Kinetic fluid moments closure for a magnetized plasma with collisions 12.12.17

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Abstract

A novel method aimed at a kinetic moments closure for a magnetized plasma with collisions is proposed. The velocity distribution function for each species is expanded in 8 Gaussian Radial Basis Functions (GRBFs) which are essentially shifted Maxwellians at eight representative 3D-velocity points of drift. The vector of 8 fluid moments (for particle density, 3 particle fluxes, total energy density, and 3 energy fluxes) has an 8x8 analytic linear matrix relation to the vector of 8 GRBF density weighs in 3D-real space. The 8 fluid moments with sources for each species are advanced in time while the 8 GRBF weighs are determined from the 8x8 inverse matrix. The two closure moments (for the stress tensor and the energy weighted stress tensor) are linearly determined from the GRBF weighs. Most importantly the velocity moments of the nonlinear Coulomb Fokker-Planck collision operator [Rosenbluth et al, Phys. Rev 107, 1957] are evaluated from the GRBF weights. Generalization from 8 to 12, 16, 20 .., in an energy weighted moment hierarchy is straightforward. The electric field follows from a generalized vorticity (quasi-neutral charge conservation) equation. A *strong* drift ordering approximation can be applied to eliminate any spuriously unstable high frequency cyclotron motions. A novel *weak* drift ordering two-time step scheme avoids the vorticity equation by following ion cyclotron motion in time to get the electric field with ion gyro-averaging. Inclusion of low-beta magnetic perturbations is straightforward.

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I. INTRODUCTION

It is well known that the kinetic (6D) equations for a plasma in a magnetic field with collisions can be represented by an infinite number of velocity or fluid moments. The symmetric moment hierarchy in a conservative time advance form is given in our principal reference text by Hazeline and Meiss[1] (see[Ref. 1, p.214]. The time advance of the first few (or finite number) of the lowest moments depends on the higher neglected moments which must be expressed in terms of the lower advanced moments: the so called *fluid closure* problem. The famous Braginskii[2] closure for a collisional magnetized ion and electron plasma truncates at 5 moments per species [particle density, 3 particle (or momentum) flux, and total energy density]. Closure for the collision moments and dependent transport fluxes are based on a short mean free path approximation questionable for our intended application: numerical code simulation of the turbulent tokamak edge and "scrap-off-layer". Ref. [3] extends and reviews previous work following the Braginshii closure while detailing the drift ordering, quasi-neutrality, and low-beta approximations commonly used in such codes (e.g. BOUT++[4], GBS[5]) This reference serves as a point of comparison and contrast with the "kinetic" fluid moments closure and methods proposed here. The kinetic closure here allows for arbitrary collisionality, flow strength, as well as deviation from drifted Maxwellian velocity distributions. Limitation to drift motions much less than ion cyclotron frequency and long perpendicular field length scales much greater than the ion gyroradius still follow from a *strong* drift approximation. A novel *weak* drift approximation is proposed to include gyro-averaging by following the ion cyclotron motion in time.

In this work, kinetic closure is first illustrated for an 8 moment system (particle density, 3 particle fluxes, total energy density, and 3 total energy fluxes) per species conservatively time advanced with external sources (see [Ref. 1, p. 213]) . The velocity distribution function for each species is expanded in 8 *Gaussian Radial Basis Functions* (GRBFs) which are essentially drifted Maxwellians at eight "representative" 3D-velocity points of drift. The vector of 8 fluid moments has a pre-computed 8x8 analytic linear matrix G_{8x8} relation to the vector of 8 GRBF density weighs at each point in 3D-real space. The 8 GRBF weights are computed from the time advanced moments using the inverse matrix G_{8x8}^{-1}. The independent components of the stress (and energy weighted

stress) tensor "closure moments" are found from the 8 GRBF weights provided by analytically pre-computed matrices P_{6x8} and R_{6x8}. Just as important, the velocity moments of the nonlinear Coulomb (inverse square) Fokker-Planck collision operator are evaluated from the GRBF weights with a large 8x8 pre-computed matrix C_{8x8}. In fact our choice of the GRBF representation was strongly motivated by the recent demonstration by Hirvijoki et al[6] that the Rosensluth potential form[7] (see also [Ref. 1 p. 182]) of the Landau collision operator can be analytically represented with GRBFs without recourse to numerical velocity derivatives.

The 8 moment system can be extended as needed to 12 (16, 20, ...) time evolved moment equations by adding energy $(1/2\ m\ v^2)^N$ factors to the energy density and 3 energy flux moments and two closure stress tensors with N=1 (2, 3, …) respectively. However numerical practicality rests on an efficient choice for the corresponding number of "representative" velocity vector points. It is assumed any numerical simulation starts with a given "target" toroidally symmetric local temperature and parallel field flow velocity (2D spatial) profiles for each species. For example, the velocity vector points can be shifted by the target local particle flow velocity and then normalized to the target local thermal velocity. The 3D "cloud" of shifted and normalized velocity points are the same for each species with normalized speeds ranging up to O(1). The (up-front once) pre-computed "advance" matrices (G_{8x8}, G_{8x8}^{-1}, P_{6x8} , R_{6x8}, and C_{8x8}) are functions of the fixed target local temperature and velocity shift profiles. If the actual evolved temperature and velocity flow profiles move too far from the targets, the advance matrices may need to be recomputed. Final results in a quasi-stationary state are expected to be independent of the details for the velocity point cloud or the toroidally symmetric target profiles. It is expected the 8-moment truncation will quickly saturate going to 12- (or 16-) moments.

Generalized cross magnetic field *drift* fluxes inversely proportional to the magnetic field are introduced. The generalized drift fluxes retain dependence on the full divergence of the stress tensors in addition to the electric and collisional forces. The electric potential can be obtained from a time advanced generalized *vorticity* equation combining the particle drift flux and polarization density flux in the quasi-neutral charge continuity equation. The explicitly computed polarization flux much less than the generalized drift flux, justifies the low frequency *weak* drift approximation. To preclude

any spuriously unstable high frequency cyclotron modes, a *strong* drift approximation then sets all cross field fluxes equal to *drift* fluxes without explicit time advance. An alternative use of the *weak* drift approximation replaces the equivalent vorticity equation with a novel two-time-step *leap frog* method to get the electric field with ion gyro-averaging by following the ion cyclotron motion. Extension to low-beta perpendicular magnetic perturbations with a parallel Ampere's law is straightforward.

The formulations for the GRBF representation, the moments hierarchy, the pre-computed time advance matrices relating GRBF weights to the moments (including the collisional moments), and the generalized drift fluxes, vorticity equation, and methods to get the electric field are given in Section II. Appendix A provide some details on the conservation properties of the GRBF nonlinear Rosenbluth collision matrix. Section III provides a discussion comparing and contrasting the GRBF kinetic closure presented here with the convectional Braginshii closure as formulated in Ref. [3].

