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Chapter Four: Turbulence Model Selection

"*An ideal model should introduce the minimum amount of complexity while capturing the essence of the relevant physics." David Wilcox, 2006.*

Wilcox summarized it quite well: an "ideal" turbulence model ought to require a minimized number of equations and variables, but at the same time, it should be able to model features that are important for designers, researchers, engineers, physicists, mathematicians, and so forth. However, the literature generally shows that turbulence models tend to become more complex as their ability to capture physics increases. A somewhat analogous view for turbulence models is that they ought to capture as many of the physical features displayed in turbulent flows, while having a minimized number of *ad hoc* fits, stitches, and patches. Furthermore, such model should be purely based on variables that reflect the correct physics. For example, if the turbulent kinetic energy k refers to the energy of the *large* integral eddies, would it be acceptable to couple k with the *small* Kolmogorov eddies? These issues, as well as how turbulence models were derived, their applicability, and lack thereof, are covered in the sections that follow.

4.1 The *R* Mechanics Behind Reynolds Stress Transport Models

At this point, Chapter 4 extends the theory and modeling presented in Chapter 2 in Sections 2.4 and 2.5. It is now a matter of moving forward towards closure, based on the desired Reynolds-averaged Navier-Stoke (RANS) modeling complexity; there are zero-, one-, and two-equation, as well as the so-called second-order closure models. The latter are also referred as second-moment closure models, stress transport models, Reynolds stress models (RSM), and Reynolds stress transport (RST) models. Note that these are not the same as the "shear stress transport" (SST) models.

The following sections extend turbulence modeling by zeroing in on the unanswered issues that remain from Chapter 2. This is particularly so for the Reynolds stress tensor R, to point out what is still needed to have a full set of n independent turbulence equations, thereby solving n unknowns. Then, it is shown that applying the Boussinesq approximation is not sufficient for closure, so the path forward is to perform a set of mathematical operations that lead to additional independent equations, with the aim of providing closure to the turbulence problem. In particular, the R tensor is derived mathematically in its exact form from the un-operated Navier-Stokes momentum equation by taking its first moment based on the fluctuating velocity, and then a time-average is performed. The resultant R tensor is used in RSM models, and is therefore part of the motivation. But most importantly, the k PDE is derived from R by taking its trace, and it is this k PDE that is used in most zero-, one-, and two-equation RANS models (e.g., Prandtl, k-ε, k-ω, SST, etc.).

4.2. The Introduction of the *R* Tensor into Turbulence Modeling

Recall that if the kinematic viscosity is constant and the gravitational term is negligible (or is lumped with the *P* term), the original (un-operated) Navier-Stokes equation is simply,

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=\nu\frac{\partial^2 u_i}{\partial x_j^2}-\frac{1}{\rho}\frac{\partial P}{\partial x_i}.$$

Recall that the RANS momentum equation was derived in Chapter 2. In its vector-tensor notation, the equation appears as,

$$\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}=-\frac{\partial \bar{P}}{\partial x_i}+2\frac{\partial\left(\nu\bar{S}_{ji}\right)}{\partial x_j}-\frac{\partial\left(\overline{u'_j u'_i}\right)}{\partial x_j}$$

where $\bar{S}$ is the mean strain rate tensor for an incompressible fluid,

$$\bar{S}_{ij}=\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right).$$

At this point, $\bar{S}$ can be included into the momentum equation, and the vector-tensor equation can be expanded in the *x*-, *y*-, and *z*-directions of a Cartesian system. The PDE for the *x*-momentum is

$$\frac{\partial \bar{u}}{\partial t}+\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{\mathrm{v}}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}+\nu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)$$
$$-\left(\frac{\partial \overline{u'u'}}{\partial x}+\frac{\partial \overline{u'\mathrm{v}'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right),$$

and similarly for the *y*-momentum,

$$\frac{\partial \bar{\mathrm{v}}}{\partial t}+\left(\bar{u}\frac{\partial \bar{\mathrm{v}}}{\partial x}+\bar{\mathrm{v}}\frac{\partial \bar{\mathrm{v}}}{\partial y}+\bar{w}\frac{\partial \bar{\mathrm{v}}}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial y}+\nu\left(\frac{\partial^2 \bar{\mathrm{v}}}{\partial x^2}+\frac{\partial^2 \bar{\mathrm{v}}}{\partial y^2}+\frac{\partial^2 \bar{\mathrm{v}}}{\partial z^2}\right)$$
$$-\left(\frac{\partial \overline{\mathrm{v}'u'}}{\partial x}+\frac{\partial \overline{\mathrm{v}'\mathrm{v}'}}{\partial y}+\frac{\partial \overline{\mathrm{v}'w'}}{\partial z}\right),$$

and for the *z*-momentum,

$$\frac{\partial \overline{w}}{\partial t}+\left(\overline{u}\frac{\partial \overline{w}}{\partial x}+\overline{\mathrm{v}}\frac{\partial \overline{w}}{\partial y}+\overline{w}\frac{\partial \overline{w}}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial z}+\nu\left(\frac{\partial^2 \overline{w}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}\overline{w}+\frac{\partial^2 \overline{w}}{\partial z^2}\right)$$
$$-\left(\frac{\partial \overline{w'u'}}{\partial x}+\frac{\partial \overline{w'\mathrm{v}'}}{\partial y}+\frac{\partial \overline{w'w'}}{\partial z}\right).$$

The primed terms are a direct consequence of the time-averaging that was performed in Chapter 2; an inspection of the last term in parenthesis on the RHS of each of the three *x*-, *y*-, and *z*-momentum equations shows how the *R* tensor is assembled. In particular, each of the three PDEs contributes three primed quantities, for a total of nine terms, namely,

$$\vec{\vec{\mathrm{R}}}=-\begin{bmatrix}\overline{u'u'} & \overline{u'\mathrm{v}'} & \overline{u'w'}\\ \overline{\mathrm{v}'u'} & \overline{\mathrm{v}'\mathrm{v}'} & \overline{\mathrm{v}'w'}\\ \overline{w'u'} & \overline{w'\mathrm{v}'} & \overline{w'w'}\end{bmatrix}=-\overline{\mathrm{u}_i'\mathrm{u}_j'}\cdot$$

The three terms on the top column of the tensor correspond to the *x*-momentum Reynolds stresses, followed by the middle column with its three terms corresponding to the *y*-momentum, and likewise for the bottom column.

R has many useful properties, including positive definiteness. Fortunately, the tensor is symmetric as well, so only six of the nine Reynolds stress terms are unknown. For example,

$$R_{xy}=R_{yx}\,.$$

This is where the linear Boussinesq approximation provides additional information, e.g., more information that can be substituted onto the RANS momentum equations. Recall the definition for the Boussinesq approximation, viz.,

$$R_{ij}\equiv 2\nu_t\overline{S}_{ij}-\frac{2}{3}kI_{ij}=\nu_t\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)-\frac{2}{3}k\delta_{ij}\,.$$

As shown in Example 16 in Chapter 2, the individual components are rather straightforward to express from the Boussinesq approximation. For convenience, the entire set of nine components are summarized here (though there are only six independent components). The three *x*-momentum Reynolds stresses are

$$R_{xx}=-\overline{\mathrm{u}'\mathrm{u}'}=2\nu_t\frac{\partial \overline{u}}{\partial x}-\frac{2}{3}k\,,$$

$$R_{xy}=-\overline{\mathrm{u}'\mathrm{v}'}=\nu_t\left(\frac{\partial \overline{u}}{\partial y}+\frac{\partial \overline{\mathrm{v}}}{\partial x}\right)\text{, and}$$

$$R_{xz} = -\overline{\mathrm{u}'\mathrm{w}'} = \nu_t \left(\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x} \right),$$

while the three *y*-momentum Reynolds stresses are

$$R_{yx} = -\overline{\mathrm{v}'\mathrm{u}'} = \nu_t \left(\frac{\partial \overline{\mathrm{v}}}{\partial x} + \frac{\partial \overline{u}}{\partial y} \right),$$

$$R_{yy} = -\overline{\mathrm{v}'\mathrm{v}'} = 2\nu_t \frac{\partial \overline{\mathrm{v}}}{\partial y} - \frac{2}{3}k \text{, and}$$

$$R_{yz} = -\rho\overline{\mathrm{v}'\mathrm{w}'} = \mu_t \left(\frac{\partial \overline{\mathrm{v}}}{\partial z} + \frac{\partial \overline{w}}{\partial y} \right).$$

Finally, the three *z*-momentum Reynolds stresses are

$$R_{zx} = -\overline{\mathrm{w}'\mathrm{u}'} = \nu_t \left(\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{u}}{\partial z} \right),$$

$$R_{zy} = -\overline{\mathrm{w}'\mathrm{v}'} = \nu_t \left(\frac{\partial \overline{w}}{\partial y} + \frac{\partial \overline{\mathrm{v}}}{\partial z} \right) \text{, and}$$

$$R_{zz} = -\overline{\mathrm{w}'\mathrm{w}'} = 2\nu_t \frac{\partial \overline{w}}{\partial z} - \frac{2}{3}k.$$

Now it is possible to incorporate the *R* terms into the RANS PDE. For example, the *x*-direction Cartesian RANS equation transforms into a more manageable expression,

$$\frac{\partial \overline{u}}{\partial t} + \left(\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{\mathrm{v}}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} \right) = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2} \right)$$
$$+ \frac{\partial\left[\nu_t\left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{u}}{\partial x} \right) - \frac{2}{3}k \right]}{\partial x} + \frac{\partial\left[\nu_t\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{\mathrm{v}}}{\partial x} \right) \right]}{\partial y} + \frac{\partial\left[\nu_t\left(\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x} \right) \right]}{\partial z}$$

The same can be applied to the *y* and *z* momentum equations, resulting in the following two PDEs, respectively,

$$\frac{\partial \overline{\mathrm{v}}}{\partial t} + \left(\overline{u}\frac{\partial \overline{\mathrm{v}}}{\partial x} + \overline{\mathrm{v}}\frac{\partial \overline{\mathrm{v}}}{\partial y} + \overline{w}\frac{\partial \overline{\mathrm{v}}}{\partial z} \right) = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} + \nu\left(\frac{\partial^2 \overline{\mathrm{v}}}{\partial x^2} + \frac{\partial^2 \overline{\mathrm{v}}}{\partial y^2} + \frac{\partial^2 \overline{\mathrm{v}}}{\partial z^2} \right)$$
$$+ \frac{\partial\left[\nu_t\left(\frac{\partial \overline{\mathrm{v}}}{\partial x} + \frac{\partial \overline{u}}{\partial y} \right) \right]}{\partial x} + \frac{\partial\left[\nu_t\left(\frac{\partial \overline{\mathrm{v}}}{\partial y} + \frac{\partial \overline{\mathrm{v}}}{\partial y} \right) - \frac{2}{3}k \right]}{\partial y} + \frac{\partial\left[\nu_t\left(\frac{\partial \overline{\mathrm{v}}}{\partial z} + \frac{\partial \overline{w}}{\partial y} \right) \right]}{\partial z}$$

and

$$\frac{\partial \bar{w}}{\partial t}+\left(\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{\mathrm{v}}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial z}+\nu\left(\frac{\partial^2 \bar{w}}{\partial x^2}+\frac{\partial^2 \bar{w}}{\partial y^2}+\frac{\partial^2 \bar{w}}{\partial z^2}\right)$$
$$+\frac{\partial\left[\nu_t\left(\frac{\partial \bar{w}}{\partial x}+\frac{\partial \bar{u}}{\partial z}\right)\right]}{\partial x}+\frac{\partial\left[\nu_t\left(\frac{\partial \bar{w}}{\partial y}+\frac{\partial \bar{\mathrm{v}}}{\partial z}\right)\right]}{\partial y}+\frac{\partial\left[\nu_t\left(\frac{\partial \bar{w}}{\partial z}+\frac{\partial \bar{w}}{\partial z}\right)-\frac{2}{3}k\right]}{\partial z}.$$

Notice that at this point, the six unknown, independent *R* components are resolved thanks to the Boussinesq approximation. But in doing so, the process introduced two new variables: *k* and ν_t! *So, at this point, closure is closer, but not fully achieved*. Fortunately, patience is a good trait of turbulence modelers!

In particular, the above three PDEs can *almost* be solved numerically for the time-averaged turbulence velocities, $\bar{u}$, $\bar{\mathrm{v}}$, and $\bar{w}$. All that is needed at this point for closure is as follows: 1) an expression for *k* must somehow be derived for the large, energy-bearing eddies and 2) a relationship for ν_t is developed, usually by using dimensional arguments as functions of *k* and some characteristic length or an equivalent. Certainly, most modern expressions for ν_t are based on dimensional arguments that usually involve *k*, but not always—it all depends on the pedigree of the model.

In any case, this is how many RANS turbulent transport models are developed—the number of transport variables determines the number of transport equations that are required for mathematical closure. This means that if there are *n* unknown transport variables, there must be *n* independent transport equations to solve the turbulent flow, and that if the transport equations add additional *m* unknowns, the model must also supply *m* additional auxiliary equations.

At this point, the next step is to develop a rigorous mathematical expression for *R*, which will then be used to derive *k*.

4.3 Mathematical Derivation of *R*

The *R* tensor is derived mathematically in its exact form as follows. The first step is to take the un-operated "laminar" Navier-Stokes momentum equation

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=\nu\frac{\partial^2 u_i}{\partial x_k \partial x_k}-\frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

and apply a so-called first moment based on the fluctuating velocities u_i' and u_j'. Basically, each term in the Navier-Stokes equation is multiplied by the fluctuating velocities, the instantaneous u_i and *P* variables are replaced with the appropriate Reynolds decomposition expressions, the terms are time-averaged, and the final step is to simplify the terms. This is accomplished by multiplying each term with u_i' and then with u_j', and thereafter adding the two, e.g.,

$$u_i' \frac{\partial u_j}{\partial t} + u_j' \frac{\partial u_i}{\partial t}.$$

Then, the instantaneous velocities u_i and u_j are replaced with the Reynolds decomposition velocity; and likewise for the pressure term. Recall that the Reynolds velocity decomposition is

$$u_i = \overline{u}_i + u_i',$$

while that of pressure is

$$P = \overline{P} + P'.$$

For example,

$$u_i' \frac{\partial u_j}{\partial t} + u_j' \frac{\partial u_i}{\partial t} \quad \rightarrow \quad u_i' \left[\frac{\partial \left(\overline{u}_j + u_j' \right)}{\partial t} \right] + u_j' \left[\frac{\partial \left(\overline{u}_i + u_i' \right)}{\partial t} \right].$$

The final steps are to perform time averaging (the bar operation) and to simplify the expressions.

Due to the number of mathematical steps performed during this approach, it is easier to do the RANS equation term by term, and then assemble the results. Another advantage of solving each term individually is that it is easier to see and to appreciate how new turbulence physics terms arise (e.g., this is where the turbulence production, dissipation, and turbulence triple velocity correlation transport terms first appear). The step-by-step approach is covered in what follows, with the most details shown for the momentum accumulation term, to avoid repetition.

