

Optimization Letters manuscript No.
(will be inserted by the editor)

1 Approximating Two-Stage Chance-Constrained 2 Programs with Classical Probability Bounds

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5 Received: date / Accepted: date

6 **Abstract** We consider a joint-chance constraint (JCC) as a union of sets, and
7 approximate this union using bounds from classical probability theory. When these
8 bounds are used in an optimization model constrained by the JCC, we obtain
9 corresponding upper and lower bounds on the optimal objective function value.
10 We compare the strength of these bounds against each other under two different
11 sampling schemes, and observe that a larger correlation between the uncertainties
12 tends to result in more computationally challenging optimization models. We also
13 observe the same set of inequalities to provide the tightest upper and lower bounds
14 in our computational experiments.

15 **Keywords** chance-constrained optimization · Bonferroni inequalities · union
16 bounds · stochastic optimization · approximations

17 1 Introduction

18 Consider a two-stage stochastic program with a first-stage decision variable
19 $x = (x_t)_{t \in T}$. Let $w = (w_t)_{t \in T}^\omega$ denote a random vector under realization

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20 $\omega = \omega_1, \omega_2, \dots, \omega_N$, where $N = |\Omega|$. We assume these N realizations are i.i.d.
 21 (for example, generated via Monte Carlo sampling) and equally likely. After mak-
 22 ing the first stage decision x , the uncertainty ω unfolds, and in the second stage we
 23 make a recourse decision $y = (y_t)_{t \in T}^\omega$ based on the first stage decision and the real-
 24 ization of w . We are interested in satisfying the following probabilistic constraint:

$$25 \quad \mathbb{P}(x_t \leq y_t^\omega + w_t^\omega, \forall t \in T) \geq 1 - \varepsilon, \quad (1)$$

26 where $0 < \varepsilon < 1$. Typically ε is small; for example, 0.01 or 0.05.

27 Constraints such as (1) are known as joint chance constraints (JCCs), and
 28 were first introduced in [5]. Optimizing over a JCC is known to be NP-hard; see,
 29 e.g., [15]. Further, chance-constrained optimization models are often computationally
 30 intractable; see, e.g., [12]. Existing approaches to solve chance constrained
 31 optimization models include decomposition methods [12, 13], sample average approx-
 32 imations [1, 14], and formulating deterministic equivalents [6, 18, 19]. We are
 33 interested in deterministic approximations of JCCs in a manner we describe below.

34 Let A_t denote the set of scenarios that we “fail” at t ; i.e., $A_t = \{\omega : x_t >$
 35 $y_t^\omega + w_t^\omega\}$. The complement of A_t is the set of scenarios that we “succeed” at
 36 t ; i.e., $A_t^c = \{\omega : x_t \leq y_t^\omega + w_t^\omega\}$. Then, we have $\mathbb{P}(x_t \leq y_t^\omega + w_t^\omega, \forall t \in T) =$
 37 $\mathbb{P}(\bigcap_{t \in T} A_t^c) = 1 - \mathbb{P}(\bigcup_{t \in T} A_t)$. Hence, we can rewrite the JCC in equation (1) as
 38 follows:

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) \leq \varepsilon. \quad (2)$$

39 We study approximations of the LHS of equation (2). If we approximate the
 40 LHS using a quantity larger than $\mathbb{P}(\bigcup_{t \in T} A_t)$ the feasible region representing an
 41 optimization model with equation (2) as a constraint is restricted, and thus the
 42 approximate model gives a lower (upper) bound under maximization (minimiza-
 43 tion). If we approximate the LHS using a quantity smaller than $\mathbb{P}(\bigcup_{t \in T} A_t)$ the
 44 feasible region representing an optimization model with equation (2) as a con-
 45 straint is enlarged and thus the approximate model gives a upper (lower) bound
 46 under maximization (minimization). Henceforth, we consider maximization as our
 47 objective.

48 In part, we are motivated by the work in [18], where individual chance con-
 49 straints are approximated using classical inequalities, such as Chebyshev’s inequal-
 50 ity, Bernstein’s inequality and Hoeffding’s inequality. Our work also approximates
 51 chance constraints using classical inequalities, but there are at least two major dif-
 52 ferences. First, the work in [18] considers only *individual* chance constraints while
 53 we study *joint* chance constraints. As a result, Chebyshev’s inequality, Bernstein’s

inequality and Hoeffding's inequality cannot be directly applied. Second, the work in [18] considers weighted sums of first-stage decision variables and is a stochastic program without recourse. The work in [22] is related as well, but also considers only stochastic programs with first-stage decisions. We consider a general two-stage stochastic program with recourse and second-stage costs. In contrast to [18], as we show later in this article, we derive upper and lower bounds on chance-constrained models rather than the exact deterministic equivalents of the chance constraint itself.