II. FORMULATION

A. GRBF representation of the energy weighted time advanced moment hierarchy

The Vlasov equation for a plasma in an electric $\vec{E}$ and magnetic field $\vec{B}$ with collisions $C(f)$ and source S is given by

$$\partial f / \partial t + \vec{\nabla} \cdot (\vec{\upsilon} f) + (Ze/m)(\vec{E} + \vec{\upsilon} / c \times \vec{B}) \cdot \vec{\nabla}_{\upsilon} f = C(f) + S \qquad [1]$$

$f(\vec{x}, \vec{\upsilon}, t)$ is the 6D kinetic distribution function. Ion and electron species labels are suppressed. The notation is standard or otherwise follows Ref. [1]. The distribution function is to be represented as a sum of weighted GRBF's or Maxwellians drifting at representative velocity points $\upsilon_i(\vec{x})$ [4]:

$$f(\vec{x}, \vec{\upsilon}, t) = \sum_i w_i(\vec{x}, t) f_i^{GRBF}(\vec{x}) = \sum_i w_i(\vec{x}, t)[(\gamma(\vec{x})/\pi)^{3/2} \exp[-\gamma(\vec{x})(\vec{\upsilon} - \vec{\upsilon}_i(\vec{x}))^2] \qquad [2]$$

where $\gamma = 1/\upsilon_{th}^2$ with $\upsilon_{th} = \sqrt{2T_0(\vec{x})/m}$ thermal velocity. The density weights $w_i(\vec{x}, t)$ index $i = [1, i_M]$ where i_M the number of independent moments time advanced. The velocity points are shifted by local velocity flows $\vec{u}_0(\vec{x})$ and normed to the local thermal velocity: $\vec{\upsilon}_i(\vec{x}) = \upsilon_{th}(\vec{x})[\hat{\vec{\upsilon}}_i' + \hat{\vec{u}}_0(\vec{x})]$ to make them efficiently "representative". The local

"target" temperature T_0 and flow velocity shifts $\vec{u}_0$ are considered constant or quasi-static in time and toroidally symmetric. Normally only the ion flow velocity along the toroidally symmetric equilibrium field $\vec{B}_0$ approaching the ion thermal speed is worth consideration. However, parallel electron sheath flow velocities approaching thermal speeds near the diverter plate may be of concern. The shifted and normed velocity points $\hat{\vec{\upsilon}}_i'$ are independent of $(\vec{x},t)$ and the same for each species. Going forward with the moment hierarchy, perturbations in the magnetic field are ignored: $\vec{B} \equiv \vec{B}_0$ and the electric field is electrostatic $\vec{E} = -\vec{\nabla}\Phi$. Parallel and perpendicular directions are with respect to the local unperturbed equilibrium magnetic field $\vec{B}_0$. In Section E below, a simple extension to include the effects of perpendicular magnetic perturbations should suffice for low-β plasmas.

The first 4 moments for the particle density n and particle fluxes $\vec{\Gamma} = n\vec{V}$ are given by the particle and momentum conservation equations

$$\partial n / \partial t + \vec{\nabla} \cdot \vec{\Gamma} = S_n \qquad [3]$$

$$m\partial\vec{\Gamma} / \partial t + \vec{\nabla} \cdot \overleftrightarrow{P} - ZenE - Ze\vec{\Gamma} \times \vec{B}_0 / c = \vec{F}_C + \vec{S}_M \qquad [4]$$

where $\vec{F}_C$ is the exchange (friction) force and $\overleftrightarrow{P}$ is the symmetric stress tensor. It is useful to note $\overleftrightarrow{P} = p\overleftrightarrow{I} + \overleftrightarrow{\Pi} + mn\vec{V}\vec{V}$ where $p = nT$ is the pressure, and $mn\vec{V}\vec{V}$ is the dynamic stress. The second 4 energy moment conservation equations follow:

$$\partial U / \partial t + \vec{\nabla} \cdot \vec{Q} - Ze\vec{\Gamma} \cdot \vec{E} = \Delta_C + S_E \qquad [5]$$

where $U = 3/2p + mnV^2/2$ is the total energy density, Δ_C is the total energy exchange, and $\vec{Q}$ is the energy flux moment advanced by

$$\partial\vec{Q} / \partial t + \vec{\nabla} \cdot \overleftrightarrow{R} - (Ze/m)[U\vec{E} + \overleftrightarrow{P} \cdot \vec{E}] - (Ze/mc)\vec{Q} \times \vec{B}_0 = \vec{G}_C + S_E \ . \qquad [6]$$

$\overleftrightarrow{R}$ is an energy weighted symmetric stress tensor, and $\vec{G}_C$ is an energy weighted friction. It is useful to note $\vec{Q} = \vec{q} + (5/2)p\vec{V} + \overleftrightarrow{\Pi} \cdot \vec{V} + (mnV^2/2)\vec{V}$ where $\vec{q}$ is the heat flux. The first 8 time advance moments $[n, \vec{\Gamma}, U, \vec{Q}]$ are linearly related to the 8 GRBF weights w_i by

$$n = \int d\upsilon^3 f = \sum_i w_i \qquad [7a]$$

$$\vec{\Gamma} = \int d\upsilon^3 \vec{\upsilon} f = \upsilon_{th} \sum_i w_i \hat{\vec{\upsilon}}_i \qquad [7b]$$

$$U = \int d\upsilon^3 m\upsilon^2 / 2 f = (m\upsilon_{th}^2 / 2) \sum_i w_i [\hat{\upsilon}_i^2 + 3/2] \qquad [7c]$$

$$\vec{Q} = \int d\upsilon^3 (m\upsilon^2 / 2) \vec{\upsilon} f = (m\upsilon_{th}^2 / 2) \upsilon_{th} \sum_i w_i [\hat{\upsilon}_i^2 + 3/2] \hat{\vec{\upsilon}} \qquad [7d].$$

where $\hat{\vec{\upsilon}}_i = (\hat{\vec{\upsilon}}_i' + \hat{\vec{u}}_0)$. Inversion of the 8x8 matrix G_{8x8} implied by Eq. [7] provides the GRBF weights from the time advance moments. To ensure the $[G_{8x8}]^{-1}$ inverse exists, the representative $\hat{\vec{\upsilon}}_i'$ should have differing speeds $|\hat{\vec{\upsilon}}_i'|$ to avoid any rotational symmetry. The now time advanced weights are then used to the evaluate the linear closure moments:

$$\vec{\vec{P}} = \int d\upsilon^3 m \vec{\upsilon}\vec{\upsilon} f = m\upsilon_{th}^2 \sum_i w_i [\hat{\vec{\upsilon}}_i \hat{\vec{\upsilon}}_i + \vec{\vec{I}} / 2] \qquad [8a]$$

$$\vec{\vec{R}} = \int d\upsilon^3 (m\upsilon^2 / 2) \vec{\upsilon}\vec{\upsilon} f = \upsilon_{th}^2 (m\upsilon_{th}^2 / 2) \sum_i w_i [(\hat{\upsilon}_i^2 + 7/2) \hat{\vec{\upsilon}}_i \hat{\vec{\upsilon}}_i + (\hat{\upsilon}_i^2 / 2 + 5/4) \vec{\vec{I}}] \qquad [8b]$$

The matrices G_{8x8} and $[G_{8x8}]^{-1}$ implied by Eq. [7], and P_{6x8} and R_{6x8} implied by Eq. [8] are to be pre-computed. If the evolving quasi-steady toroidally symmetric temperature T_0 and flow velocity shifts u_0 profiles move too far from the starting "target" profiles, these matrices (as well as collision matrices C_{8x8} below) will need to be re-computed (perhaps many times.)

Generalization to the next tranches of energy weighted time advance and closure moments is straightforward:

$$[U_N, \vec{Q}_N] = \int d\upsilon^3 [m\upsilon^2 / 2, (m\upsilon^2 / 2)\vec{\upsilon}](m\upsilon^2 / 2)^N f \qquad [9a]$$

$$[\vec{\vec{P}}_N, \vec{\vec{R}}_N] = \int d\upsilon^3 [m\vec{\upsilon}\vec{\upsilon}, (m\upsilon^2 / 2)\vec{\upsilon}\vec{\upsilon}](m\upsilon^2 / 2)^N f \qquad [9b]$$

