SAND2018-1461C

Probability Distribution of von Mises Stress in the Presence of Pre-Load

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von Mises Stress in Random Environments

- Serious issue in many field, including in launch and re-entry
- Prediction is still pretty primitive
  - Single degree of freedom model (Miles' equation, 1954)
  - RMS von Mises stress for arbitrary zero-mean random load (Segalman et al 2000a)
  - Approximate probability distribution for Gaussian loads (Segalman et al 2000b)
  - Approximate probability distribution for a single Gaussian load in the presence of pre-load (Tibbits 2011)

Work Presented Here

- An effort to approximate the probability distributions for von Mises stresses given
  - Multiple independent loads, each a weakly stationary Gaussian process with zero mean
  - A field of static pre-loads

Dynamic Loads are Characterized

- Let $F(t)$ be an $\mathbb{R}^d$-valued, weakly stationary Gaussian process of zero mean and having correlation matrix

$$r_{FF}(\tau) = E\left[F(t)F(t+\tau)^T\right]$$

a $d \times d$ matrix.

- The matrix of two-sided spectral densities $S_{FF}(\omega)$ is defined so that

$$S_{FF}(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} r_{FF}(\tau)e^{-i\omega\tau}\,d\tau$$

Assuming a Linear Structure (A BIG Assumption)

- From $S_{FF}(\omega)$ plus the structure's frequency response functions we can derive the cross spectral density matrix of modal displacement

$$S_{qq}(\omega) = \overline{H_q(\omega)}\, S_{FF}(\omega) H_q(\omega)^T$$

- From $S_{qq}(\omega)$ we can evaluate Γ_{qq}, the zero-time-lag covariance matrix of modal displacement.

$$\Gamma_{qq} = E[q(t)\,q(t)^T] = \int_{-\infty}^{\infty} S_{qq}(\omega)\, d\omega$$

von Mises Stress

- Stress:

$$\sigma(t,x) = \sigma_0(x) + \sum_n q_n(t)\Psi_n(x) = \sigma_0(x) + \Psi(x)q(t)$$

- von Mises stress

$$p^2(t,x) = \left(\Psi(x)q(t)+\sigma_0(x)\right)^T A \left(\Psi(x)q(t)+\sigma_0(x)\right)$$

….where

$$A = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

von Mises Stress

- Expanding

$$p^2(t,x) = \left(\Psi(x)q(t)\right)^T A\left(\Psi(x)q(t)\right) + \sigma_0^T(x)A\sigma_0(x)$$
$$+\left(\Psi(x)q(t)\right)^T A\sigma_0(x) + \sigma_0^T(x)A\left(\Psi(t)q(t)\right)$$

- For future convenience, define $B(x) = \Psi^T(x)\,A\,\Psi(x)$

so $$p^2(t,x) = q(t)^T\,B(x)\,q(t) + \sigma_0^T(x)A\sigma_0(x)$$
$$+\left(\Psi(x)q(t)\right)^T A\sigma_0(x) + \sigma_0^T(x)A\left(\Psi(t)q(t)\right)$$

- Note that

$$p_{RMS}^2(x) = E[p^2(t)] = \mathrm{Tr}(\,B(x)^T\,\Gamma_{qq}\,) + \sigma_0^T(x)A\sigma_0(x)$$

About this Intermediate Result

$$p_{RMS}^2(x) = \mathrm{Tr}(B(x)^T \Gamma_{qq}) + \sigma_0^T(x) A \sigma_0(x)$$

$$p_{RMS}(x) = \sqrt{\mathrm{Tr}\left(B^T(x)\Gamma_{qq}\right) + \sigma_0^T(x) A \sigma_0(x)}$$

$$\leq \sqrt{\mathrm{Tr}\left(B^T(x)\Gamma_{qq}\right)} + \sqrt{\sigma_0^T(x) A \sigma_0(x)}$$

- Provides information about time average von Mises stress independent of the nature of the probability distribution $p(t)$
- Tells very little about the likelihood of high stress levels.