4.3.1 The momentum accumulation term

The overall procedure begins by taking the momentum term and multiplying it by the fluctuating velocities:

$$\underbrace{\frac{\partial u_i}{\partial t}}_{\text{starting point}} \rightarrow \underbrace{u_i' \frac{\partial u_j}{\partial t} + u_j' \frac{\partial u_i}{\partial t}}_{\text{mult. momentum term with } u'} \rightarrow \underbrace{u_i' \left[\frac{\partial \left(\overline{u}_j + u_j' \right)}{\partial t} \right] + u_j' \left[\frac{\partial \left(\overline{u}_i + u_i' \right)}{\partial t} \right]}_{\text{Substitute Reynolds decomposition velocity}}$$

$$\rightarrow \underbrace{\overline{u_i' \left[\frac{\partial \left(\overline{u}_j + u_j' \right)}{\partial t} \right]} + \overline{u_j' \left[\frac{\partial \left(\overline{u}_i + u_i' \right)}{\partial t} \right]}}_{\text{Apply time average}} \rightarrow \text{simplify.}$$

Note that the above procedure is applied to the other momentum terms as well. The simplification of the accumulation term is as follows. Take the last expression from above and expand the partial differential term,

$$\overline{u_i'\left(\frac{\partial \overline{u}_j}{\partial t}\right)} + \overline{u_i'\left(\frac{\partial u_j'}{\partial t}\right)} + \overline{u_j'\left(\frac{\partial \overline{u}_i}{\partial t}\right)} + \overline{u_j'\left(\frac{\partial u_i'}{\partial t}\right)} \qquad (\text{multiplication}).$$

Recall from Section 2.5 that time-averaging the product of a fluctuating velocity and a time-averaged velocity is zero,

$$\overline{v'\overline{v}} = 0,$$

and that

$$\overline{\frac{\partial}{\partial t} v} = \frac{\partial}{\partial t}\overline{v}.$$

Thus, the first and third terms are zero,

$$\cancel{\overline{u_i'\left(\frac{\partial \overline{u}_j}{\partial t}\right)}} + \overline{u_i'\left(\frac{\partial u_j'}{\partial t}\right)} + \cancel{\overline{u_j'\left(\frac{\partial \overline{u}_i}{\partial t}\right)}} + \overline{u_j'\left(\frac{\partial u_i'}{\partial t}\right)}.$$

The two terms can be lumped into a single term by recognizing that they are formed from the derivative of a product, namely that

$$\overline{u_i'\left(\frac{\partial u_j'}{\partial t}\right)} + \overline{u_j'\left(\frac{\partial u_i'}{\partial t}\right)} = \overline{\left[\frac{\partial\left(u_i'u_j'\right)}{\partial t}\right]} \qquad (\text{derivative product rule})$$

Finally, recalling the definition for the Reynolds stress *R*, the above expression reduces

$$\overline{\left[\frac{\partial\left(u_i'u_j'\right)}{\partial t}\right]} = -\frac{\partial \tilde{R}_{ij}}{\partial t} \quad \left(\begin{array}{l}\text{Recall } \tilde{R}_{ij} = -\overline{u_i'u_j'}.\\ \text{Substitute definition for Reynolds stress.}\end{array}\right).$$

The key result for the transient term is the transformation of u into the stress tensor $\tilde{R}_{ij}$,

$$\frac{\partial u_j}{\partial t} \rightarrow -\frac{\partial \tilde{R}_{ij}}{\partial t}.$$

4.3.2 The convective term

Just as was done to the transient (accumulation) term above, now take the convective term and perform the same procedure. In particular, take $u_k \dfrac{\partial u_i}{\partial x_k}$, multiply it by the fluctuating velocities,

$$\overline{\mathrm{u}_i'\left(u_k \frac{\partial u_j}{\partial x_k}\right) + \mathrm{u}_j'\left(u_k \frac{\partial u_i}{\partial x_k}\right)}.$$

Next, substitute the Reynolds decomposition velocity, and apply the time average operator:

$$\overline{\mathrm{u}_i'\left(\bar{\mathrm{u}}_k + \mathrm{u}_k'\right)\frac{\partial\left(\bar{\mathrm{u}}_j + \mathrm{u}_j'\right)}{\partial x_k}} + \overline{\mathrm{u}_j'\left(\bar{\mathrm{u}}_k + \mathrm{u}_k'\right)\left[\frac{\partial\left(\bar{\mathrm{u}}_i + \mathrm{u}_i'\right)}{\partial x_k}\right]}.$$

At this point, it is now a matter of simplifying the expression. In this case, use the FOIL multiplication acronym (Front, Outside, Inside, Last) to expand each of the two terms:

$$\left[\overline{\mathrm{u}_i'\bar{\mathrm{u}}_k\left(\frac{\partial \bar{\mathrm{u}}_j}{\partial x_k}\right)} + \overline{\mathrm{u}_i'\bar{\mathrm{u}}_k\left(\frac{\partial \mathrm{u}_j'}{\partial x_k}\right)} + \overline{\mathrm{u}_i'\mathrm{u}_k'\left(\frac{\partial \bar{\mathrm{u}}_j}{\partial x_k}\right)} + \overline{\mathrm{u}_i'\mathrm{u}_k'\left(\frac{\partial \mathrm{u}_j'}{\partial x_k}\right)}\right]$$
$$+\left[\overline{\mathrm{u}_j'\bar{\mathrm{u}}_k\left(\frac{\partial \bar{\mathrm{u}}_i}{\partial x_k}\right)} + \overline{\mathrm{u}_j'\bar{\mathrm{u}}_k\left(\frac{\partial \mathrm{u}_i'}{\partial x_k}\right)} + \overline{\mathrm{u}_j'\mathrm{u}_k'\left(\frac{\partial \bar{\mathrm{u}}_i}{\partial x_k}\right)} + \overline{\mathrm{u}_j'\mathrm{u}_k'\left(\frac{\partial \mathrm{u}_i'}{\partial x_k}\right)}\right].$$

Recalling that

$\overline{\bar{\mathrm{v}}\mathrm{v}'} = 0 \text{ and } \overline{\mathrm{v}'\bar{\mathrm{v}}} = 0$ and rearranging,

$$\left\{\left[\cancel{\overline{\mathrm{u}_i'\bar{\mathrm{u}}_k\left(\frac{\partial \mathrm{u}_j'}{\partial x_k}\right)}} + \cancel{\overline{\mathrm{u}_j'\bar{\mathrm{u}}_k\left(\frac{\partial \mathrm{u}_i'}{\partial x_k}\right)}}\right] + \left[\overline{\mathrm{u}_i'\mathrm{u}_k'\left(\frac{\partial \mathrm{u}_j'}{\partial x_k}\right)} + \overline{\mathrm{u}_j'\mathrm{u}_k'\left(\frac{\partial \mathrm{u}_i'}{\partial x_k}\right)}\right]\right\}$$
$$+\left\{\overline{\mathrm{u}_j'\mathrm{u}_k'\left(\frac{\partial \bar{\mathrm{u}}_i}{\partial x_k}\right)} + \overline{\mathrm{u}_i'\mathrm{u}_k'\left(\frac{\partial \bar{\mathrm{u}}_j}{\partial x_k}\right)} + \cancel{\overline{\mathrm{u}_i'\bar{\mathrm{u}}_k\left(\frac{\partial \bar{\mathrm{u}}_j}{\partial x_k}\right)}} + \cancel{\overline{\mathrm{u}_j'\bar{\mathrm{u}}_k\left(\frac{\partial \bar{\mathrm{u}}_i}{\partial x_k}\right)}}\right\}.$$

Now combine the derivatives in anticipation of obtaining $\tilde{R}_{ij}$,

$$\left\{ \overline{\bar{u}_k \left[\frac{\partial \left(u'_i u'_j\right)}{\partial x_k} \right]} + \overline{u'_j u'_k \left(\frac{\partial \bar{u}_i}{\partial x_k}\right)} + \overline{u'_i u'_k \left(\frac{\partial \bar{u}_j}{\partial x_k}\right)} + \overline{u'_k \left[\frac{\partial \left(u'_i u'_j\right)}{\partial x_k}\right]} \right\}$$

$$= -\bar{u}_k \frac{\partial \tilde{R}_{ij}}{\partial x_k} - \tilde{R}_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tilde{R}_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + \overline{\frac{\partial \left(u'_i u'_j u'_k\right)}{\partial x_k}} \quad \left(\text{Recall } \tilde{R}_{ij} = -\overline{u'_i u'_j}\right)$$

The key results for the convective term are

$$u_k \frac{\partial u_i}{\partial x_k} \rightarrow \underbrace{-\bar{u}_k \frac{\partial \tilde{R}_{ij}}{\partial x_k}}_{\text{Stress convection}} \underbrace{-\tilde{R}_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tilde{R}_{jk}\frac{\partial \bar{u}_i}{\partial x_k}}_{\text{Production term: "eddy factory"}} + \underbrace{\overline{\frac{\partial \left(u'_i u'_j u'_k\right)}{\partial x_k}}}_{\text{triple velocity correlation}}.$$

The second term on the RHS accounts for the production of eddies. It should come as no surprise that operators applied onto the momentum convection term unveil a "hidden" eddy-production term, which accounts for the eddies that arise from flow instabilities in the mean flow. That is, eddy production is a consequence of the nonlinear convective term. It is also very insightful that production is the product of the stress and velocity gradient of the mean flow. Velocity gradients are largest near the wall, and this is precisely where the production term is the largest, in the general ballpark range of $0 < y^+ < 25$, and thereafter drops exponentially as y^+ increases; this is applicable for *Re* in the range of 3,000 to 40,000 [Mansour, Kim, and Moin, 1988; Wilcox, 2006].

4.3.3 The viscous term

Now apply the same procedure onto the viscous term, $\nu \dfrac{\partial^2 u_i}{\partial x_k \partial x_k}$, such that

$$\nu \overline{\left[u'_i \left(\frac{\partial^2 u_j}{\partial x_k \partial x_k}\right) + u'_j \left(\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right)\right]} = \nu \overline{u'_i \frac{\partial^2 \left(\bar{u}_j + u'_j\right)}{\partial x_k \partial x_k}} + \nu \overline{u'_j \frac{\partial^2 \left(\bar{u}_i + u'_i\right)}{\partial x_k \partial x_k}}.$$

Recalling that both $\overline{v'\bar{v}}$ and $\overline{v' \dfrac{\partial \bar{v}}{\partial x}}$ are equal to zero, and expanding the terms, reduces to

$$\nu \left(\cancel{\overline{u'_i \frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_k}}} + \overline{u'_i \frac{\partial^2 u'_j}{\partial x_k \partial x_k}} + \cancel{\overline{u'_j \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k}}} + \overline{u'_j \frac{\partial^2 u'_i}{\partial x_k \partial x_k}} \right).$$

Applying the mathematical trick that

$$\overline{u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k}} = \frac{\partial \overline{\left(u_i' \frac{\partial u_j'}{\partial x_k} \right)}}{\partial x_k} + \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}$$

and in an analogous manner,

$$\overline{u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k}} = \frac{\partial \overline{\left(u_j' \frac{\partial u_i'}{\partial x_k} \right)}}{\partial x_k} + \overline{\frac{\partial u_j'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}}.$$

Note also that

$$\overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} = \overline{\frac{\partial u_j'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}}.$$

Substituting the above mathematical equivalents onto the time-averaged, Reynolds decomposed, first-moment viscous term results in the following expression,

$$\nu \left\{ \left[\frac{\partial \overline{\left(u_i' \frac{\partial u_j'}{\partial x_k} \right)}}{\partial x_k} + \frac{\partial \overline{\left(u_j' \frac{\partial u_i'}{\partial x_k} \right)}}{\partial x_k} \right] - 2 \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \right\}.$$

The first two terms in the above expression can be lumped,

$$\nu \left\{ \left[\frac{\partial^2 \overline{\left(u_i' u_j' \right)}}{\partial x_k \partial x_k} \right] - 2 \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \right\},$$

and in anticipation of swapping the product of the two fluctuating velocities for the Reynolds stress,

$$\nu \left\{ \left[\overline{\frac{\partial}{\partial x_k} \left(\frac{\partial}{\partial x_k} \left(u_i' u_j' \right) \right)} \right] - 2 \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \right\} = -\nu \frac{\partial^2 \tilde{R}_{ij}}{\partial x_k \partial x_k} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}.$$

The key results for the viscous term are therefore

$$\nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \quad \rightarrow \quad -\nu \underbrace{\frac{\partial^2 \tilde{R}_{ij}}{\partial x_k \partial x_k}}_{\text{Viscous stress diffusion}} - \underbrace{2\nu \overline{\frac{\partial \mathrm{u}_i'}{\partial x_k} \frac{\partial \mathrm{u}_j'}{\partial x_k}}}_{\text{Eddy dissipation}}.$$

The second term on the RHS accounts for eddy dissipation (decay). It comes as no surprise that operators applied onto the momentum viscous term unveil a “hidden” eddy-damping term. This “new” term fosters the decay of eddies, until all their turbulent energy reverts to the mean flow as heat, thereby causing all the eddy’s swirling motion to cease. That is, eddy decay is a consequence of the viscous damping term, which applies the brakes onto eddies.

4.3.4 The pressure term

Finally, apply the same procedure to the pressure term, $\frac{1}{\rho}\frac{\partial P}{\partial x_i}$.

Here, there are no instantaneous velocities, but there is an instantaneous pressure that can be decomposed using Reynolds decomposition,

$$P = \overline{\mathrm{P}} + P'.$$

Other than this change, the procedure is the same as was used for the other three momentum terms. Namely,

$$\overline{\mathrm{u}_i' \frac{\partial P}{\partial x_j} + \mathrm{u}_j' \frac{\partial P}{\partial x_i}} = \overline{\mathrm{u}_i' \frac{\partial\left(\overline{\mathrm{P}} + P'\right)}{\partial x_j} + \mathrm{u}_j' \frac{\partial\left(\overline{\mathrm{P}} + P'\right)}{\partial x_i}}.$$

Now take the derivatives, and recall that because $\overline{\mathrm{v}'\overline{\mathrm{P}}}$ is zero, $\overline{\mathrm{v}' \frac{\partial \overline{\mathrm{P}}}{\partial x}}$ must also be zero, thereby yielding

$$\overline{\cancel{\mathrm{u}_i' \frac{\partial \overline{\mathrm{P}}}{\partial x_j}} + \mathrm{u}_i' \frac{\partial P'}{\partial x_j} + \cancel{\mathrm{u}_j' \frac{\partial \overline{\mathrm{P}}}{\partial x_i}} + \mathrm{u}_j' \frac{\partial P'}{\partial x_i}}.$$

Using the property that $\overline{a+b} = \overline{a} + \overline{b}$, leads to the simplification of the pressure term as

$$\overline{\mathrm{u}_i' \frac{\partial P'}{\partial x_j}} + \overline{\mathrm{u}_j' \frac{\partial P'}{\partial x_i}}.$$

The key result for the pressure term is that

$$\frac{1}{\rho}\frac{\partial P}{\partial x_i} \;\to\; \underbrace{\frac{1}{\rho}\left(\overline{u_i'\frac{\partial P'}{\partial x_j}}+\overline{u_j'\frac{\partial P'}{\partial x_i}}\right)}_{\text{Stress transport due to pressure-velocity fluctuations}}.$$

4.3.5 Assembly of the R Reynolds stress PDE

At last, assembling the above key results that were derived earlier, and dividing by -1, results in the sought-after expression for the Reynolds stress $\tilde{R}_{ij}$ PDE:

$$\underset{1}{\frac{\partial \tilde{R}_{ij}}{\partial t}}+\underbrace{\overline{u}_k\frac{\partial \tilde{R}_{ij}}{\partial x_k}}_{2}=\underbrace{-\left(\tilde{R}_{ik}\frac{\partial \overline{u}_j}{\partial x_k}+\tilde{R}_{jk}\frac{\partial \overline{u}_i}{\partial x_k}\right)}_{3}+\underbrace{2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{4}+\underbrace{\frac{\partial}{\partial x_k}\left(\nu\frac{\partial \tilde{R}_{ij}}{\partial x_k}\right)}_{5}$$
$$+\underbrace{\frac{1}{\rho}\left(\overline{u_i'\frac{\partial P'}{\partial x_j}}+\overline{u_j'\frac{\partial P'}{\partial x_i}}\right)}_{6}+\underbrace{\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right)}_{7}$$

A description of the Reynolds stress PDE terms is as follows:

Term 1 = transient stress change (accumulation),

Term 2 = stress convection,

Term 3 = production terms that arise from the product of the stress $\tilde{R}$ (calculated via the Boussinesq approximation) and the mean velocity gradients. The production term quantifies the rate at which the mean flow imparts energy onto eddies,

Term 4 = stress dissipation rate. This represents the rate at which the stress-generated eddies are converted back into internal energy, where the dissipation stress tensor is defined as,

$$\varepsilon_{ij} \equiv 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}},$$

Term 5 = viscous (molecular) diffusion of stress,

Term 6 = turbulent stress transport associated with the eddy pressure and velocity fluctuations, and

Term 7 = diffusive stress transport resulting from the triple-correlation eddy velocity fluctuations.