The number of representative velocity points is increased to i_M=8+4N with an increasing number of velocity angles and speed selections available. The analytic i_M x i_M time advance matrices implied by analogy to Eq. [7] $G_{i_M \times i_M}$ and $[G_{i_M \times i_M}]^{-1}$ as well as the 2(N+1) 6 x i_M matrices implied by analogy to Eq, [8] P_{6xi_M} and R_{6xi_M} are again pre-computed. Most importantly the form of any additional 4N time advance equations for $[U_N, Q_N]$ (i.e. like Eqs. [5,6] for $[U_0, \vec{Q}_0]$) with tensor closures $[\vec{\vec{P}}_N, \vec{\vec{R}}_N]$ (i.e. like $[\vec{\vec{P}}_0, \vec{\vec{R}}_0]$) remains

unchanged: the number of terms is the same and the one stiff term $(Ze/mc)\vec{Q}_N \times \vec{B}_0$ driving fast ion cyclotron motion (like $(Ze/mc)\vec{\Gamma} \times \vec{B}_0$ in Eq. 4) remains in place. This allow a simple matrix inversion for an implicit advance of the stiff terms for all the flux $\vec{F}$ moments: $[\partial\vec{F}/\partial t - (Ze/mc)\vec{F} \times B_0]_{implicit} = [\ \ ]_{\exp licit}$. If the 8-moment truncation fails to quickly saturate going to 12- (or 16-) moments, it would seem that the proposed scheme is unlikely to be practical. For example, the GRBF representation of the next (N=1) tranche of 12 time advance moments is given by

$$U_1 = (m\upsilon_{th}^2/2)^2 \sum_i w_i[\hat{\upsilon}_i^4 + 3\hat{\upsilon}_i^2 + 3/4] \qquad [9c]$$

$$\vec{Q}_1 = (m\upsilon_{th}^2/2)^2 \upsilon_{th} \sum_i w_i[\hat{\upsilon}_i^4 + 4\hat{\upsilon}_i^2 + 5/2]\vec{\hat{\upsilon}}_i \qquad [9d]$$

and the closure moments by

$$\vec{\vec{P}}_1 = m\upsilon_{th}^2(m\upsilon_{th}^2/2)\sum_i w_i[(\hat{\upsilon}_i^2 + 7/2)\vec{\hat{\upsilon}}_i\vec{\hat{\upsilon}}_i + (\hat{\upsilon}_i^2/2 + 5/4)\vec{\vec{I}}] \qquad [9e]$$

$$\vec{\vec{R}}_1 = \upsilon_{th}^2(m\upsilon_{th}^2/2)^2 \sum_i w_i[(\hat{\upsilon}_i^4 + 9\hat{\upsilon}_i^2 + 63/4\vec{\hat{\upsilon}}_i\vec{\hat{\upsilon}}_i + (1/2\hat{\upsilon}_i^4 + 7/2\hat{\upsilon}_i^2 + 35/8)\vec{\vec{I}}] \qquad [9f]$$

There are several reasons why the energy weighted moment hierarchy is clearly simpler and numerically more efficient than the general symmetric moment hierarchy (see Ref [1] p. 214): The first 13,40,121…symmetric moments have a decreasing 10, 20, 35…fraction of independent components yielding the corresponding number of independent representative velocity points. The symmetrization of the right hand sides leads to a rapidly increasing number of terms and multiplies. Going beyond the divergence of 2-tensor closure moments (in Eq. [4] and [6] with generalization in Eq. [9b]) to divergence of 3-,4-, 5-…tensor closures in the general closure requires an additional loop over Christoffel matrices at each stage[6].

B. GRBF representation of the nonlinear Rosenbluth collisional moments

Hirvijoki et al [6] recently demonstrated that the Rosenbluth potential form [7] (see also [Ref. 1 p. 182]) of the Landau collision operator can be analytically represented with GRBFs without recourse to numerical velocity derivatives. The equilibration of two widely separated 3D velocity space "balls" to a single Maxwellian "ball" $\partial f(\vec{\upsilon},t)/\partial t = C(f) \rightarrow C(f_{Max}) = 0$ was illustrated. Good number, momentum, and energy

conservation was demonstrated with O(10^3) "collocated" velocity points evaluting $f(\vec{v},t)$ represented by O(10^3) GRBFs. The novelty here is that while integration over O(10^3) "co-located" velocity point to get the moments of the collision operator ($\vec{F}_C$ in Eq. [4], Δ_C in Eq. [5], and G_C (and generalizations G_{CN}) in Eq. [6]), only a few (maybe only 8) GRBF weights are likely to be sufficiently accurate. Let $C_{ab}(\vec{v})$ correspond to the collision operator for species "a" on "b" with $\vec{v}$ the velocity space of species "a". In the GRBF representation

$$C_{ab}(\vec{v}) = \sum_{k,l} C_{ab}^{k,l}(\vec{v}) w_k^a w_l^b \qquad [10]$$

with

$$\vec{F}_{Ca} = \sum_{k,l} [\int dv^3 m\vec{v} C_{ab}^{k,l}(\vec{v})] w_k^a w_l^b = \sum_{k,l} \vec{F}_{Ca}^{k,l} w_k^a w_l^b = -\vec{F}_{Cb} \qquad [11a]$$

$$\Delta_{Ca} = \sum_{k,l} [\int dv^3 (mv^2/2) C_{ab}^{k,l}(\vec{v})] w_k^a w_l^b = \sum_{k,l} \Delta_{Ca}^{k,l} w_k^a w_l^b = -\Delta_{Cb} \qquad [11b]$$

$$\vec{G}_{CNa} = \sum_{k,l} [\int dv^3 \vec{v} (mv^2/2)^{(1+N)} (C_{ab}^{k,l}(\vec{v}) + C_{aa}^{k,l}(\vec{v}))] w_k^a w_l^b = \sum_{k,l} \vec{G}_{CNa}^{k,l} w_k^a w_l^b \qquad [11c]$$

The very expensive nonlinear collision matrices $[\vec{F}_{Ca}^{kl}(\vec{x}), \Delta_{Ca}^{kl}(\vec{x}), \vec{G}_{CNa}^{kl}(\vec{x})]$ are to be pre-computed. A detailed formulation of a test code for computation of $C_{ab}^{k,l}(\vec{v})$ in terms of Rosenbluth potentials is given in Appendix A. The accuracy of number conservation $\int dv^3 \sum_{k,l} C_{ab}^{k,l}(\vec{v}) \sim 0$ is tested.

C. The drift approximation and the evolution of the electrostatic potential

The most naive path to find $\vec{E} = -\vec{\nabla}\Phi$ from the electrostatic potential Φ is via Poisson's equation $-\nabla^2\Phi = 4\pi e(Zn_i - n_e)$ for local charge imbalance. However, this implies working on the very short Debye length scales. It is possible to work with an artificially much larger Debye length, then show the final results are insensitive to smaller lengths. A more conventional approach is to enforce the quasi-neutral approximation $Zn_i = n_e$ and then extract the Φ from the quasi-neutral charge conservation $\vec{\nabla}\cdot\vec{j} = \vec{\nabla}\cdot(Ze\vec{\Gamma}_i - e\vec{\Gamma}_e) = 0$. The most well traveled path in plasma physics is

to cross the B-field with the flux equations to extract the cross-B or perpendicular drift fluxes. For example, B crossed on Eq. [4] defines an *generalized* drift flux $\vec{\Gamma}_{d\perp}$:

$$\vec{\Gamma}_{d\perp} \equiv nc\vec{b}_0 \times \vec{\nabla}_\perp \Phi / B_0 + c\vec{b}_0 \times [\vec{\nabla} \cdot \overleftrightarrow{P} - \vec{F}_{C\perp} - S_{M\perp}] / (ZeB_0) \qquad [12a]$$

$$\vec{\Gamma}_\perp \equiv \vec{\Gamma}_{d\perp} + 1/\omega_c \partial(\vec{b}_0 \times \vec{\Gamma}_\perp)/\partial t \qquad [12b]$$

where $\omega_c = (ZeB_0 / mc)$ is the very high cyclotron frequency. There is no approximation in Eq [12b]. The *weak* drift approximation substitutes $\vec{\Gamma}_{d\perp}$ for $\vec{\Gamma}_\perp$ in the time derivative term of Eq. [12b]

$$\vec{\Gamma}_\perp \cong \vec{\Gamma}_{d\perp} + 1/\omega_c \partial(\vec{b}_0 \times \vec{\Gamma}_{d\perp})/\partial t \qquad [12c]$$