Back to More General Problem: Probability Distributions for Combined Stress

Elements of the process

- Gaussian Loads

……..→Gaussian Modal Coordinates

…………..→Gaussian Modal Stresses

- Many changes of variable to put dynamic stresses and static stresses in commensurate form
- A very approximate assumption
- Completing the square
- Integration strategy
- A couple of examples

Stress Processes

- Noting that matrix Γ_{qq} is square $N_M \text{ x } N_M$ and positive semi-definite, we may decompose it $\Gamma_{qq} = QX^2Q^T$ where $\dim(X) = N_R$ and $Q^T Q = I_{N_R}$
- Note that because Γ_{qq} has no time or spatial dependence, neither do Q or X.
- Make change of variable $\beta(t) = X^{-1}Q^T q(t)$
- If the elements of $F(t)$ are Gaussian, so are the elements of $\beta(t)$.
- By construction $E[\beta(t)\beta(t)^T] = I_{N_R}$ so the elements of $\beta(t)$ are IID Gaussian processes.

Back to the Problem of Pre-stress
Try to put

- Recall

$$p^2(t,x) = \left(\Psi(x)q(t)\right)^T A\left(\Psi(x)q(t)\right) + \sigma_0^T(x)A\sigma_0(x)$$
$$+\left(\Psi(x)q(t)\right)^T A\sigma_0(x) + \sigma_0^T(x)A\left(\Psi(t)q(t)\right)$$

- In the absence of pre-stress

$$p^2(t,x) = p_R^2(x,t) = \left(\Psi(x)q(t)\right)^T A\left(\Psi(x)q(t)\right)$$
$$= \beta(t)^T C(x)\beta(t)$$

where $C(x) = X^T Q^T B(x) Q X$

- Decompose $C(x) = R(x)D^2(x)R(x)^T$ where $R(x)$ is a rectangular matrix

Introduce a new Gaussian Random Variable (I apologize for the complexity)

- Define $y(t,x) = R^T(x)\beta(t)$ then

  $p_R^2(x,t) = y(t,x)^T D(x)^2 y(t,x)$
- The dimension of $D(x)^2$ is the number of independent "stress processes"
- The combined von Mises stress is

  $$p^2(t,x) = y(t,x)^T D(x)^2 y(t,x)$$
  $$+2\beta(t)^T \left(X^T Q^T \Psi(x)^T \right) A\sigma_0(x) + \sigma_0(x)^T A\sigma_0(x)$$
- How to reconcile terms with $\beta(t)$ and those with $y(t,x)$?

The Big Approximation

- Recall that by definition $y(t,x) = R^T(x)\beta(t)$ where $R^T(x)$ is a rectangular matrix.
- Let's approximate $\beta(t) \approx R(x)y(t,x)$
- The von Mises stress is now

$$p^2(t,x) = y^T(t,x)D(x)^2 y(t,x) - 2y(t,x)^T D(x)^2 \gamma(x) + p_0(x)^2$$

where $p_0^2(x) = \sigma_0^T(x)A\sigma_0(x)$, $\gamma(x) = G(x)\sigma_0(x)$,
and $G(x) = -D(x)^{-2} R(x)^T X Q^T \Psi(x)^T A$

- Note the diagonal nature of the above equation.

Completing the Square

$$p^2(t,x) = y^T(t,x)D(x)^2 y(t,x) - 2y(t,x)^T D(x)^2 \gamma(x) + p_0(x)^2$$

- Adding and subtracting appropriate constants

$$p^2(t,x) = (y(t) - \gamma(x))^T D(x)^2 (y(t) - \gamma(x)) + Y_0(x)^2$$

where $Y_0(x)^2 = p_0(x)^2 - \gamma(x)^T D(x)^2 \gamma(x)$

- Define an N dimensional ellipsoid in y space

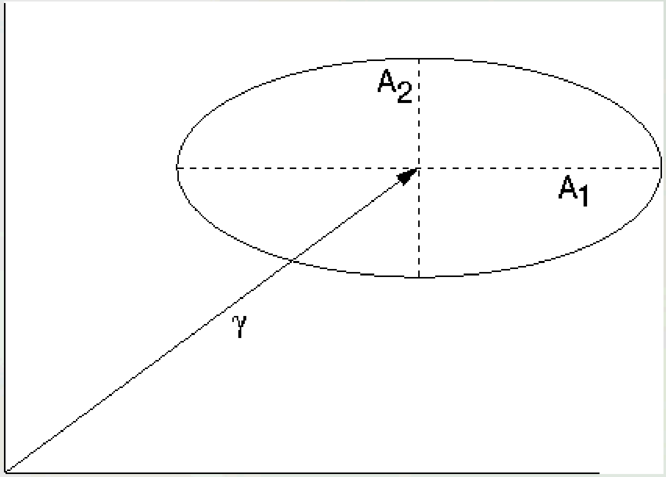
$$Z(\{D\},\gamma,Y_0,Y) = \left\{ y : ((y-\gamma)^T D^2 (y-\gamma)) \le Y^2 - Y_0^2 \right.$$

