EGBTER: Capturing degree distribution, clustering coefficients, and community structure in a single random graph model

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***Abstract*—Random graph models are important constructs for data analytic applications as well as pure mathematical developments, as they provide capabilities for network synthesis and principled analysis. Several models have been developed with the aim of faithfully preserving important graph metrics and substructures. With the goal of capturing degree distribution, clustering coefficient, and communities in a single random graph model, we propose a new model to address shortcomings in a progression of network modeling capabilities. The Block Two-Level Erdős-Rényi (BTER) model (of Seshadhri et al.), designed to allow prescription of expected degree and clustering coefficient distributions, neglects community modeling, while the Generalized BTER (GBTER) model (of Bridges et al.), designed to add community modeling capabilities to BTER, struggles to faithfully represent all three characteristics simultaneously. In this work, we fit BTER and two GBTER configurations to several real-world networks and compare the results with that of our new model, the Extended GBTER (EGBTER) model. Our results support that EBGTER adds a community-modeling flexibility to BTER, while retaining a satisfactory level of accuracy in terms of degree and clustering coefficient. Our insights and empirical testing of previous models as well as the new model are novel contributions to the literature.**

I. Introduction

Graphs provide natural representations of objects and their relationships, and are now a prevalent tool in the applied sciences, including biology, physics, and computer science [1–12]. Random graph models, which prescribe probability distributions over a set of graphs, have become a mainstay for both applied and pure graph theory [13–20]. They give probabilistic machinery admitting principled network simulation capabilities that facilitate statistical analysis in the face of data that is scarce, time-consuming, or impractical to apprehend. Complex networks, especially those arising from natural representations of real world data, exhibit structural idiosyncrasies, which are measured and deduced using many heterogeneous graph metrics, e.g., distributions of degrees, clustering coefficients, and motifs as well as measures of size, centrality, and modularity, to name a few. Consequently, a major focus in random graph models has been on identifying those graph metrics that are important in real-world graph representations and developing models that preserve these metrics [14, 17, 18, 21–30].

This paper identifies and addresses gaps in a progression of graph models presented recently in the research literature. Building on the famous Erdös-Rényi (ER) and Chung-Lu (CL) models, the Block Two-Level Erdös-Rényi (BTER) introduced by Seshadhri et al. (2012) [17] was designed to allow specification of degree distribution and clustering coefficient per degree. For understanding social networks, clustering coefficient is an important metric as it is driven by the nature of real-world relationships. Initial tests show accurate modeling of clustering coefficients per degree as well as the degree distribution by BTER. Further work of Kolda et al. [25] discuss scalable implementations of BTER.

Similarly, understanding community structure (number, size, within- and between-community densities) and membership (who i.e., which nodes, interacts with whom) is important for network analysis, as it shows important relationships and their changes. As BTER does not allow specification of communities, Bridges et al. (2015) introduced a Generalized BTER model, (GBTER) [19, 20] adding user flexibility to prescribe community membership and community edge densities, while retaining the flexibility to also prescribe (most) degree distributions in expectation. As initial efforts with GBTER focused on aiding anomaly detection algorithms (specifically, GBTER's probabilistic formulation allowed p-value computations, which were used for identifying anomalous changes in node degree and community membership in time-varying graphs), GBTER's efficacy in modeling communities while preserving degree or clustering coefficient properties is unknown. In this work we investigate how well GBTER configurations model many real world networks. We identify shortcomings of this model and provide quantitative results and explanations for why the model fails to preserve desired characteristics.

In light of these findings, we introduce an improved model, the Extended GBTER (EGBTER), and exhibit it's modeling capabilities on the same set of networks. An advantage of the GBTER and EGBTER contributions (as well as Stochastic Block Model [16]) is that one can model not just community structure (in terms of number of communities, size and density of each, and interactions between communities) but also specify membership.

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Our contributions provide qualitative comparisons of the BTER model and two configurations of the GBTER model, as well as quantifiable head-to-head results by using three graph metrics that reflect edge, clustering, and community structure statistics on several real-world networks. Our testing gives insight to the modeling capabilities of these relatively new models, and, broadly speaking, shows that BTER often succeeds in faithfully reproducing clustering metrics at the expense of community structure, while GBTER performs conversely.