where $\vec{\Gamma}_{\perp pol} = 1/\omega_c \partial(\vec{b}_0 \times \vec{\Gamma}_{d\perp})/\partial t$ is defined here as the polarization flux. As long as the time steps (dt) are large enough to avoid following the cyclotron motion, then $\vec{\Gamma}_{\perp pol} << \vec{\Gamma}_{\perp d}$. Substituting Eq. [12c] for the ions into the quasi-neutral charge conservation $\vec{\nabla} \cdot \vec{j} = 0$, dropping the small electron polarization so that $\Gamma_{\perp e} \cong \Gamma_{d\perp e}$,and also dropping electron viscosity and dynamic stress consistent with the small electro mass $m_e << m_i$ $\overleftrightarrow{P}_e \cong p_e \overleftrightarrow{I}$, we arrive at a generalized vorticity equation:

$$\partial \Omega_i / \partial t = \vec{\nabla} \cdot \vec{j}_\parallel + \vec{\nabla}_\perp \cdot c\vec{b}_0 \times [\vec{\nabla} \cdot (\overleftrightarrow{P} + p_e \overleftrightarrow{I}) / B_0 - (S_{Mi} + S_{Me}) / B_0] \qquad [13a]$$

where $\vec{j}_0 = \vec{b}_0 j_\parallel$ and the *extended* vorticity is defined by

$$\Omega_i \equiv \vec{\nabla}_\perp \cdot (1/\omega_{ci} B_0)[Zen\vec{\nabla}_\perp \Phi + \vec{\nabla}_\perp \cdot \overleftrightarrow{P}_i - \vec{F}_{iC\perp} - S_{iM\perp}] \qquad [13b]$$

(Dropping small electron mass terms is not essential and could easily added back at little cost for the methods proposed here.) We refer to the *generalized* viscosity because the "traditional" vorticity in Ref. [3] is defined by $\varpi \equiv \vec{\nabla}_\perp \cdot (1/\omega_{ci} B_0)[Zen\vec{\nabla}_\perp \Phi + \vec{\nabla}_\perp p_i]$. It would appear that *generalized* vorticity equation Eq [13] is actually equivalent to Eq. [76] of Ref [3]. Keeping the moment equations in conservative (rather than convective form) and avoiding any breakdown of the stress tensor into pressure, viscous stress and dynamic stress, leads to the simpler form. Furthermore there is no need to discuss the "gyroviscous cancellation approximation" (see Ref. [3] p. 47530) or neglect any other

terms. Often simulation codes make the so called "Boussinesq" approximate $\vec{\nabla}_\perp \cdot (1/\omega_{ci}B_0)[Zen\vec{\nabla}_\perp \Phi \approx (Zen/\omega_{ci}B_0)\nabla_\perp^2 \Phi$ to get the more easily invertible $\nabla_\perp^2 \Phi$.

There is an entirely analogous *drift approximation* for the perpendicular energy flux (like Eqs [12]):

$$\vec{Q}_{d\perp} \equiv c\vec{b}_0 \times [(U\vec{E} + \vec{P} \cdot \vec{E}) - m(\vec{\nabla} \cdot \vec{R} - G_{C\perp} - S_{Q\perp})/Ze]/B_0 \qquad [14a]$$

and again without approximation from B cross Eq. [6]

$$\vec{Q}_\perp \equiv \vec{Q}_{d\perp} + 1/\omega_c \partial(\vec{b}_0 \times \vec{Q}_\perp)/\partial t \qquad [14b]$$

with a *weak* drift approximation analogous to Eq. [12c]

$$\vec{Q}_\perp \cong \vec{Q}_{d\perp} + 1/\omega_c \partial(\vec{b}_0 \times \vec{Q}_{d\perp})/\partial t \qquad [14c]$$

and similarly for the $\vec{Q}_{N\perp}$'s. Going forward after the generalized vorticity Eq. [13] is used to determine the electric field, the *strong* drift approximation sets all $[\vec{\Gamma}_\perp, \vec{Q}_{N\perp}]$ to $[\vec{\Gamma}_{d\perp}, \vec{Q}_{Nd\perp}]$ with no explicit time evolution for any perpendicular flux moment. This may be needed to avoid any spuriously unstable cyclotron modes. The parallel fluxes $[\vec{\Gamma}_\parallel, \vec{Q}_{N\parallel}]$ are advanced in time per Eqs. [4], [6], and analogs. The electron inertia term may be retained in the parallel momentum equation Eq.[4], however high frequency "electrostatic Alfven modes" $\omega/\omega_{ci} \sim (k_\parallel/k_\perp)\rho_* \sqrt{m_i/m_e}$ maybe spuriously unstable. [Here $k_\parallel$ is a parallel and $k_\perp$ is a perpendicular wave number and $\rho_* = (c_s/\omega_{ci})/a$ is ion gyroradius relative to the minor radius.] This could be a problem for the *weak* drift approximation which required frequencies less than the ion cyclotron frequency. Only the strong drift approximated electron continuity equation $\partial n_e/\partial t + \vec{\nabla}_\perp \cdot \vec{\Gamma}_{d\perp e} + \vec{\nabla} \cdot \vec{\Gamma}_\parallel = S_n$ is needed to advance the density for both species with quasi-neutraliy $Zn_i = n_e$. The ion continuity Eq. [3] is implicit in the vorticity Eq. [13] (equivalent to $\vec{\nabla} \cdot \vec{j} = 0$) but not explicitly used.

D. Evolution of the electrostatic potential with gyro-averaging

The most bothersome aspect of the *strong* drift approximation (for the ions in particular) is that gyro-averaging to properly treat and cut-off short wave motion on the ion gyro-radius ($\rho_i = \upsilon_{thi}/\omega_{ci}$) scale is precluded. (This is likely also the case using the

weak drift approximation consistently, i.e. Eqs. [12c], [14c], flux analogs.). In the standard gyrokinetic approximation (see Ref. [1], p. 136), the gyro-averaging is done on the perpendicular space. Here we propose approximating the ion gyro-averaging by following the ion cyclotron time scales. A new time step starts with a given electric field $\vec{E}$. The electrons are time-stepped with the *strong* drift approximation as above with no need to evolve the ion density and continuity. The remaining ion equations for $[U_i, \vec{\Gamma}_i, \vec{Q}_{iN}](\vec{E})$ are half-stepped with fluxes in the implicit form. In particular, the perpendicular part of Eq. [4] looks like

$$[\partial\vec{\Gamma}_{\perp i}(\vec{E})/\partial t - \omega_{ci}\vec{\Gamma}_{\perp i}(\vec{E})\times b_0]_{implicit} = [\omega_{ci}\vec{\Gamma}_{d\perp i}(\vec{E}_\perp)\times b_0]_{explicit} \qquad [15a]$$

where $\vec{\Gamma}_{d\perp i}(\vec{E}_\perp)$ is given by Eq. [12b] (with $nc\vec{b}_0 \times \vec{\nabla}_\perp \Phi / B_0 = nc\vec{E}_\perp \times \vec{b}_0 / B_0$). The next-step perpendicular electric field $\vec{E}'_\perp$ is obtained from $\vec{\Gamma}_{d\perp i}(\vec{E}'_\perp)$ using the *weak* drift approximation Eq. [12c] crossed with B and converted to implicit form:

$$[\partial\Gamma_{di}(\vec{E}'_\perp)/\partial t - \omega_{ci}\Gamma_{di}(\vec{E}'_\perp)\times\vec{b}_0]_{implicit} = [\omega_{ci}\Gamma_{di}(\vec{E}_\perp)\times\vec{b}_0]_{explicit} \qquad [15b]$$

Clearly Eq. [15a] and Eq. [15b] form a "leap-frog" system. Using B cross Eq. [12a],

$$\vec{E}'_\perp = \vec{\Gamma}_{\perp d}(\vec{E}'_\perp)[B_0/nc] + [\vec{\nabla}\cdot\vec{P}(\vec{E}) - \vec{F}_{c\perp}(\vec{E}) - S_{M\perp}]/(nZe) \qquad [15c]$$

