

Electromagnetic Roots for Radar

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Outline (more-or-less)

- Maxwell's Equations
- Wave Propagation Equation
- Plane-Wave Propagation
- Plane-Wave Reflection
- Radar Range/Delay
- Dielectrics
- Point Sources and Reflections
- Complicated Scattering
- Born Approximation
- Antenna Basics

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Maxwell's Equations

Maxwell's equations relate electric fields and magnetic fields. They underpin all electrical, optical, and radio technologies.

$\nabla \cdot \mathbf{D} = \rho$	(1) Gauss' Law
$\nabla \cdot \mathbf{B} = 0$	(2) Gauss' Law for magnetism
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(3) Faraday's Law
$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$	(4) Ampere-Maxwell Law

$\mathbf{D} = \epsilon \mathbf{E}$ = Electric Displacement field
 $\mathbf{B} = \mu \mathbf{H}$ = Magnetic Induction field

Let there be light.
 \mathbf{E} = Electric Field
 \mathbf{H} = Magnetic Field
 ρ = charge density
 \mathbf{J} = current density
 ϵ = permittivity
 μ = permeability

Everything starts here.

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Vector Calculus Identities/Formulæ

$$\begin{aligned}
 \mathbf{A} \cdot \mathbf{C} &= \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \\
 \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \\
 \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \\
 \nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} \\
 \nabla \cdot \nabla \Phi &= \nabla^2 \Phi \\
 \nabla \cdot \nabla \times \mathbf{A} &= \mathbf{0} \\
 \nabla \times \nabla \times \mathbf{A} &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\
 \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})
 \end{aligned}$$

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l} \quad \text{Stokes theorem}$$

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S} \quad \text{Divergence theorem}$$

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Free-Space Propagation

In free-space there are no currents or charges, and no losses.

Maxwell's equations can be manipulated to

$$\nabla^2 \mathbf{E} = -\mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and in turn, using some identities, to

$$\nabla^2 \mathbf{E} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly, for the magnetic field

$$\nabla^2 \mathbf{H} = \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

In Cartesian coordinates, each component of the vectors \mathbf{E} and \mathbf{H} satisfy a scalar wave equation.

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We further identify

$$c = \frac{1}{\sqrt{\mu \epsilon}} = \text{Propagation velocity}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \text{Characteristic wave impedance}$$

In free-space

$$\begin{aligned}
 \epsilon &= \epsilon_0 \approx 8.854 \times 10^{-12} \text{ F/m} \\
 \mu &= \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \\
 c &= c_0 = 299,792,458 \text{ m/s} \\
 \eta &= \eta_0 \approx 377 \text{ ohms}
 \end{aligned}$$

Note that these are second-order differential equations, with solutions that are sinusoids.

Free-Space Propagation

Taking the Inverse Fourier Transform of both sides yields the Helmholtz equations

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = \mathbf{0}$$

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = \mathbf{0}$$

where we also define

$$\begin{aligned}
 f &= \text{Temporal frequency in Hz (cycles/sec.)} \\
 \omega &= 2\pi f = \text{Angular frequency in radians/sec.}
 \end{aligned}$$

$$k = \frac{\omega}{c} = \text{Wavenumber in radians/meter}$$

We further define

$$\lambda = \frac{c}{f} = \frac{2\pi}{k} = \text{Wavelength in meters}$$

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Poynting's theorem shows that the direction and magnitude of energy flow is

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}$$

As seen in the next few slides, Maxwell's equations reveal that \mathbf{E} and \mathbf{H} are perpendicular to each other, and both are also perpendicular to the direction of travel.

The orientation of \mathbf{E} defines the "polarization" of the plane-wave.

These 'waves' travel, with a free-space velocity of propagation

Solutions have phase that is a function of both time and space.

Sinusoidal Plane-Wave Propagation

A propagating wave with a planar wave-front is a plane-wave.