2 Approximating a JCC

The classical inclusion-exclusion formula for the union of sets, $A_t, t \in T$, is

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) = S_1 - S_2 + \dots + (-1)^{T-1} S_T,$$

where, $S_k = \sum_{1 \leq i_1 < \dots < i_k \leq T} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k})$. We are interested in approximating the probability of this union using S_1 and S_2 alone.

One of the earliest known bounds on the union are due to Bonferroni [2]:

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) \leq S_1, \tag{3a}$$

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) \geq S_1 - S_2. \tag{3b}$$

Tighter bounds are available from Result 3 and Result 1, respectively, of Sathe et al. [24]:

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) \leq S_1 - \frac{2}{T} S_2 \tag{4a}$$

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) \geq \frac{S_1 + 2S_2}{T^2}. \tag{4b}$$

The upper bound in equation (4a) is the tightest possible bound for the union by the use of $S_i, i \leq 2$. Independently, Prékopa et al. have shown the tightness and sharpness (attainability) of this bound, as well as some of the other bounds we present, through the use of linear programming [3, 20, 21, 23]. The lower bound in (4b) is not the tightest possible bound, however it has the advantage that both S_1 and S_2 have exponents of one. The tightest possible lower bound for the union

75 by the use of $S_i, i \leq 2$ is available from Dawson and Sankoff [7], and has an
76 exponent of two for S_1 :

$$\mathbb{P}\left(\bigcup_{t \in T} A_t\right) \geq \frac{S_1^2}{S_1 + 2S_2}. \quad (5)$$

77 The bound in equation (5) is equivalent to the bound from (4b) under certain
78 conditions which require apriori information on S_1 and S_2 ; see, e.g., [9,11] for
79 more details. Assuming an exact linearization, we can obtain a piecewise-linear
80 approximation with $|K|$ pieces for the the quadratic constraint, $S_1^2 - \varepsilon S_1 \leq 2\varepsilon S_2$,
81 as follows:

$$\alpha_k S_1 + \beta_k \leq 2\varepsilon S_2, k = 0, 1, \dots, |K| - 1, . \quad (6)$$

82 where $\alpha_k = C(2k + 1) - \varepsilon, \beta_k = -C^2(k^2 + k)$. Here $C = \frac{T}{|K|}$ and we use the fact
83 that $0 \leq S_1 \leq T$. Later, in this article we study the efficiency of this lineariza-
84 tion. Further, bounds presenting a non-linear combination of S_1 and S_2 are also
85 available; see, e.g., [8,10,27].

86 3 Optimizing over JCCs

87 Consider the following chance-constrained optimization model:

$$\max_{x,y} \sum_{t \in T} (R_t x_t - \mathbb{E}[B_t y_t^\omega]) \quad (7a)$$

$$\text{s.t. } \mathbb{P}(y_t^\omega + w_t^\omega \geq x_t, \forall t \in T) \geq 1 - \varepsilon \quad (7b)$$

$$0 \leq y_t^\omega \leq \Delta, \forall t \in T, \omega \in \Omega \quad (7c)$$

$$x_t \geq 0, \forall t \in T. \quad (7d)$$

88 Here, R_t and B_t are some non-negative deterministic parameters. JCC in equa-
89 tion (7b) can be rewritten exactly using a big- M formulation as follows,

$$x_t - y_t^\omega - w_t^\omega \leq M_t^\omega z^\omega, \forall t \in T, \omega \in \Omega \quad (8a)$$

$$\frac{1}{N} \sum_{\omega \in \Omega} z^\omega \leq \varepsilon. \quad (8b)$$

90 Here, $z^\omega = 1$ if we fail in scenario ω and 0 otherwise; and, M_t^ω is large enough
91 that constraint (8a) is vacuous when $z^\omega = 1$. Equation (8b) restricts the fraction
92 of failures to be no more than ε . Failing in scenario ω implies we fail in at least one
93 t . Let $u_t^\omega = 1$ denote we fail in scenario ω at time t . Further, let $v_{tt'}^\omega = 1, t' > t$,
94 denote we fail in scenario ω at both time t and t' . Then, we have the following
95 relations: $\mathbb{P}(A_t) = \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega$ and $\mathbb{P}(A_t \cap A_{t'}) = \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega$. Further,

$$S_1 = \sum_{t \in T} \mathbb{P}(A_t) = \sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega, \quad (9a)$$

$$S_2 = \sum_{t, t' \in T, t' > t} \mathbb{P}(A_t \cap A_{t'}) = \sum_{t, t' \in T, t' > t} \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega. \quad (9b)$$