Informed by our analysis of BTER and GBTER, we introduce the EGBTER model, a natural combination of BTER and GBTER, addressing the limitations of both. More specifically, EGBTER is crafted to circumvent degree distribution problems within user-defined communities that are caused by the simplicity of the ER process used in the GBTER model, but retains the ability to specify community membership and density. We compare EGBTER both qualitatively and quantitatively to the previous two models. Our results suggest that EGBTER provides more balanced modeling of degree distribution, clustering coefficient, and community structure.

We note that many other random graph models seek to preserve similar characteristics. E.g., the Stochastic Block Model [16] seeks to faithfully preserve density between specified nodes. The dK-Orbis models [18] prescribe distribution of edges between d-tuples of node degrees and have exhibited preservation of community structure and degree distribution for large enough d (but do not allow specification of which nodes constitute each community). Comparison of EGBTER against other models besides BTER and GBTER is necessary for greater understanding, but is outside of scope for this work.

II. Notation & Previous Models

We use classical terms and notation except in the case of *clustering coefficient*, where we use *average local clustering coefficient* in place of clustering coefficient of a node. We shall deal exclusively with simple graphs. See Table I for notation and definitions used throughout. We use *density* to refer to the percentage of total possible edges present in a network. A triangle refers to a complete graph on three nodes. In the literature, notions of community and community structure vary. In this work, we use the algorithm of Louvain et al. [31] to produce a partition of V. We refer to each set of nodes in this partition as a community and the partition itself as the community structure.

Next, we describe the previous models, BTER and GBTER. Both are built on combinations of two influential and historical models, ER and CL, which are described in the Appendix.

BTER Model: The BTER model uses a two-step edge insertion process. In the first stage, nodes are partitioned based on the given expected degree sequence, and an ER model is sampled in the subgroups with parameter chosen to preserve the clustering coefficient per degree (CCPD). Next a CL model is applied to ensure the expected degree sequence is attained.

Inputs: (1) Expected degree distribution $\{d\}$, (2) Expected CCPD $\{cc_d\}$.

TABLE I: Notation and metrics used throughout.

Notation	Description
v_i	i^{th} vertex
$G[X]$	vertex induced subgraph of G on $X \subset V$
d_i or $deg_G(v_i)$	degree of v_i in graph G
$\{d\}$	degree distribution
n_d	number of nodes of degree d
$RMSE\{d\}$	root mean squared error of degree distribution
$cc(v_i) = \frac{2L_i}{d_i(d_i-1)}$	v_i local clustering coefficient (CC)[1]
$cc_d = \frac{1}{n_d}\sum_{d_i=d} cc(v_i)$	average local CC for nodes of degree d
$\{cc_d\} = \{(d, cc_d)\}$	CC per degree (CCPD) distribution[2]
$\{cc_{d,C_k}\}$	cc_d of $G[C_k]$
$RMSE\{cc_d\}$	root mean squared error of CCPD distribution
$\{cc_{d,C_k}\}$	$G[C_k]$ CCPD distribution[3]
$CL(w_i, w_j) = \sum_k \frac{w_i w_j}{\sum w_k}$	Chung-Lu probability of adding edge $v_i v_j$
C_k	k^{th} community of G
$Q = \sum_i (e_{ii} - a_i^2)$	modularity[4]
A_L	L^{th} grouping formed by BTER model
$\epsilon_i = \max\{0, d_i - (\lvert C_k\rvert - 1)p_k\}$	$v_i \in C_k$ expected excess degree after ER process[5]
$E_i = D_i - d_i$	v_i expected excess degree after within-C_k CL process

[1] L_i is the number of links among neighbors of v_i. Quantity $cc(v_i)$ is the average probability that two neighbors of v_i are also neighbors of one another.
[2] CCPD distribution is the set of tuples (d, cc_d). If there are no nodes of degree d then $cc_d = 0$.
[3] This is the CCPD distribution of the induced subgraph $G[C_k]$.
[4] Note that e_{ij} is the fraction of total egdes that connect nodes in community C_i to community C_j and $a_i = \sum_j e_{ij}$.
[5] With d_i the expected degree of v_i, ϵ_i is the remaining expected degree of v_i after generating edges from the internal ER process on C_k.