Note that Terms 1, 2, and 5 are described exactly in a mathematical and physical sense, and thus have no need for the dreaded “drastic surgery” described by Wilcox. Term 3 is reasonably modeled (often enough) using the Boussinesq approximation. However, if more complex flows need to be modelled, then Boussinesq can be replaced with a more appropriate expression, such as those that include rotational and higher order terms.

On the other hand, the remaining three terms: dissipation, the pressure-velocity fluctuation correlation, and the eddy fluctuations due to the triple velocity correlation, are not known, and thus require "drastic surgery". As so fondly described by Wilcox [Wilcox, 2006],

> "In essence, Reynolds averaging is a brutal simplification that loses much of the information contained in the Navier-Stokes equation."

And this certainly applies to the *R* PDE. Note that Wilcox applied the term "drastic surgery" to both the *k* and ε PDEs as well. However, the transformation of the *k* PDE requires some drastic engineering measures, but certainly not to the degree of the ε PDE, as will be shown later.

Note that RSM models only use six transport PDEs because *R* is symmetric, so the nine PDEs reduce to six. Further, RSM involves no *k* PDE because the stresses are calculated individually. On the other hand, RANS assumes that *k* is the sum of the diagonal stresses,

$$\frac{1}{2}\overline{\mathrm{u}_i'\mathrm{u}_i'} = \frac{1}{2}\left(\overline{u'^2} + \overline{\mathrm{v}'^2} + \overline{w'^2}\right).$$

As far as the historical records show, Chou was the first to develop the *R* PDE, as expressed in more modern times [Chou, 1940 (refer to Equations 3.1 and 3.4)], and various terms were further elaborated in later papers [Chou, 1945; Chou and Chou, 1995].

4.4 Development of the k PDE

As formidable as the Reynolds stress tensor PDE appears, it can be vastly simplified by taking the mathematical trace of all its terms, thereby generating a new expression for *k*. This is achieved as follows. First, consider the trace of tensor $\tilde{R}_{ij}$,

$$\mathrm{tr}\left(\tilde{R}_{ij}\right) = \tilde{R}_{ii} \equiv -\overline{\mathrm{u}_i'\mathrm{u}_i'}.$$

Recalling the definition for *k* first proposed by Prandtl in 1945,

$$k = \frac{1}{2}\overline{\mathrm{u}_i'\mathrm{u}_i'}$$

or

$$\overline{\mathrm{u}_i'\mathrm{u}_i'} = 2k.$$

Therefore,

$\tilde{R}_{ii} = -\overline{\mathrm{u}_i'\mathrm{u}_i'} = -2k$. Later on, this expression will be substituted onto the trace of *R*.

Taking the trace of the entire *R* PDE, term by term, yields

$$\underbrace{\frac{\partial \tilde{R}_{ii}}{\partial t}}_{1}+\underbrace{\overline{\mathrm{u}}_k \frac{\partial \tilde{R}_{ii}}{\partial x_k}}_{2}=\underbrace{2\tilde{R}_{ij}\frac{\partial \overline{\mathrm{u}}_i}{\partial x_j}}_{3}+\underbrace{\varepsilon_{ii}}_{4}+\underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tilde{R}_{ii}}{\partial x_k}\right)}_{5}+\underbrace{\frac{2}{\rho}\left(\overline{\mathrm{u}_i' \frac{\partial P'}{\partial x_i}}\right)}_{6}+\underbrace{\frac{\partial}{\partial x_k}\left(\overline{\mathrm{u}_i'\mathrm{u}_i'\mathrm{u}_k'}\right)}_{7}.$$

Note that the two production terms are combined into a single term. At this point, $\tilde{R}_{ii}$ can be replaced with its equivalent (-2*k*), and simplifying the new expression, the exact *k* PDE is obtained at last,

$$\underbrace{\frac{\partial k}{\partial t}}_{1}+\underbrace{\overline{\mathrm{u}}_i \frac{\partial k}{\partial x_i}}_{2}=\underbrace{\tilde{R}_{ij}\frac{\partial \overline{\mathrm{u}}_i}{\partial x_j}}_{3}-\underbrace{\varepsilon_{ii}}_{4}+\underbrace{\frac{\partial\left(\nu \frac{\partial k}{\partial x_i}\right)}{\partial x_i}}_{5}-\underbrace{\frac{1}{\rho}\frac{\partial\left(\overline{P'\mathrm{u}_i'}\right)}{\partial x_i}}_{6}-\underbrace{\frac{1}{2}\frac{\partial}{\partial x_i}\left(\overline{\mathrm{u}_j'\mathrm{u}_j'\mathrm{u}_i'}\right)}_{7}.$$

The *k* PDE includes the following terms:

Term 1 = transient change (accumulation),

Term 2 = convection,

Term 3 = production term that arises from the product of the Boussinesq stress $\tilde{R}_{ij}$ and the mean velocity gradients. This quantifies the rate at which the mean flow imparts energy onto eddies,

Term 4 = dissipation rate. This represents the rate at which *k* is converted back into internal energy,

Term 5 = viscous (molecular) diffusion of turbulence energy,

Term 6 = turbulent transport associated with the eddy pressure and velocity fluctuation, and

Term 7 = diffusive turbulent transport resulting from the eddy triple correlation velocity fluctuations.

Note that Terms 1, 2, and 5 are described exactly in a mathematical and physical sense, with no need for “drastic surgery” conversions. The production term is reasonably modeled using the Boussinesq approximation or some form of nonlocal, nonequilibrium approaches [Speziale and Eringen, 1981; Hamba, 2005; Schmitt, 2007; Hamlington and Dahm, 2009; Wilcox, 2006; Spalart, 2015]. On the other hand, the remaining three terms, dissipation, eddy fluctuation due to the triple velocity correlation, and the pressure-velocity fluctuation correlation are not known, and thus require the dreaded “drastic surgery” described by Wilcox. On the brighter side, this situation applies to three terms in the *k* PDE, whereas the ε PDE requires the surgery for six terms, perhaps making the *k* PDE 50% more palatable!

The expression for Term 4, $\varepsilon_{ii}=\varepsilon,$ depends on which model is used. For example, Prandtl’s one-equation model assumes that

$$\varepsilon = \frac{C_D k^{3/2}}{\ell} .$$

On the other hand, the standard k-ε (SKE) model takes the moment approach to derive an ε PDE, analogous to the development of the *k* PDE.

For Terms 6 and 7, dimensional arguments loosely based on gradient transport are employed (refer to Chapter 2), in the manner of

$$-\overline{u'_i \phi'} = \nu_t \frac{\partial \overline{\phi}}{\partial x_i} .$$

In this case, the terms are lumped, with the following bold (and desperate!) assertion that

$$-\frac{1}{\rho}\overline{P' u'_i} - \frac{1}{2}\overline{u'_j u'_j u'_i} \approx \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i} .$$

Of course, neither $\overline{P' u'_i}$ nor $\overline{u'_j u'_j u'_i}$ have much to do with ν_t and the spatial gradient of *k*. To say the least, the analytical expression of the two terms is currently unknown, so they are reconfigured as a wishful and fanciful expression with no physical or mathematical basis, other than having the appropriate dimensional arguments! Presumably, the constant-valued "correction factor" σ_k helps the user make adjustments if experimental data should be available. In any case,

$$\underbrace{-\frac{1}{\rho}\frac{\partial\left(\overline{P' u'_i}\right)}{\partial x_i}}_{6} \underbrace{-\frac{1}{2}\frac{\partial}{\partial x_i}\left(\overline{u'_j u'_j u'_i}\right)}_{7} = -\frac{\partial\left(\frac{1}{\rho}\overline{P' u'_i} + \frac{1}{2}\overline{u'_j u'_j u'_i}\right)}{\partial x_i} \approx \frac{\partial\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right)}{\partial x_i} .$$

On the other hand, what else can turbulence researchers do when presented with such a dilemma? It is extremely difficult to obtain accurate measurements for experimental pressure and triple correlation terms. Fortunately, such approximation may be fairly harmless under simple turbulent flows. For example, DNS at *Re*=3,200 showed that Term 7 is of the same magnitude as the production term for $y^+ < 7$ (the viscous sublayer), while production was much larger for $10 < y^+ < 100$. For $y^+ > 100$, the two approached each other asymptotically [Mansour, Kim, and Moin, 1988]. On the other hand, the pressure gradient term (Term 6) was relatively smaller than the production term for $y^+ < 50$, and was of comparable magnitude for $y^+ > 50$. It is noted that these differences are expected to be magnified as *Re* increases. In summary, the lumping of Terms 6 and 7 to form the product of the turbulent kinematic viscosity and the spatial gradient of the turbulent kinetic energy can result in significant issues in complex flows involving high Re and two-phase flows [Mansour, Kim, and Moin, 1988; Sawko, 2012].

Nevertheless, it is unfortunate that the triple velocity correlation is "wished away" with a term that does not reflect its behavior. This is especially true, because the triple velocity correlation originates from the convective term, similarly to the crucial production term. Therefore, a more rigorous form of estimating its turbulent behavior is highly desirable.

Regardless, the k PDE is now solvable with the aforementioned transformations, approximations, and with an expression for the turbulent kinematic viscosity, ν_t,

$$\frac{\partial k}{\partial t} + \bar{\mathrm{u}}_i \frac{\partial k}{\partial x_i} = \tilde{R}_{ij} \frac{\partial \bar{\mathrm{u}}_i}{\partial x_j} - \varepsilon + \frac{\partial \left(\nu \frac{\partial k}{\partial x_i} \right)}{\partial x_i} + \frac{1}{\sigma_k} \frac{\partial \left(\nu_t \frac{\partial k}{\partial x_i} \right)}{\partial x_i}.$$

Notice that the RANS zero-, one-, and two-equation models assume ν_t is *isotropic*. Once an expression for ν_t is known (e.g., Prandtl-Kolmogorov for the k-ε model, $\frac{k}{\omega}$ for the k-ω model, etc.), then the entire set of equations required for closure is complete, and computational turbulence modeling may proceed at last!

Generally (with some exceptions), the lower the number of transport equations used, the faster the model will run for simple turbulent flows; by contrast, more complex behaviors can be analyzed as a higher the number of transport equations is used, but at the expense of longer computational time—at least in theory.

In any case, now that the k PDE has been derived, it is noted that its full expression, albeit an approximation, is rather complex. A full expansion of the k PDE shows just how complex k behavior can be

$$\begin{aligned}
&\frac{\partial k}{\partial t} + \bar{\mathrm{u}} \frac{\partial k}{\partial x} + \bar{\mathrm{v}} \frac{\partial k}{\partial y} + \bar{\mathrm{w}} \frac{\partial k}{\partial z} = \\
&\left\{ \left[\nu_t \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{u}}{\partial x} \right) - \frac{2}{3} k \right] \frac{\partial \bar{\mathrm{u}}}{\partial x} + \nu_t \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{\mathrm{v}}}{\partial x} \right) \frac{\partial \bar{\mathrm{u}}}{\partial y} + \nu_t \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) \frac{\partial \bar{\mathrm{u}}}{\partial z} \right\} \\
&+ \left\{ \nu_t \left(\frac{\partial \bar{\mathrm{v}}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \right) \frac{\partial \bar{\mathrm{v}}}{\partial x} + \left[\nu_t \left(\frac{\partial \bar{\mathrm{v}}}{\partial y} + \frac{\partial \bar{\mathrm{v}}}{\partial y} \right) - \frac{2}{3} k \right] \frac{\partial \bar{\mathrm{v}}}{\partial y} + \nu_t \left(\frac{\partial \bar{\mathrm{v}}}{\partial z} + \frac{\partial \bar{w}}{\partial y} \right) \frac{\partial \bar{\mathrm{v}}}{\partial z} \right\} \\
&+ \left\{ \nu_t \left(\frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} \right) \frac{\partial \bar{w}}{\partial x} + \nu_t \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{\mathrm{v}}}{\partial z} \right) \frac{\partial \bar{w}}{\partial y} + \left[\nu_t \left(\frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z} \right) - \frac{2}{3} k \right] \frac{\partial \bar{w}}{\partial z} \right\} - \varepsilon \\
&+ \frac{\partial}{\partial x} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x} \right] + \frac{\partial}{\partial y} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] + \frac{\partial}{\partial z} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial z} \right].
\end{aligned}$$

To better understand k, consider it as a fundamental metric of the energy contained within the swirling, dynamic, 3D, high-vorticity flow sheets that shape the eddies. It is this energy that shapes the fluid so that it rolls into curls, eddies, and all sorts of coherent structures—this is Reynolds' *eddying motion*. The eddy velocity is based on clusters of fluid that move in a coherent fashion, with fluctuating velocity u', and it is the square of the fluctuating velocity that provides a measure of the eddy's turbulent kinetic energy. If a mathematical definition of a root mean square (RMS) for a generalized n-space involving n velocities is considered, then

$$u' \equiv \sqrt{\frac{1}{n}\left(\overline{u_1'^2} + \ldots + \overline{u_n'^2}\right)}.$$

Thus, u' can be expressed in its generalized, anisotropic form in 3D space as a velocity RMS [Spiegel, 1999; SimScale, 2018],

$$u' = \sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}.$$

Because the square of the three velocities is summed and then divided by three, the quantity inside the square root represents an *average* velocity in this situation.

The velocity fluctuations can be expressed as a function of the turbulence kinetic energy, indicating that higher velocity fluctuations occur for the more energetic eddies,

$$u' = \sqrt{\frac{2}{3}k}\ .$$

If the above two equations are solved for *k*, then

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right),$$

which is the exact expression for the turbulent kinetic energy that was used by Prandtl in the development of his one-equation turbulence transport model [Prandtl, 1945]. Note that the above expression has a factor of 2 instead of 3, so it does not strictly follow the mathematical definition of an RMS, though it is quite often referred as such in the literature.

As noted later on in Chapter 4, Kolmogorov assumed a slight variation for the expression of *k* a few years earlier than Prandtl; refer to Kolmogorov's k-ω model. Nevertheless, Prandtl's definition prevailed, and is used predominantly in the literature.

At this point, *k* can be expressed as a function of u' for isotropic eddies because

$$\overline{u'^2} \approx \overline{v'^2} \approx \overline{w'^2}\ ,$$

and therefore,

$$k = \frac{3}{2}\overline{u'^2}\ .$$

This expression for *k* arises more rigorously from the integral of the energy spectral density across all the eddy wave numbers represented by the Greek letter kappa (κ, not to be confused with "*k*"). The function is shown in Figure 3.1 in Chapter 3, and its integration yields the total kinetic energy *k* held by the entire spectrum of eddies, from the largest to the smallest,

$$k = \int_0^\infty E(\kappa) d\kappa \ = \ \frac{3}{2}\overline{u'^2},$$

where the wavenumber

$$\kappa = \ \frac{2\pi}{\tilde{\lambda}}, \text{ e.g., } \frac{1}{\kappa} = \frac{\tilde{\lambda}}{2\pi} \ \equiv \text{ eddy size.}$$

The factor of 2π comes into play because κ is in terms of *radians* per unit length.

