot \mathbf{v}(\mathrm{x}) d\boldsymbol{y} d\boldsymbol{x}\\ =& -\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times (\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) + \mathbf{u}(\boldsymbol{y},\boldsymbol{x}))\cdot (\mathbf{v}(\mathrm{y}) - \mathbf{v}(\mathrm{x})) d\boldsymbol{y} d\boldsymbol{x}.\end{aligned}$$

Here we note that when $\mathbf{v} \in Ker(B)$,

$$((\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}),\mathbf{v}) = -\frac{C}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \times (\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) + \mathbf{u}(\boldsymbol{y},\boldsymbol{x}))\cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} d\boldsymbol{x} = 0.$$

We can therefore define right hand side as a functional on the quotient space:

$$F([\mathbf{v}]) := ((\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}),[\mathbf{v}]) = ((\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}),\mathbf{v}). \tag{23}$$

To apply the Lax-Milgram theorem and show the well-posedness of the variational problem $B([\mathbf{w}],[\mathbf{v}]) = ((\mathcal{C}_{\boldsymbol{\alpha}}\mathbf{u}),[\mathbf{v}])$, it suffices to show the coervicity and boundedness of the operator $B(\mathbf{w},\mathbf{v})$:

Lemma 2

$$\forall [\mathbf{u}],[\mathbf{v}] \in L^2/Ker(B),\ B([\mathbf{u}],[\mathbf{v}]) \le C_1 ||[\mathbf{u}]||_{L^2/Ker(B)} ||\mathbf{v}||_{L^2/Ker(B)}, \tag{24}$$

$$\forall [\mathbf{u}] \in L^2/Ker(B),\ B([\mathbf{u}],[\mathbf{u}]) \ge C_2 ||[\mathbf{u}]||^2_{L^2/Ker(B)}, \tag{25}$$

for two constants $C_1, C_2 > 0$.

Proof Note that for each $\boldsymbol{x}$, $\boldsymbol{y}$, the matrix $\mathbf{A}$ is a symmetric matrix and can be diagonalized:

$$\begin{aligned}\mathbf{A} =& \begin{bmatrix} \alpha_2^2+\alpha_3^2 & -\alpha_1\alpha_2 & -\alpha_1\alpha_3 \\ -\alpha_1\alpha_2 & \alpha_1^2+\alpha_3^2 & -\alpha_2\alpha_3 \\ -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & \alpha_1^2+\alpha_2^2 \end{bmatrix}\\ =& S \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha_1^2+\alpha_2^2+\alpha_3^2 & 0 \\ 0 & 0 & \alpha_1^2+\alpha_2^2+\alpha_3^2 \end{bmatrix} S^T\\ =& S\Lambda S^{-1},\end{aligned}$$

where $S(\boldsymbol{x},\boldsymbol{y})$ is an orthogonal matrix function. For (24), with Holder's inequality we have

$$\begin{aligned}B([\mathbf{u}],[\mathbf{v}]) =& B(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbf{u}(\boldsymbol{y}) - \mathbf{u}(\boldsymbol{x})) S\Lambda S^T (\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x}))^T d\boldsymbol{y} d\boldsymbol{x}\\ \le& \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|(\mathbf{u}(\boldsymbol{y}) - \mathbf{u}(\boldsymbol{x}))S\sqrt{\Lambda}\right|^2 d\boldsymbol{y} d\boldsymbol{x}\right)^{1/2} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|(\mathbf{v}(\boldsymbol{y}) - \mathbf{v}(\boldsymbol{x}))S\sqrt{\Lambda}\right|^2 d\boldsymbol{y} d\boldsymbol{x}\right)^{1/2}\\ =& \sqrt{B(\mathbf{u},\mathbf{u})B(\mathbf{v},\mathbf{v})} = ||[\mathbf{u}]||_{L^2/Ker(B)} ||[\mathbf{v}]||_{L^2/Ker(B)}.\end{aligned}$$

Here we note that

$$\sqrt{\Lambda} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2} & 0 \\ 0 & 0 & \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2} \end{bmatrix}$$

is well-defined. On the other hand, with the definition of the $|||[\cdot]|||_{L^2/Ker(B)}$ norm we can show (25). We have then finished the proof.

5 Numerical tests

In this section, we validate the nonlocal Helmholtz decomposition by testing a manufactured example. We focus on the functions whose domains belong to $\mathbb{R}^2$ and later are embedded into $\mathbb{R}^3$. More precisely, let

$$\boldsymbol{x} = (x_1, x_2, 0)' \quad \text{and} \quad \boldsymbol{y} = (y_1, y_2, 0)' \in \Omega,$$

where Ω is a subset of $\mathbb{R}^2$ and is embedded into $\mathbb{R}^3$. The two-point vector function $\mathbf{u} : \Omega \times \Omega \to \mathbb{R}^3$, the one-point scalar function $\varphi : \Omega \to \mathbb{R}$, and the one-point vector function $\mathbf{w} : \Omega \to \mathbb{R}^2 \subset \mathbb{R}^3$ (that is, the range of $\mathbf{w}$ could be embedded into $\mathbb{R}^3$). Hence, according to the definitions of nonlocal operators (2) and (4) we have

$$(\mathcal{G}_{\boldsymbol{\alpha}}\varphi)(\boldsymbol{x},\boldsymbol{y}) : \Omega \times \Omega \to \mathbb{R}^3 \quad \text{and} \quad (\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}) : \Omega \times \Omega \to \begin{pmatrix} 0 \\ 0 \\ \mathbb{R} \end{pmatrix}.$$

These test functions of φ and $\mathbf{w}$ are chosen to be

$$\varphi(\boldsymbol{x}) := x_1^2 \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, \tag{26}$$

and the nonlocal kernel $\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})$ is chosen to be

$$\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{\delta^{3/2}} \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}.$$