The electric field of a linearly polarized plane wave is given by

$$\mathbf{E}(t, \mathbf{r}) = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$$

where

$\mathbf{E}_0 = E \hat{\mathbf{E}}_0$ = Polarization vector
 $\mathbf{k} = k \hat{\mathbf{k}}$ = Direction of propagation
 $\mathbf{r} = r \hat{\mathbf{r}}$ = Field observation point

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Sinusoidal Plane-Wave Propagation

If traveling in the direction of the z -axis, with an electric field oriented parallel to the x -axis, our field reduces to simply

$$\mathbf{E} = E_x \hat{\mathbf{x}}$$

with

$\hat{\mathbf{k}} = \hat{\mathbf{r}} = \hat{\mathbf{z}}$
 $\hat{\mathbf{E}}_0 = \hat{\mathbf{x}}$

and the field equation reduces to

$$\frac{\partial^2}{\partial z^2} E_x = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x$$

with a solution

$$E_x^+(t, z) = e_1 \cos(\omega t - kz)$$

and another solution

$$E_x^-(t, z) = e_2 \cos(\omega t + kz)$$

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Wave front (right travelling)

\mathbf{E} and \mathbf{H} fields are related as

$$\hat{\mathbf{z}} \times \mathbf{E} = \eta \mathbf{H}$$

$$\hat{\mathbf{z}} \times \mathbf{H} = -\frac{1}{\eta} \mathbf{E}$$

Forward/right travelling

Backward/left travelling

Propagation in a Dielectric

In a dielectric

$$\epsilon = \epsilon_r \epsilon_0$$

$$\mu = \mu_r \mu_0$$

where

ϵ_r = relative permittivity
 μ_r = relative permeability

These relative quantities are typically greater than one.

Complex values denote propagation is lossy.

Frequency-dependence implies a "dispersive" media, where the echo may 'not' be a faithful reproduction of the incident signal.

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In a lossless dielectric, it remains true, but with different numerical values, that

$$c = \frac{1}{\sqrt{\mu \epsilon}} = \text{Propagation velocity}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \text{Characteristic wave impedance}$$

with \mathbf{E} and \mathbf{H} fields still related as

$$\hat{\mathbf{z}} \times \mathbf{E} = \eta \mathbf{H} \quad \text{and} \quad \hat{\mathbf{z}} \times \mathbf{H} = -\frac{1}{\eta} \mathbf{E}$$

with comparable electric field solutions

$$E_x^+(t, z) = e_1 \cos(\omega t - kz)$$

$$E_x^-(t, z) = e_2 \cos(\omega t + kz)$$

Note additionally that k and λ are affected.

FIELDS IN A PERFECT CONDUCTOR

(Time-varying fields)

In a conductor, we observe electric fields causing charge motion, i.e. a current, with density calculated by Ohm's law

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ = conductivity.

Actually a part of Maxwell's original set of equations

In a perfect conductor

$$\sigma \rightarrow \infty$$

Free charges placed within a conductor will disperse towards the conductor surface, instantaneously, leaving none in the interior, until the total electric field inside the conductor is zero.

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At the conductor boundary:

Applications of Stokes' theorem to a perfect conductor boundary shows that tangential electric field must be zero at the boundary.

Application of Gauss' law for magnetic fields yields that the normal electric field may exist, but must do so with a corresponding surface charge density.

Applications of Stokes' theorem to a perfect conductor boundary also shows that a tangential magnetic field may exist, but must do so with a corresponding surface current density.

Application of Gauss' law for magnetic fields yields that the normal component is zero.

Bottom line: the surface of a perfect conductor cannot support tangential electric fields, or normal magnetic fields.

Plane-Wave Reflection from Perfect Conductor

Let the forward-travelling field encounter a perfectly conducting planar surface at normal incidence after distance z_0 .

At distance z_0 the forward-travelling wavefront will have travelled a time

$$t_0 = \frac{z_0}{c}$$

Recall that at the conductor surface, the tangential electric field must be zero, and the normal magnetic field must also be zero.

Boundary Conditions

For these boundary conditions to be met, we must also have generated at the surface a backward travelling wave such that

$$E_x^+(t_0, z_0) + E_x^-(t_0, z_0) = 0$$

This backward travelling field will take an additional time t_0 to reach the forward wave starting point $z = 0$

Consequently, the backward travelling field is related to the forward-travelling field by

$$E_x^-(2 \frac{z_0}{c_0}, 0) = -E_x^+(0, 0)$$

$$E_x^-(t, 0) = -E_x^+(t - 2 \frac{z_0}{c_0}, 0)$$

Radar Echo

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Plane-Wave Reflection from Perfect Conductor

We observe that the incident and reflected fields are

$$E_x^+(t, z) = e_1 \cos(\omega t - kz)$$

$$E_x^-(t, z) = -e_1 \cos(\omega t + kz - 2kz_0)$$

$z < z_0$

Note that the ratio of the magnitudes of these fields is constant (unity), and independent of frequency.