The next-step parallel electric field $\vec{E}'_\parallel = -\nabla_\parallel \Phi'$ is obtained by inverting the quasi-neutral Poisson equation $\vec{\nabla}\cdot\vec{E} = 0$:

$$\vec{\nabla}\cdot(\vec{b}_0 E'_\parallel) = -\vec{\nabla}_\perp \cdot \vec{E}'_\perp \qquad [15d]$$

The generalized vorticity equation Eqs. [13] for advancing the electric field has been "leaped-over". $\vec{\Gamma}_{\perp i}$ is time evolved in the same way as every other ion moment responding to the same instantaneous electric field and cyclotron motion. Again only the electron density continuity equation is used with $\partial n_e/\partial t + \vec{\nabla}_\perp \cdot \vec{\Gamma}_{d\perp e} + \nabla\cdot\vec{\Gamma}_{\parallel e} = S_n$

Substituting into the unused Eq. [3] for the ions and subtracting the quasi-neutral Eq. [3] for the electrons, the quasi-neutral charge continuity is $\vec{\nabla}\cdot\vec{j} = \nabla\cdot[Ze\vec{\Gamma}_i - e(\vec{\Gamma}_{\perp de} + \vec{\Gamma}_{\parallel e})]$. There is a paradox: Since $\vec{\nabla}\cdot\vec{E} = 0$ Eq. [15d] was used rather than $\vec{\nabla}\cdot\vec{j} = 0$, the latter is unlikely to hold with any accuracy. However the same paradox arises in the more

conventional way of finding the electric field from the vorticity equation $\vec{\nabla}\cdot\vec{j}=0$ Eq. [13]: $\vec{\nabla}\cdot\vec{E}=0$ is unlikely to hold with any accuracy.

E. Extentions to include magnetic perturbations for a low-β plasma

For a low beta plasma typical of the tokamak edge and scape-off-layer plasma, a straightforward and consistent way to include small perpendicular magnetic field perturbations $|\delta\vec{B}_\perp| << B_0$ while deprecating any importance of parallel perturbations $\delta B_\parallel$ follows from the *ansatz*

$$\delta\vec{B}=\vec{\nabla}\times(\vec{b}_0\delta A_\parallel) \qquad [16a]$$

neglecting $\delta\vec{A}_\perp$ entirely. This form of the low-β approximation satisfies $\vec{\nabla}\cdot\delta\vec{B}=0$ (with $\vec{\nabla}\cdot\vec{B}_0=0$ of course). Again, parallel and perpendicular directions refer to the unperturbed magnetic field $\vec{b}_0$ direction. The parallel electric field is $E_\parallel=-\nabla_\parallel\Phi-1/c\partial\delta A_\parallel/\partial t$ and the perpendicular field remains electrostatic with $\vec{E}_\perp=-\vec{\nabla}_\perp\Phi$ (consistent with neglect of $\delta\vec{A}_\perp$). Using the evolving internal parallel current density moment $j_\parallel$, Ampere's law to obtain $\delta A_\parallel$ is then written

$$\nabla^2\delta A_\parallel-\vec{b}_0\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{b}_0\delta A_\parallel)=-4\pi(j_\parallel-\vec{j}_0\cdot\vec{b}_0) \qquad [16b]$$

Eq. [16b] is consistent with Eq. [28] of Ref. [3], if the left-hand-side is interpreted as the the "fast" derivative $\nabla_f^2\delta A_\parallel$. $\vec{j}_0=\nabla\times\vec{B}_0/(4\pi/c)$ is the equilibrium current. It follows exactly from Eq. [16a] that

$$\delta\vec{B}_\perp=(\vec{\nabla}\delta A_\parallel)\times\vec{b}_0+\delta A_\parallel[(\vec{\nabla}\times\vec{b}_0)-\vec{b}_0(\vec{b}_0\cdot\vec{\nabla}\times b_0)] \qquad [16c]$$

Eq. [16b] is consistent with Eq. [26] of Ref. [3] if the right-hand-side is the defined use of the "fast" derivative $(\vec{\nabla}_f\delta A_\parallel)\times\vec{b}_0$. It also follows exactly that

$$\delta B_\parallel=\delta A_\parallel(\vec{b}_0\cdot\vec{\nabla}\times\vec{b}_0) \qquad [16d]$$

Where it should be clear that $\delta B_\parallel << |\vec{B}_\perp|$.

The perturbed field can be simply added to the flux equations with $\vec{B}_0$ replaced by $\vec{B}_0 + \delta\vec{B}$ in Eqs. [4,6] . By the same "cross $\vec{B}_0$" steps, the generalized drift fluxes Eqs. [12a,14a] acquire "magnetic flutter" additions:

$$\vec{\Gamma}_{d\perp} \Rightarrow \vec{\Gamma}_{d\perp}^{(12a)} + \Gamma_{\parallel}\delta\vec{B}_{\perp} / B_0 - \vec{\Gamma}_{\perp}\delta B_{\parallel} / B_0 \approx \vec{\Gamma}_{d\perp}^{(12a)} + \Gamma_{\parallel}\delta\vec{B}_{\perp} / B_0 \qquad [17a]$$

$$\vec{Q}_{d\perp} \Rightarrow \vec{Q}_{d\perp}^{(14a)} + Q_{\parallel}\delta\vec{B}_{\perp} / B_0 - \vec{Q}_{\perp}\delta B_{\parallel} / B_0 \approx \vec{Q}_{d\perp}^{(14a)} + Q_{\parallel}\delta\vec{B}_{\perp} / B_0 \qquad [17b]$$

where $\delta B_{\parallel}$ terms can be safely neglected, e.g. $\vec{\Gamma}_{\perp}\delta B_{\parallel} / B_0 \sim \vec{\Gamma}_{d\perp}\delta B_{\parallel} / B_0 << \Gamma_{\parallel}\delta\vec{B}_{\perp} / B_0$. Using the weak drift approximation Eq. [12c], the generalized vorticity equation Eq. [13a] in Sec. C now includes the magnetic flutter current

$$\partial\Omega_i / \partial t = \vec{\nabla}\cdot\vec{j}_{\parallel} + \vec{\nabla}_{\perp}\cdot(j_{\parallel}\delta\vec{B}_{\perp} / B_0) + \vec{\nabla}_{\perp}\cdot c\vec{b}_0 \times [\vec{\nabla}\cdot(\vec{P} + p_e\vec{I}) / B_0 - (S_{Mi} + S_{Me}) / B_0] \qquad [18a]$$

with the generalized vorticity including the magnetic field perturbation:

$$\Omega_i = \vec{\nabla}_{\perp}\cdot(1/\omega_{ci}B_0)\{Zen[\vec{\nabla}_{\perp}\Phi + (\Gamma_{\parallel}^{i} / nc)\vec{b}_0 \times \delta\vec{B}_{\perp}] + \vec{\nabla}_{\perp}\cdot\vec{P} - \vec{F}_{iC\perp} - S_{iM\perp}\} \qquad [18b]$$