96

97 We can now use the approximations presented in Section 2 for the JCC in (1)
98 as follows:

$$x_t - y_t^\omega - w_t^\omega \leq M_t^\omega u_t^\omega, \quad \forall t \in T, \omega \in \Omega \quad (10a)$$

$$v_{tt'}^\omega \leq u_t^\omega, \quad \forall t, t' \in T, t' > t, \omega \in \Omega \quad (10b)$$

$$v_{t,t'}^\omega \leq u_{t'}^\omega, \quad \forall t, t' \in T, t' > t, \omega \in \Omega \quad (10c)$$

$$v_{tt'}^\omega \geq u_t^\omega + u_{t'}^\omega - 1, \quad \forall t, t' \in T, t' > t, \omega \in \Omega \quad (10d)$$

$$u_t^\omega \in \{0, 1\}, \quad \forall t \in T, \omega \in \Omega \quad (10e)$$

$$v_{tt'}^\omega \in \{0, 1\}, \quad \forall t, t' \in T, t' > t, \omega \in \Omega, \quad (10f)$$

and one of the following inequalities:

$$\sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega \leq \varepsilon \quad (11a)$$

$$\sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega - \sum_{t, t' \in T, t' > t} \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega \leq \varepsilon \quad (11b)$$

$$\sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega - \frac{2}{T} \sum_{t, t' \in T, t' > t} \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega \leq \varepsilon \quad (11c)$$

$$\frac{1}{T^2} \left(\sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega + 2 \sum_{t, t' \in T, t' > t} \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega \right) \leq \varepsilon \quad (11d)$$

$$\left(\sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega \right)^2 \leq \left(\sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega + 2 \sum_{t, t' \in T, t' > t} \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega \right) \varepsilon \quad (11e)$$

$$\alpha_k \sum_{t \in T} \frac{1}{N} \sum_{\omega \in \Omega} u_t^\omega + \beta_k \leq 2\varepsilon \sum_{t, t' \in T, t' > t} \frac{1}{N} \sum_{\omega \in \Omega} v_{tt'}^\omega, \forall k = 0, 1, \dots, |K| - 1. \quad (11f)$$

99

100 Equation (10a) is analogous to equation (8a). Equations (10b)-(10d) represent
101 a McCormick envelope [16], and the tightest possible formulation achieving $v_{tt'}^\omega = 1$
102 if and only if $u_t^\omega = u_{t'}^\omega = 1$. Further, the constraint matrix (10b)-(10d) is totally
103 unimodular and thus we can relax the binary restrictions on v . The six inequalities
104 in equation (11) represent the approximations resulting from equations (3a), (3b),
105 (4a), (4b), (5) and (6), respectively.

106 The following proposition provides a sufficiently large value for M_t^ω :

107 **Proposition 1** *A sufficiently large value for M_t^ω for the optimization model (7)*
 108 *is $M_t^\omega = \Delta + w_t^{w(\lfloor N\varepsilon \rfloor + 1, t)} - w_t^\omega, \forall t \in T, \omega \in \Omega$. Here, Δ is an upper bound on*
 109 *$y_t^\omega, \forall t \in T, \omega \in \Omega$; $w_t^{w(l, t)}$ denotes the l^{th} largest realization of w at time t ; and,*
 110 *$\lfloor \cdot \rfloor$ rounds its argument down to the nearest integer.*

111 *Proof* The proof mirrors that of Proposition 5.2.1 of [25]. By constraint (8b) we can
 112 remove at most $\lfloor N\varepsilon \rfloor$ scenarios for every t . Further, we have $y_t^\omega \leq \Delta, \forall t \in T, \omega \in \Omega$.
 113 Hence, $x_t \leq \Delta + w_t^{w(\lfloor N\varepsilon \rfloor + 1, t)}, \forall t \in T, \omega \in \Omega$ is a valid inequality. Thus, we have
 114 $M_t^\omega \geq \Delta + w_t^{w(\lfloor N\varepsilon \rfloor + 1, t)} - w_t^\omega, \forall t \in T, \omega \in \Omega$.

