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The Cross Spectrum in Multiple Input Multiple Response Vibration Testing

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Abstract

Random vibration tests have been conducted for over 5 decades using vibration machines which excite a test item in uniaxial motion. With the advent of multi shaker test systems, excitation in multiple axes and/or at multiple locations is feasible. For random vibration testing, both the auto spectrum of the individual controls and the cross spectrum, which defines the relationship between the controls, define the test environment. This is a striking contrast to uniaxial testing where only the control auto spectrum is defined.

In a vibration test the energy flow proceeds from drive excitation voltages to control acceleration auto and cross spectral densities and finally, to response auto and cross spectral densities. This paper examines these relationships, which are encoded in the frequency response function. Following the presentation of a complete system diagram, examination of the relationships between the excitation and control spectral density matrices is clarified. It is generally assumed that the control auto spectra are known from field measurements, but the control cross spectra may be unknown or uncertain. Given these constraints, control algorithms often prioritize replication of the field auto spectrum. The system dynamics determine the cross spectrum. The Nearly Independent Drive Algorithm, described herein, is one approach.

A further issue in Multi Input Multi Response testing is the link between cross spectrum at one set of locations and auto spectra at a second set of locations. The effect of excitation cross spectra on control auto spectra is one important case, encountered in every test. The effect of control cross spectra on response auto spectra is important since we may desire to adjust control cross spectra to achieve some desired response auto spectra. The relationships between cross spectra at one set of locations and auto spectra at another set of locations is examined with the goal of elucidating the advantages and limitations of using control cross spectra to define response auto spectra.

Keywords

Control spectra; cross spectral density; multi-axis vibration, multi-input multi-output (MIMO) control

Introduction

Most environmental vibration tests are conducted using a single vibration machine which excites the test item in a single direction. Three sequential tests, one in each of the X, Y, and Z axes are considered sufficient to ensure that a test item will survive the relevant field vibration environment. Test requirements, derived from field vibration data, are used to derive the test specifications. In the context of random vibration these specifications are typically auto spectral densities. Auto spectral densities define the squared magnitude of the vibration level (usually accelerations) as a function of frequency. This methodology is mature, a product of refinement over the last 70 years. It has always been recognized that there are significant limitations to the "single axis" excitation approach. Field environments inherently excite a test body in multiple axes simultaneously and, in addition, field environments sometimes involve multiple, simultaneously active, vibration sources.

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Multiple degrees of freedom and the multiple source issues encouraged development of vibration systems using multiple actuators to simultaneously excite a test item. Methodologies developed over the past three decades have matured to the point where multiple excitation, multiple control point vibration tests are readily conducted [1][2][3]. These tests utilize either a "rigid body" fixture, like the Team Corporation "Tensor" systems [1], or a distributed set of vibration machines [4]. The rigid fixture approach seeks to excite the test item simultaneously in three to six degrees of freedom, while the distributed exciter approach seeks to meet the multiple vibration source requirement in addition to exciting the body in at least several degrees of freedom.

Random vibration test control requires the replication of a specified auto spectral densities at one or more points on the test body. These points are referred to as "control locations". Random voltages applied to vibration exciters are acted on by the exciter, fixture, and system dynamics to produce the control auto spectra. For a single input, single control point test, the auto spectrum is all that is required. [3] Multiple Input-Multiple Response Tests have a set of auto spectra, each defining the magnitude profile of a response as a function of frequency at a specific location and/or degree of freedom. With multiple control auto spectra, the relationships between the control auto spectra, called cross spectra, are of vital importance. These relationships are defined by the relevant cross spectral densities. The result is a spectral density matrix, whose diagonal terms are auto spectra and whose cross terms are the cross spectra. The number of cross spectra terms grows rapidly with the number of control points. For two controls there is a single cross spectrum. For three control locations there are three cross spectra (1-2, 1-3, and 2-3). In general, for n control locations, there are $(n^2 - n)/2$ independent cross spectra.

When the voltage excitation signals are independent random processes, the control cross spectra are created by the system dynamics. The relevant dynamics includes the vibration machines, fixtures, and test item. By artificially relating the voltage excitation signals new, "artificial" dynamics are introduced, and these dynamics interact with the system dynamics to produce a modified test environment. Different excitation cross spectra can thus expose the test item to profoundly different environments.

At present, the typical single control axis vibration test is conducted using an auto spectrum composed of straight line segments on a log amplitude-log frequency scale. Multiple axis- multiple control location tests might be able to use this type of specification for the auto spectra, but it may not be practical for cross spectra specifications. The problem has been clearly elucidated by Daborn [4]. As we proceed through this paper some of the issues related to specification of cross spectra will be clarified.

The development of a reasonable laboratory replication of the field environment must deal with a set of issues, the primary ones including:

1. Test item boundary conditions usually differ between field and laboratory.
2. The measured field environment is known through a limited set of measurements and will differ between similar field events.
3. The field and laboratory test items exhibit different, though similar, dynamics.

All of these issues influence the environmental specification process and the definition of auto and cross spectral densities.

This document systematically examines issues related to the transfer of energy from independent excitations, to coupled excitations (drives), thence to control locations and ultimately to response locations. The objective is an understanding of the relationships between excitations, drives, controls, and responses. The drives interact with the drive to control frequency response function (FRF) to produce the control spectra. In a similar manner, the control spectra react with the control to response FRF to produce the response spectra. Moving from the domain of spectra to that of spectral density, the interactions between controls are defined by the control cross spectral densities. In the same manner, the interactions between responses are described by response cross spectral densities.