Generative process:

1) Node groupings are implicitly assigned by putting nodes of degree d into groups of size $d+1$ and denoted by A_L.
2) Build an ER graph in each A_L of size $d+1$ nodes with $p_{A_L} = cc_d^{1/3}$.
3) Build a CL graph on the entire network with node weights $w_i := \max\{0, d_i - cc_d^{1/3}(|C_k| - 1)\}$.

See Fig. 1 for a visual depiction of the edge generation process. For a more detailed description, see the works of Kolda et al. and Seshadhri et al. [17, 25].

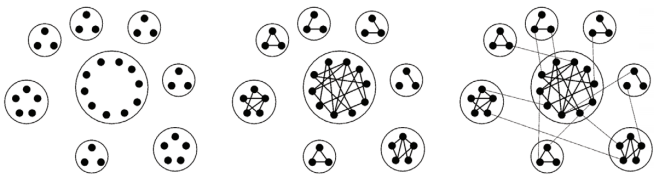


Fig. 1: BTER process depicted, (Left to right) node groupings, within-group ER process, CL process. Figure from Seshadhri et al. [17].

GBTER Model: The Generalized BTER (GBTER) model was introduced by Bridges et al. [19, 20] and arose with a

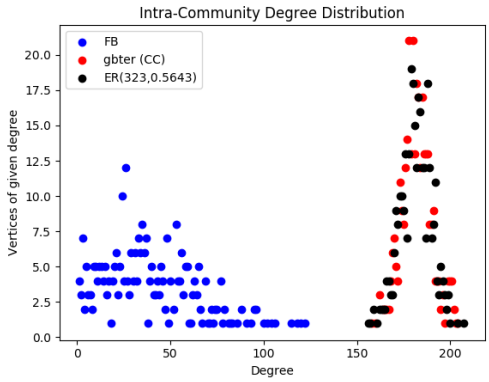


Fig. 2: Particular within-community degree distribution comparison using a Facebook ego network[32], GBTER (modeling within-community CC), and the corresponding ER process from the GBTER within-community simulation.

goal of detecting multi-level anomalies in a sequence of time-varying graphs. It generalizes the BTER model by giving the user flexibility to partition the nodes into groupings (to model communities if desired, and specifically which nodes participate in each community) and to specify the ER-density parameter of each community (e.g., to model within-community density or within-community clustering coefficient). Both are implicitly defined in the first two stages of the BTER generation process, but explicitly prescribed as inputs for GBTER. GBTER then uses a CL model to attain the expected degree sequence. The edge generation below combines both steps (ER then CL) into a single probability for each edge. Setting the GBTER partition and ER density parameter to be that of BTER reduces the GBTER configuration to that of BTER.

Inputs:

1) Expected degree distribution, $\{d_i : v_i \in G\}$
2) A community partition $P = \dot{\bigcup}_k C_k$ (community membership specified)
3) Expected density parameter, p_k for each community C_k (modeling within-community density or within-community CCPD)

Generative process:

1) Each edge is added with probability

$$P(v_i v_j \in E) = \begin{cases} p_k + (1-p_k)CL(\epsilon_i, \epsilon_j), & \text{if} v_i, v_j \in C_k \\ CL(\epsilon_i, \epsilon_j), & \text{otherwise,} \end{cases} \quad (1)$$

where $\epsilon_i = \max\{0, d_i - p_k(|C_k| - 1)\}$.

III. Extended GBTER (EGBTER) Model

With the goal of capturing degree distribution,

CCPD, and community membership and density in a single graph model, we propose the EGBTER model. Unlike the previous two models, which use ER generation within groupings of nodes (to fulfill CCPD or community density expectations) and CL generation across all nodes (to fulfill the expected degree distribution), the EGBTER model exchanges the within-community ER process (used in the GBTER model) for a BTER analogue. The motivation for doing so is to achieve greater overall accuracy by addressing the deficiencies that may occur at the within-community stage. While GBTER allows modeling of dense sub-regions (communities), the ER processes poorly model within-community degree distributions, as well as clustering coefficients; for example, see Figs. 2, 3, 4. The critical insight is that (a) degree distribution and CCPD is important in modeling graphs and these metrics are heavily influenced by edge generation within tight-knit communities, (b) neither are modeled well by ER (and therefore not by GBTER), but (c) both are modeled well by BTER at the graph level; hence, we hypothesize that using a BTER process inside each specified community (and CL process between communities) will give better overall modeling of degrees, CCPD, and community membership and density. Explicit details are given below.