115 *Remark 1* Sathe et al. [24] provide a proof that the inequalities in equation (4)
 116 can be further improved; specifically, the bound in equation (4a) can be improved
 117 if $S_2 < S_1(T - 1)$, and the bound in equation (4b) can be improved if $2S_2 < S_1$.
 118 However, for the optimization models we consider, S_1 and S_2 are not known apriori,
 119 and hence these tighter bounds cannot be used.

120 *Remark 2* We can use an extended variable formulation of constraint (10a), with
 121 the u_t^ω variable, by using the so-called mixing set [15, 17]. Thus, if $u_t^{\omega(l, t)} = 1$, then
 122 we have $u_t^{\omega(l-1, t)} = \dots u_t^{\omega(1, t)} = 1$; and, vice-versa if $u_t^{\omega(l, t)} = 0$, then we have
 123 $u_t^{\omega(l+1, t)} = \dots u_t^{\omega(N, t)} = 0$. This leads to the following valid inequality: $u_t^{\omega(l+1, t)} \leq$
 124 $u_t^{\omega(l, t)}, \forall l = 1, 2, \dots, N-2$. We note that if the second-stage variables, y_t , are linked
 125 to each other, or have a cost coefficient depending on ω , this constraint is not valid.

126 4 A Case Study

127 In this section, we assess the computational performance of the approximations
 128 in Section 3 using model (7). We use $|T| = 24$ and solve model (7) for $\varepsilon = 0.01$
 129 and $\varepsilon = 0.03$. We sample two batches of 250 and 500 scenarios for w_t^ω both
 130 using two methods we describe later, a sufficiently large value of M_t^ω given by
 131 Proposition 1, $\Delta=30$, and B and R values from [26]. We use $|K| = 5000$ for
 132 the linearization in equation (6). As our aim is to compare the strength of the
 133 developed approximations against each other, we attempt to solve all problems to
 134 a MIP gap of 0%; however, we use a maximum time limit of 35 minutes. All tests
 135 in this article were carried out with GAMS 24.8.5 using CPLEX 12 [4] on an Intel
 136 Core i7 2.8 GHz processor with 16 GB of memory.

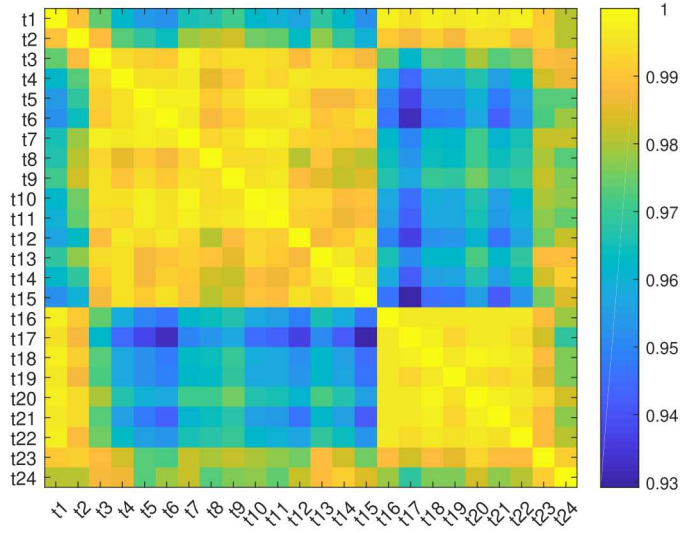
137 4.1 First Batch of Samples

138 In Tables 1-4, we compare the six models for two classes of sampling that we
 139 describe below. Bold-numbers indicate the relevant bound obtained when the
 140 optimization model could not be solved to a 0% MIP gap in the 2100 sec-
 141 onds time limit; in that case the “Lower bound” and the “Upper bound” are
 142 not equal. We choose the more conservative of the bounds to identify the rel-
 143 evant approximation, indicated in bold. The MIP gap reported by GAMS is
 144 $100 \frac{\text{Upper bound} - \text{Lower bound}}{\text{Upper bound}} \%$. We define the “Gap from optimal” in the same
 145 spirit as the MIP gap: $100 \frac{|\text{Bound} - \text{Optimal}|}{\max\{\text{Bound}, \text{Optimal}\}} \%$, where “Bound” is the relevant
 146 approximation.