Furthermore, the field equation is linear, meaning that any signal that can be written as the sum of sinusoids will exhibit the same reflection characteristics, which means pretty much any signal we can realistically create.

These observations combine to yield the following:

A fundamental tenet of monostatic radar is that any generated/transmitted field in free-space that encounters a reflecting boundary will echo a faithful reproduction (in shape) of the incident signal, to arrive at its origin with a round-trip time delay of

$$t_{\text{delay}} = 2 \frac{z_0}{c} \quad \text{In free-space}$$

True for all frequencies.

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Plane-Wave Reflection from Dielectric Boundary

The boundary conditions are that the tangential components of E and H must be continuous at the dielectric interface.

We define

$$\Gamma = \frac{E_x^r(z_0)}{E_x^i(z_0)} = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \text{Reflection coefficient}$$

$$\tau = \frac{E_x^t(z_0)}{E_x^i(z_0)} = \frac{2\eta_1}{\eta_1 + \eta_0} = \text{Transmission coefficient}$$

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Note that for a perfect "match"

$$\eta_1 = \eta_0$$

and for a perfect conductor

$$\eta_1 = 0$$

With respect to power, we observe

$$|\Gamma|^2 = \text{Relative reflected power}$$

$$\frac{\eta_0}{\eta_1} |\tau|^2 = \text{Relative transmitted power}$$

Plane-Wave Reflection from Dielectric Boundary

For oblique angles, and lossy dielectrics, reflections and transmission properties are readily calculated.

Furthermore, familiar optical properties of reflection, refraction, and Snell's law apply.

Similarly, for interfaces other than a plane, diffraction applies.

The "index of refraction" is still defined as

$$n = \frac{c_0}{c} = \sqrt{\mu_r \epsilon_r}$$

Spherical Wavefronts

Consider a radiating field in a lossless dielectric driven by a forcing function; a "ping" in both time and space, namely

$$\nabla^2 E(t, \mathbf{r}) - \mu \epsilon \frac{\partial^2}{\partial t^2} E(t, \mathbf{r}) = -\delta(r) \delta(t)$$

This has a solution

$$E(t, \mathbf{r}) = E_0(\mathbf{r}) \frac{\delta(t - \frac{r}{c})}{4\pi r}$$

This field is travelling in a radial direction, with diminishing field strength.

Furthermore, recall that fields are perpendicular to direction of travel.

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Observations:

- Recall that fields are perpendicular to the direction of travel
- A small finite-dimension area becomes more planar as r increases
- Power/Energy density diminishes as $1/r^2$
- Total power/energy crossing the sphere's surface remains constant; independent of sphere size.

Mythical Point Target

Consider a reflecting object that

- Occupies a point in space
- Intercepts a portion of a radiated field, and
- Emanates a reflected field from that point towards a receiver with finite total power

Consider the point reflector intercepting a propagating field with power density ψ W/m².

Let the point reflector reradiate a field with a power density as seen by a receiver of

$$\zeta / (4\pi r^2)$$

The point then has a "Radar Cross Section" of

$$RCS = \sigma = \frac{\zeta}{\psi} \text{ m}^2$$

Using real targets that approximate point target reflectors is an indispensable tool for radar performance evaluation.



Courtesy NASA

Mythical Point Target

For real radar-hardware testing and evaluation, we like to use targets that mimic a point reflector to some extent.

These targets are typically large with respect to wavelength, so geometrical optics principles apply.

The RCS of these "canonical" targets can be calculated with relatively high accuracy and precision... with some caveats.