Finally, the development of the *k* PDE has a colorful history. The earliest version is attributed to Chou because he derived the *R* PDE and then proceeded to "*contract* the indices…to find the equation of '*energy transport*'" [Chou, 1940]. Note that "tensor contraction" is a generalized form of the trace. Chou's 1940 *k* PDE is as follows,

$$-R_{kj}\varepsilon_{kj} + \frac{1}{2}\frac{\partial\left[\overline{\left(\overline{\mathrm{u}}_j + \mathrm{u}_j'\right)q^2}\right]}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\left(\overline{\mathrm{u}_k' P'}\right)}{\partial x_k} + \frac{1}{2}\nu\nabla^2 q^2 - \nu g^{mn}\frac{\partial \mathrm{u}_k'}{\partial x_m}\frac{\partial \mathrm{u}_k'}{\partial x_n},$$

where, in Chou's notation,

$$R_{kk} \equiv q^2 = k = \overline{\mathrm{u}_i'\mathrm{u}_i'},$$

$\varepsilon_{kj} \equiv S_{ij}$ $\left(\text{Chou chose } \varepsilon \text{ as the deformation tensor}\right)$, and

$$g^{mn} = 1.$$

Upon rearrangement and some substitution,

$$-2R_{kj}S_{kj} + \frac{\partial\left[\overline{\left(\overline{\mathrm{u}}_j + \mathrm{u}_j'\right)q^2}\right]}{\partial x_j} = -\frac{2}{\rho}\frac{\partial\left(\overline{\mathrm{u}_k' P'}\right)}{\partial x_k} + \nu\nabla^2 k - 2\nu\frac{\partial \mathrm{u}_k'}{\partial x_m}\frac{\partial \mathrm{u}_k'}{\partial x_n},$$

where

$$\frac{\partial\left[\overline{\left(\overline{\mathrm{u}}_j q^2 + \mathrm{u}_j' q^2\right)}\right]}{\partial x_j} = \frac{\partial\overline{\left(\overline{\mathrm{u}}_j k\right)}}{\partial x_j} + \frac{\partial\overline{\left(\mathrm{u}_j'\mathrm{u}_i'\mathrm{u}_i'\right)}}{\partial x_j} = \overline{\mathrm{u}}_j\frac{\partial k}{\partial x_j} + k\frac{\partial \overline{\mathrm{u}}_j}{\partial x_j} + \frac{\partial\overline{\left(\mathrm{u}_j'\mathrm{u}_i'\mathrm{u}_i'\right)}}{\partial x_j}.$$

Furthermore, employing the modern definition for the dissipation symbol ε,

$$\varepsilon_{diss} = 2\nu\frac{\partial \mathrm{u}_k'}{\partial x_m}\frac{\partial \mathrm{u}_k'}{\partial x_n}.$$

Therefore, Chou's *k* PDE in more modern terms, and reorganized, is as follows,

$$\bar{\mathrm{u}}_j \frac{\partial k}{\partial x_j} + k\frac{\partial \bar{\mathrm{u}}_j}{\partial x_j} = 2R_{kj}S_{kj} - \varepsilon + \frac{\partial\left(\nu \frac{\partial k}{\partial x_i}\right)}{\partial x_i} - \frac{2}{\rho}\frac{\partial\left(\overline{\mathrm{u}'_k P'}\right)}{\partial x_k} - \frac{\partial\overline{\left(\mathrm{u}'_j \mathrm{u}'_i \mathrm{u}'_i\right)}}{\partial x_j}.$$

Of course, now that turbulence modelers have more information, the last two terms in Chou's formulation can be lumped using dimensional arguments based on gradient transport (as discussed earlier in this section), whereby

$$-\frac{1}{\rho}\overline{\mathrm{u}'_i P'} - \frac{1}{2}\overline{\mathrm{u}'_j \mathrm{u}'_j \mathrm{u}'_i} \approx \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}.$$

If so, then Chou's *k* PDE would be the equivalent of the following expression,

$$\bar{\mathrm{u}}_j \frac{\partial k}{\partial x_j} + k\frac{\partial \bar{\mathrm{u}}_j}{\partial x_j} = 2R_{ij}\frac{\partial \bar{\mathrm{u}}_i}{\partial x_j} - \varepsilon + \frac{\partial\left(\nu \frac{\partial k}{\partial x_i}\right)}{\partial x_i} + \frac{2}{\sigma_k}\frac{\partial\left(\nu_t \frac{\partial k}{\partial x_i}\right)}{\partial x_i}.$$

Comparing Chou's *k* PDE with the modern version shows that there are a significant number of terms in common. Note that the transient term is missing in Chou's formulation because he attempted to apply average values "with respect to time over a period τ".

In summary, the RANS closure process is sketched from a "bird's eye view" in Figures 4.1A and B, with the hope that the overall approach is clearer, once the full details of the derivations are covered!

1. Navier Stokes $\xrightarrow{\text{TIME-AVERAGE}}$ RANS :

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}-\frac{1}{\rho}\frac{\partial P}{\partial x_i}\xrightarrow{\text{TIME-AVERAGE}}\frac{\partial \overline{u}_i}{\partial t}+\overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j}=\nu\frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j}-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i}-\frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

2. RANS $\xrightarrow{\text{MULT. BY U}', \text{TIME AVE.}}$ Exact Reynolds stress tensor PDE :

$$(RANS)\xrightarrow{\text{MULT. BY U}', \text{TIME AVE.}}\frac{\partial \overline{u_i' u_j'}}{\partial t}+\overline{u}_k\frac{\partial \overline{u_i' u_j'}}{\partial x_k}=$$

$$-\left(\overline{u_i' u_k'}\frac{\partial \overline{u}_j}{\partial x_k}+\overline{u_j' u_k'}\frac{\partial \overline{u}_i}{\partial x_k}\right)+2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}+\frac{\partial}{\partial x_k}\left(\nu\frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right)+\frac{1}{\rho}\left(\overline{u_i'\frac{\partial P'}{\partial x_j}}+\overline{u_j'\frac{\partial P'}{\partial x_i}}\right)+\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right)$$

3. Exact Reynolds stress tensor PDE $\xrightarrow{\text{TRACE}}$ Exact k PDE :

$$(Exact\ Reynolds\ stress\ tensor\ PDE)\xrightarrow{\text{TRACE}}\frac{\partial k}{\partial t}+\overline{u}_i\frac{\partial k}{\partial x_i}=\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}-\varepsilon_{ii}+\frac{\partial\left(\nu\frac{\partial k}{\partial x_i}\right)}{\partial x_i}$$

$$-\frac{1}{\rho}\frac{\partial\left(\overline{P' u_i'}\right)}{\partial x_i}-\frac{1}{2}\frac{\partial}{\partial x_i}\left(\overline{u_j' u_j' u_i'}\right)$$

Figure 4.1A. Bird's eye view of the RANS, *k*, and *R* PDE derivation, Part 1.

4. Exact k PDE $\xrightarrow{\text{ENGINEERING APPROXIMATIONS}}$ Solvable k PDE :

$$(Exact\ k\ PDE)\xrightarrow{\text{ENGINEERING APPROXIMATIONS}}\frac{\partial k}{\partial t}+\overline{u}_i\frac{\partial k}{\partial x_i}=\tilde{R}_{ij}\frac{\partial \overline{u}_i}{\partial x_j}-\varepsilon+\frac{\partial\left(\nu\frac{\partial k}{\partial x_i}\right)}{\partial x_i}+\frac{1}{\sigma_k}\frac{\partial\left(\nu_t\frac{\partial k}{\partial x_i}\right)}{\partial x_i}$$

$$\text{where } \tilde{R}_{ij}=-\overline{u_i' u_j'}=-\begin{bmatrix}\overline{u'u'} & \overline{u'v'} & \overline{u'w'}\\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'}\\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'}\end{bmatrix}\equiv\nu_t\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)-\frac{2}{3}k\delta_{ij}\quad\text{(Boussinesq Approximation)}$$

Figure 4.1B. Bird's eye view of the RANS, *k*, and *R* PDE derivation, Part 2.

4.5 The Choice of Transport Variables

So, why the various choices of transport variables and which might be best? When viewed solely from a pure dimensionless perspective (as many turbulence researchers have often done), the situation reduces to the following issue: How can a mathematical expression for ν_t be derived such that it is formed solely from relevant transport variables? This reasoning, of course, is diametrically opposite to the approach where physical principles associated with turbulence are considered, so model issues routinely appear. Nevertheless, the dimensionless approach has resulted in plenty of great numerical simulations, and it is difficult to argue with success! Thus, as discussed previously, *k* is used because its square root yields a reasonable transport eddy scale—a turbulence velocity. So, researchers reason that if analytical expressions for ν_t and *k* are not available, then, what combination of *k* and some other transport variable can form ν_t, thereby reaching mathematical closure? In other words, closure is sought such that

$$\nu_t = \nu_t\left(k;\omega,\varepsilon,\ell,t...\right).$$

In SI units, ν_t is in units of m^2/s. Thus, the goal is to find a second transport variable that is 1) internally-consistent with *k*, 2) has a plausible chance of describing the eddy behavior as transport variable "*x*", and 3) conveniently combines with *k* to form an expression that has units of m^2/s,

$$\nu_t = \nu_t\left(k,x\right)$$

This is very convenient, as the introduction of the second transport variable is required to form part of the expression for ν_t*; therefore, no more variables are typically introduced, so closure is easier to approach.*

Some examples of this dimensionless approach are shown in Table 1 the following pairs of variables in the curly brackets. Certainly, this approach is not confined to 1 or two transport variables, and models with three variables can be found in the literature [Wilcox, 2006].

Prandtl used the following expressions for his one transport-variable model [Prandtl, 1945],

$$\nu_t = C_D\frac{k^2}{\varepsilon}$$

and

$$\varepsilon = C_D\frac{k^{3/2}}{\ell}.$$

Table 1. Transport variable development for turbulence kinematic viscosity relationships.

Transport Variable Pair	Variable Formulation	Turbulence Kinematic Viscosity Expression

$\{k,\ \omega\}$	$\omega \approx \frac{k^{1/2}}{\ell}$	$\nu_t \approx \frac{k}{\omega}$ [Kolgomorov, 1942]
$\{k,\ \varepsilon\}$	$\varepsilon \approx \frac{k^{3/2}}{\ell}$	$\nu_t \approx \frac{k^2}{\varepsilon}$ [Chou, 1945; Taylor suggested $\varepsilon \approx \frac{k^{3/2}}{\ell}$ in 1935.]
$\{k,\ \ell\}$	$\ell \approx \frac{k^{3/2}}{\varepsilon}$	$\nu_t \approx k^{1/2}\ell$ [Rota, 1951, 1968; Taylor suggested $\varepsilon \approx \frac{k^{3/2}}{\ell}$ in 1935.]
$\{k,\ t\}$	$t \approx \frac{\ell}{k^{1/2}}$	$\nu_t = c_\mu kt$ (t=turbulence dissipation time) [Zeierman and Wolfshtein, 1986].

In general, the above models can be summarized with a single equation, such that

$$\nu_t = \alpha k^\beta x^\gamma$$

where

k = kinetic energy (first transport variable),

x = second transport variable, and

α, β, γ are constants chosen such that ν_t is in units of length squared per unit time.

Values for the constants are summarized in Table 2 for several well-known turbulence models. Other formulations can be found in the literature [Rodi, 1993; Andersson et al., 2012], such as

$$\phi = k^\alpha \ell^\beta.$$

Table 2. Transport variables for turbulence equations.

ν_t	***x* description**	α	β	γ
$\nu_t = \frac{k}{\omega}$	x = ω, eddy frequency. (Kolmogorov, k-ω)	0	1	-1
$\nu_t = k^{1/2}\ell$	x = ℓ, eddy length scale. Prandtl 1-Eqn. model with constitutive relationship for ℓ; Rotta two-equation model [Rotta, 1962].	0	1/2	1

	$\ell = \ell(y) = \kappa y = 0.41y$			
$\nu_t = C_\mu \frac{k^{3/2}}{\ell}$	x = ℓ, eddy length scale, 2-Eqn. model.	C_μ=0.09	3/2	-1
$\nu_t = C_\mu \frac{k^2}{\varepsilon}$ (Prandtl-Kolmogorov)	*x* = ε, dissipation rate (Taylor, Chou, SKE, RNG KE, RKE)	C_μ=0.09; C_μ=0.09$f(\mathrm{Re})$	2	-1
$\nu_t = \alpha\ell$	*x* = ℓ, eddy length scale; α is a constant in units of "average" eddy velocity.	---	0	1
$\nu_t = C_\mu kt$	*x* = *t* (or τ), eddy time scale [Zeierman and Wolfshtein, 1986; Speziale, Abid, and Anderson, 1992]	C_μ=0.09	-1/2	1
$\nu_t = \gamma \frac{k}{\omega}$	*x* = ω^2, where ω^2 is the mean vorticity scale for the energetic (large) eddies [Saffman, 1970].	0	1	-2

4.6 Top RANS Turbulence Models

4.6.1 Zero-equation models

The zero-equation models (also known as algebraic turbulence models) require no additional transport closure equations, and hence the naming convention, "zero".

In general, zero-equation models tend to have the following advantages:

- Fastest computationally,
- Easiest to code and tend to be very robust numerically,
- Great for developing analytical solutions and theoretical insights from PDEs (e.g., the Prandtl mixing-length model can be readily substituted into a RANS PDE),
- Form the simplest turbulence models, while providing some physical insights,
- Because of the method the models were developed and calibrated, these models reasonably predict certain complex turbulent flows (e.g., Baldwin-Lomax is reasonable for turbomachinery, aerospace, and applications that have attached thin boundary layers).

Some general disadvantages of the zero-equation models are as follows:

- Very limited applicability. They may be great for thin boundary layers, but fall flat outside of their intended, limited space!
- Successful mostly for very simple flows. (Wilcox found such a model as a "pleasant surprise despite its theoretical shortcomings".)
- Provide no transport of turbulent scales (no velocity, length, or some other appropriate variable).

Despite the complexity of turbulent flows, algebraic models serve a unique role in turbulence modeling, and there are dozens of such models. Provided the caveats are understood, three well-known models include Prandtl mixing length, Baldwin-Lomax, and Cebeci-Smith.

4.6.2 Prandtl's one-equation model

For one-equation models, a turbulence transport variable is needed; such variable is predominantly the turbulence kinetic energy, *k*, in conjunction with auxiliary closure expressions. Ludwig Prandtl developed the first one-equation *full-closure* turbulent transport model, where *k* was his transport choice [Prandtl, 1945].

Note that the first two-equation model was originated by Kolmogorov three years earlier (refer to Section 4.6.3.1). Kolmogorov used a *k* transport PDE (using *b* for his naming convention), but did not include a full set of transport parameters. Prandtl's *k* transport formulation continues to be the modern standard for the *k* PDE, and is expressed as follows,

$$\frac{\partial k}{\partial t} + \bar{\mathrm{u}}_{\mathrm{j}} \frac{\partial k}{\partial x_j} = R_{ij} \frac{\partial \bar{\mathrm{u}}_{\mathrm{i}}}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right],$$

where

$$R_{ij} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3} k \delta_{ij} .$$

Prandtl provided closure by defining dissipation, the turbulent kinematic viscosity, and the eddy length scale, respectively,

$$\varepsilon = \frac{C_D k^{3/2}}{\ell},$$

$\nu_t = \ell k^{1/2}$, and

$\ell = \kappa \mathrm{y}$.

The closure coefficients are

$\sigma_k = 1.0$,

$\kappa = 0.41$ = Kolmogorov's constant,

$C_D = 0.3$ (for shear flows), and

$C_D \approx C_\mu = 0.085$ (in general).

4.6.3 Two-equation models

As a result of the ubiquitous advent of computers in the workplace, the two-equation turbulence models gained popularity, and formed the basis for much of the turbulence simulations over the past 40 years. As their name implicates, these models are primarily based on finding two key variables that are self-consistent and suitable for forming an expression having the same units as ν_t; these variables must also uniquely describe the turbulence scale (eddy) behavior. For example, turbulent flow has eddies with length scales (e.g., ℓ, λ, η) that move about chaotically at some velocity *u'*. By the same token, the eddies also decay within certain times *t* or frequency ω, dissipate with some magnitude ε, accelerate about with acceleration *a*, and so forth.

Practically all two-equation turbulence models employ the Boussinesq approximation and use the *k* transport variable. Note that since the early 1980s, the accuracy of the *k* PDE has been compared successfully with many experiments and DNS simulations, but the same cannot be said for many choices for the second transport equation. Though they are generally suitable for many applications, two-equation models are notoriously inaccurate for non-equilibrium, non-isotropic flows, and hence inclusion of nonlocal, nonequilibrium approaches instead of the traditional form of the Boussinesq approximation [Speziale and Eringen, 1981; Hamba, 2005; Schmitt, 2007; Hamlington and Dahm, 2009; Wilcox, 2006; Spalart, 2015]. Furthermore, the introduction of a second transport variable PDE (e.g., ε, ω), can cause numerical difficulties and yield anomalous results if not coded and coupled correctly.