About Our N-Dimensional Ellipsoid

$$Z(\{D\},\gamma,Y_0,Y) = \left\{ y : ((y-\gamma)^T D^2 (y-\gamma)) \le Y^2 - Y_0^2 \right.$$

Ellipsoid centered at $\gamma(x)$ with semi-axes

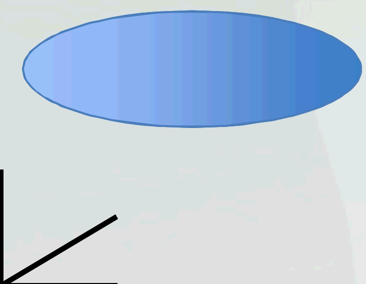
$$A_r = \sqrt{\frac{Y^2 - Y_0^2}{D_r^2}}$$



Now we can calculate probabilities

- The probably of von Mises stress being less than value Y is

$$F_Y = P(p \le Y) = P(p^2 \le Y^2) =$$



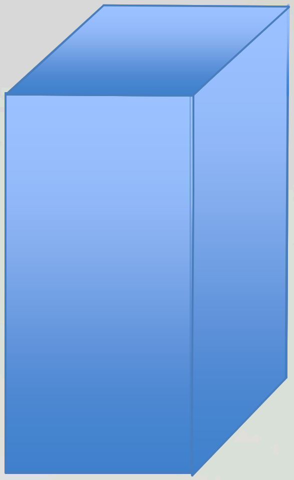
$$\begin{cases} 0 & \text{for} \quad Y \le Y_0 \\ \int_{Z(\{D\},\gamma,Y_0,Y)} \prod \rho_r(y_r)\,dy_r & \text{for} \quad Y > Y_0 \end{cases}$$

where $\rho_r(y_r) = \dfrac{1}{\sqrt{2\pi}} e^{-y_r^2/2}$

- A numerical technique for evaluating these is presented outlined and detailed in the paper.

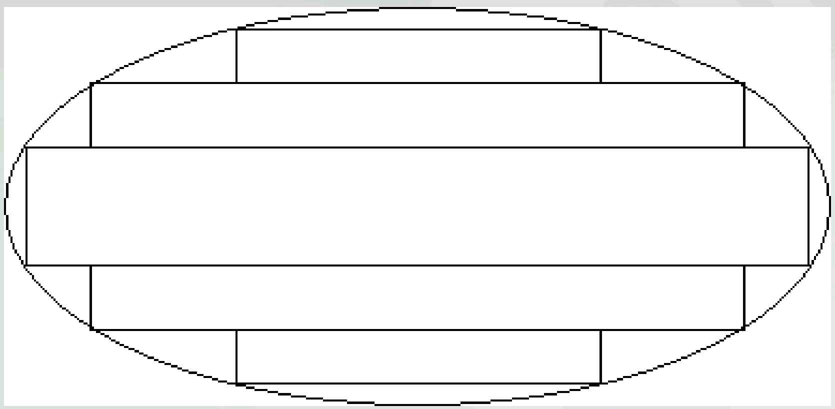
Outline of Integration Strategy

- Note that we can perform the integrals over N-dimensional boxes in closed form

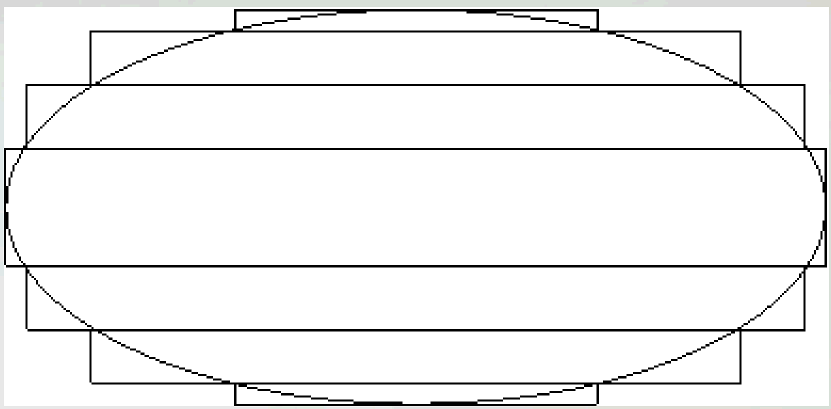


$$\int_{B_\lambda} \prod_{r=1}^{N_P} \rho_r(y_r)\,dy_r = \prod_{r=1}^{N_P} \left[\Phi(y_{r,\max}) - \Phi(y_{r,\min}) \right]$$