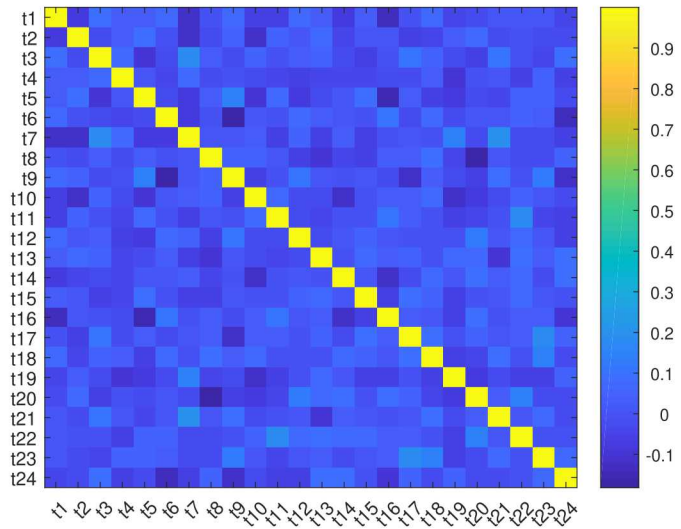
147 First, we sample 250 scenarios for w from an ARMA(2, 2) model. This results
 148 in a large degree of correlation in the w_t^ω , see Figure 1a. Table 1 compares the
 149 six models for this sampling. The true optimal value for $\varepsilon = 0.01$ is 8,634.1 while
 150 that for $\varepsilon = 0.03$ is 9,154.9. We note, from the computational time presented in
 151 the “Time” column of the table, that models with constraints (4a) and (4b) are
 152 respectively harder to solve than with constraints (3a) and (3b), since the former
 153 constraints are tighter (i.e., constraints (4a) and (4b) dominate constraints (3a)
 154 and (3b), respectively). Further, the mixed-integer quadratic constrained model
 155 with constraint (5) and its linearization with constraint (6) give nearly the same
 156 upper bound; although the linearized constraint performs slightly better. In this
 157 case, the Bonferroni bound from equation (3a), and the Sathe et al. bound from
 158 equation (4a), are both within 3.3% of the optimal; while the Dawson and Sankoff
 159 bound, with the linearization, from equation (6) is at most 0.8% far from the
 160 optimal. These two bounds are the tightest lower and upper bounds, respectively.

161 Second, we sample 250 scenarios for w from a normal random number genera-
 162 tor, using the same hourly means and variances as the ARMA model. This results
 163 in an almost zero correlation in the w_t^ω , see Figure 1b. Table 2 compares the six
 164 models for this sampling. We have a similar trend as in Table 1, but ten of the
 165 twelve models in Table 2 are slightly easier to solve (see the “MIP gap” column)
 166 than Table 1. We also note that except for the Dawson and Sankoff bound, the
 167 other bounds are all relatively closer to the optimal (see the “Gap from optimal”
 168 column) than in Table 1. Again, the Bonferroni bound from equation (3a), which
 169 is almost the same as the Sathe et al. bound from equation (4a), and the Dawson
 170 and Sankoff bound, with the linearization, from equation (6) provide the tightest
 171 lower and upper bounds, respectively.

172 Now, we increase the number of sampled scenarios to 500. Tables 3 and 4
 173 present results analogous to Tables 1 and 2, respectively. With the larger prob-



(a)



(b)

Fig. 1: Correlation between 250 scenarios of w_t^ω for 24 hours, when w is sampled using (a) an ARMA(2,2) process, and (b) a normal (Gaussian) distribution. Both samples have the same hourly means and variances.

174 lem size for both the ARMA and Gaussian processes, the Dawson and Sankoff
 175 bound from equation (5) is extremely challenging to solve due to the mixed-integer

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	8,351.3	8,351.3	0%	2	3.3%
	(3b)	21,282.8	21,282.8	0%	12	59.4%
	(4a)	8,351.3	8,365.8	0.1%	2100	3.3%
	(4b)	8,339.6	10,682.1	21.9%	2100	19.2%
	(5)	8,339.7	8,726.7	4.5%	2100	1.1%
	(6)	8,688.9	8,702.1	0.2%	2100	0.8%
0.03	(3a)	8,374.6	8,374.6	0%	2	8.5%
	(3b)	22,353.2	22,353.2	0%	14	59.0%
	(4a)	8,339.6	8,755.4	4.7%	2100	8.9%
	(4b)	8,339.6	13,321.2	37.4%	2100	31.3%
	(5)	9,137.3	9,311.4	1.9%	2100	1.7%
	(6)	9,074.4	9,252.2	1.9%	2100	1.1%

Table 1: Comparison of approximations from Sections 2 and 3 using samples from an ARMA model with 250 scenarios. Bold-numbers indicate the relevant bound obtained when the optimization model could not be solved to a 0% MIP gap in the 2100 seconds time limit. Hence, the tightest lower and upper bounds for $\varepsilon = 0.01$ are 8,351.3 and 8,702.1, respectively while the true optimal value is 8,634.1. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 8,374.6 and 9,559.2, respectively while the true optimal value is 9,154.9.