Consequently, for $\mathcal{G}_{\boldsymbol{\alpha}}\varphi$ we get

$$(\mathcal{G}_{\boldsymbol{\alpha}}\varphi)(\boldsymbol{x},\boldsymbol{y}) = \frac{-1}{\delta^{3/2}} \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|} \left(y_1^2 - x_1^2\right) = \frac{1}{\delta^{3/2}\,|\boldsymbol{y}-\boldsymbol{x}|} \begin{pmatrix} -(x_1^2-y_1^2)(x_1-y_1) \\ -(x_1^2-y_1^2)(x_2-y_2) \\ 0 \end{pmatrix}.$$

For $\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}$, it gives

$$(\mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w})(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\delta^{3/2}|\boldsymbol{y}-\boldsymbol{x}|} \begin{pmatrix} 0 \\ 0 \\ (y_1-x_1)(y_2^2-x_2^2) \end{pmatrix}.$$

The proposed $\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{G}_{\boldsymbol{\alpha}}\varphi + \mathcal{C}^*_{\boldsymbol{\alpha}}\mathbf{w}$ is equal to

$$\mathbf{u}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\delta^{3/2}|\boldsymbol{y}-\boldsymbol{x}|}\begin{pmatrix} -(x_1^2-y_1^2)(x_1-y_1) \\ -(x_1^2-y_1^2)(x_2-y_2) \\ (y_1-x_1)(y_2^2-x_2^2) \end{pmatrix}. \tag{27}$$

Now for given $\mathbf{u}$ in (27), we will follow the proof of Theorem 1 to solve for the pair φ and $\mathbf{w}$ defined in (26).

1. **Find** φ. The corresponding nonlocal equation is,

$$\begin{cases} (\mathcal{L}_{\boldsymbol{\alpha}}\varphi)(\boldsymbol{x}) = \dfrac{2}{\delta^{3/2}}\displaystyle\int_{B_\delta(\boldsymbol{x})}(x_1^2-y_1^2)|\boldsymbol{y}-\boldsymbol{x}|\,d\boldsymbol{y}, \text{ in } \Omega, \\ \text{volumetric Dirichlet B.C. for } \varphi, \end{cases}$$

   where $d\boldsymbol{y}$ is restricted on the plane $\mathbb{R}^2$.
2. **Find** $\mathbf{w}$. The corresponding nonlocal equation is,

$$\begin{cases} -\dfrac{2}{\delta^3}\displaystyle\int_{B_\delta(\boldsymbol{x})}(\mathbf{w}(\boldsymbol{y})-\mathbf{w}(\boldsymbol{x}))\,d\boldsymbol{y} + \dfrac{2}{\delta^3}\displaystyle\int_{B_\delta(\boldsymbol{x})}\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}\otimes\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}d\boldsymbol{y} \\ \quad = \dfrac{2}{\delta^{3/2}}\begin{pmatrix} \displaystyle\int_{B_\delta(\boldsymbol{x})}\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}(y_1-x_1)(y_2-x_2)(y_2^2-x_2^2)d\boldsymbol{y} \\ \displaystyle\int_{B_\delta(\boldsymbol{x})}-\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}(y_1-x_1)^2(y_2^2-x_2^2)d\boldsymbol{y} \\ 0 \end{pmatrix} \\ \quad = \dfrac{2}{\delta^{3/2}}\begin{pmatrix} \displaystyle\int_{B_\delta(\boldsymbol{x})}\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}(y_1-x_1)(y_2-x_2)(y_2^2-x_2^2)d\boldsymbol{y} \\ \displaystyle\int_{B_\delta(\boldsymbol{x})}-\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}(y_1-x_1)^2(y_2^2-x_2^2)d\boldsymbol{y} \end{pmatrix}, \text{ in } \Omega, \\ \text{volumetric Dirichlet B.C. for } \mathbf{w}, \end{cases}$$

   where $d\boldsymbol{y}$ is again restricted on the plane $\mathbb{R}^2$.

5.1 Convergence of nonlocal to local

We also test the asymptotic convergence of nonlocal to local when δ goes to zero. The nonlocal and domain settings remain the same the previous one, and the manufactured solutions for local limit are set to be the followings

$$\varphi_\ell(\boldsymbol{x}) := x_2^4 \quad \text{and} \quad \mathbf{w}_\ell(\boldsymbol{x}) := \begin{pmatrix} 0 \\ x_1^4 \\ 0 \end{pmatrix}, \tag{28}$$

hence the local limiting solution of $\mathbf{u}$ is

$$\mathbf{u}_\ell(\boldsymbol{x}) := \nabla(\varphi_\ell)(\boldsymbol{x}) + \nabla\times(\mathbf{w})(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 4x_2^3 \\ 4x_1^3 \end{pmatrix}. \tag{29}$$

Therefore, the benchmark problem for nonlocal solutions φ and $\mathbf{w}$ is

1. **Find φ.** The corresponding nonlocal equation is

$$\begin{cases} (\mathcal{L}_{\boldsymbol{\alpha}}\varphi)(\boldsymbol{x}) = 12x_2^2, \text{ in } \Omega, \\ \text{volumetric Dirichlet B.C. for } \varphi. \end{cases} \tag{30}$$

2. **Find w.** The corresponding nonlocal equation is

$$\begin{cases} \mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{C}_{\boldsymbol{\alpha}}^{*}\mathbf{w}\right)(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 12x_1^2 \\ 0 \end{pmatrix}, \text{ in } \Omega, \\ \text{volumetric Dirichlet B.C. for } \mathbf{w}. \end{cases} \tag{31}$$

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