We can approximate N-Dimensional Ellipsoids as the Union of N-Dimensional Boxes



Set B_L contained in Z



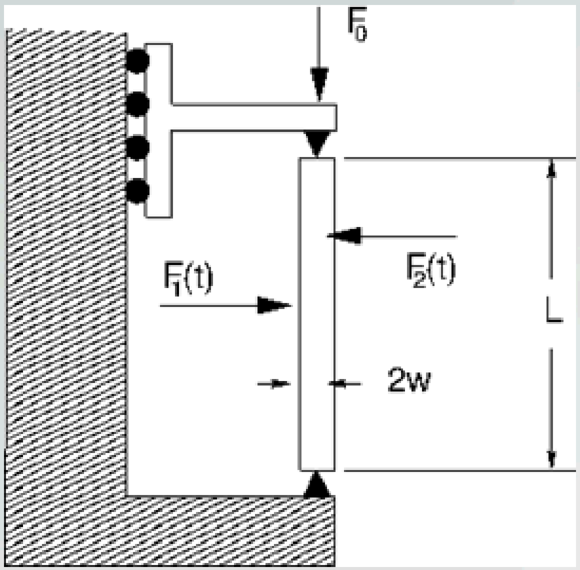
Set B_U containing Z

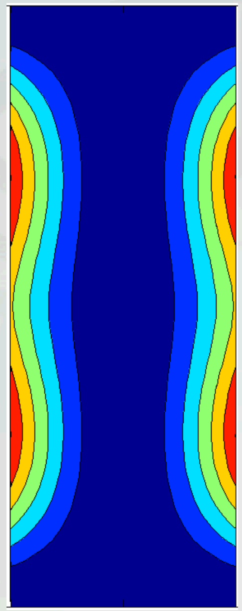
$$\int_{B_L} \prod \rho_r(y_r)\mathrm{d}y_r \le \int_{Z(\{D\},\gamma,Y_0,Y)} \prod \rho_r(y_r)\mathrm{d}y_r$$