For the two-equation model closure, *k* is usually chosen, as well as one additional transport variable. Typically, the second closure equation involves one of the following variables: ω, ε, τ (time), ζ (enstropy), *a* (acceleration), and so forth. The list can be as extensive as the researcher's imagination, so long as dimensionless analysis allows the combination of the two transport variables to form a mathematical expression that has units of length squared per unit time—the units for ν_t.

4.6.3.1 The Kolmogorov 1942 k-ω

The first two-equation transport model ever developed is fast-approaching the century mark [Kolmogorov, 1942]. Kolmogorov's original model is discussed here for various reasons. First, it was the precursor of the modern k-ω turbulence models, forms the basis for other two-equation RANS models, and sheds light on the appropriateness of the k-ε models. Though developed too early to take advantage of modern computational power, Kolmogorov's model contributed to modern analysis of turbulence by inferring that turbulence can be modeled in momentum-like terms with transport variables that reflect key turbulence scales such as velocity and length.

Kolmogorov began his model by noting that turbulence flows consist of "turbulent pulsations" (eddies) that range from large to small scales that are superimposed onto the mean flow. The large scales are the integral eddies, which correspond to the length scale ℓ found in modern turbulence research (Kolmogorov originally used "*L*"). He also used "λ" to refer to the smallest eddy scale at which the fluid viscosity dampens the eddies; in modern notation, λ is used for the Taylor eddies. In Kolmogorov's honor, the smallest eddies are called "Kolmogorov eddies", with modern notation using the symbol "η". Kolmogorov proceeded with his two-equation model development by noting that the large eddies constantly extract energy from the mean flow, and in turn, their energy is transferred onto smaller eddies. Finally, only the smallest eddies (the Kolmogorov eddies) participate in a decay process whereby they transform from tiny chaotic structures onto tiny laminar regions as the dissipation of their energy is brought forth by the viscous force of the fluid. Kolmogorov described the transport of the turbulent kinetic energy as a "stream of energy, flowing constantly". That is, he envisioned that a continuous transfer of energy existed in the fluid continuum. The process is now called "cascading". *These phenomena insights are crucial, as dissipation applies to Kolmogorov eddies, while energy transfer from the mean flow involves all three scales—integral, Taylor, and Kolmogorov*. In fact, Kolmogorov referred to his second transport variable (ω) as applicable to all three scales [Kolmogorov, 1942],

> "A fundamental characteristic of the turbulent motion at all scales is the quantity ω which stands for the rate of dissipation in unit volume and time."

Thus, if equilibrium is assumed, then the decay rate equals the production rate, and ω rightfully describes the energy dissipation rate under equilibrium conditions as experienced throughout the cascade—from the integral to the Kolmogorov scales. This physical description of turbulence energy flow strongly suggests which variables are appropriate, and most importantly, *which form an internally-consistent pair of variables for the development of two-equation turbulence models*.

Kolmogorov continued his model development by postulating that ω is a "fundamental" quantity that is applicable to the entire length spectrum, because the eddy energy transfer is assumed constant—in equilibrium. Certainly, this need not always be the case in all turbulent flows, but is a reasonable approximation under many situations—a good starting point. Kolmogorov envisioned this fundamental quantity ω as representing the dissipation rate in a flow volume per unit time, or as a "mean frequency". Thus, ω is construed from an equilibrium point of view, applicable to the entire eddy spectrum, whereby turbulent energy transfer occurs as a "continuous stream of energy" from the mean flow, and that cascades throughout all the eddy scales. This approach is consistent with Hinze, who considered that "ε is on the one hand practically equal to the dissipation in the highest-wavenumber range and on the other hand practically equal to the work done by the energy-containing eddies, which is the energy supplied to the smaller eddies" [Hinze, 1987]. Saffman considered ω^2 as the mean vorticity scale for the energetic (large) eddies [Saffman, 1970].

Thus, *k* and ω are self-consistent turbulence transport properties: *k* represents the large eddies that typically hold about 80% of the total turbulent kinetic energy, while ω considers the energy dissipation of the large eddies (as well as the smaller eddies). Said briefly, both k and ω are associated with the same eddy scale size, and are thus a self-consistent pair of transport variables.

Kolmogorov defined his turbulence model using three partial differential equations, as follows. He started with the time-averaged velocity for the RANS-based momentum equation,

$$\frac{D\bar{\mathrm{v}}_i}{Dt} = F_i - \frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho} + b\right) + A\sum_j \frac{\partial}{\partial x_j}\left[\frac{b}{\omega}\left(\frac{\partial \bar{\mathrm{v}}_i}{\partial x_j} + \frac{\partial \bar{\mathrm{v}}_j}{\partial x_i}\right)\right].$$

Then, the turbulent mean frequency ω is modeled with the following transport equation,

$$\frac{D\omega}{Dt} = -\frac{7}{11}\omega^2 + A'\sum_j \frac{\partial}{\partial x_j}\left[\frac{b}{\omega}\frac{\partial \omega}{\partial x_j}\right],$$

while the averaged root-mean-square (RMS) of the fluctuating velocities, b, is

$$\frac{Db}{Dt} = -b\omega + \frac{1}{3}A\frac{b}{\omega}\varepsilon + A''\sum \frac{\partial}{\partial x_j}\left[\frac{b}{\omega}\frac{\partial b}{\partial x_j}\right].$$

As will be explained shortly b is a measure of k. But before doing so, note that Kolmogorov used the following notation,

$\frac{D}{Dt}$ = substantial derivative,

F_i = external force,

$\bar{\mathrm{v}}_i$ = time-averaged velocity,

$\bar{p}$ = time averaged pressure,

$b \equiv \frac{1}{3}\sum_j \overline{\mathrm{v}_j'^2}$ = RMS expression for fluctuation velocity, and

$$\varepsilon = \sum_{i,j}\left(\frac{\partial \bar{\mathrm{v}}_i}{\partial x_j} + \frac{\partial \bar{\mathrm{v}}_j}{\partial x_i}\right)^a.$$

In this context, a, A, A', A'', and c are closure constants that have yet to be determined experimentally. Note that Kolmogorov's ε symbol in the third partial differential equation (the b transport PDE) is not to be confused with the modern symbol for dissipation (the first k-ε model was developed three years *after* Kolmogorov's k-ω turbulence model). Letting $a = 1$ shows that in his context, ε is actually twice the modern mean strain rate tensor,

$$\varepsilon = \sum_{i,j}\left(\frac{\partial \bar{\mathrm{v}}_i}{\partial x_j} + \frac{\partial \bar{\mathrm{v}}_j}{\partial x_i}\right) \equiv 2\vec{\vec{S}}.$$

For the sake of clarity, Kolmogorov's ε will *not* be used outside of this section. Furthermore, in Kolmogorov's context, *b* is the turbulent kinetic energy *k*, whence

$$\omega \equiv c\frac{b^{1/2}}{L} \approx c\frac{k^{1/2}}{\ell}$$

where

ℓ = characteristic eddy length

and

$$b \equiv \frac{1}{3}\sum_j \overline{\mathrm{v}_j'^2} = \frac{1}{3}\left(\overline{u'^2} + \overline{\mathrm{v}'^2} + \overline{w'^2}\right).$$

Thus, Kolmogorov's turbulent kinetic energy is

$$k \approx b = \frac{1}{3}\left(\overline{u'^2} + \overline{\mathrm{v}'^2} + \overline{w'^2}\right),$$

which is an RMS in the strict sense, mathematically speaking. As noted in Chapter 4, Prandtl used a slightly different definition [Prandtl, 1945; Spalding, 1991],

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{\mathrm{v}'^2} + \overline{w'^2}\right).$$

Thus, there is a slight difference amongst the Kolmogorov and Prandtl turbulent kinetic energy definitions. For clarity, it is pointed out that the modern literature uses the Prandtl notation fairly exclusively, and is therefore followed in this book.

As noted by various researchers, Kolmogorov's ω transport equation does not include a production term [Spalding, 1991; Bulicek and Malek, 2000; Wilcox, 2006; Mielke and Naumann, 2015]. The literature indicates that this issue can be addressed to diverse degrees of success by including an ω source boundary at the wall [Spalding, 1991], or a fixed nonhomogeneous Dirichlet boundary [Bulicek and Malek, 2000], as well as periodic boundaries [Mielke and Naumann, 2015].

Wilcox was the first to introduce an ω production term into the ω transport PDE, as is reflected in his 1988 k-ω model, where the production term P_ω is

$$P_\omega = \alpha\frac{\omega}{k}R_{ij}\frac{\partial \overline{\mathrm{u}}_{\mathrm{i}}}{\partial x_j}.$$

However, it is noted that Kolmogorov did include a *b* production term that is a function of ω, namely

$$P_k = \frac{1}{3}A\frac{b}{\omega}\varepsilon = \frac{2}{3}A\frac{b}{\omega}\overline{S}$$

(recall Kolmogorov's non-standard definition for ε).

Because the Kolmogorov transport equations are coupled and P_k is a function of ω, then at least there is a connection between turbulent kinetic energy production and ω.

Note that ω is in units of inverse time, and is therefore a fundamental turbulence frequency associated with the time it takes for eddies to transfer their energy. Based on dimensionless arguments, the frequency is a function of the turbulent kinetic energy *k* and the dissipation ε,

$$\omega \approx \frac{\varepsilon}{k}.$$

Wilcox expressed Kolmogorov's ω transport model in more modern terms [Wilcox, 2006], and added an ω production term (the first term on the RHS of the arrow), viz.,

$$-\frac{7}{11}\omega^2 + A'\sum_j \frac{\partial}{\partial x_j}\left[\frac{b}{\omega}\frac{\partial \omega}{\partial x_j}\right] \;\rightarrow\; \gamma\frac{\omega}{k}R_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - \beta\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma\nu_t\right)\frac{\partial \omega}{\partial x_j}\right]$$

whereby

$$\beta = \frac{7}{11},$$

$A' = \sigma$, and

$$\nu_T = \frac{b}{\omega}.$$

4.6.3.2 Wilcox 1988 k-ω

The 1988 model is included here to show the evolutionary progress of the k-ω models. Though the 1988 k-ω can be quite sensitive to free stream boundary conditions [Menter, 1992; Menter, Kuntz, and Langtry, 2003; Wilcox, 2006], *it cannot be overemphasized that this issue was fixed in the 2006 k-ω version. To be clear, the 2006 k-ω supersedes the 1988 and 1998 k-ω versions.*

Based on Prandtl's *k* transport and Kolmogorov's ω transport, Wilcox developed an improved k-ω model, which is now known as the 1988 Wilcox k-ω turbulence model [Wilcox, 1988A; Wilcox, 1993]. A similar version, but with disalignment between the Reynolds stress tensor and the mean strain rate, was presented on the same year [Wilcox, 1988B]. Wilcox also developed a 1988 k-ω variant that is suitable for laminar to turbulent transition [Wilcox, 1994]. The literature indicates that there are many k-ω hybrid models [Saffman, 1970; Wilcox, 1993; Bredberg, 2000; Wilcox, 2006; Gorji et al., 2014; NASA1, 2018], so the specific version should always be referenced.

The 1998 model is elegant because it captures the major elements of transport for both *k* and ω, while having neither limiters nor blending functions, and requiring only a minimal number of closure coefficients. The 1988 k-ω is very useful for low *Re* and near wall boundary layers.

The turbulent kinetic energy transport for the 1988 k-ω is

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\beta^* k\omega+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma^*\nu_t\right)\frac{\partial k}{\partial x_j}\right],$$

while the turbulent frequency ω (specific dissipation rate) is

$$\frac{\partial \omega}{\partial t}+\bar{u}_j\frac{\partial \omega}{\partial x_j}=\gamma\frac{\omega}{k}R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma\nu_t\right)\frac{\partial \omega}{\partial x_j}\right].$$

The model has the following six closure coefficients [Wilcox, 1988A; Wilcox, 1988B; Wilcox, 1993; Bredberg, 2000],

$$\gamma = \frac{5}{9},$$

$$\gamma^* = 1,$$

$$\beta = \frac{3}{40},$$

$$\beta^* = \frac{9}{100},$$

$\sigma = \frac{1}{2}$, and

$$\sigma^* = \frac{1}{2}.$$

Its final closure relationship is

$$\nu_t = \frac{k}{\omega}.$$

4.6.3.3 1998 k-ω

A decade later, Wilcox overhauled his 1988 version to generate the 1998 k-ω model [Wilcox, 2004]. On the one hand, the 1998 version resulted in significant advances for k-ω models as Wilcox experimented with various recent turbulence modeling advances. For example, this was the first time whereby he added a cross-diffusion term, albeit it had a strong ω function to the third power; by comparison, note that the 2006 model has a functionality to the first power. It is very likely that the lower power dependency significantly decreased k-ω numerical stiffness. In addition, other 1998 model advancements included the inclusion of a vorticity tensor, and this is certainly ideal for modeling rotating coherent structures. To say the least, these ideas fostered Wilcox's attempt to improve the 1988 k-ω, and many of these concepts would later prove very beneficial in his future k-ω models, especially the 2006 k-ω.

But, on the other hand, there was some de-evolution in k-ω development, mainly because the 1998 model included a disconcerting number of closure coefficients and blending functions. And yes, this direction went against Wilcox's fundamental belief that "an ideal model should introduce the minimum amount of complexity while capturing the essence of the relevant physics." For these reasons, and because the 2006 k-ω model replaces the 1998 model, a description of the 1998 version is not included here.

4.6.3.4 2006 k-ω

Fortunately, a little less than a decade after his 1998 model, in a stroke of genius, Wilcox not only reduced the number of blending functions in half, but also improved its modeling characteristics in a significant manner [Wilcox, 2006; Wilcox, 2007]. For example, he added a revised cross-diffusion term to vastly reduce boundary-condition sensitivity to the free stream. In addition, the new cross-diffusion model had a weaker ω dependency (first- vs. third-power dependency compared with the 1998 version). Then, Wilcox went further by adding a stress limiter to better simulate flow detachment, incompressible flows, as well as transonic flows; neither the 1988 or 1998 versions included the limiter. Finally, the 2006 model only requires six closure coefficients (not including the limiter).

The 2006 transport *k* and ω models are, respectively,

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\beta^* k\omega+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma^*\frac{k}{\omega}\right)\frac{\partial k}{\partial x_j}\right]$$

and

$$\frac{\partial \omega}{\partial t}+\bar{u}_j\frac{\partial \omega}{\partial x_j}=\alpha\frac{\omega}{k}R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma\frac{k}{\omega}\right)\frac{\partial \omega}{\partial x_j}\right]+\sigma_d\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}.$$

The Reynolds stresses (*R*) are obtained from the Boussinesq relationship.

The last term of the ω PDE model is the cross-diffusion term, and was included to minimize sensitivity to boundary conditions. Near the wall, the leading coefficient is 0, so cross diffusion is suppressed. The coefficient is of higher consequence for free shear flows, where ω production is increased via cross-diffusion.

The 2006 model has the following closure relationships. Starting with the turbulent kinematic viscosity,

$$\nu_t=\frac{k}{\tilde{\omega}}.$$

The specific dissipation rate (frequency) has the following stress-limiter for improving separated flows and incompressible flows up to transonic *Ma*,

$$\tilde{\omega} = \max\left(\omega,\ C_{lim}\sqrt{\frac{2S_{ij}S_{ij}}{\beta^*}}\right)$$

The stress limiter only becomes active for large mean strain rate (S_{ij}) magnitudes, such as occur for strongly-separated flows and high *Ma* flows.

The blending function, f_β is defined as

$$f_\beta = \frac{1+85\chi_\omega}{1+100\chi_\omega}.$$

The blending function ranges from a minimum of 0.85 as χ approaches large values, to a peak value of 1.0 as χ approaches 0. Thus, the blending function approaches 1.0 for near wall flows, and 0.85 for shear flows; see Figure 4.2.