where $\vec{\Gamma}_{\perp}\delta B_{\parallel} / B_0 << \Gamma_{\parallel}\delta\vec{B}_{\perp} / B_0$ has been dropped. Note that the combination $\vec{b}_0 \times [\vec{\nabla}_{\perp}\Phi + (\Gamma_{\parallel}^{i} / nc)\vec{b}_0 \times \delta\vec{B}_{\perp}] \approx \vec{b}_0 \times [\vec{\nabla}_{\perp}\Phi - (\Gamma_{\parallel}^{i} / nc)\vec{\nabla}_{f\perp}\delta A_{\parallel}]$, which also appears in the extended $\vec{\Gamma}_{d\perp}$ drift flux Eq. [17a], is reminiscent of the generalized potential δU often used in $\delta f -$ gyrokinetic codes for microturbulence where perpendicular derivatives are "fast" and parallel derivatives are "slow": $\vec{b}_0 \times \vec{\nabla}_{f\perp}\delta U = \vec{b}_0 \times \vec{\nabla}_{f\perp}[\delta\Phi - (\upsilon_{\parallel} / c)\delta A_{\parallel}]$ where the parallel particle velocity has been replaced by a fluid velocity $\Gamma_{\parallel} / n$. The *strong* drift approximation again sets all $[\vec{\Gamma}_{\perp}, \vec{Q}_{N\perp}]$ to $[\vec{\Gamma}_{d\perp}, \vec{Q}_{Nd\perp}]$ with no explicit time evolution for any perpendicular flux moment. $Ze(\vec{b}_0 \cdot \delta\vec{\Gamma}_{d\perp}^{(12a)} \times \delta\vec{B}_{\perp})$ and $(Ze / mc)(\vec{b}_0 \cdot \delta Q_{d\perp}^{(14a)} \times \delta\vec{B}_{\perp})$ should be added to the right-hand-side of the explicitly time advanced parallel momentum and energy flux Eqs. [4] and [6[respectively.

III. SUMMARY AND DISCUSSION

The novel GRBF kinetic fluid moment closure scheme presented here is in marked contrast to the commonly used Braginskii [2] collisional two fluid closure commonly used in tokamak edge and scrape-off-layer plasma codes. The detailed closed form of the equations with the strong drift approximation is detailed in Ref. [3]. In the

primary case, 8 moments $[n,\vec{\Gamma},U,\vec{Q}]$ for each species are time advanced (see Eq. [7]) in conservative form [e.g. $\vec{\nabla}\cdot(n\vec{V})$, $\vec{\nabla}\cdot\vec{P}$] in place of 5 moments (equivalent to $[n,\vec{\Gamma},U]$) in convective form [e.g. $\vec{V}\cdot\vec{\nabla}n$, $m\vec{V}\cdot\vec{\nabla}\vec{V}$]. In contrast for the Braginskii closure, neither the primary kinetic closure moments for stress and energy weighted stress tenor $[\vec{\vec{P}},\vec{\vec{R}}]$ (see Eq. [8]) nor the collisional source moments $[\vec{F}_C,\Delta_C,\vec{G}_C]$ (see Eq. [11]) depend *explicitly* on gradients of time advance lower moments (like $[n,\vec{\Gamma},U]$). For example in the Braginskii closure, the most troublesome closure for the viscous stress tenor $\vec{\vec{\Pi}}=-\vec{\vec{\eta}}:\vec{\nabla}\vec{V}+..$ (e.g. see Eq. [14] of Ref. [3]) is broken free from the total stress tenor , and the heat conduction part of the closure for $\vec{Q}$ (as a closure moment) has $q_{\parallel}=-\kappa_{\parallel}\nabla_{\parallel}T+..$ etc. Most importantly, the GRBF kinetic closure is not limited to high collisionality.

As in the conventional approach (like Ref. [3]), the electric field is found from a quasi-neutral current continuity equation $\vec{\nabla}\cdot\vec{j}=0$ with a g*eneralized* vorticity equation (see Eq. [13]). The *weak* drift approximation is used where the cross field ion flux is broken into a drift and polarization fluxes $\vec{\Gamma}_{\perp}\approx\vec{\Gamma}_{d\perp}+\vec{\Gamma}_{\perp pol}$ with $\vec{\Gamma}_{\perp pol}=1/\omega_c\partial(\vec{b}_0\times\vec{\Gamma}_{d\perp})/\partial t$ Eq. [12c] and $\Gamma_{d\perp}$ is a *generalized* drift (see Eq. [12a]) with $\vec{\nabla}\cdot\vec{\vec{P}}$ replacing $\vec{\nabla}\cdot(p\vec{\vec{I}})$ in the usual diamagnetic drifts. This avoids difficult to verify approximations involving $\vec{\nabla}\cdot\vec{\vec{\Pi}}$ and $\vec{\nabla}\cdot(mn\vec{V}\vec{V})$ in Ref. [3] and complicated general expression of the gyroviscous force in Ref. [8]. Compare the simple looking vorticity Eq. [13] with equation Eq. [76] and Apprendix D derivation of Ref. [3]. The *strong* drift approximation $\vec{\Gamma}_{\perp}\approx\vec{\Gamma}_{d\perp}$ and $\vec{Q}_{\perp}\approx\vec{Q}_{d\perp}$ (Eq. [14]) easily applied to the electrons with the electron continuity Eq. [3] evolving the quasi neutral density $n=n_e=n_i$ The vorticity equation Eq. [13] subsumes the ion continuity equation. Some preliminary test codes results with linearized adiabatic electrons and GRBF ion equations suggest spurious high frequency ion cyclotron modes may obtain. Going beyond the vorticity equation which rests on *weak* drift approximation, it may be necessary to apply the *strong* drift approximation to eliminate such spurious modes. As in Ref. [3], the ions are then purely "drift kinetic" which has no ion gyro-radius cut-off at short perpendicular wave lengths.

Only the parallel flux moment equations $[\vec{\Gamma}_{\parallel},\vec{Q}_{\parallel}]$ Eqs. [4] (including electron inertia) and [6], as well as the (electron) density n_e and total energy moments U Eqs. [3] and [4], are explicitly time evolved (including electron inertia).

To retain the ion gyro-radius cut-off and gyro-averaging of the electric potential, the $\vec{\nabla}\cdot\vec{j}=0$ and vorticity equation (as commonly used, e.g. Ref. [3]) is completely abandoned in favor of finding a gyro-averaged electric field from the quasi-neutral Poisson equation $\vec{\nabla}\cdot\vec{E}=0$. A novel two-half-step "leap-frog" method (see Eq. [15]) is proposed which avoids the *strong* drift approximation for the ions: gyro-averaging obtains form following the ion-cyclotron motion implicit in evolving the perpendicular flux $[\vec{\Gamma}_{\perp},\vec{Q}_{\perp}]$ equations Eqs. [4] and [6] .

Simple extensions of Sections A and C to include the magnetic field perturbations for a low-β plasma were provided in Sec. E. $\vec{B}_0$ is replacing $\vec{B}_0+\delta\vec{B}$ in the perpendicular flux equations Eqs. [4] and [6] and the ease of derivation follows by referring parallel and perpendicular field directions to the unperturbed field direction $\vec{b}_0$. "Magnetic flutter" fluxes $[\Gamma_{\parallel}\delta\vec{B}_{\perp}/B_0, Q_{\parallel}\delta\vec{B}_{\perp}/B_0]$ are naturally added to the extended drift fluxs $[\vec{\Gamma}_{d\perp},\vec{Q}_{d\perp}]$ (see Eq. [17]). A simple *ansatz* approximation that $\delta B_{\parallel}=\delta A_{\parallel}(\vec{b}_0\cdot\vec{\nabla}\times\vec{b}_0)$ with $\delta A_{\perp}=0$ rigorously follows Faraday's law for the electric field and allows for a rigorously defined Ampere's law for $\delta A_{\parallel}$ (see Eq. [16b]). For example, the "fast" derivatives used in the Ref. [3] Ampere's law are now clearly defined : $\nabla_f^2\delta A_{\parallel}=\nabla^2\delta A_{\parallel}-\vec{b}_0\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{b}_0\delta A_{\parallel})$. The effects of $\delta B_{\parallel}$ are deprecated.