Inputs:

1) A vertex partition $P = \dot{\bigcup}_k C_k$ (community membership specified)
2) Expected within-community degree of each node, d_i
3) Expected global degree of each node, D_i, with $d_i \leq D_i$
4) Within-community CCPD distribution $\{cc_{d,C_k}\}$ for all C_k

Generative process:

1) For each community, C_k, place nodes of global degree D into within-community groupings of size $D+1$.
2) For each within-community grouping, store $p = \sqrt[3]{cc_d}$ for ER process. Here, cc_d is the average CCPD of nodes of degree d, where d is the minimum within-community degree of the grouping.
3) Compute and store the expected within-community and between-community excess degrees, $\epsilon_i = \max\{0, d_i - p_L(D)\}$ and $E_i = D_i - d_i$, respectively. Community density, p_L, is the ER probability, and D is the number of neighbors of v_i in this grouping.
4) Within each community C_k run a BTER process—ER with parameter p_k and CL with node weight ϵ_i for all nodes $i \in C_k$.
5) For all nodes $v_i \in V$, run global CL process with node weight E_i.

IV. Experiment

We compare the BTER, GBTER and EGBTER models on faithful preservation of degree distribution, CCPD distribution community structure for seven real-world networks, all of which are briefly described in the Appendix. For degree distribution and CCPD evaluation, the root mean squared error is computed against the original graph's corresponding feature sequence. For community structure, we compare modularity and present a few telling visualizations of real and simulated graphs. Modularity is a graph metric taking values in $[-0.5, 1]$ with higher scores indicating stronger community structure in relation to the given partition; i.e, high modularity indicates high density of edges within communities and sparse edges across communities. Borrowing vocabulary from the clustering literature [33] modularity is internal validation, i.e., unsupervised, metric. Modularity is computed after designating

the community structure of the simulated network using the algorithm of Louvain et al. [31]. See Table I for metric formulas. For each metric, we macro-averaged results across 100 simulations for each model using the same input parameters.

BTER model generation: For the BTER model, we use the online Matlab implementation [34] provided by BTER authors, Kolda et al. [17, 25, 28]. Step-by-step instructions for fitting BTER to real-world networks are provided online [34]. The supplied code measures/estimates degree and CCPD distributions before sampling edges for the ER and CL processes. We use the default value of any additional parameters. The edge list of each network was exported to Python where graph metrics were computed using the Networkx package [35].

GBTER model generation: By design, GBTER admits ample flexibility of configuration, in particular, in defining communities members and prescribing within-community ER probabilities. Bridges et al. configured the model by learning p_k as within-community density of community C_k, but also mentioned that within-community CC could be modeled by setting $p_k = cc_K^{1/3}$. We configure GBTER with each of the above two possibilities. To partition vertices into communities (for GBTER and EGBTER fitting), community assignments are determined by using the Louvain community detection algorithm [31] in Python [36].

For each real-world network, a Networkx graph object G is created. We find $\{C_k\}$ with Louvain's algorithm, and for each C_k we measure/compute density and average local CC on $G[C_k]$. We also computed $\epsilon_i = \max\{0, d_i - p_k(|C_k| - 1)\}$ for each node v_i. For edge insertion, we iterate over all possible edges and add the edge with probability given by Eqn. 1.

EGBTER model generation: To generate an EGBTER network, we measure inputs from the original network, and for each C_k we generate and store the BTER community groupings, A_L, using global degrees, D_i, to determine A_L. In each A_L we use $p_L = \sqrt[3]{cc_d}$ of the node with lowest within-community degree ($\{d_i\}$) in A_L to compute $\{\epsilon_i\}$ (Refer to Sec III). We then sample edges as done in the scalable BTER model implementation [25].