Section 2 examines how independent or coupled drives interact with the Frequency Response Function to produce the control spectra and the control spectral density. A central issue here is the relationships between the excitations and the frequency response function. Together these generate the control spectra and the control spectral density matrix.

There are no inherent mathematical constraints on the auto spectral density. At each frequency and each control location the ASD is simply a real number. The situation is different with the cross spectral densities. In the first place, the magnitudes of

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[3] Random vibration tests are conducted using vibration with a Gaussian Probability Density. These signals are readily generated given a required auto spectral density.

the cross spectral terms are bounded by the relevant auto spectra. Further (as if this wasn't already enough), all physically realizable spectral density matrices are positive definite. This positive definite character is examined in Section 3.

Specifying the cross spectra for a laboratory vibration test is a significant issue. Cross spectral magnitudes and phases quantify the relative motions between control locations. When the excitation platform is a rigid body like the Team Tensor Table [1], the magnitudes and phases of the cross spectrum couple motions in each degree of freedom. For example, motion in the x direction could be independent of, or correlated with, motion in y. The dynamics of the rigid table does not "favor" one cross spectral configuration over another. The situation is quite different when motions of a dynamically active test item are considered. In the field the test item moves in a pattern defined by the interactions of the field excitation, the boundary conditions, and the linear and nonlinear dynamics of the complete assembly. Points on the test body move in unison, with phase and magnitude relationships defined by the dynamics, at system modal frequencies. Reproducing this motion in the laboratory, requires that the dynamics of the laboratory test body play a significant role in the definition of the cross spectrum.

In a manner, analogous to the excitation-control relationship, the controls react with the system FRF to produce the response spectra and the corresponding response auto and cross spectra. Like the excitation to control situation, the control to response relationship involves both the coupling between the controls (control cross spectra) and the coupling encoded in the FRF.

The above relationships, while mathematically reasonably straightforward, can be confusing. Herein the following issues are examined:

1. The equations relating excitation spectra to the control spectral density matrix and then, the equations relating control to response spectral densities.
2. To be physically realizable, a spectral density matrix must be positive definite.
3. Implications of strongly coupled controls (high cross spectral relationships).
4. The Nearly Independent Drive Algorithm.
5. When Independent Drives Fail.

Spectra, Spectral Density, and Frequency Response for MIMO Systems

Consider the multiple-excitation multiple response system depicted in Figure 1. N vibration machines excite a test item and associated fixture. The response is controlled at M locations on the item.

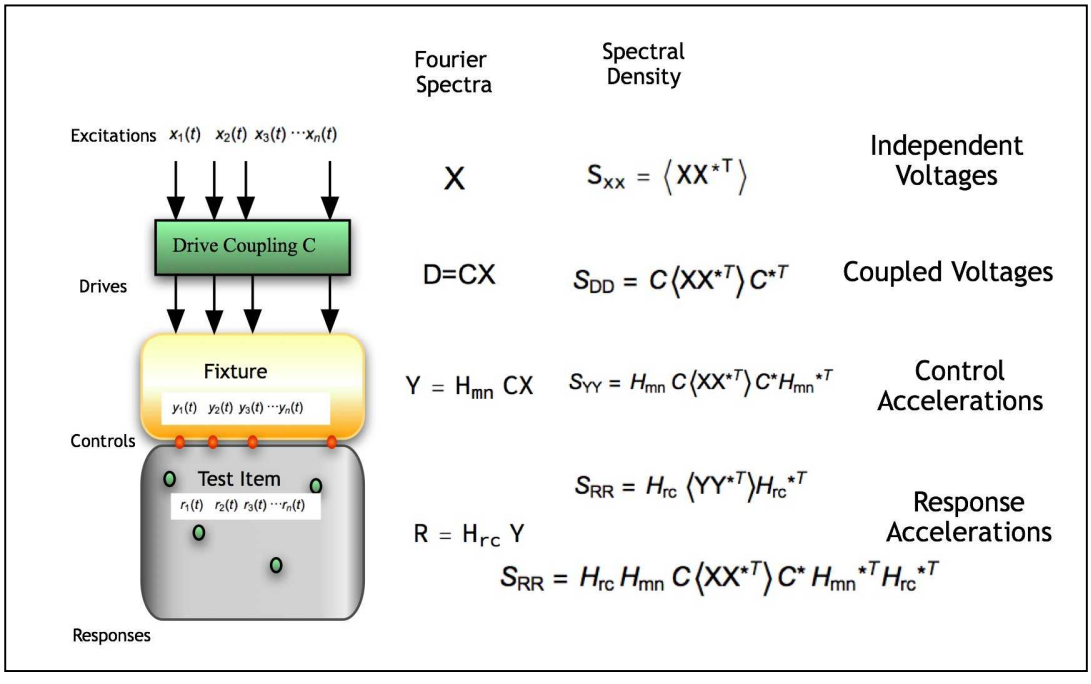

Figure 1 A General Multiple Input Multiple Response System Illustrating the Path from Excitations X to Responses R

Figure 1 illustrates the complete path from excitation spectra to response spectral densities. N drive voltages, each representing a band limited random process, excite the system. The corresponding time domain voltages are $x_i(t)$ and the Fourier Spectra are $X_i(t)$. Input excitations are coupled through a coupling matrix C and these coupled drives CX are applied to the power

amplifier driving the vibration machines. For a linear system, the frequency response function H_{mn} maps the frequency domain drive voltages to the frequency domain control accelerations $Y(f)$. The frequency response function H_{rc} maps the control accelerations $Y(f)$ to the Response Accelerations $R(f)$. The equations are shown, in both the spectral and spectral density domains, in Figure 1. The vector of excitation spectra is:

$$X = \{X_1, X_2, \cdots, X_N\} \tag{1}$$

Using Expected Values, the corresponding Diagonal Excitation Spectral Density Matrix is:

$$\boldsymbol{S}_{xx} = \langle X^H X \rangle \tag{2}$$

where $\langle \quad \rangle$ indicates expected values and the ^{H} is the conjugate transpose (Hermitian).