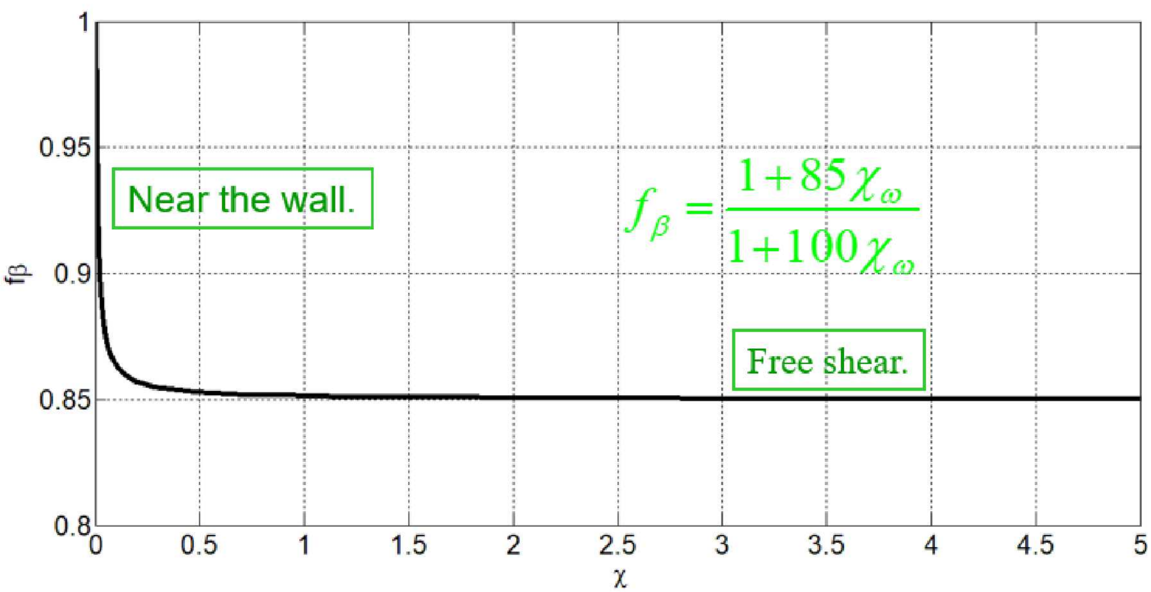

Figure 4.2. The 2006 k-ω blending function.

χ is the so-called "non-dimensional vortex stretching", and is defined as

$$\chi_\omega \equiv \left|\frac{\Omega_{ij}\Omega_{jk}S_{ki}}{(\beta^*\omega)^3}\right|.$$

Notice the introduction of the vorticity tensor Ω, which is an ideal function for expressing rotational fluid mechanics. Because turbulent coherent structures involve 3D sheets that fold with spiral-like curvature, involving the mean rotational tensor is highly intuitive.

The mean rotational and mean strain rate tensors, respectively, are defined as

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial \overline{\mathrm{u}}_i}{\partial x_j} - \frac{\partial \overline{\mathrm{u}}_j}{\partial x_i}\right)$$

and

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{\mathrm{u}}_i}{\partial x_j} + \frac{\partial \overline{\mathrm{u}}_j}{\partial x_i}\right).$$

The inclusion of χ_ω is an attempt to quantify vortex stretching as the energy cascade proceeds from the larger to the smaller eddies; rightly so, χ_ω= 0 for 2D flows. However, the vortex stretching phenomenon is much more complex than anticipated. For example, recent investigations for homogeneous isotropic flows show that the larger eddies tend to be perpendicular to the fastest-stretching eddies, while the orientation of the larger eddies compared with the stretching associated with the smaller vortices tends to be at 45° [Hirota et al., 2017].

To compensate for near wall and free shear effects, Wilcox uses the following relationship based on the product of the *k* and ω spatial gradients,

$$\sigma_d = \begin{cases} 0, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} \le 0 \ (\text{near the wall}) \\ \sigma_{do}, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} > 0 \ (\text{free shear}) \end{cases}$$

The model has the following closure coefficients,

$$\alpha = \frac{13}{25},$$

$$\beta = \beta_0 f_\beta,$$

$$\beta^* = \frac{9}{100},$$

$$\beta_0 = 0.0708,$$

$$\sigma = \frac{1}{2},$$

$\sigma^* = \dfrac{3}{5}$, and

$$\sigma_{do} = \frac{1}{8}.$$

The stress limiter coefficient is

$$C_{lim} = \frac{7}{8}.$$

Based on the above definitions and dimensional arguments, the following quantities can be obtained, is so desired,

$$\ell = \frac{k^{1/2}}{\omega},$$

$$t = \frac{1}{\omega},$$

and

$$\varepsilon = \beta^{*} \omega k.$$

Wilcox proudly tested his 2006 model on approximately 100 relevant test cases, including

- Attached boundary layers,
- Free shear flows (far wake, mixing layer; plane, round, and radial jets),
- Backward-facing steps,
- Shock-separated flows,
- Mach number ranging from compressible to hypersonic flows, and
- Fluids undergoing heat transfer.

Therefore, the model is recommended for free shear flows, including far wakes, mixing layer, strongly-separated flows, as well as plane, round, and radial jets [Wilcox, 2006]. It is also suitable for flows with a high degree of swirl [Rodriguez, 2011]. Comparisons with experimental data are typically very good, and earlier-model k-ω issues involving free shear flows, strongly-separated flows, and high *Ma* flows have been greatly improved. *For these reasons, the 2006 k-ω model is strongly-recommended vs. the 1988 and 1998 versions.* Furthermore, the 2006 k-ω is a good, all-around model for both near-wall and free stream turbulence, and is therefore a great contender vs. Menter's SST model.
Finally, note that the 2006 k-ω clearly supersedes the 1988 and 1998 k-ω models because it eliminates issues regarding sensitivity to boundary conditions and versatility [Wilcox, 2006]. Indeed, unlike its 1988 and 1998 predecessors, the 2006 k-ω is suitable for both free shear and near-wall flows, subsonic, supersonic, and hypersonic flows. Therefore, modern analysts and researchers should not use the earlier versions if the 2006 k-ω is available. To do otherwise is at the expense of finding "issues" that were corrected long ago by Wilcox. This is pointed out here explicitly, as there are many papers in the 2010-2018 time frame that still use the 1988 k-ω model, and whose work is therefore not cited here as a courtesy.

4.6.3.5 The standard k-ε model

The two-equation transport model, now known as the standard k-ε (SKE) turbulence model, traces its origins to Pei Yuan Chou's eddy decay research, whose ideas were later adapted and extended by various researchers. *Chou developed the notion that k can be associated with vorticity (ω) decay.* Then, what we now refer in more modern times as the dissipation ε term based on the derivatives of the fluctuating velocities, was derived independently by Chou. Chou *associated ε decay with the Taylor length scale, λ* [Chou, 1940; Chou, 1945]. Ironically, Chou proceeded to develop a vorticity transport dissipation model based on ω for Taylor eddies (not Kolmogorov eddies); the present authors consider this as highly ironic because the origin of the k-ε model is based on the development of a k-ω formulation! In Chou's words,

> "Since Taylor's scale of micro-turbulence λ plays a very important role in the decay of turbulence, it is necessary to find the equation which governs the behaviour of this fundamental length. This equation is provided by the decay of the vorticity."

Note that Chou referred "dissipation" as "decay". His original ω PDE transport model is found in Equation 1.4 of his original paper [Chou, 1945], and is reproduced here for convenience, to trace how Chou's original concept for ω transport evolved onto the ε transport model. Note that Chou uses w_i and U_i to denote the velocity fluctuations and mean velocities, respectively, and that τ represents his Reynolds stress. For convenience, τ is replace with R. Chou's ω transport is

$$\frac{\partial \omega_{ik}}{\partial t} + U_j \frac{\partial \omega_{ik}}{\partial x_j} + \frac{\partial U_j}{\partial x_k}\frac{\partial w_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\frac{\partial w_k}{\partial x_j} + w_j \frac{\partial \omega_{ik}}{\partial x_j} + \frac{\partial w_j}{\partial x_k}\frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}\frac{\partial w_k}{\partial x_j} + w_j \frac{\partial \Omega_{ik}}{\partial x_j}$$

$$+ \frac{\partial w_j}{\partial x_k}\frac{\partial U_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}\frac{\partial U_k}{\partial x_j} = -\frac{1}{\rho}\left(\frac{\partial^2 R_{ji}}{\partial x_j \partial x_k} - \frac{\partial^2 R_{jk}}{\partial x_j \partial x_i}\right) + \nu \nabla^2 \omega_{ik}.$$

Ω is the mean vorticity expressed as

$$\Omega_{ik} = \frac{\partial U_i}{\partial x_k} - \frac{\partial U_k}{\partial x_i},$$

while the ω vorticity fluctuation is defined as

$$\omega_{ik} = \frac{\partial w_i}{\partial x_k} - \frac{\partial w_k}{\partial x_i}.$$

Chou also developed the first transport version of the Reynolds stress *R* PDE in 1940, whereby he introduced eddy decay (dissipation) [Chou, 1940 (refer to Equations 3.1 and 3.4)],

$$\frac{\partial R_{ik}}{\partial t} = w_j'\left(\frac{\partial U_i}{\partial x_j}w_k' + \frac{\partial U_k}{\partial x_j}w_i'\right) + U_j\frac{\partial R_{ik}}{\partial x_j} + w_j'\frac{\partial R_{ik}}{\partial x_j} = -\frac{1}{\rho}\left(\frac{\partial p'}{\partial x_i}w_k' + \frac{\partial p'}{\partial x_k}w_i'\right)$$

$$-\frac{1}{\rho}\left(\frac{\partial R_{ij}}{\partial x_j}w_k' + \frac{\partial R_{kj}}{\partial x_j}w_i'\right) + \nu\nabla^2 R_{ik} - \nu g_{mn}\left(\overline{\frac{\partial w_i'}{\partial x_m}\frac{\partial w_k'}{\partial x_n}} + \overline{\frac{\partial w_i'}{\partial x_n}\frac{\partial w_k'}{\partial x_m}}\right).$$

Years later, he elaborated and expanded various terms in the *R* PDE [Chou, 1945; Chou and Chou, 1995]. With a desire to focus on the issue of dissipation, Chou's decay term is

$$\frac{\partial R_{ik}}{\partial t} \approx -\nu g_{mn}\left(\overline{\frac{\partial w_i'}{\partial x_m}\frac{\partial w_k'}{\partial x_n}} + \overline{\frac{\partial w_i'}{\partial x_n}\frac{\partial w_k'}{\partial x_m}}\right) = 2\nu g_{mn}\overline{\frac{\partial w_i'}{\partial x_m}\frac{\partial w_k'}{\partial x_n}} = \varepsilon_{ik}.$$

It is noted that this is the modern dissipation ε_{ik} term, with the metric tensor $g_{mn} = 1$.

Chou proceeded by deriving a *relationship for eddy decay associated with the Taylor length scale, λ,*

$$\varepsilon_{ik} = 2\nu g_{mn}\overline{\frac{\partial w_i'}{\partial x_m}\frac{\partial w_k'}{\partial x_n}} = -\frac{2\nu}{3\lambda^2}(k-5)q^2 g_{ik} + \frac{2\nu k}{\lambda^2}\overline{w_i'w_k'},$$

where

$$q^2 = \overline{w_j'w_j'}.$$

Note, though, that whereas Chou developed transport PDEs for k, ω and τ (i.e., R), and an expression for ε=ε(k,λ, $\overline{w_i'w_k'}$), he did not develop a transport PDE for ε [Chou, 1940].

Now, researchers developed triple velocity correlations in the hope that those equations could be used to achieve closure. Instead, the triple velocity correlations generated fourth order velocity correlations, and so forth onto the next higher order, indefinitely [Chou, 1940]. As stated by Chou, "…and hence lead to nowhere". At this point, Chou chose to end the endless cycle by stating that,

$$\overline{u_i'u_j'u_k'u_l'} \approx \frac{1}{2}\left(\overline{u_i'u_j'}\,\overline{u_k'u_l'} + \overline{u_i'u_k'}\,\overline{u_l'u_j'} + \overline{u_i'u_l'}\,\overline{u_j'u_k'}\right).$$

His justification is based on either assuming that the fluctuating velocities can be represented as sinusoidal functions of time, or as "an independent hypothesis".

In 1941, Millionshtchikov developed a more generalized and rigorous expansion for the fourth-order velocity correlation based on homogeneous isotropic theory [Millionshtchikov, 1941]. Remarkably, its expression is nearly the same as Chou's approximation.

$$\overline{u_i'u_j'u_k'u_l'} \approx \overline{u_i'u_j'}\,\overline{u_k'u_l'} + \overline{u_i'u_k'}\,\overline{u_j'u_l'} + \overline{u_j'u_k'}\,\overline{u_i'u_l'} = R_{ij}R_{kl} + R_{ik}R_{jl} + R_{jk}R_{il}.$$

The next progression in the k-ε model surfaced when the first ε transport PDE was finally completed by B. I. Davydov in 1961 [Davydov, 1961],

$$\frac{\partial \varepsilon}{\partial t}+\bar{u}_i\frac{\partial \varepsilon}{\partial x_i}+2\nu\frac{\partial \bar{u}_i}{\partial x_k}\left(\overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}}+\overline{\frac{\partial u_l'}{\partial x_i}\frac{\partial u_l'}{\partial x_k}}\right)+2\nu\frac{\partial^2\bar{u}_i}{\partial x_k\partial x_l}\overline{u_k'\frac{\partial u_i'}{\partial x_l}}+\nu\frac{\partial\left[\overline{u_j'\left(\frac{\partial u_i'}{\partial x_k}\right)^2}\right]}{\partial x_j}$$

$$+2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}}+2\nu\frac{\partial}{\partial x_k}\left(\overline{\frac{\partial p'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}}\right)+2\nu^2\overline{\left(\frac{\partial^2 u_i'}{\partial x_k\partial x_l}\right)^2}=\nu\frac{\partial^2\varepsilon}{\partial x_k^2}.$$

Note that a systematic, term-by-term comparison shows that Davydov's 1961 exact ε PDE has the same nine terms as the Hanjalic 1970 exact ε PDE [Hanjalic, 1970]. For convenience, Hanjalic's exact ε PDE is shown later in this section.
In any case, after some surgery on his ε PDE, Davydov reduced the exact PDE to the following simplified expression,

$$\frac{\partial \varepsilon}{\partial t}+\bar{u}_i\frac{\partial \varepsilon}{\partial x_i}+\frac{\partial\left(u_i'\varepsilon\right)}{\partial x_i}+\alpha\frac{\varepsilon}{k}R_{ik}\frac{\partial \bar{u}_i}{\partial x_k}+\gamma\frac{\varepsilon^2}{k}=\nu\frac{\partial^2\varepsilon}{\partial x_k^2},$$

where some of Davydov's notations are expressed in more modern terminology,

$$\varepsilon_{Davydov}\equiv Q\equiv\nu\left(\frac{\partial \overline{\mathrm{v}_i'}}{\partial x_k}\right)^2,$$

$$C_k\equiv\nu\overline{u_i'\left(\frac{\partial u_i'}{\partial x_k}\right)^2}=\overline{u_k'\varepsilon},$$

$$R_{ij}=\overline{\mathrm{u}_i'\mathrm{u}_j'},$$

$R\equiv R_{ii}\equiv k\sim\frac{1}{t}$, and

$$\gamma=4.$$

Davydov choose *N* to represent the turbulent kinematic viscosity,

$$N\equiv\nu_t\equiv-R_{ik}\frac{\partial \bar{u}_i}{\partial x_k}\left(\frac{\partial \bar{u}_l}{\partial x_m}\right)^{-2}.$$

Again, a systematic, term-by-term comparison shows that except for one term, Davydov's 1961 simplified, modified ("drastic surgery") ε PDE is nearly the same expression as the 1970 Hanjalic simplified ε PDE [Hanjalic, 1970]. For convenience, the Hanjalic expression is shown later in this section. In any case, the only difference between the two PDEs is that Davydov considers the dissipation of diffusion involving velocity fluctuations,

$$\frac{\partial\left(u_i'\varepsilon\right)}{\partial x_i},$$

whereas Hanjalic considered dissipation of diffusion involving dissipation,

$$\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_j}\right).$$

Of course, using dimensional arguments based on gradient transport (refer to Chapter 2), Davydov's term becomes the same as Hanjalic's, with the exception of the σ_ε coefficient. Namely, Davydov's dissipation of diffusion term becomes,

$$-\frac{\partial}{\partial x_j}\left(\overline{\mathrm{u}'_j\phi'}\right)_{Davydov} = -\frac{\partial}{\partial x_j}\left(\overline{\mathrm{u}'_j\varepsilon}\right)_{Davydov} = \frac{\partial}{\partial x_j}\left(\nu_t\frac{\partial\varepsilon}{\partial x_j}\right)_{Davydov}.$$

In this context, the comparison of both simplified ε PDEs shows that

$$\alpha_{Davydov} \rightarrow C_{1,\mathrm{Hanjalic}}$$

and

$$\lambda_{Davydov} \rightarrow C_{2,\mathrm{Hanjalic}}.$$

Nevertheless, unlike Hanjalic, Davydov solved $\frac{\partial\left(u_i'\varepsilon\right)}{\partial x_i}$ via an additional transport equation.