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Complicated Scattering

We now presume that some incident electric field results in a scattered, or reflected, electric field, with

$$E^i(t, \mathbf{r}) = \text{Incident field}$$

$$E^r(t, \mathbf{r}) = \text{scattered/reflected field}$$

The total field is the sum of both, namely

$$E^{tot}(t, \mathbf{r}) = E^i(t, \mathbf{r}) + E^r(t, \mathbf{r})$$

Scattering occurs from dielectric changes, which causes changes in propagation velocity. For convenience we acknowledge this with the model

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2} - \nu(\mathbf{r})$$

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The wave equations can be manipulated to the Lippmann-Schwinger integral equation

$$E^r(t, \mathbf{r}) = \iint \frac{\delta(t - \tau - \frac{|\mathbf{r} - \mathbf{p}|}{c})}{4\pi |\mathbf{r} - \mathbf{p}|} \nu(\mathbf{p}) \frac{c^2}{\partial \tau^2} E^{tot}(\tau, \mathbf{p}) d\tau d\mathbf{p}$$

problem

Since the total field also contains the scattered field, this becomes an equation that needs to be solved, which is not tractable except for the simplest of geometries.

Born Approximation

See development by Cheney & Borden, and Cheney & Borden.

To make the problem tractable, we ignore the scattered field on the right side of the equation and approximate the problem as

$$E^r(t, \mathbf{r}) \approx \iint \frac{\delta(t - \tau - \frac{|\mathbf{r} - \mathbf{p}|}{c})}{4\pi |\mathbf{r} - \mathbf{p}|} v(\mathbf{p}) \frac{\partial^2}{\partial \tau^2} E^i(\tau, \mathbf{p}) d\tau d\mathbf{p}$$

Incident field only

Born Approximation

This is the equivalent to assuming that the scattered/reflected field is generally small/weak compared to the incident field.

While this makes the problem tractable, it leads to some errors in rendering radar data, often called multipath 'artifacts.'

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Born Approximation - Artifacts

Near range

Ray trace

Side of movement

Ground

Direct return, Single bounce

Double bounce

Triple bounce

Jet engine inlets often exhibit characteristic multipath effects.

This image of a tank seems to suggest 3 cannon barrels. However careful analysis shows that along with the direct return, we have multipath effects of double and triple bounces involving the ground.

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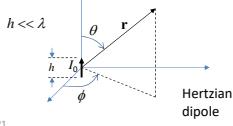
Antenna Basics – Hertzian Dipole

See development in Ramo, Whinnery, & Van Duzer

The task of creating propagating E and M fields from signal voltages/currents is the function of an antenna.

Typical antenna design/analysis begins with establishing a current density \mathbf{J} as a forcing function to generate the fields.

Consider a short linear current element of length h and current strength I_0



$h \ll \lambda$

$H_\phi = \frac{I_0 h}{4\pi} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin \theta$

$E_r = \frac{I_0 h}{4\pi} e^{-jkr} \left(\frac{2\eta}{r^2} + \frac{2}{j\omega \epsilon r^3} \right) \cos \theta$

$E_\theta = \frac{I_0 h}{4\pi} e^{-jkr} \left(\frac{j\omega \mu}{r} + \frac{\eta}{r^2} + \frac{1}{j\omega \epsilon r^3} \right) \sin \theta$

In the far-field, where r is large,

$E_\theta \approx \frac{j\omega \mu h \sin \theta}{4\pi r} e^{-jkr}$

$H_\phi \approx \frac{E_\theta}{\eta}$

Field components are perpendicular to each other, and to the direction of travel

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Antenna Basics – Hertzian Dipole

For our Hertzian dipole

$$P(\mathbf{r}) = \frac{\eta k^2 (I_0 h)^2}{32\pi^2 r^2} \sin^2 \theta = \text{Power density}$$

$$W = \frac{\eta \pi (I_0 h)^2}{3 \lambda^2} = \text{Total radiated power}$$

The power density related to that of an isotropic antenna is calculated as

$$p(\mathbf{r}) = \frac{3}{2} \sin^2 \theta$$

The power radiated in some directions has been enhanced at the expense of other directions

→ Antenna Gain

More complicated antennas can be analyzed by treating them as collections of infinitesimal Hertzian dipoles, and superposing the results.

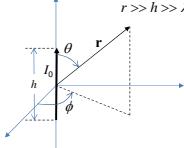
As a practical matter, at large distances, we may assume the following

1. Differences in the radius vectors to the elemental dipoles are unimportant in their effect on *magnitudes*.
2. All field components decreasing faster than $1/r$ are negligible.
3. Differences in the radius vectors to the elemental dipoles 'are' important for their *phase*, but may be approximated.