The coupling matrix $\boldsymbol{C}$ allows linear combinations of the drives to be used as excitations.

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ c_{12} & c_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{3}$$

The frequency response or transfer function matrices couple the drives $\boldsymbol{C}X$ to the controls $\boldsymbol{Y}$.

$$\boldsymbol{Y} = \boldsymbol{H}_{mn}\boldsymbol{C}X \tag{4}$$

Or, in the format of spectral densities

$$\boldsymbol{S}_{yy} = \boldsymbol{H}_{mn}\boldsymbol{C}\boldsymbol{S}_{xx}\boldsymbol{C}^H\boldsymbol{H}_{mn}^H \tag{5}$$

In a similar manner, the controls are coupled to the responses.

$$\boldsymbol{S}_{rr} = \boldsymbol{H}_{rc}\boldsymbol{S}_{yy}\boldsymbol{H}_{rc}^H \tag{6}$$

Equations (1) - (6) define the spectral and spectral density relationships linking excitations, drives, controls, and responses. In a typical test, at least the diagonal of the control spectra, i.e., the auto spectra at the control locations, is defined based on field data. The control cross spectra may or may not be defined from field data.

Properties of the Spectral Density Matrix

Spectral density matrices are formed from the expected value of a vector or matrix product of the form:

$$\boldsymbol{S}_{rr} = \langle \boldsymbol{A}^H \boldsymbol{A} \rangle \tag{7}$$

This matrix product has several important properties. First, $\boldsymbol{S}_{rr}$ is symmetric. Second, $\boldsymbol{S}_{rr}$ is positive definite. For a positive definite matrix $\boldsymbol{A}$ the product:

$$X^H \boldsymbol{A} X \geq 0 \tag{8}$$

For any vector X. If, for some X the equality is true, the matrix $\boldsymbol{A}$ is positive semi-definite. Properties of positive definite matrices include:

1. All eigenvalues of a positive definite matrix are positive.
2. All pivots of a positive definite matrix are positive.
3. All leading minors of a positive definite matrix are positive.
4. If $\boldsymbol{A} \in n \times n$ is positive definite and $X \in \ n \times k$ has rank k then $X^H \boldsymbol{A} X$ is also positive definite.
5. For symmetric positive definite matrices, like the spectral density matrix, the eigenvectors are orthogonal, and the eigenvalues are real.

The diagonal terms of $\boldsymbol{A}^H\boldsymbol{A}$ are the squared magnitudes of the corresponding rows (or terms, if $\boldsymbol{A}$ is a vector) of $\boldsymbol{A}$. The off diagonal terms quantify the degree to which one row (or term) of $\boldsymbol{A}$ resembles another. The cross, or off diagonal terms, as dot products have magnitudes bounded by:

$$a_{ij} = \sqrt{|a_{ii}||a_{jj}|} \tag{9}$$

In the spectral density context, $\boldsymbol{A}$ is a complex matrix, and each off diagonal term has a real and imaginary part or, equivalently, a magnitude and phase. The off diagonal terms of $\boldsymbol{A}$ are of the form:

$$a_{ij} = e^{i\theta_{ij}} \sqrt{\gamma^2 |a_{ii}||a_{jj}|} \tag{10}$$

where γ^2 is the ordinary "coherence" between row i and row j and its maximum value is unity. θ_{ij} is the angle between the complex quantities in rows i and j.

If the candidate control spectral density matrix $\boldsymbol{S}_{yy}$ is created from some compilation of measured field vibration data, there is no a priori assurance that the construction is positive definite. Of course, if the field measurement itself is either a set of Fourier spectra or a spectral density matrix then the requirement is satisfied. But field data are often noisy, modified, or enveloped. In these cases, it is easy to violate the positive definite requirement.

Given a non-positive definite symmetric candidate matrix a "hopefully nearby" positive definite matrix can be constructed by first performing an eigenvalue decomposition, setting the negative (hopefully only one small) eigenvalue to a small positive number, and reconstructing the matrix. The new matrix is positive definite by construction and is in some sense the "closest" positive definite matrix to the candidate. Other approaches are also possible, Smallwood [5] recommends decreasing the coherence in the cross terms by a few percent until the result gives a positive definite matrix.

General Approaches to Specification of Cross Spectral Density

As noted above, the proposed Control Spectral Density Matrix must be symmetric, positive definite, with off diagonal terms whose magnitudes are bounded by Equation (10). Ideally, the CSD (control spectral density) matrix could be entirely derived from field data, but there are significant issues.

Field data are often limited to estimated auto spectra at a set of predetermined measurement locations. Relative phase between locations may or may not be available. For single input tests the auto spectral control (perhaps with limiting) is specified for nearly all random vibration tests. It is natural to extend this procedure to MIMO tests by specifying the auto spectrum at $3 \leq n \leq 6$ control locations. If good field measurements of cross spectra are available this argument can be extended to specification of the cross spectra, but there are deeper issues which can make this difficult to implement in practice. Typically, auto spectra are specified as a suite of straight line values on a log-log plot. These straight lines in some sense "envelope" the field data and also, optimistically, attempt to allow some leeway to account for different dynamics of the field and laboratory test systems. The combined effects of this enveloping and a probable difference between laboratory and field boundary conditions forces the laboratory control force to unrealistic levels at test item resonant frequencies because the vibration machine, unlike most field structures, is capable of imparting very high force levels even when the mechanical impedance of the test item is low. To mitigate this issue, response auto spectra are limited at various judiciously selected locations.