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Appendix A: Formulation of the nonlinear Rosenbluth GRBF collision operator

The Landau Rosenbluth, MacDonald [7] and Judd Coulomb collision operator for species "a" with velocity $\vec{\upsilon}$ colliding on "b" can be written as (see Ref. [1] p183)

$$C_{ab} = [(\gamma_{ab}/2)/m_a][\partial^2/\partial\upsilon_\alpha\partial\upsilon_\beta(f_a\partial^2 G_b/\partial\upsilon_\alpha\partial\upsilon_\beta)] - 2(1+m_a/m_b)\partial/\partial\upsilon_\alpha(f_a\partial H_b/\partial\upsilon_\alpha)] \qquad [A.1]$$

where the famous Rosenbluth potenials are defined by $G_s(\vec{\upsilon}) = \int d\upsilon'^3\vec{\upsilon}' f_s(\vec{\upsilon}')u$ and $H_s(\vec{\upsilon}) = \int d\upsilon'^3\vec{\upsilon}' f_s(\vec{\upsilon}')/u$ where $u = |\vec{\upsilon} - \vec{\upsilon}'|$. Using the identities $\partial^2 G_s/\partial\upsilon_\alpha\partial\upsilon_\alpha = 2H_s$ and $\partial^2 H_s/\partial\upsilon_\alpha\partial\upsilon_\alpha = -4\pi f_s$, Eq. [A.1] can be written[6] in a form with fewer high derivatives as

$$C_{ab} = L_{ab}[(m_a/m_b)f_a f_b + \mu_{ab}(\partial f_a/\partial\upsilon_\alpha)(\partial\varphi_b/\partial\upsilon_\alpha) - (\partial^2 f_a/\partial\upsilon_\alpha\partial\upsilon_\beta)(\partial^2\psi_b/\partial\upsilon_\alpha\partial\upsilon_\beta)] \qquad [A.2]$$

where $\varphi_b = -H_b/4\pi = (n_b/\upsilon_{thb})\Phi(s_b)$, $\psi_b = -G_b/8\pi = (n_b\upsilon_{thb})\Psi(s_b)$, $\mu_{ab} = m_a/m_b - 1$, and $s_b = \upsilon/\upsilon_{thb} = \sqrt{\gamma_b}\upsilon$. In Guassian units $L_{ab} = 4\pi\gamma_{ab}/m_a = (4\pi)^2(e_a^2 e_b^2/m_a^2)\ln\Lambda$ with $\ln\Lambda$ the Coulomb logarithm. Most importantly there are well know analytic forms for the Rosenbluth potentials: $\Phi(s) = erf(s)/s$ and $\Psi(s) = [s + 1/(2s)]erf(s) + \exp(-s^2)/2\pi$. G corresponds to velocity space diffusion and H corresponds to drag.

A code has been written to test the particle (density) conservation properties of the GRBF collision moments matrix $C_{ab}^{k,l}(\vec{\upsilon}_a)$ defined in Eq. [10]: $C_{ab}(\vec{\upsilon}) = \sum_{k,l} C_{ab}^{k,l}(\vec{\upsilon})w_k^a w_l^b$. The details of the test are formulated here. Following Eq. [1] and expanding Eq. [A.1], we have

$$C_{ab}^{k,l}(\vec{\upsilon}_j) = L_{qb}[(m_a/m_b)F_a^k F_b^l + \mu_{ab}(\partial F_a^k/\partial\upsilon_\alpha)(\partial\varphi_b^l/\partial\upsilon_\alpha) - (\partial^2 F_a^k/\partial\upsilon_\alpha\partial\upsilon_\beta)(\partial^2\psi_b^l/\partial\upsilon_\alpha\partial\upsilon_\beta)] \qquad [A.3]$$

where

$$\begin{aligned}
&F_a^k(s_a^{j,k}) = (\gamma_a/\pi)^{3/2}\exp[-(s_a^{j,k})^2] \\
&\partial F_a^k/\partial\upsilon_\alpha = (s_a^{j,k}/\upsilon_{tha})(\vec{\hat{\upsilon}}_a^j - \vec{\hat{\upsilon}}_a^k)_\alpha[\partial F_a^k(s_a^{j,k})/\partial s_a^{j,k}] \\
&\partial^2 F_a^k/\partial\upsilon_\alpha\partial\upsilon_\beta = \gamma_a\delta_{\alpha\beta}[(1/s_a^{j,k})\partial F_a^k(s_a^{j,k})/\partial s_b^{j,l}] + \\
&\quad \gamma_a[\hat{\upsilon}_a^j - \hat{\upsilon}_a^k]_\alpha[\hat{\upsilon}_a^j - \hat{\upsilon}_a^k]_\beta(1/s_a^{j,k})\partial[(1/s_a^{j,k})\partial F_a^k(s_a^{j,k})/\partial s_a^{j,k})]/\partial s_a^{j,k}
\end{aligned} \qquad [A.4]$$

and where $s_a^{j,k} = |\vec{\hat{\upsilon}}_a^j - \vec{\hat{\upsilon}}_a^k|$. As in the text $\vec{\hat{\upsilon}}_a^k = \vec{\hat{\upsilon}}'^k + \vec{\hat{u}}_{0a}$ with k=1,i_{max} with the velocities normed to υ_{tha}. However $\vec{\hat{\upsilon}}_a^j = \vec{\hat{\upsilon}}''^j + \vec{\hat{u}}_{0a}$ corresponds to a much larger collisional moment integration grid $\vec{\hat{\upsilon}}''^j$ with j=1,j_{max} and centered on the normed colliding species "a" flow velocity $\hat{u}_{0a}$. Similarly

$$
\begin{aligned}
&F_b^l(s_b^{j,l}) = (\gamma_b/\pi)^{3/2}\exp[-(s_b^{j,l})^2] \\
&\partial\varphi_b^l/\partial\upsilon_\alpha = \gamma_b[\hat{\vec{\upsilon}}_a^j(\upsilon_{tha}/\upsilon_{thb}) - \hat{\vec{\upsilon}}_b^l][\partial\Phi(s_b^{j,l})/\partial s_b^{j,l}] \\
&\partial^2\psi_b^l/\partial\upsilon_\alpha\partial\upsilon_\beta = \gamma_b\upsilon_{thb}\delta_{\alpha\beta}[(1/s_b^{j,l})\partial\Psi(s_b^{j,l})/\partial s_b^{j,l}] + \\
&\quad \gamma_b\upsilon_{thb}[\hat{\vec{\upsilon}}_a^j(\upsilon_{tha}/\upsilon_{thb}) - \hat{\vec{\upsilon}}_b^l]_\alpha[\hat{\vec{\upsilon}}_a^j(\upsilon_{tha}/\upsilon_{thb}) - \hat{\vec{\upsilon}}_b^l]_\beta(1/s_b^{j,l})\partial[(1/s_b^{j,l})\partial\Psi(s_b^{j,l})/\partial s_b^{j,l})]/\partial s_b^{j,l}
\end{aligned}
\qquad [A.5]
$$

with $s_b^{j,l} = |\vec{\hat{\upsilon}}_a^j(\upsilon_{tha}/\upsilon_{thb}) - \vec{\hat{\upsilon}}_b^l|$ and as in the text $\vec{\hat{\upsilon}}_b^l = \vec{\hat{\upsilon}}'^l + \hat{u}_{0b}$. Note that unlike $s_a^{j,k}$, the flow velocities in $s_b^{j,l}$: $\vec{u}_{0a}/\upsilon_{thb} - \vec{u}_{0b}/\upsilon_{thb}$ is not zero in general.