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Antenna Basics – Linear Aperture

Consider a line-antenna that is long compared to a wavelength, but with constant-strength current



$r \gg h \gg \lambda$

The field pattern can be calculated as an integral of a line of infinitesimal Hertzian dipoles, resulting in the form

$$E_\theta(\mathbf{r}) \approx e^{jkr} E_{\text{peak}} \text{sinc} \left(\frac{h}{\lambda} \cos \theta \right)$$

where $\text{sinc}(\xi) = \frac{\sin(\pi \xi)}{\pi \xi}$

Longer antenna, shorter wavelength, mean more/higher gain.

Wave-fronts are still spherical, but strength varies with direction.

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Antenna Basics – Far-Field Pattern

Note that the current density with shape

$$x(l) = \text{rect} \left(\frac{l}{h} \right) = \begin{cases} 1 & |l/h| < 1/2 \\ 1/2 & |l/h| = 1/2 \\ 0 & |l/h| > 1/2 \end{cases}$$

has far-field pattern shape

$$X \left(\frac{\cos \theta}{\lambda} \right) = h \text{sinc} \left(h \frac{\cos \theta}{\lambda} \right)$$

These constitute a Fourier Transform pair

$$x(l) \Leftrightarrow X \left(\frac{\cos \theta}{\lambda} \right)$$

It is generally true that the far-field antenna pattern shape is the Fourier Transform of the current distribution on the radiator.

and can be shaped accordingly

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Antenna Basics – Area Aperture

In two dimensions

$E_\theta(r) \approx e^{jkr} E_{peak} \text{sinc}\left(\frac{h}{\lambda} \cos \theta\right) \text{sinc}\left(\frac{w}{\lambda} \cos \phi\right)$

Shape is 2D Fourier Transform of current density

The main lobe of this response has nominal angular beamwidths of

$$\theta_{bw} = \frac{\lambda}{h}$$

$$\phi_{bw} = \frac{\lambda}{w}$$

The power density within this main beam has been enhanced with respect to an isotropic antenna by

$$P(\mathbf{r}) \Big|_{\theta=\pi/2, \phi=\pi/2} \approx \frac{4\pi}{\theta_{bw} \phi_{bw}} = \frac{4\pi h w}{\lambda^2} = \frac{4\pi}{\lambda^2} A$$

Actual aperture area

Larger-area antenna, shorter wavelength, mean more/higher gain, narrower beam

Antenna Basics – Gain and Effective Area

Real antennas radiate only a fraction of the power with which they are supplied.

The ratio of total radiated-power to supplied-power is the antenna efficiency.

The antenna power gain in the center of its main beam is approximated as

$$G_A \approx \xi \frac{4\pi}{\theta_{bw} \phi_{bw}} = \xi \frac{4\pi}{\lambda^2} A$$

efficiency

Just as current densities can cause radiated fields, so too can radiated fields cause current densities. This is the duality nature of antennas.

The sensitivity of a receiving antenna versus direction is the same as the field shape for field generation.

The power generated by an antenna, useable to subsequent processing, is based on the power density incident. Specifically, received power is the incident power density multiplied by

$$A_e = \frac{\lambda^2}{4\pi} G_A = \text{Antenna effective area}$$

$$= \xi A$$

Bigger antennas are more sensitive

Antenna Basics – System Parameters

From a systems standpoint, the important parameters of an antenna are

- Frequency of operation
 - Bandwidth
- Gain versus angles
 - Mainlobe beamwidths
 - Pattern shape in all dimensions
 - Sidelobes
 - Efficiency
- Phase center

Where on the physical structure is the center of the spherical wavefront?

Answers the question "range from where?"

Antenna – Examples

Various internet sources

Section Summary

- Maxwell's equations are the root of all radar behavior and operation
- Radar is about how radiated fields interact with, and reflect from, dielectric boundaries
- A "point target" is a useful fiction, and can be physically approximated for radar analysis
- The Born approximation makes radar analysis tractable, but comes at a price of artifacts in the data rendering
- Antennas are the transducer between signal voltages/currents and EM fields
- The far-field pattern is related to aperture current distribution by Fourier transform

Select References

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