Hanjalic also noted the similarities between Davydov's exact and simplified ε PDEs and those included in his research [Hanjalic, 1970], but not to the extent discussed here.

Just as with Kolmogorov's k-ω model, little progress was made towards Davydov's k-ε modelling approach until nearly a decade after its initial development [Harlow and Nakayama, 1967; Harlow and Nakayama, 1968]; this was primarily due to the lack of computational power at the time.

A somewhat distinct approach from Davydov, and more analogous to Chou's vorticity decay transport, was undertaken in 1968. The formulation of this alternative ε transport PDE is as follows [Harlow and Nakayama, 1967; Harlow and Nakayama, 1968],

$$\frac{\partial\varepsilon}{\partial t} + \overline{\mathrm{u}}_k\frac{\partial\varepsilon}{\partial x_k} = \frac{a\Delta\nu_t}{s^2}\frac{\partial\overline{\mathrm{u}}_j}{\partial x_k}\frac{\partial\overline{\mathrm{u}}_j}{\partial x_k} - \frac{2\nu\Delta'\varepsilon}{s^2} + \frac{a_2\Delta}{s^2}\frac{\partial}{\partial x_k}\left(\nu_t\frac{\partial k}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left[\left(\nu + a_3\nu_t\right)\frac{\partial\varepsilon}{\partial x_k}\right]$$
$$+a_4\frac{\partial}{\partial x_k}\left(\frac{\Delta\nu_t}{2}\frac{\partial\varphi}{\partial x_k}\right),$$

where ε is in s^{-2}. The auxiliary constants and expressions are

$a \approx 2,$

$a_2 \approx a_3 \approx a_4 \approx 1,$

$\beta = 5,$

$\beta' = 2\beta,$

$\delta = 0.01,$

$\Delta = \beta\left(1+\delta\xi\right),$

$\Delta' = \beta'\left(1+\delta\xi\right),$

$\xi = \dfrac{\nu_t}{\nu},$

$s = \lambda\sqrt{1+\delta\xi},$

λ= Taylor length scale, and

$\varphi = \dfrac{p'}{\rho}.$

As shown next, Hanjalic extended the modified Davydov ε PDE by incorporating a new approach using dimensional arguments involving velocity and length scales for calculating ν_t, a term to compute turbulent diffusion of dissipation based on dimensional arguments in gradient transport (as discussed earlier in this section), and a polynomial function based on the distance from the wall [Hanjalic, 1970]. Similarly to Chou's efforts, [Chou, 1940; Chou and Chou, 1995], Hanjalic obtained a fluctuating velocity PDE whereby "the time averaged parts are subtracted from the time dependent general Navier-Stokes equation (2.2)" [Hanjalic, 1970]. For convenience, the fluctuating velocity PDE is included here,

$$\frac{\partial \mathrm{u}_i'}{\partial t} + \bar{u}_k \frac{\partial \mathrm{u}_i'}{\partial x_k} = -\mathrm{u}_k' \frac{\partial\left(\bar{u}_i + \mathrm{u}_i'\right)}{\partial x_k} - \frac{\partial\left(\overline{\mathrm{u}_i'\mathrm{u}_k'}\right)}{\partial x_k} - \frac{\partial\left(\nu \dfrac{\partial \mathrm{u}_i'}{\partial x_k}\right)}{\partial x_k} - \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i}.$$

Hanjalic [Hanjalic, 1970] then made the assertion that if the fluctuating velocity PDE is

> "differentiated with respect to x_ℓ, then multiplied by $2\nu \dfrac{\partial u_i'}{\partial x_\ell}$ and time averaged, it transforms into the exact transport equation for the variable $\nu \overline{\dfrac{\partial u_i'}{\partial x_\ell}\dfrac{\partial u_i'}{\partial x_\ell}} \equiv \varepsilon$ …which represents the dissipation of turbulent kinetic energy."

Like the earlier work presented by Davydov [Davydov, 1961], Hanjalic noted that ε could be modeled with a turbulence transport PDE. Hanjalic also postulated that the length scale for ε could be approximated with dissipating eddies smaller than those considered by Rotta [Rotta, 1951], to achieve a relationship whereby

$$\nu_t = k^{1/2}\ell = \nu_t(k,\varepsilon).$$

Namely, Hanjalic stated,

> "One could, perhaps, argue that the integral scale, employed in Rotta's equation, is weighted much with the lowest wave numbers and is therefore not representative of the scale of the energy containing eddies which are mostly responsible for both the energy dissipation and diffusion processes."

This assertion extended the idea that ε could be associated with a smaller length scale, rather than the Taylor length scale selected by Chou for dissipation. In particular, Hanjalic associated ε with a small scale L_ε, whereby

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_i'}{\partial x_l}} = c_\varepsilon \frac{k^{3/2}}{L_\varepsilon},$$

and, solving for a scale length,

$$L_\varepsilon = c_\varepsilon \frac{k^{3/2}}{\varepsilon},$$

and thus,

$$\nu_t = k^{1/2}\ell = k^{1/2}\left(c_\varepsilon \frac{k^{3/2}}{\varepsilon}\right) = c_\varepsilon \frac{k^2}{\varepsilon} = \nu_t(k,\varepsilon),$$

where c_ε is the modern C_μ. Therefore, the above relationship collapses to the Prandtl-Kolmogorov relationship, which applies for the *larger* eddies, not the *smaller* eddies.

Though L_ε was defined as a "dissipation length scale", Hanjalic understood that the following scales are different: the ℓ (integral length), Chou's *L*=λ, and L_ε. Indeed, he noted [Hanjalic, 1970],

> "Although these scales are not the same in general, it is reasonable to expect them to vary in a similar fashion. Thus, considering the degree of approximation implied by the method so far, it may be assumed that $L \approx L_\varepsilon$. It remains to specify a unique length scale which will represent both L and L_ε."

He then prescribed a third-order polynomial for calculating a new, geometry-dependent length-scale, L_a. In particular, his new length scale is intended to "represent both L and L_ε", and is based on a polynomial function based on the distance from the wall.

That said, Hanjalic compared his model with a plane smooth channel, the boundary layer on a flat plate, a plane wall jet, a plane free jet, and a plane with a mixing layer. Except for the plane wall jet, whose predicted value was within 20 to 30% of the experimental value, the other four cases showed results that compared very well with experimental data, usually to within 10% or less. However, only one predicted value was compared with the experimental data for each of the five cases (refer to Hanjalic's Table 4.4, page 189); when Hanjalic made more detailed comparisons, the plane mixing layer simulation vs. experimental data was not as good.

However, to be clear, L *represents the length of the energy carrying eddies, whereas* $L_\varepsilon \approx \eta$, *because most of the dissipation occurs near the small scales.* Of course, eddy size depends heavily on fluid velocity, physical properties, and system size. Nevertheless, LIKE calculations for air and water systems for *Re* in the range of 3,000 to 700,000 shows that η~7x10^{-5} to 1x10^{-4} m, λ~1x10^{-3} to 1x10^{-2} m, and ℓ~2x10^{-3} to 5x10^{-2} m. Thus, to assert that $L \approx L_\varepsilon$ is clearly incorrect, as $\eta \ll \lambda \ll \ell$.

At this point, the derivation of the "exact" ε PDE is warranted, so it may shed additional light on the k-ε model. However, its derivation takes several dozen pages, assuming that one is careful and lucky enough not to make a mathematical error! Nevertheless, based on Hanjalic's version, and modernized somewhat, its entire expression is [Hanjalic, 1970],

$$\underbrace{\frac{\partial \varepsilon}{\partial t}}_{1} + \underbrace{\bar{\mathrm{u}}_{\mathrm{j}}\frac{\partial \varepsilon}{\partial x_j}}_{2} = \underbrace{-2\nu \frac{\partial \bar{\mathrm{u}}_{\mathrm{i}}}{\partial x_j}\left(\overline{\frac{\partial \mathrm{u}_{\mathrm{i}}'}{\partial x_k}\frac{\partial \mathrm{u}_{\mathrm{j}}'}{\partial x_k}} + \overline{\frac{\partial \mathrm{u}_{\mathrm{k}}'}{\partial x_i}\frac{\partial \mathrm{u}_{\mathrm{k}}'}{\partial x_j}}\right)}_{3} \underbrace{- 2\nu \frac{\partial^2 \bar{\mathrm{u}}_{\mathrm{i}}}{\partial x_k \partial x_j}\overline{\mathrm{u}_k' \frac{\partial \mathrm{u}_{\mathrm{i}}'}{\partial x_j}}}_{4} \underbrace{- 2\nu \overline{\frac{\partial \mathrm{u}_{\mathrm{i}}'}{\partial x_k}\frac{\partial \mathrm{u}_{\mathrm{i}}'}{\partial x_m}\frac{\partial \mathrm{u}_{\mathrm{k}}'}{\partial x_m}}}_{5}$$

$$\underbrace{-2\nu^2 \overline{\frac{\partial^2 \mathrm{u}_{\mathrm{i}}'}{\partial x_k \partial x_m}\frac{\partial^2 \mathrm{u}_{\mathrm{i}}'}{\partial x_k \partial x_m}}}_{6} + \underbrace{\nu \frac{\partial \left(\frac{\partial \varepsilon}{\partial x_j}\right)}{\partial x_j}}_{7} \underbrace{- \nu \frac{\partial \left(\overline{\mathrm{u}_{\mathrm{j}}' \frac{\partial \mathrm{u}_{\mathrm{i}}'}{\partial x_m}\frac{\partial \mathrm{u}_{\mathrm{i}}'}{\partial x_m}}\right)}{\partial x_j}}_{8} \underbrace{- 2\frac{\nu}{\rho}\frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial \mathrm{u}_{\mathrm{j}}'}{\partial x_m}}\right)}{\partial x_j}}_{9}$$

where

Term 1 = transient change of dissipation (accumulation term),

Term 2 = *mean* flow convection dissipation,

Term 3 = dissipation production term that arises from the product of the gradients of the fluctuating and mean velocities,

Term 4 = dissipation production term that generates additional dissipation based on the fluctuating and mean velocities,

Term 5 = the so called "destruction" rate for dissipation, which is associated with eddy velocity fluctuation gradients,

Term 6 = another "destruction" term for dissipation, which arises from eddy velocity fluctuation diffusion, Term 7 = term to account for the viscous diffusion associated with dissipation. Note that this is the only positive term on the RHS,

Term 8 = diffusive turbulent transport resulting from the eddy velocity fluctuations, and finally,

Term 9 = dissipation of turbulent transport arising from eddy pressure and fluctuation velocity gradients.

The above comprise the nine exact dissipation terms associated with the ε transport PDE. However, besides the two terms that form the substantial derivative (Terms 1 and 2) and the viscous diffusion of dissipation (Term 7), the remaining six terms are completely unknown, to a point whereby the PDE requires Wilcox's infamous "drastic surgery" to transform it into a *vastly different* equation that can be modelled (solved) numerically [Wilcox, 2006], or "surgically modified beyond recognition" [Schobeiri and Abdelfattah, 2013]; others would agree, though perhaps using slightly less colorful assertions [Myong and Kasagi, 1990; Rodi, 1993; Andersson et al., 2012]. And this "drastic surgery" is exactly what Hanjalic performed on the exact ε PDE, when he provided analytical expressions for the six PDE terms, some of which are still in use today.

Moving forward a few years, the k-ε model was revamped onto its more modern format in the early '70s. This was a time period when various theoretical and computational efforts by Jones, Launder, and Sharma showed excellent potential vs. experimental data [Jones and Launder, 1972; Launder et al., 1973; Jones and Launder, 1973; Launder and Sharma, 1974]. During this crucial three-year period, the k-ε model evolved towards the following formulation, which is now commonly called the SKE model. The *k* PDE is written as

$$\frac{\partial k}{\partial t} + \bar{\mathrm{u}}_{\mathrm{j}} \frac{\partial k}{\partial x_j} = R_{ij} \frac{\partial \bar{\mathrm{u}}_{\mathrm{i}}}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right],$$

while the ε PDE is transformed into the following,

$$\underbrace{\frac{\partial \varepsilon}{\partial t}}_{\substack{1\\ \text{transient}}} + \underbrace{\bar{\mathrm{u}}_{\mathrm{j}} \frac{\partial \varepsilon}{\partial x_j}}_{\substack{2\\ \text{convection}}} = \underbrace{C_1 \frac{\varepsilon}{k} R_{ij} \frac{\partial \bar{\mathrm{u}}_{\mathrm{i}}}{\partial x_j}}_{\substack{\text{3 and 4}\\ \text{production}}} - \underbrace{C_2 \frac{\varepsilon^2}{k}}_{\substack{\text{5 and 6}\\ \text{decay}}} + \underbrace{\frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]}_{\substack{7\text{–}9\\ \text{diffusion}}}.$$

As written in their original 1974 paper, the model includes a Prandtl-Kolmogorov-like formulation for closure that involves a turbulent Reynolds number (Re_T) fit [Launder and Sharma, 1974],

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

via the damping function,

$$C_\mu = 0.09 e^{-\left[3.4/\left(1+\mathrm{Re}_T/50\right)^2\right]},$$

Launder and Sharma defined Re_T as

$$\mathrm{Re}_T = \frac{\rho k^2}{\mu \varepsilon}.$$

Note that Re_T is based on the large eddies (refer to Chapter 3), and that Launder and Sharma did not include C_μ in their Re_T expression.

The final closure coefficients underwent "reoptimization" for high *Re* applications [Launder et al., 1973; Launder and Sharma, 1974; Wilcox, 2006],

$$C_1 = C_{\varepsilon 1} = 1.44,$$

$$C_2 = C_{\varepsilon 2} = 1.92\left(1.0 - 0.3e^{-\mathrm{Re}_T^2}\right),$$

$\sigma_\varepsilon = 1.3$, and

$$\sigma_k = 1.0$$

The above formulation for the ε PDE shows that the nine terms for the "exact" PDE were collapsed onto just five terms. Note that in the "transformed" ε PDE, only Terms 1, 2, and 7 survived in an intact manner—the remaining terms, bear little, if any resemblance to the exact equation! In other words, a comparison of the previous two ε PDE equations shows an utter lack of resemblance once the transformative "drastic surgery" occurred, namely:

$$\underbrace{-2\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}}\right)\frac{\partial \overline{u}_i}{\partial x_j}}_{3} \underbrace{- 2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j}\overline{u_k'\frac{\partial u_i'}{\partial x_j}}}_{4} \rightarrow \underbrace{C_{\varepsilon 1}\frac{\varepsilon}{k}R_{ij}\frac{\partial \overline{u}_i}{\partial x_j}}_{\substack{\text{3 and 4}\\ \text{production}}}.$$

Ditto,

$$\underbrace{-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}}_{5} \underbrace{- 2\nu^2\overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m}\frac{\partial^2 u_i'}{\partial x_k \partial x_m}}}_{6} \rightarrow \underbrace{-C_{\varepsilon 2}\frac{\varepsilon^2}{k}}_{\substack{\text{5 and 6}\\ \text{decay}}}.$$

and ditto,

$$\underbrace{\nu\frac{\partial\left(\frac{\partial \varepsilon}{\partial x_j}\right)}{\partial x_j}}_{7} \underbrace{- \nu\frac{\partial\left(\overline{u_j'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)}{\partial x_j}}_{8} \underbrace{- 2\frac{\nu}{\rho}\frac{\partial\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}}\right)}{\partial x_j}}_{9} \rightarrow \underbrace{\frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]}_{\substack{\text{7–9}\\ \text{diffusion}}}.$$

Certainly, some of this monkey business also goes on with k-ω models as well, but not to the degree whereby only three terms out of nine can be modeled directly, and the remaining terms involve astonishing transformations that bear no resemblance to the "exact" terms.