For each edge, we sample from a trinomial distribution to determine which process generated the edge (ER, within-C_k CL, or global CL process). Weights for each CL process are given by the sum of the excess expected degree sequence divided by total weight for all three processes. In the ER process, sampling gives rise to duplicates so in each grouping A_L, if we wish to add $\binom{|A_L|}{2}p_L$ distinct edges, we must perform $w(A_L) = \binom{|A_L|}{2}\log\{(1-p_L)^{-1}\}$ samples in expectation. See the work of Kolda et al. [25] for a proof. The ER process total weight is $\sum_{A_L} w(A_L)$ divided by total weight across all three processes.

Once we determine which process generated the edge, we proceed to sample edge endpoints according to the corresponding edge criteria for that process. We repeat the weighted calculation above but on the level of A_L, C_k or globally depending on predetermined edge process. For example, suppose in the first step we determine that our edge is generated in some ER process. Next we sample A_L (a group of nodes) with $P(A_L) \propto w(A_L)$. Once A_L is determined, we sample two endpoints, without replacement, uniformly from $\{v_i \in A_L\}$ since the ER process has uniform edge insertion probability.

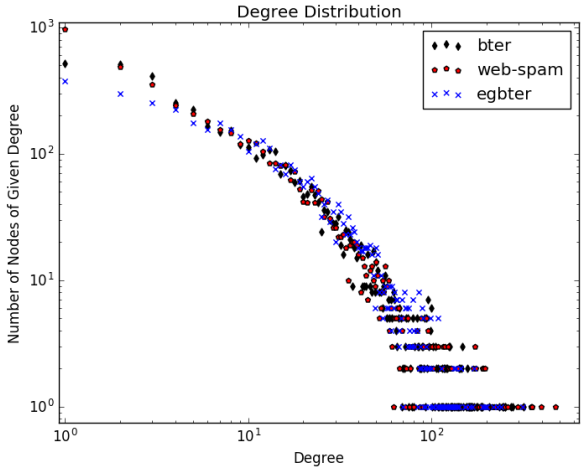
TABLE II: Color hierarchy (best to worst) is blue, light blue, light red, red, for each network & metric. Across all networks, BTER is usually first or second for degree & CC distributions, yet usually poor in modularity; EGBTER is usually best or second in all three metrics. All metrics are macro-averaged across 100 random realizations for each model.

	Model	$RMSE\{d\}$	Q	$RMSE\{cc_d\}$
bio-dmela	true	NA	0.4530	NA
	bter	59.3526	0.3445	0.0113
	gbter	112.1429	0.4335	0.0105
	gbter^{CC}	127.8613	0.7087	0.0118
	egbter	61.7373	0.4620	0.0181
facebook	true	NA	0.8350	NA
	bter	2.4670	0.7160	0.0973
	gbter	10.2906	0.6648	0.2021
	gbter^{CC}	16.7792	0.8728	0.2617
	egbter	3.9274	0.8448	0.1748
bn-fly	true	NA	0.4188	NA
	bter	5.1272	0.5565	0.0358
	gbter	15.7820	0.4497	0.0464
	gbter^{CC}	18.4049	0.6520	0.0799
	egbter	7.4551	0.5149	0.0307
ca-GrQc	true	NA	0.8628	NA
	bter	75.9245	0.8093	0.3202
	gbter	84.0651	0.6567	0.4004
	gbter^{CC}	111.2112	0.9237	0.4688
	egbter	82.4652	0.8617	0.3476
inf-power	true	NA	0.9357	NA
	bter	151.1312	0.7322	0.0461
	gbter	184.9929	0.7711	0.0755
	gbter^{CC}	236.2863	0.9266	0.1990
	egbter	190.5787	0.9399	0.0346
web-spam	true	NA	0.5002	NA
	bter	21.3190	0.5310	0.0590
	gbter	48.7568	0.4529	0.0984
	gbter^{CC}	57.9175	0.7656	0.2235
	egbter	29.1909	0.5693	0.0566
webbase-2001	true	NA	0.9354	NA
	bter	126.9450	0.7437	0.0494
	gbter	178.8203	0.7868	0.0600
	gbter^{CC}	225.2118	0.4467	0.2136
	egbter	134.7270	0.9367	0.0340