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	9,100.8	9,100.8	0%	1	2.7%
	(3b)	21,606.6	21,606.6	0%	18	56.7%
	(4a)	9,102.0	9,113.3	0.1%	2100	2.7%
	(4b)	9092.3	11,365.5	20%	2100	17.7%
	(5)	9,434.3	9,486.3	0.5%	2100	1.4%
	(6)	9,421.5	9,452.3	0.3%	2100	1.1%
0.03	(3a)	9,124.3	9,124.3	0%	2	7.7%
	(3b)	22,762.1	22,762.1	0%	21	56.6%
	(4a)	9,124.8	9,198.4	0.8%	2100	7.7%
	(4b)	9,092.3	13,907.6	34.9%	2100	28.9%
	(5)	9,092.3	10,062.6	9.6%	2100	1.8%
	(6)	9,092.3	10,004.8	9.1%	2100	1.2%

Table 2: Comparison of approximations from Section 2 using samples from a Gaussian model with 250 scenarios. Bold-numbers indicate the relevant bound obtained when the optimization model could not be solved to a 0% MIP gap in the 2100 seconds time limit. Hence, the tightest lower and upper bounds for $\varepsilon = 0.01$ are 9,100.8 and 9,449.9, respectively while the true optimal value is 9,353.2. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 9,124.3 and 9,994.8, respectively while the true optimal value is 9,884.0.

176 quadratic constraint. However, this constraint offers a strong upper bound, for
177 both instances, when linearized as in equation (6). Next, equation (4b) remains

178 challenging in both instances as we also observed for the 250 scenario case; while,
 179 equation (3a) offers a fast and efficient lower bound.

180 Empirically, we observe that the Bonferroni lower bound and the Dawson and
 181 Sankoff linearized upper bound provide the tightest bounds. Also, in the absence
 182 of correlated structures, the models are computationally easier (smaller MIP gaps)
 183 and the gaps from the optimal are smaller as well. This suggests the intuitive fact
 184 that a larger correlation in the uncertainties leads to harder to solve models, due
 185 to non-zero higher order moments.

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	8,453.4	8,453.4	0%	1	2.9%
	(3b)	21,582.9	21,582.9	0%	129	59.7%
	(4a)	8,701.0	8,701.0	0%	1717	0%
	(4b)	10,462.7	11,318.4	7.5%	2100	23.1%
	(5)	8,348.9	40,116.9	79.2%	2100	78.3%
	(6)	8,348.9	8,772.9	4.8%	2100	0.8%
0.03	(3a)	8,542.5	8,542.5	0%	3	7.3%
	(3b)	22,570.6	22,570.6	0%	175	59.2%
	(4a)	8,348.9	9,396.1	11.1%	2100	9.4%
	(4b)	8,348.9	15,127.8	44.8%	2100	39.1%
	(5)	8,348.9	41,151.4	79.8%	2100	77.6%
	(6)	8,348.9	9,352.9	10.7%	2100	1.5%

Table 3: Comparison of approximations from Sections 2 and 3 using samples from an ARMA model with 500 scenarios. Bold-numbers indicate the relevant bound obtained when the optimization model could not be solved to a 0% MIP gap in the 2100 seconds time limit. Hence, the tightest lower and upper bounds for $\varepsilon = 0.01$ are 8,701.0 and 8,772.9, respectively while the true optimal value is 8,701.0. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 8,542.5 and 9,352.9, respectively while the true optimal value is 9,211.3.

186 4.2 Second Batch of Samples

187 Now, we re-run the computational results presented above using a different batch
 188 of scenarios for the two sampling classes. The new samples have half the mean
 189 and half the variance as the original samples. This class of instances results in a
 190 significantly more computationally challenging optimization problem; for the true
 191 problem only the $\varepsilon = 0.01$, $|\Omega| = 250$ instances could be solved to a 0% MIP gap
 192 within eight hours. Thus, we need to change the definition of ‘‘Gap from optimal’’.
 193 We use the following definition to obtain the conservative gap: $\frac{\text{Bound} - z_{LB}^*}{\text{Bound}}$, when
 194 ‘‘Bound’’ is an upper bound; and $\frac{z_{UB}^* - \text{Bound}}{z_{UB}^*}$, when ‘‘Bound’’ is a lower bound.