Extending vibration testing to multiple degrees of freedom exacerbates this problem. There are significant differences in the dynamics of nominally identical test items, and the test items are subjected to very different boundary conditions in the field and laboratory. Additionally, of course, nothing in the field produces straight line spectra. Both Daborn [4] and Smallwood [5], have addressed this issue, albeit from somewhat different viewpoints.

There is no universal way of addressing these issues. Their resolution depends on details of the specific system. However, there are some viable general approaches. These include:

1. Control the auto spectra at designated control locations and let the system dynamics determine the cross spectra. The Nearly Independent Drive Algorithm, described in more detail below, is one embodiment of this approach.
2. Control the auto spectra at designated control locations and adjust the cross spectra for minimum excitation voltages. This is Smallwood's minimum drive algorithm [5]. In practice, minimum drive and nearly independent drives achieve similar results.
3. Control both auto and cross spectra to measured field levels. This is best achieved in situations where motion, at the control locations, in each degree of freedom, are nearly independent, and implies low cross spectral coherence. This is approximately true for some component configurations. Daborn [4] describes successful application of this technique in some example cases.
4. Control auto spectra and cross spectra coherence (no phase control) to field levels. In this framework, the control system works to achieve the desired coherence between controls and allows the phase to be determined by the natural system dynamics. This has been achieved recently with some success, although not all control systems have this capability.
5. Force all control cross spectra to be near zero.

6. Don't control cross spectra directly, rather, specify, for say, n drives, auto spectral control at more than n control locations. The cross spectra "adapt" to produce the specified auto spectra. Daborn [4] also describes an application of this approach. Since auto spectra are often known at a fair suite of field locations and auto spectral control avoids the rather messy difficulty of specifying cross spectral phase, this is an attractive approach, though it is somewhat limited as noted below.

The Nearly Independent Drive Algorithm and the effect of cross spectra on response auto spectra are now examined in more detail.

The Nearly Independent Drive Algorithm

Consider a MIMO system described by $\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X}$. The auto spectral terms in $\boldsymbol{S}_{yy}$ are specified, but not the cross spectra. Excite the system with independent drives. The response auto spectrum is the sum of the contributions from each of the independent drives conditioned by the appropriate term in $\boldsymbol{H}$.

$$\begin{aligned} \boldsymbol{S}_{yyd} &= \boldsymbol{H}_d \boldsymbol{H}_d^H \boldsymbol{S}_{xxd} \\ \boldsymbol{S}_{xxd} &= (\boldsymbol{H}_d \boldsymbol{H}_d^H)^{-1} \boldsymbol{S}_{yyd} \end{aligned} \tag{11}$$

Where the d subscript indicates that we are dealing only with the diagonal terms. Using Equation (11), solve for $\boldsymbol{S}_{xxd}$. If all of the $\boldsymbol{S}_{xxd}$ are positive, then the independent drive magnitudes are simply the square root of the respective diagonal auto spectra in $\boldsymbol{S}_{xxd}$. However, some $\boldsymbol{S}_{xxd}$ terms may be negative. This means that the requested auto spectra $\boldsymbol{S}_{yyd}$ cannot be achieved with independent drives. In essence too much interaction between drives is encoded in $\boldsymbol{H}$ for fully independent drives to work. Set the drive corresponding to negative auto spectrum to zero and compute a new trial $\boldsymbol{S}_{yy0}$ from

$$\begin{aligned} \boldsymbol{Y}_0 &= \boldsymbol{H} X_{d0} \\ \boldsymbol{S}_{yy0} &= \boldsymbol{Y}_0 \boldsymbol{Y}_0^H \end{aligned} \tag{12}$$

Where X_{d0} are the independent drives corresponding to $\boldsymbol{S}_{xxd}$ with the negative drive values set to zero. Some of the diagonal values in $\boldsymbol{S}_{yy0}$ will be too large. Scale these values using a diagonal matrix of real valued scaling constants $\boldsymbol{S}_c$

$$\boldsymbol{S}_c = \sqrt{\frac{\boldsymbol{S}_{yy0}}{\boldsymbol{S}_{yyd}}} \tag{13}$$

Such that

$$\widehat{\boldsymbol{Y}} = \boldsymbol{S}_c \boldsymbol{Y}_0 \tag{14}$$

The new response spectral density

$$\begin{aligned} \boldsymbol{S}_{yyc} &= \widehat{\boldsymbol{Y}} \boldsymbol{S}_c^2 \widehat{\boldsymbol{Y}}^H \\ X_d &= \boldsymbol{H}^{-1} \boldsymbol{S}_c \widehat{\boldsymbol{Y}} \end{aligned} \tag{15}$$

has the desired auto spectral values in $\boldsymbol{S}_{yyc}$ and the drives X_d are coupled. Either the inverse of $\boldsymbol{H}$ or the pseudo inverse are used in Equation (15) as appropriate. This process from Equations (11) through (15) couples the drives "just enough" to achieve the desired $\boldsymbol{S}_{yyd}$.