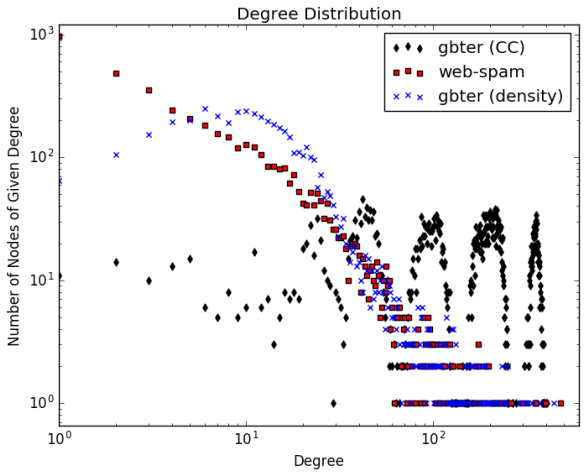
V. Results

Table II summarizes the numerical output of our simulations.

Degree Distribution Results: We see that the BTER model performs the best in $RMSE\{d\}$ across all seven networks and the EGBTER model performs second best in six of seven networks. Configuring GBTER to model within-community CC in each community produces the worst error across all networks. This is because for the networks used, within-community CC exceeded within-community density, causing

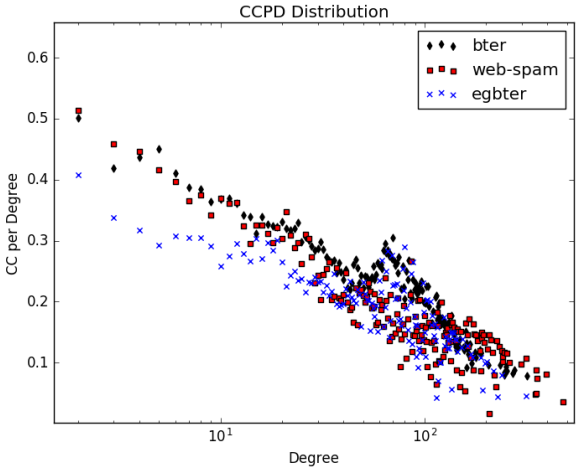


(a) Comparison of web-spam [37], BTER and EGBTER degree distributions.

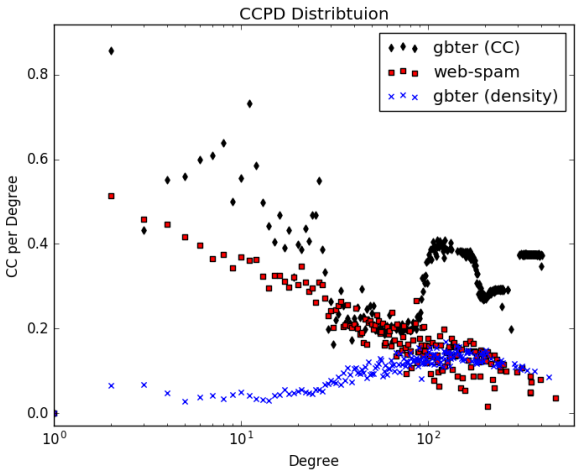


(b) Comparison of web-spam and GBTER models degree distributions.

Fig. 3: Note the similar fit of the BTER and EGBTER model in matching the degree distribution for this particular network. GBTER (CC) model's degree distribution is dominated by within-community ER models causing unrealistic fit.



(a) Comparison of web-spam [37], BTER and EGBTER CCPD distributions.



(b) Comparison of web-spam and GBTER models' CCPD distributions.

Fig. 4: Visual depiction of web-spam [37] degree distribution vs. simulated analogs.