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	9,005.1	9,005.1	0%	1	3.7%
	(3b)	21,503.7	21,503.7	0%	75	56.5%
	(4a)	8866.9	8,889.3	1.3%	2100	5.1%
	(4b)	8,866.9	11,071.9	19.9%	2100	15.6%
	(5)	8,866.9	40,126.1	77.9%	2100	76.7%
	(6)	9,343.6	9,390.3	0.5%	2100	0.5%
0.03	(3a)	9,148.2	9,148.2	0%	3	7.4%
	(3b)	22,565.4	22,565.4	0%	46	56.2%
	(4a)	8,866.9	9,315.3	4.8%	2100	10.2%
	(4b)	8,866.9	13,711.9	35.3%	2100	27.9%
	(5)	8,866.9	41,187.8	78.5%	2100	76.0%
	(6)	8,866.9	9,990.9	11.2%	2100	1.2%

Table 4: Comparison of approximations from Sections 2 and 3 using samples from a Gaussian process with 500 scenarios. Bold-numbers indicate the relevant bound obtained when the optimization model could not be solved to a 0% MIP gap in the 2100 seconds time limit. Hence, the tightest lower and upper bounds for $\varepsilon = 0.01$ are 9,005.1 and 9,390.3, respectively while the true optimal value is 9,346.4. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 9,148.2 and 9,990.9, respectively while the true optimal value is 9,874.1.

195 Here, z_{UB}^* and z_{LB}^* are the upper and lower bounds on the true optimal value;
 196 when $z_{UB}^* = z_{LB}^*$ this definition is the same as the one we used in Section 4.1.

197 Tables 5, 6, 7 and 8 are analogous to Tables 1, 2, 3 and 4, respectively. Our
 198 empirical findings are the same as those we report in Section 4.1, but there are
 199 a few differences. First, except for the $\varepsilon = 0.03$ case in Table 8, the Dawson and
 200 Sankoff bound from equation (5) is not as challenging as before. In fact, in Table 7
 201 equation (5) results in the tightest upper bound for $\varepsilon = 0.01$. Second, the gap from
 202 optimal in Table 7 is not consistently larger than that in Table 8. This could be
 203 because of the relatively large range in which the true optimal values lies.

204 4.3 Discussion on Computation Results

205 When the true optimal solution is not exactly known (i.e., $z_{UB}^* \neq z_{LB}^*$), the “Gap
 206 from optimal” could appear to be large. In this sense, the results from Section 4.1
 207 are more useful than those from Section 4.2. However, our empirical observations
 208 from the two computational experiments do not significantly differ. First, nearly
 209 always the Bonferroni inequality and the Dawson and Sankoff inequality, respec-
 210 tively provide the tightest lower and upper bounds. Second, the lower correlation
 211 model tends to give smaller gaps from optimal than the larger correlation model

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	4,561.9	4,561.9	0%	1	2.2%
	(3b)	20,557.3	20,557.3	0%	18	77.3%
	(4a)	4,547.4	4,583.2	0.8%	2100	2.5%
	(4b)	4,547.4	7,404.2	38.5%	2100	37.0 %
	(5)	4,547.4	4,848.3	6.2%	2100	3.8%
	(6)	4,622.6	4,808.6	3.8%	2100	3.1%
0.03	(3a)	4,596.6	4,596.6	0%	1	8.4%
	(3b)	21,092.5	21,092.5	0%	21	76.6%
	(4a)	4,585.8	4,850.5	5.5%	2100	8.6%
	(4b)	4,547.4	10,416.8	56.3%	2100	52.6%
	(5)	4,547.4	5,311.9	14.3%	2100	7.1%
	(6)	4,699.1	5,243.9	10.3%	2100	5.9%

Table 5: Analogous results to Table 1. The tightest lower and upper bounds for $\varepsilon = 0.01$ are 4,561.9 and 4,808.6, respectively while the true optimal value is 4,663.9. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 4,596.6 and 5,243.9, respectively while the true optimal value is in the range [4,935.0—5,016.9].