An Example of Nearly Independent Drives in a 3x3 System

Consider a three by three system described by the transfer function matrix in Equation (16).

$$\boldsymbol{H} = \begin{bmatrix} e^{i\theta_{11}} h_{11} & e^{i\theta_{12}} h_{12} & e^{i\theta_{13}} h_{13} \\ e^{i\theta_{21}} h_{21} & e^{i\theta_{22}} h_{22} & e^{i\theta_{23}} h_{23} \\ e^{i\theta_{31}} h_{31} & e^{i\theta_{32}} h_{32} & e^{i\theta_{33}} h_{33} \end{bmatrix} \tag{16}$$

Now, if the cross terms $h_{ij}\ \forall\ i \neq j$ are small relative to the diagonal terms then fully independent drives work. Consider the case in which a single cross term, say h_{32}, is significant. The revised $\boldsymbol{H}$ matrix is:

$$\boldsymbol{H} = \begin{bmatrix} e^{i\theta_{11}} h_{11} & 0 & 0 \\ 0 & e^{i\theta_{22}} h_{22} & 0 \\ 0 & e^{i\theta_{32}} h_{32} & e^{i\theta_{33}} h_{33} \end{bmatrix} \tag{17}$$

Application of the nearly independent drive algorithm yields drives:

$$\boldsymbol{X}_d = \begin{bmatrix} \frac{\sqrt{S_{yy1}}}{h_{11}} \\ \frac{\sqrt{S_{yy2}}}{h_{22}} \\ -\frac{e^{i\theta_{32}-i\theta_{33}}h_{32}\sqrt{S_{yy2}}}{h_{22}h_{33}} + \frac{e^{i\theta_{32}-i\theta_{33}}\sqrt{S_{yy3}}}{h_{33}} \end{bmatrix} \tag{18}$$

Drives one and two, X_{d1} and X_{d2}, are independent. Drive three, X_{d3}, is coupled to drive two through the last term, in red, in Equation (18). The situation is depicted graphically in Figure 2.

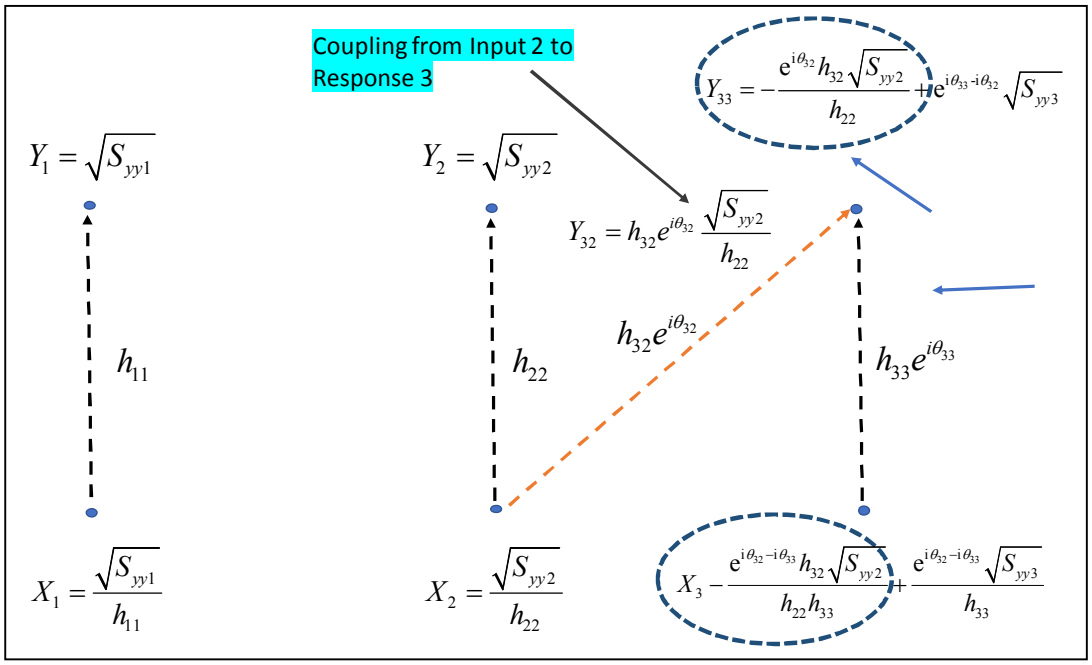

Figure 2 Coupled Drives Compensate for System Dynamics

The coupling term, shown by the orange arrow, means that drive two induces more response at control location three than $\sqrt{S_{yy3}}$. To alleviate this coupling, drive three is modified by the circled term, which effectively cancels the coupling from drive two. The drive and response spectral densities are:

$$\boldsymbol{S}_{xx} = \begin{bmatrix} \frac{S_{yy1}}{h_{11}^2} & \frac{\sqrt{S_{yy1}}\sqrt{S_{yy2}}}{h_{11}h_{22}} & -\frac{e^{-i(\theta_{32}-\theta_{33})}\sqrt{S_{yy1}}(h_{32}\sqrt{S_{yy2}}-h_{22}\sqrt{S_{yy3}})}{h_{11}h_{22}h_{33}} \\ \frac{\sqrt{S_{yy1}}\sqrt{S_{yy2}}}{h_{11}h_{22}} & \frac{S_{yy2}}{h_{22}^2} & \frac{e^{-i(\theta_{32}-\theta_{33})}\sqrt{S_{yy2}}(h_{32}\sqrt{S_{yy2}}-h_{22}\sqrt{S_{yy3}})}{h_{22}^2h_{33}} \\ -\frac{e^{i(\theta_{32}-\theta_{33})}\sqrt{S_{yy1}}(h_{32}\sqrt{S_{yy2}}-h_{22}\sqrt{S_{yy3}})}{h_{11}h_{22}h_{33}} & -\frac{e^{i(\theta_{32}-\theta_{33})}\sqrt{S_{yy2}}(h_{32}\sqrt{S_{yy2}}-h_{22}\sqrt{S_{yy3}})}{h_{22}^2h_{33}} & \frac{(h_{32}\sqrt{S_{yy2}}+h_{22}\sqrt{S_{yy3}})^2}{h_{22}^2h_{33}^2} \end{bmatrix} \tag{19}$$