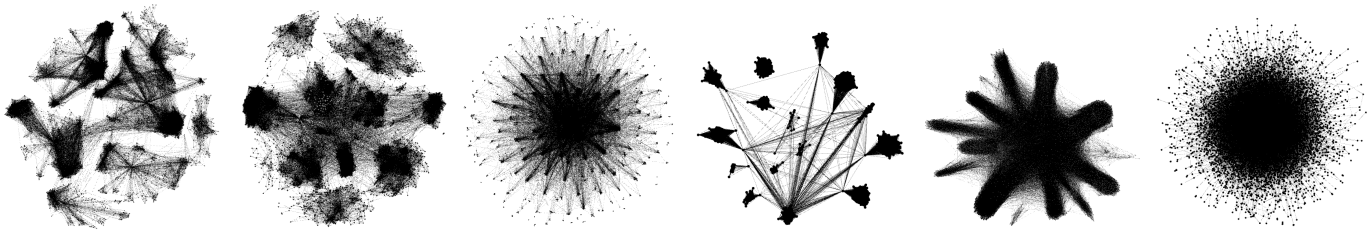
the model to overstate the number of edges in each community in the ER process. When this happens, the local surplus in each community translates to a global degree distribution that is a superposition of the overstated ER degree distributions from each of the communities. This arise from the implicit assumption in GBTER that communities are well-modeled internally by an ER model. See Fig. 2 and Fig. 3b for visualizations of GBTER degree distributions when within-community CC exceeds within-community density.

Clustering Coefficient per Degree Distribution Results: EGBTER does the best in four of seven networks on minimizing $RMSE\{cc_d\}$ (global CCPD distribution). This is to be expected as we incorporate an analog of the BTER process on the community level to adhere to the within-community CCPD distribution. Despite incorporating community structure into our model, we gain a slight advantage in $\{cc_d\}$. This is not quite clear from Fig. 4a, however is evident from the numerical computation (Table II).

The deficiency of GBTER in modeling $\{cc_d\}$ can be seen in Fig. 4b. The CCPD of high degree nodes is grossly overstated. As with the previously observed edge surplus (Fig. 3b), we attribute this to the ER process used within communities in the GBTER model.

Community Structure Results:
On modularity, EGBTER performed the best in five of seven networks with GBTER (modeling within-community density) coming in second. This indicates that EGBTER does better in preserving the overall level of community structure in the network than the other models.

Note that when modeling within-community CC, GBTER tends to overstate the overall level of network division while modeling within-community density tends to have the op-



(a) Original FB network. (b) EGBTER FB simulation. (c) BTER FB simulation. (d) GBTER FB simulation (fitting model to within-community CC). (e) GBTER FB simulation (fitting model to within-community density). (f) bio-dmela BTER visualization.

Fig. 5: Visualizations of FB network and simulated networks according to model. Following quantitative modularity results, EGBTER qualitatively simulates community structure best. We also include a BTER visualization of a biological protein interaction network (f), as BTER tended to exhibit either a star pattern (c) or a dense core (f). All visualizations were produced in Gephi [38].

posite effect. This agrees with previous discussion on edge metrics where we saw that the degree distribution was a superposition of ER degree distributions from each community when modeling within-community CC. See Fig. 5 for network visualizations.

The BTER model does not perform as well as EGBTER and GBTER in preserving modularity, only performing better than the GBTER model when modeling within-community CC. This is unsurprising as it does not allow user prescription of community structure. Visualizations (using the software Gephi [38]) of the outputs generated by the BTER model tend to exhibit an organized star-shaped collection of dense ER graphs together with, in some cases, a large dense core (see Fig. 5c and Fig. 5f). The visualization BTER produced depended heavily on which subprocess generated the majority of edges in the underlying simulated network. The star-shaped collection arose when the ER process was the dominant edge generation process, while the dense core arose when the CL model was the dominant edge generation process. Each distinct type of visualization agrees with previous work on both the BTER model [17] and on average distances in CL graphs [14, 39].

VI. Conclusion

The EGBTER model presented here showed strong overall performance across metrics measuring degree distribution, clustering coefficients, and community structure. The BTER model uses an automated grouping process that does well to match both degree and CCPD distribution but fails to preserve community structure at the coarsest scale. This problem is addressed in the GBTER model which imposes community membership and size. However, this comes at a cost to CCPD, as was found to be the case in both configurations of GBTER tested here.