$$\le \int_{B_U} \prod \rho_r(y_r)\mathrm{d}y_r$$

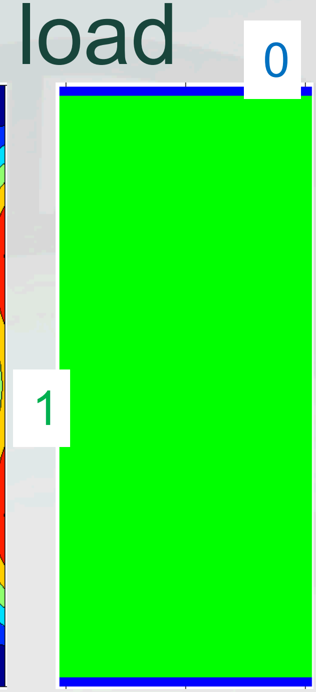
Some Example Calculations

Example Calculation 1: A simply supported beam column with end load

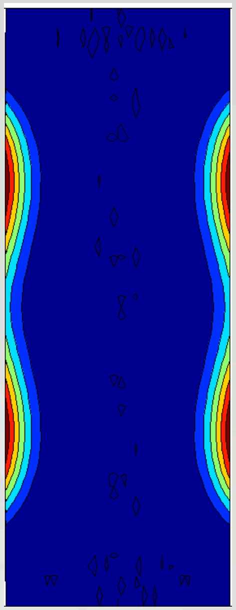




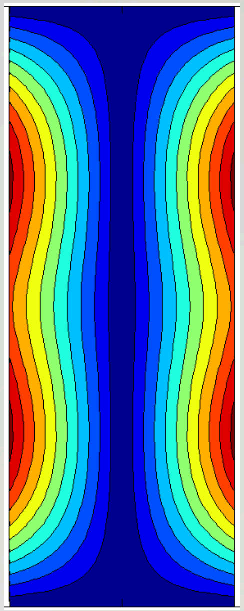
RMS von Mises



Number Processes



50 % Probability

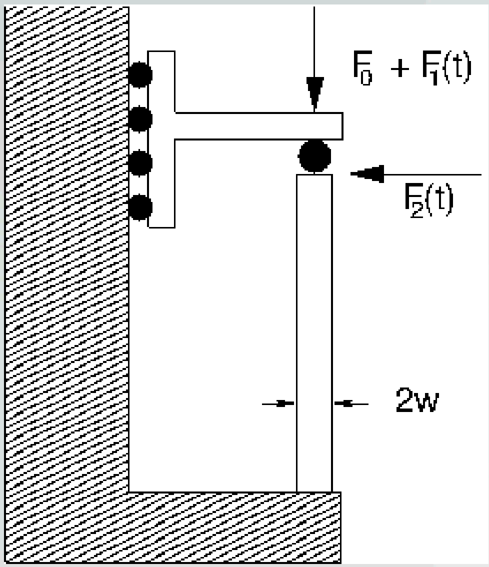


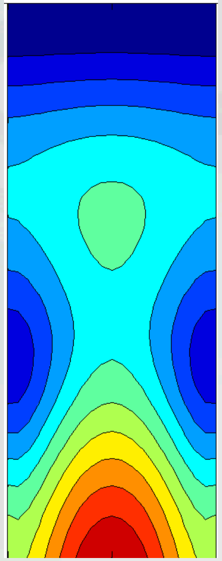
95 % Probability

F_1 band limited and excites first bending mode

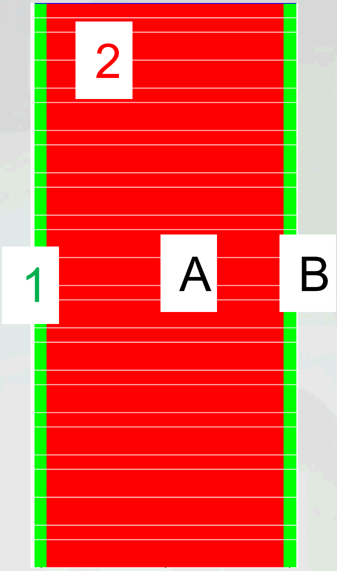
F_2 band limited and excites second bending mode

Example Calculation 2: A cantilevered beam-column with end load

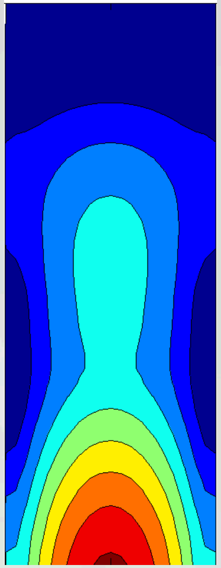




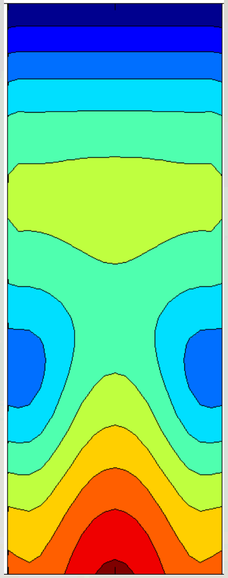
RMS von Mises



Number Processes



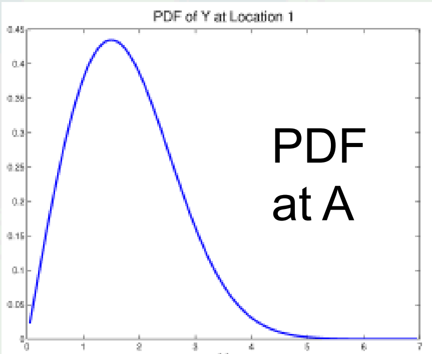
50 % Probability

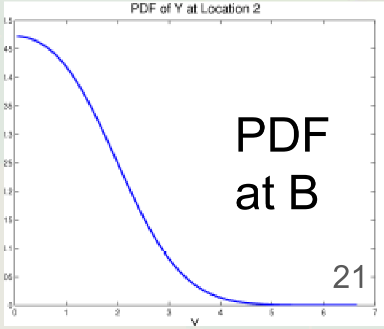


95 % Probability

F_1 band limited and excites first 2 axial modes

F_2 band limited and excites the first bending mode





Summary

- Despite the complex derivation, this approach is straight-forward and numerically efficient to implement.
- We have no idea how good (or bad) the core approximation is.
- This whole process is restricted to Gaussian loads.