And

$$\boldsymbol{S}_{yy} = \begin{bmatrix} S_{yy1} & e^{i(\theta_{11}-\theta_{22})}\sqrt{S_{yy1}}\sqrt{S_{yy2}} & e^{i(\theta_{11}-\theta_{32})}\sqrt{S_{yy1}}\sqrt{S_{yy3}} \\ e^{-i(\theta_{11}-\theta_{22})}\sqrt{S_{yy1}}\sqrt{S_{yy2}} & S_{yy2} & e^{i(\theta_{22}-\theta_{32})}\sqrt{S_{yy2}}\sqrt{S_{yy3}} \\ e^{-i(\theta_{11}-\theta_{32})}\sqrt{S_{yy1}}\sqrt{S_{yy3}} & e^{-i(\theta_{22}-\theta_{32})}\sqrt{S_{yy2}}\sqrt{S_{yy3}} & S_{yy3} \end{bmatrix} \tag{20}$$

In Equation (20) the diagonal (auto spectral) terms are just as desired. Now the cross spectral terms are nonzero, and, in fact, have unity coherence.

There is a fundamental relationship here. The magnitudes of the diagonal terms of $\boldsymbol{S}_{yy}$ are defined by the requested control auto spectral values. The cross spectral terms, have magnitudes defined by the various combinations of auto spectra and phases defined by the dynamics in $\boldsymbol{H}$. The drives are no longer independent, and their coupling is defined by the dynamics.

Note the "phase sensitive" character of the excitation via the term $e^{i(\theta_{32}-\theta_{33})}$. Phase sensitive excitations are not typical in single axis random vibration tests though they are present in single axis transient waveform control. The phase sensitive excitation here implies that, for Multiple Input Multiple Output systems the effect of nonlinearity on control is likely to be more strident than in the single axis case because nonlinear behavior can easily contaminate the precise, rapidly changing phase relationships generated by linear resonances.

When is coupling between off diagonal drives and responses likely? The most significant coupling occurs at modal frequencies. At modes, the various control locations are coupled through the dynamics of the structure, which includes the test item, fixture, and vibration machines. Of course, at low frequencies, for some configurations, rigid body motions could also induce coupling. For a relatively small test item affixed to a large rigid table, as may often be the case with the Team Tensor, coupling between drives at the table-test item interface is unlikely. If the table is the control location, then independent drives should work well. The situation is quite different for a test item in a nearly free-free configuration excited by smaller vibration machines. In this case, structural modes can strongly couple motions at control locations.

Controlling Response Auto Spectra Using Cross Spectra Between Control Locations

With specified control auto spectral densities, the control cross spectra can be adjusted to effect the response auto spectral densities. To some degree this relationship can allow control of response auto spectra by modifying control cross spectra because the relative phases between the controls have a sometimes dramatic effect on response spectra and spectral density.

There are numerous possibilities. Consider three cases:

1. The independent drive algorithm is utilized. Via the algorithm control auto and cross spectral densities are defined so response spectral density is determined by the FRF from control to response, $\boldsymbol{H}_{rc}$.
2. Based on field measurements, auto and cross spectra are controlled. Again, the response spectral density is determined by $\boldsymbol{H}_{rc}$.
3. Control Auto spectra are specified. Control cross spectra are adjusted so that a combination of the control cross spectral densities and $\boldsymbol{H}_{rc}$ determine response auto spectra.

Here we are interested in case 3. It is quite likely, given field measurement considerations, that auto spectra at control points are known along with auto spectra at some response points. Cross spectra are not available. It is desirable to adjust the control cross spectral densities to approximate the response auto spectra.

The dynamics of the test body interact with the control auto and cross spectra. Consequently, cross spectra adjustment produces a range of auto spectral responses. Not all response auto spectra are achievable. Even what seem to be "reasonable" values may not be achievable. An equally serious issue, noted in Section 1, is the differing, albeit similar, dynamics of field and laboratory test items. The equations relating control and response spectra and spectral density are:

$$\begin{aligned} \boldsymbol{Y}_r &= \boldsymbol{H}_{rc} X_c \\ \boldsymbol{S}_{rr} &= \boldsymbol{H}_{rc}^H \boldsymbol{S}_{cc} \boldsymbol{H}_{rc} \end{aligned} \qquad (21)$$

We can imagine diverse situations. If we are driving a small component excited by the Team Tensor System where a large mass is excited by 12 vibration machines, we are exciting the test system with rigid body motion in six degrees of freedom. $\boldsymbol{H}_{rc}$ then describes how the test body responds to motions in these various degrees of freedom. For example, $\boldsymbol{H}_{rc}(1,2)$ could describe the relationship between the Y response at a given point on the test item to an excitation in X. When the test item is a rigid body, as it is at low frequency, motion in Y is independent of motion in X so no adjustment of phase between X and Y is going to change the test item response in X or Y. At very high frequencies test item response is likely decoupled from control in either X or Y so again the phase between X and Y control is moot. In the intermediate frequency range, the test body dynamics often "couples" motion between different degrees of freedom. The coupling between degrees of freedom is responsible for the "cross motion" in single axis vibration tests.

Note that the case for rotations is somewhat different. Rotations about the origin do not couple to translations. However, if the rotational accelerations are large and the test item is sizeable, rotation does induce linear motion. So, rotation about the Z axis, for example, will produce a gradient in the X accelerations. This may be significant and does induce coupling between translational and rotational degrees of freedom. It is also worth noting that the space of rotations and translations of a rigid body is a non-Euclidian 6 manifold.