This work was concerned with simultaneous faithful preservation of particular graph characteristics. Performance and scalability are out of the scope of our consideration. Testing showed that in networks with high modularity, internal and global degree and clustering coefficients were nearly identical, except for high degree nodes. Recent work [28, 40] has shown CCPD distribution can be estimated via sampling and scales to large networks. Future research is needed to uncover explicit relationships and network properties that would allow for autonomous and scalable generation of EGBTER graphs without the direct reliance on the measured inputs from a seed graph. Overall, we hope our contributions will enhance the modeling and analysis capabilities graph theory brings to the many diverse applications.

Appendix (Supplementary Material)

Here we describe the metrics used to describe how well different graph models preserve important network characteristics. The three metrics we use are root mean squared error (RMSE) of both degree distribution and CCPD distribution, as well as modularity as a measure of community structure.

Degree Distribution: To measure how well the degree distribution of a simulated network fits the degree distribution of the original network, we use the root mean squared error (RMSE) applied to the simulated degree distribution using the degree distribution of the original network as a baseline.
CCPD Distribution: We apply the RMSE of the CCPD distribution in the same fashion as above. This measure implicitly depends on the set of all degrees produced in the simulated network as we measure the error from the set of ordered pairs $\{(d, cc_d)\}$ in the simulated network to the original network.
Modularity: To evaluate how well a particular vertex clustering (or in our case a partition) captures the community structure of a network, we shall primarily use the modularity metric formulated by Newman [41]. Modularity, denoted Q, is given by,

$$Q = \sum_i e_{ii} - a_i^2, \tag{2}$$

where e_{ij} is the fraction of total edges that connect nodes in community C_i to community C_j and $a_i = \sum_j e_{ij}$. We use this metric because it considers the vertex partition as a whole and allows us to compare different partitions of the same network. Modularity gives a sense of the overall level of community structure present in a network relative to the given partition. This is not to say that higher modularity means better

community structure but that higher modularity indicates more overall division into communities with respect to the partition used.

ER Model: The Erdős-Rényi (ER) model [42] was one of the first random graph models. The ER model takes as input, the desired number of nodes n and an edge insertion probability p. Then a graph on n nodes is generated by inserting each of the $\binom{n}{2}$ possible edges with probability p. This graph is denoted as $ER(n, p)$. As a result of this simple insertion process, in expectation we have network density, cc_{v_i} and CCPD equal to p. We also have $\binom{n}{2}p$ edges in expectation and

$$P(d_i = k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}. \tag{3}$$

This produces a roughly Gaussian shaped degree distribution, counter to that observed in real world networks [22].

CL Model: The Chung-Lu (CL) model [43] arose as an attempt to model graphs with a power law degree distribution. In the CL model, each node v_i is assigned a weight, w_i. The CL model takes as input, the list of corresponding vertex weights $\{w_i\}$. The probability of inserting edge $v_i v_j$ is given by

$$P(v_i v_j \in E) = \frac{w_i w_j}{\sum_{v_k \in V} w_k}, \tag{4}$$

which we shall denote by $CL(w_i, w_j)$. For this probability to be well defined we require that $w_i w_j \leq \sum_{v_k \in V} w_k$ for all $v_i, v_j \in V$. It is possible that the RHS of Eqn. 4 is greater than 1. For our purposes, if this occurs we set $P(v_i v_j \in E) = 1$. A common adaptation is to use $w_i = d_i$ for all $v_i \in V$. When $w_i = d_i$, this is referred to as a null model. CL generalizes ER and retains independent edge insertion while being able to match d_i in expectation. However, this process rarely closes triangles [25] and the average distance is relatively small [14, 39]. This makes this model a bad fit for CC and community structure but laid a foundation for capturing network degree distribution in expectation. See Fig. 5f for BTER visualization of a biological network. This visualization looks very similar to those of CL models.

Real World Networks Description: All networks used in this work for model evaluation can be found online either from the Network Repository [44] or from the Stanford Network Analysis Project (SNAP) [45]. Two networks come from biology (a protein interaction network [46] and a fly brain network), two are social networks (a Facebook friend network [32] and physics collaboration network [47]), two are web networks [37, 48, 49] and one is an infrastructure network [15]. Only the two social networks come from SNAP.

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