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	4,923.3	4,923.3	0%	1	2.0%
	(3b)	20,719.0	20,719.0	0%	29	75.7%
	(4a)	4,909.3	4,934.3	0.5%	2100	2.3%
	(4b)	4,909.3	7,743.7	36.6%	2100	35.1%
	(5)	4,909.3	5,221.3	5.9%	2100	3.8%
	(6)	4,998.9	5,164.3	3.2%	2100	2.7%
0.03	(3a)	4,955.3	4,955.3	0%	1	7.6%
	(3b)	21,296.9	21,296.9	0%	33	76.8%
	(4a)	4,934.3	5,176.7	4.5%	2100	7.9%
	(4b)	4,909.3	10,670.7	53.9%	2100	50.8%
	(5)	4,909.3	5,672.4	13.4%	2100	6.7%
	(6)	4,921.5	5,591.0	11.9%	2100	5.3%

Table 6: Analogous results to Table 2. The tightest lower and upper bounds for $\varepsilon = 0.01$ are 4,923.3 and 5,164.3, respectively while the true optimal value is 5,025.4. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 4,955.3 and 5,591.0, respectively while the true optimal value is in the range [5,293.1—5,361.7].

212 (this is reflected in 37 of the 48 models), as well as being computationally less
213 challenging (this is reflected in 42 of the 48 instances).

214 We conclude with a discussion on the size of the approximating problems. To
215 represent S_1 and S_2 we need $|T||\Omega|$ and $\frac{|T||T-1||\Omega|}{2}$ variables, respectively. The
216 original JCC formulation of model (8) has $|\Omega|$ binary variables. Future research
217 could examine incorporating these approximations into sequential algorithms for

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	4,591.9	4,591.9	0%	1	3.9%
	(3b)	20,663.1	20,663.1	0%	39	77.2%
	(4a)	4,691.8	4,890.1	4.0%	2100	1.8%
	(4b)	4,573.7	7,532.3	39.3%	2100	35.9%
	(5)	4,573.7	4,886.1	6.4%	2100	3.5%
	(6)	4,573.7	4,889.5	6.4%	2100	3.6%
0.03	(3a)	4,626.5	4,626.5	0%	1	11.9%
	(3b)	21,180.4	21,180.4	0%	44	76.5%
	(4a)	4,573.7	5,236.9	12.6%	2100	12.9%
	(4b)	4,573.7	11,449.8	60.0%	2100	56.5%
	(5)	4,573.7	5,492.6	16.7%	2100	9.4%
	(6)	4,573.7	5,485.0	16.6%	2100	9.2%

Table 7: Analogous results to Table 3. The tightest lower and upper bounds for $\varepsilon = 0.01$ are 4,691.8 and 4,886.1, respectively while the true optimal value is in the range [4,715.3—4,778.9]. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 4,626.5 and 5,485.0, respectively while the true optimal value is in the range [4,978.3—5,252.7].

ε	Bounding constraint	Optimal objective value			Time (seconds)	Gap from optimal
		Lower bound	Upper bound	MIP gap		
0.01	(3a)	4,893.9	4,893.9	0%	1	3.3%
	(3b)	20,667.7	20,667.7	0%	81	75.7%
	(4a)	4,876.3	5,034.2	3.1%	2100	3.6%
	(4b)	4,876.3	7,664.9	36.3%	2100	34.5%
	(5)	4,876.3	39,290.0	87.6%	2100	87.2%
	(6)	4,876.3	5,175.6	5.8%	2100	2.9%
0.03	(3a)	4,930.3	4,930.3	0%	3	11.4%
	(3b)	21,198.6	21,198.6	0%	84	76.5%
	(4a)	4,876.3	5,365.9	9.1%	2100	12.4%
	(4b)	4,876.3	10,726.9	54.5%	2100	50.6%
	(5)	4,876.3	5,788.1	15.7%	2100	8.5%
	(6)	4,876.3	5,783.4	15.6%	2100	8.4%

Table 8: Analogous results to Table 4. The tightest lower and upper bounds for $\varepsilon = 0.01$ are 4,893.9 and , respectively while the true optimal value is in the range [5,024.1—5,059.5]. Similarly, the tightest lower and upper bounds for $\varepsilon = 0.03$ are 4,930.3 and 5,783.4, respectively while the true optimal value is in the range [5,297.5—5,564.2].

218 solving chance constrained programs, and specific models under which the pre-
219 sented approximations could be speedened up.

5 Conclusions

We approximated a JCC using bounds from classical probability on the union of sets. These approximations in-turn result in upper and lower bounds on an optimization model with JCCs. The computational effort required to solve some of the approximating problems is much harder than the others, but equivalent reformulations show promise. In our case study, we found the Bonferroni inequality and the Dawson and Sankoff inequality offering the best possible bounds. Future work could explore tightening these bounds further, and a non-linear combination of S_1 and S_2 .

Acknowledgements We thank David Morton and Shabbir Ahmed for helpful discussions.

This work was supported in part by Sandia's Laboratory Directed Research and Development (LDRD) program.

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.

This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

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