The linear acceleration gradient induced by rotations and the test item dynamics are the major avenues driving the off diagonal terms in $\boldsymbol{H}_{rc}$. When the control in one degree of freedom is related to the control in another degree of freedom the off diagonal terms and the corresponding dynamics play a role in determining auto spectral response. Otherwise, they do not. This tells us that we cannot just casually set some straight-line response auto spectral density and expect to achieve it through adjustment

of the control cross spectra. Unless there are relevant dynamics, the coupling is just too small to make this achievable. In a different venue, the "Controllability" between the control cross spectrum and the response auto spectrum is just not significant. Conversely, at frequencies where the test item dynamics couple degrees of freedom, the cross spectra can have dramatic effects on the auto spectral response.

The general form of the auto spectral response due to a single cross spectral term is of the form shown in Equation (22), where a simplified case of interaction between two controls y_1 and y_2 is considered.

$$S_{rr1} = h_{11}^2 y_1^2 + h_{12}^2 y_1^2 \alpha^2 + 2h_{11}h_{12}y_1^2 \alpha \, cos[\gamma_2 - \theta_{11} + \theta_{12}] \qquad (22)$$

S_{rr1} = auto spectrum at response location 1

$h_{11}^2 y_1^2$ = auto spectrum at response location 1 induced by drive 1

$h_{12}^2 y_1^2 \alpha^2$ = auto spectrum at response location 1 induced by drive 2

$2h_{11}h_{12}y_1^2 \alpha \cos[\gamma_2 - \theta_{11} + \theta_{12}]$ = auto spectrum at response location 1 achieved thru adjustment of the phase γ_2 between drives 1 and 2

Consider each of the terms in Equation (22). $h_{11}^2 y_{11}^2$ is the response due to drive 1. $h_{12}^2 y_1^2 \alpha^2$ is the response from drive 2. The phase sensitive interaction between the drives is $2h_{11}h_{12}y_1^2 \alpha \cos[\gamma_2 - \theta_{11} + \theta_{12}]$. The control auto spectral terms are fixed by the specified control levels. The system dynamics fixes h_{11} and h_{12}. The magnitude of the cross spectral term, α is adjustable as is the phase of the cross term γ_2. The effect of the cross spectral phase is dependent on the product of h_{11} and h_{12}. Both terms must have significant magnitude for the cross spectral phase to affect the response auto spectrum. Consider three bounding cases:

1. h_{11} is large, h_{12} small. In this case the cross spectral phase has minimal effect and the response is a function of $y_{11}h_{11}$.
2. h_{12} is large, h_{11} small. The cross spectral phase has minimal effect and the response is a function of $\alpha h_{12}y_1$.
3. h_{11} and h_{12} have similar magnitudes. This case is illustrated in Figure 3 for $\alpha = 1$. The effect of phase is substantial, especially if both h_{11} and h_{12} are large.

Figure 3 illustrates the induced auto spectrum as a function of the magnitudes of h_{11} and h_{12} with $\alpha = 1$.

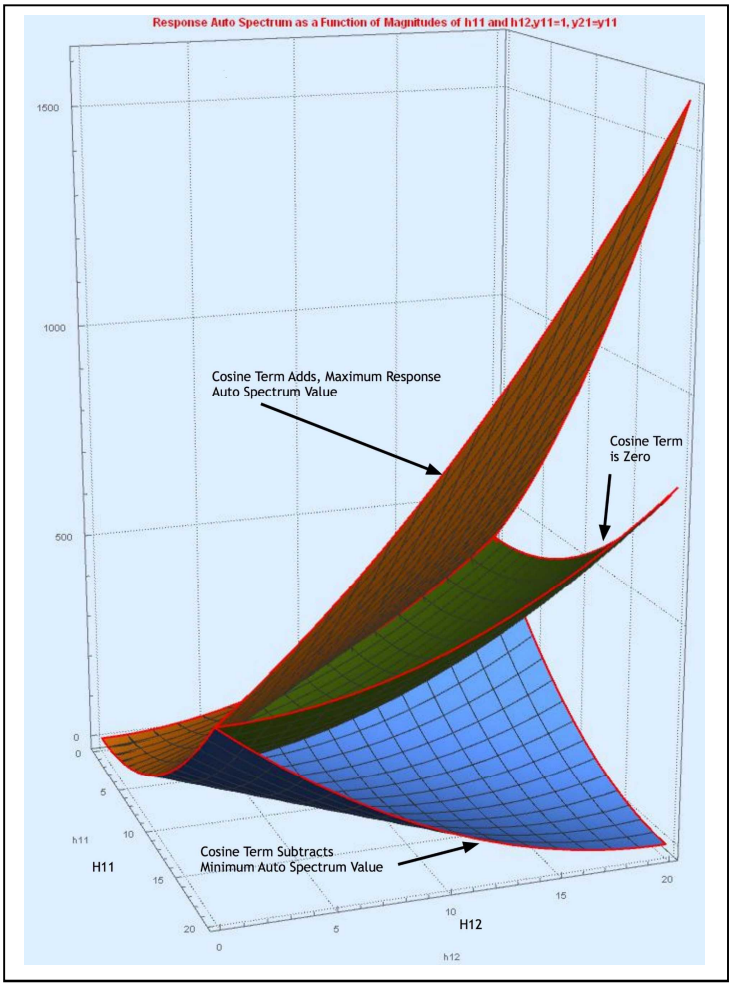

Figure 3 Induced Response Auto Spectrum as a Function of ***H*** Magnitudes

In Figure 3 the response auto spectrum in the vertical axis is plotted as a function of the FRF terms h_{11} and h_{12}.