The conservation of particles corresponds to $\int d\upsilon_a^3 C_{ab}(\vec{\upsilon}_a) = 0$. The numerical equivalent $\sum_{k,l} w_k^a w_l^b[\sum_j(\Delta\upsilon_a)^3\sum_{k,l}C_{ab}^{k,l}(\vec{\upsilon}_a^j)]$ can not be perfectly zero for an arbitrary set of weights $w_k^a w_l^b$. That would require $[\sum_j(\Delta\upsilon_a)^3\sum_{k,l}C_{ab}^{k,l}(\vec{\upsilon}_a^j)] \equiv 0$ for all k and l. To define an acceptable error, the integral over the first term in Eq. [A.3] C(1)=$[\sum_j(\Delta\upsilon_a)^3\sum_{k,l}{}^{FF}C_{ab}^{k,l}(\vec{\upsilon}_a^j)]$ which is guaranteed positive must sufficiently balance out the integral of the second and third term C(2,3)= $[\sum_j(\Delta\upsilon_a)^3\sum_{k,l}({}^{F\varphi}C_{ab}^{k,l}(\vec{\upsilon}_a^j) + {}^{F\psi}C_{ab}^{k,l}(\vec{\upsilon}_a^j))]$ as a fraction of the first. $E(k,l) = [C(1)+C(2,3)]/C(1)$ is the error in one k,l term. The test code considers the k and l grid to i_{max} = 8 moment or 8 GRBF's with the 8 representative velocities $\vec{\hat{\upsilon}}'^k$ having speeds about equally spaced up to about 2. Results here are rather insensitive to how the representative velocity points are selected. More importantly practicality requires that the integration velocity space be kept reasonably small. A 3D equally spaced cube of grids truncated to a sphere of $\vec{\hat{\upsilon}}''^j$ moment integration points with speeds up to about 6 and total j_{max} of O(1500) was deemed sufficient. It is interesting to compared with the Ref. [4] purely kinetic GRBF test case which found that [O(1500)]-*cubed* velocity point needed for good conservation as compared with the 8x8xO(1500) velocity points in the GRBF moments approach here.

First consider like species collisions where $\mu_{ab} = 0$ and the second " $F\psi$ " term vanishes. The test code demonstrated $C_{aa}^{kk}(\vec{\upsilon}_a^j) = 0$ (within round-off) for all k=l and all j as required: A single shifted Maxwellian annihilates the collision operator. It then follows that on integration the $E(k,k)$ error is zero within round-off. For the general $E(k,l)$ the average $\pm$ deviations from zero is less than 3% i.e. $\sum_{k,l} E(k,l) / \sum_{k,l} 1 \approx 2.7\%$. Of course there no is actual (like-like or like-unlike) collision term in the particle density equation or like-like collision for momentum or energy moments, i.e. non-need be computed. The higher collision moments $\vec{\upsilon}(m\upsilon^2/2)^{1+N}$ for the energy weighted friction $\vec{G}$ [11c] (and so on $(m\upsilon^2/2)(m\upsilon^2/2)^{1+N}$) with N=0,1,2.. are the only places conservation errors in like-like collisions can have any effect. Of course is possible to correct any collision operator for number, momentum and energy conservation. The momentum and energy exchanges Eqs. [11a,11b] depend on collision of unlike species to which we now turn.

For collisions with unlike species as in the "exchange terms", it is important to have the massive and slow species (i.e. ions) to be the colliding thene species "a" so that m_a / m_b is huge (like 60^2) and $\upsilon_{tha} / \upsilon_{thb}$ is very small (like 1/60). When evaluating C_{ie}, the "FF" term then nearly cancels the " $F\phi$ " "drag" term and the " $F\psi$ " "diffusion" term is small. The average deviation from zero $\sum_{k,l} E(k,l) / \sum_{k,l} 1 \approx 0.15\%$ is very small for ions colliding on electrons. In contrast, if the fast species is the colliding species "a" then $\upsilon_{tha} / \upsilon_{thb}$ is very large and hence $s_b^{j,l} = | \hat{\vec{\upsilon}}_a^j (\upsilon_{tha} / \upsilon_{thb}) - \hat{\vec{\upsilon}}_b^l |$ is very large. This makes $F_b^l(s_b^{j,l}) = (\gamma_b / \pi)^{3/2} \exp[-(s_b^{j,l})^2]$ vanishingly small and impossible to distinguish from zero even with double or quadruple precision. While it may seem strange to think of ions colliding on electrons, this is really not a problem for the momentum and energy exchange terms Eqs. [11a,11b], since $F_{Ce} = -F_{Ci}$ and $\Delta_{Ce} = -\Delta_{Ci}$. The problem of small colliding mass first appear in evaluating $\vec{G}_{Ce}$ Eq. [11c] (and higher moments) which require C_{ei} (in addition to C_{ee} for which there is no small mass ratio problem).

It would appear that use of the small mass ratio approximation (Eq. 5.57 p185 of Ref. [1]):

$$C_{ei} \approx (\gamma_{ei} n_i / 2m_e)\partial / \partial \upsilon_\alpha [U^i_{\alpha\beta} \partial f_e / \partial \upsilon_\beta] \qquad \text{[A.6]}$$

where $U^i_{\alpha\beta} = 1/u_i^3[u_i^2\delta_{\alpha\beta} - u_{i\alpha}u_{i\beta}]$ with $\vec{u}_i = \vec{\upsilon} - \vec{V}_i$ is unavoidable. The approximated ion species should be close to a drifted Maxwellian. It known for example that the small mass approximation is not accurate for the energy exchange $\Delta_{Ce} = -\Delta_{Ci}$ (Ref. [1] p. 186). The GRBF C_{ei} form is now linear in the electron weights $C_{ei}(\vec{\upsilon}^j_e) \approx \sum_{k,l} C^k_{ei}(\vec{\upsilon}^j_e) w^e_k$:

$$C^k_{ei}(\vec{\upsilon}^j_e) = (\gamma_{ei} n_i / 2m_e)[(\partial U^i_{\alpha\beta} / \partial \upsilon_\beta)(\partial F^k_e / \partial \upsilon_\alpha) + U^i_{\alpha\beta}(\partial^2 F^k_e / \partial \upsilon_\alpha \partial \upsilon_\beta)] \qquad \text{[A.7a]}$$

where (with "a"="e") $\partial F^k_e / \partial \upsilon_\alpha$ and $\partial^2 F^k_e / \partial \upsilon_\alpha \partial \upsilon_\beta$ are given by Eq. [A.4]. $\vec{\hat{\upsilon}}^j_e = \vec{\hat{\upsilon}}''^j + \vec{\hat{u}}_{0e}$ and

$$U^i_{\alpha\beta} = (1/\upsilon_{the})\{\hat{V}^j_{\alpha\beta} + [\hat{\upsilon}^j_{e\alpha}\hat{V}_{i\beta} + \hat{\upsilon}^j_{e\beta}\hat{V}_{i\alpha} + (\delta_{\alpha\beta} - 3(\hat{\upsilon}^j_{e\alpha}/\hat{\upsilon}^j_e)(\hat{\upsilon}^j_{e\beta}/\hat{\upsilon}^j_e))(\vec{\hat{\upsilon}}^j_e \cdot \vec{\hat{V}}_i)](\upsilon_{thi}/\upsilon_{the})/(\hat{\upsilon}^j_e)^2 \qquad \text{[A.7b]}$$

where $\hat{V}^j_{\alpha\beta} = [(\hat{\upsilon}^j_e)^2\delta_{\alpha\beta} - \hat{\upsilon}^j_{e\alpha}\hat{\upsilon}^j_{e\beta})]/(\hat{\upsilon}^j_e)^3$. After some tedious algebra, it can be shown that

$$\partial U^i_{\alpha\beta} / \partial \upsilon_\beta \equiv 0 \qquad \text{[A.7c]}$$

Note that the ion moments with the corresponding GRBF small mass ratio approximation $C^k_{ie}(\vec{\upsilon}^j_i) = -(\gamma_{ie} n_i / 2m_e) U^e_{\alpha\beta}(\partial^2 F^k_i / \partial \upsilon_\alpha \partial \upsilon_\beta)$ analogous to Eq. [A.7a] (with $U^e_{\alpha\beta}$ analogous to $U^i_{\alpha\beta}$ interchanging "i" for "e" in Eq. A.7b]), might be usefully compared those using the full GRBF given by Eqs. [A.3, A.4, A.5] with "a"="i" and "b=e". This is particularly the case for the friction force F_{Ci} where the small mass ratio approximation may be adequate.

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