Along the h_{11} axis, coming out of the page, h_{12} is small and the cross spectrum has no effect on the response auto spectrum. Conversely, along the h_{12} axis, h_{11} is small, and again the cross spectrum has no effect. Only when both h_{11} and h_{12} are large does the cross spectrum have a strong effect. This effect is evident along the 45 degree line where h_{12} is equal to h_{11}. Of course the effect of cross spectral adjustment is very system dependent. An example of the range of response auto spectral values achievable for a system composed of six interconnected masses with two excitations is illustrated in Figure 4.

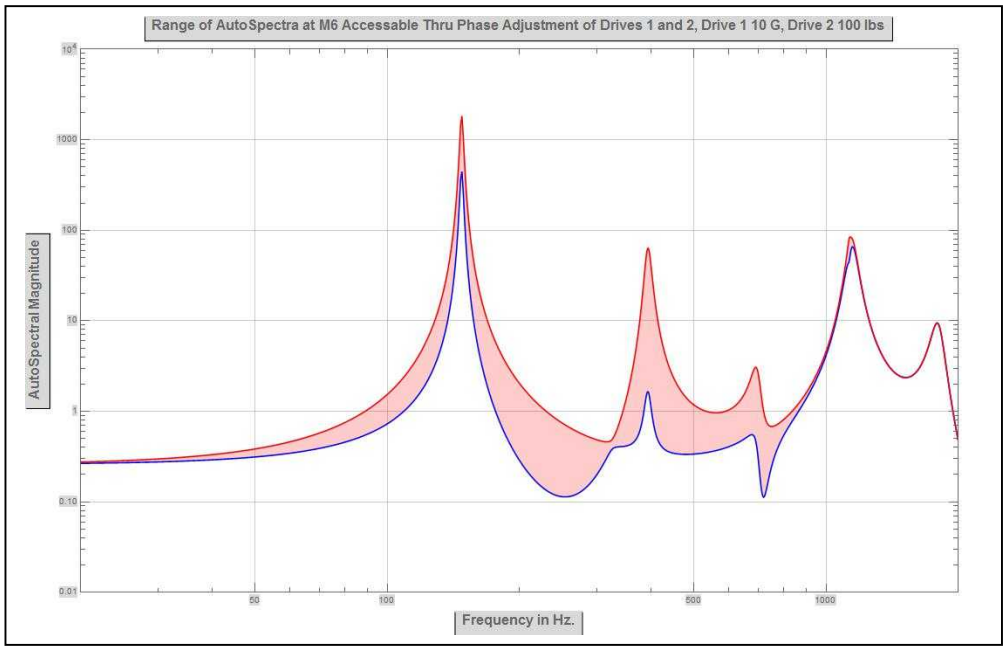

Figure 4 Range of Response Auto spectra Achieved by Adjusting Cross Spectra for a Six Mass, Two Excitation System

From Figure 4 it is evident that in some frequency ranges (below 50 Hz and above 1000 Hz.) the cross spectra has negligible effect. However, in the range of active dynamics between 100 Hz. and 600 Hz. the cross spectrum can have dramatic effect.

Conclusions

MIMO testing adds a degree of complexity and of versatility far exceeding that of single axis vibration testing. The major contrasts between, on the one hand, field and laboratory dynamics, and on the other, field and laboratory boundary conditions, are more critical in MIMO tests. No single approach is going to suffice for realistic replication of multiple degree of freedom field environments. In general, several cases can be considered.

First, a smaller component (perhaps up to several feet on a side), can readily be tested in multiple degrees of freedom using a massive vibration table such as the Team Tensor. With this type of system, the excitation in each of the translational and rotational degrees of freedom are mostly independent, at least in regards to vibration about the control axes. The standard procedure of testing with independent control spectra in each degree of freedom is reasonable and practical. Measured field data can serve as a guide to input specifications. Independent drives and controls should be possible over most of the frequency range. Note that the system geometry means that various linear combinations of the vibration drives are independent, not the individual shaker excitations.

There is a tendency to approximate a set of measured response field auto spectra at numerous locations using the 6DOF tensor as excitation. See, for example, Jacobs, Ross, et.al [6]. This can work well when, as in Jacobs [6], the inverse of the laboratory transfer function is used to derive the input in each degree of freedom. This approach is most effective for a component where the excitation is reasonably approximated by base rigid body motion in the relevant degrees of freedom.

System tests are another matter. In a system test, there may be several physically separated excitation sources and each source may operate in several degrees of freedom. Attempting to simulate this environment with a single base excited system in the laboratory is a pretty gross approximation, even when the base excitation has multiple degrees of freedom. In this case excitation from several sources is warranted. Daborn [3], has also shown excellent results with multiple, physically separated excitation sources.

The most striking issue, and one which demands further investigation, is the reconciliation of the different, but related, dynamics evidenced by field and laboratory test bodies. Use of straight line spectra to approximate field inputs, as is typical for single axis testing is a very questionable approach. A new approach, one which considers the dynamics of the specific test item, is needed.

The effects of coupling between drives or controls is through the FRF matrix $\boldsymbol{H}$. This sort of coupling is not evident in single axis tests. Significant coupling between degrees of freedom requires adjustment of the excitations to reduce the response, especially at resonances, by putting in energy that cancels some of the resonant response. The nearly independent drive algorithm described shows just how this operates.

Adjustment of the cross spectra is often proposed as a method of producing response auto spectra within a desired range. This is indeed possible, but the bounds are strongly influenced by the dynamics of the test item. Enhancing or damping existing system resonances is quite possible and reasonable, but arbitrary formulation of response auto spectra is not achievable.

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