Note that despite the claim that the dissipation eddies are small, and the intractable transformation onto a form that can be modeled, the k-ε model remains popular and enjoys substantial usage in the turbulence community. To be fair, the model has enjoyed many spectacular successes since its modern upgrades [Hanjalic, 1970; Jones and Launder, 1972; Launder et al., 1973; Jones and Launder, 1973; Launder and Sharma, 1974; Rodi, 1993; Gorji et al., 2014; Diaz and Hinz, 2015].

The SKE has successfully modeled the following,

- high *Re* pipe flow [Jones and Launder, 1973],
- pipe flow with *Pr* in the range of 0.5 to 2, 000 and Re~10,000 [Jones and Launder, 1973],
- high *Re* spinning disk under heat transfer (*Re*~$1x10^5$ to $1.5x10^6$) [Launder and Sharma, 1974],
- free shear layer flow (e.g., high *Re*, isotropic, far away from walls) [Rodi, 1993],
- flows with nonexistent to small pressure gradient [Wilcox, 2006],
- simplified case of boundary layer on a flat plate (constant pressure, far away from the wall, no adverse pressure gradient, high *Re*) [Hanjalic, 1970],
- channel flow for *Re* in the range of 6,000 to 150,000 (and higher, of course) [Hanjalic, 1970; Jones and Launder, 1973; Gorgi et al., 2014], and
- plane wall and free jet, [Hanjalic, 1970].

The SKE is not suitable for

- adverse pressure gradients [Wilcox, 1988A; Wilcox, 1993; Wilcox, 2006; Fluent, 2012],
- high curvature [Fraczek and Wroblewski, 2016],
- rotating/swirling flows [Fluent, 2012; Andersson et al., 2012; Diaz and Hinz, 2015],
- round jet [Pope, 1978; Wilcox, 2006; Fluent, 2012],
- shock mixing [Dong, Wang, and Tu, 2010],
- asymmetric diffusers [El-Behery and Hamed, 2009],
- near-wall treatment: viscous sublayer; variable-pressure boundary wall ($30 \leq y+ \leq 700$) [Wilcox, 2006; Wilcox, 2012],
- fluids at supercritical pressure [He, Kim, and Bae, 2008; Bae, Kim, and Kim, 2017],
- large strain rate (regions with large velocity gradients, stagnation points) [Fluent, 2012],
- laminar to turbulent transition for heated flow [Abdollahzadeh et al., 2017], and

- situations where overdamping adversely impacts parameters that rely on wall-based quantities (e.g., wall heat transfer, wall friction) [Zhao et al., 2017].

However, the SKE's usage trend has seen decreased levels since the 1990s, when the k-ω models increased in maturity [Wilcox, 2006] and other methods, such as LES and DNS, came into more usage. Another reason for the SKE's reduced usage may stem from a lackluster performance vs. the 2006 k-ω and SST models, when undergoing simulation of more complex turbulence systems [Wilcox, 1993; Wilcox, 2006; Menter, 1992; Kuntz, and Langtry, 2003]. In addition, the interested reader is encouraged to review Section 4.7 for additional concerns regarding the SKE.

Certainly, dozens of upgrades have been proposed to improve the SKE. Notable improvements include the realizable k-ε model and the renormalization group-theory (RNG) k-ε models, with many successes, albeit with a nagging continuation of issues, all likely due to fundamental reasons that stem from the attempt to bring small and large eddy scales into a single model. Hence, many k-ε improvements tend to gravitate towards functions that can blend the opposing scale behaviors. For example, instead of letting Re_T be based on a damping function as Hanjalic, Launder, and Sharma proposed, various authors now recommend a y^+ damping function, and they have noted better agreement with experimental data for heat transfer experiments involving supercritical pressure [Rodi and Mansour, 1993; He, Kim, and Bae, 2008; Bae, Kim, and Kim, 2017].

However, a fundamental issue for k-ε modeling remains: its association of the large turbulent kinetic energy scales with the small dissipative scales. It is very likely that so long as the model relies on the two different scales (despite numerous blending attempts), issues will crop up).

More recently, attempts to have the k-ε model include Taylor eddy behavior appear to bear much promise [Bae, 2016; Bae, Kim, and Kim, 2016; Bae, Kim, and Kim, 2017]. For these reasons, the Myong-Kasagi model is explored next.

4.6.3.6 The Myong-Kasagi k-ε model

The Myong-Kasagi (MK) k-ε model was developed from the SKE, but with some fundamental paradigm changes associated with the ε PDE. The k PDE is the same, except for a single change in the diffusive term,

$$\sigma_k = 1.4 \text{ (vs. 1.0 in the SKE)}.$$

On the other hand, the MK ε PDE has key upgrades in the production and decay terms, as follows,

$$\underbrace{\frac{\partial \varepsilon}{\partial t}}_{\text{transient}} + \underbrace{\bar{u}_j \frac{\partial \varepsilon}{\partial x_j}}_{\text{convection}} = \underbrace{C_{\varepsilon 1} f_1 \frac{\varepsilon}{k} R_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{production}} - \underbrace{C_{\varepsilon 2} f_2 \frac{\varepsilon^2}{k}}_{\text{decay}} + \underbrace{\frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]}_{\text{diffusion}}.$$

The PDE includes the following blending function modifications,

$$C_{\varepsilon 2} = 1.8 \ (MK) \text{ vs. } 1.92\left(1.0 - 0.3 e^{-\mathrm{Re}_T^2}\right) \ (SKE)$$

and

$$f_\mu = \left(1+\frac{3.45}{\mathrm{Re}_T^{1/2}}\right)\left(1\text{-}\mathrm{e}^{-y^+/70}\right) \quad (MK) \text{ vs. } \quad 0.09e^{-\left[3.4/\left(1+\mathrm{Re}_T/50\right)^2\right]} \quad (SKE).$$

Other researchers propose a slightly modified blending function that more rapidly approaches the asymptotic value [Speziale, Abid, and Anderson, 1992],

$$f_\mu = \left(1+\frac{3.45}{\mathrm{Re}_T^{1/2}}\right)\tanh\left(\frac{y^+}{70}\right).$$

The MK f_μ function was derived by adding the implied Prandtl-Kolmogorov length scale for the large and Taylor eddies,

$$L_D = L_\ell + L_\lambda = C_D\frac{k^{3/2}}{\varepsilon} + C_1\sqrt{\frac{\nu k}{\varepsilon}},$$

and thereafter multiplying the lengths with a Van-Driest-like damping function. The Taylor length was obtained using dimensional arguments, whereby

$$\varepsilon \approx \frac{\nu k}{\lambda},$$

or

$$\lambda \approx \left(\frac{\nu k}{\varepsilon}\right)^{1/2} = C_1\left(\frac{\nu k}{\varepsilon}\right)^{1/2}.$$

Taylor's theoretical research indicates that C_1 = sqrt(10) for isotropic flows.

The MK model includes the following new parameters,

$$f_1 = 1.0$$

and

$$f_2 = \left[1-\frac{2}{9}e^{(\mathrm{Re}_T/6)^2}\right]\left(1\text{-}\mathrm{e}^{-y^+/5}\right)^2.$$

The $C_{\varepsilon 1}$ term was modified slightly,

$$C_{\varepsilon 1} = 1.4 \ \ (MK) \text{ vs. } 1.44 \ \ (SKE).$$

Ironically, this effort returns the k-ε back to Chou's earlier k-ω research premises, especially because it includes the larger Taylor eddy length, thus incorporating the impact of larger eddies. Though developed in 1990, the MK adaptation of the k-ε model factors the Taylor eddy length scale in the f_μ blending function, and therefore continues to be of recent investigation [Myong and Kasagi, 1990; Speziale, Abid, and Anderson, 1992; Bae, 2016]. Not surprisingly, modeling of the Taylor eddies increases turbulent production in the viscous sublayer and the buffer layer [Bae, Kim, and Kim, 2017]. Recent comparisons with experimental data show much promise [Speziale, Abid, and Anderson, 1992; Bae, 2016; Bae, Kim, and Kim, 2016; Bae, Kim, and Kim, 2017], though there continue to be concerns due to overdamping [Zhao et al., 2017].

4.6.3.7 Menter's 2003 SST model

With the wide usage of the SKE and the development of the 1988 k-ω version, it was soon evident in the late '80s and early '90's that each model has its pros and cons, and that these are mutually exclusive in the following sense. Back then, there was a need for a reliable model suitable for turbulence aeronautics flows that was specifically able to handle flow separation and large, adverse pressure gradients. Menter shrewdly realized that the SKE resulted in reasonable flow simulations at high *Re* and away from the wall (free stream), but had shortcomings near the wall. On the other hand, the 1988 k-ω was very useful for low *Re* and near the wall boundary layer, but is more sensitive than the SKE for free stream boundary conditions. Further, the 1988 k-ω model did not work well with pressure-induced separation [Menter, 1992; Menter, 1993; Menter, Kuntz, and Langtry, 2003]. Thus, in a very fortunate situation, the shortcomings of one model are compensated by the other, and vice-versa, and without the need for additional damping functions near the wall.

So, Menter wondered, why not combine both models in such a way that each is used in the region where it excels, and in between, use some sort of weighted average, or blending? Hence the 1992 shear stress transport (SST) model was born [Menter, 1992; Menter, 1993; Menter, 1994]. The model is also known as the SST k-ω and as Menter's SST model, but is not to be confused with the "stress transport" models.

In brief, each of the two RANS models is applied onto a region where it excels over the other, as follows,

- The boundary layer is computed by the 1988 k-ω.
- SKE is used for the free stream.
- A blending function F_1 computes the asymptotic turbulent behavior between the two distinct regions.
- Another function is used as a "*stress limiter*" to consider the impact of the mean strain rate on turbulent kinematic viscosity (analogous to the 2006 k-ω stress limiter).
- The blending and stress limiter functions are based on the hyperbolic tangent, tanh.

This is shown conceptually in Figures 4.3 and 4.4.

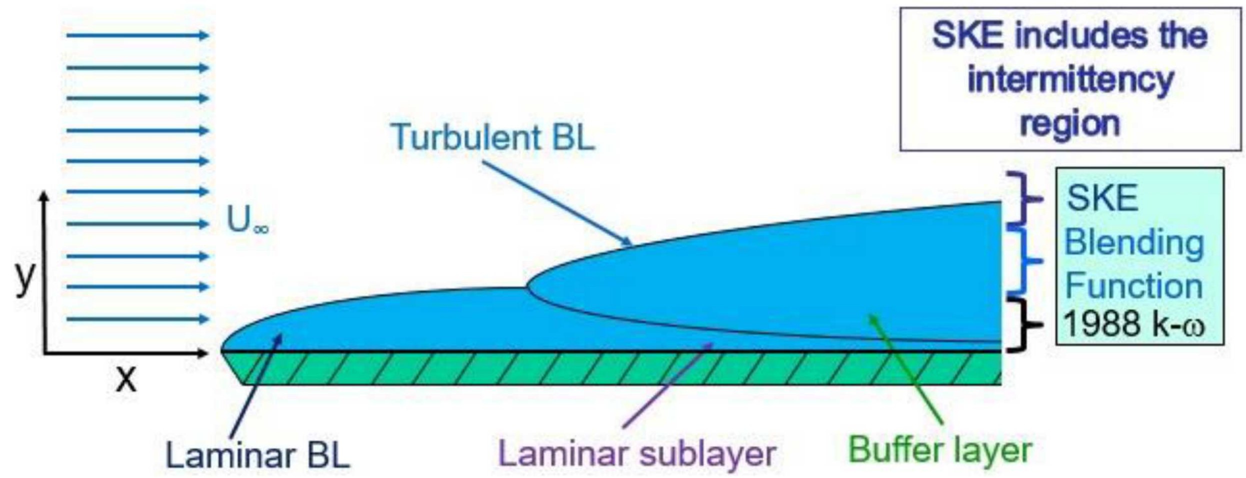

Figure 4.3. Application domains for the SST model.

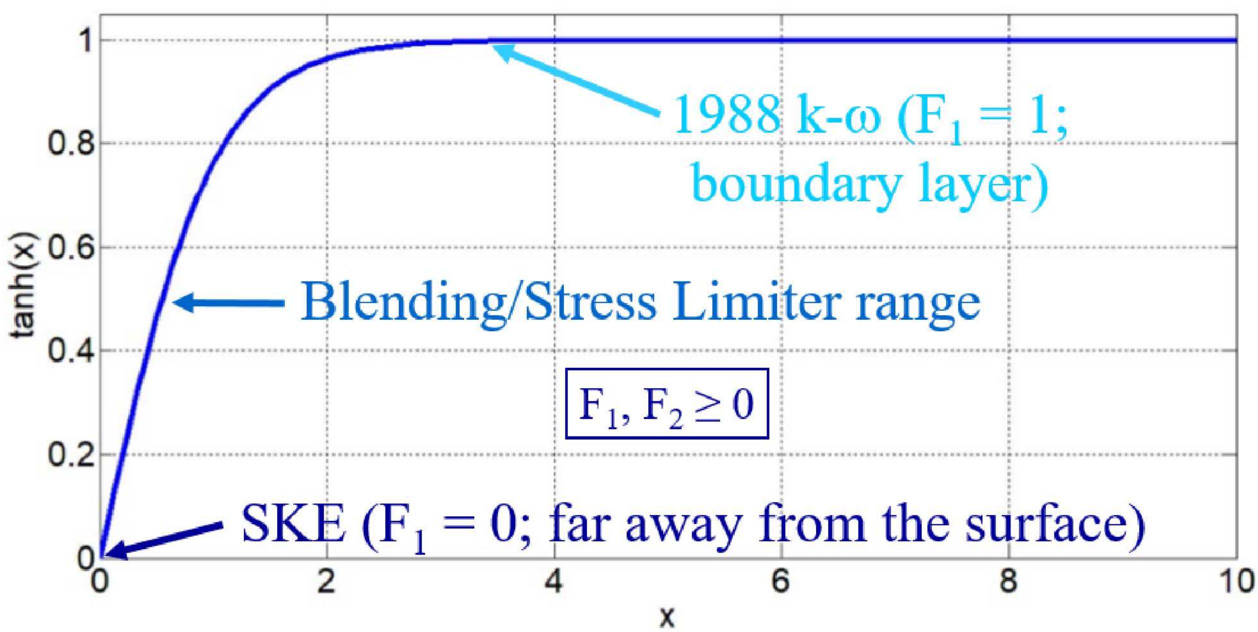

Figure 4.4. Blending function application range for SST.

Certainly, there are various SST hybrids. The model listed here is based on Menter's 2003 modifications, years after the initial debut of his 1992 SST model [Menter, Kuntz, and Langtry, 2003]. The primary differences between the original 1992 Menter model and the 2003 formulation are primarily the replacement of vorticity with *S,* and secondarily the replacement of 20 with 10 in the production limiter.

In the 2003 version, Menter used the following k and ω transport equations, respectively,

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=\tilde{P}_k-\beta^* k\omega+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma_k\nu_t\right)\frac{\partial k}{\partial x_j}\right],$$

where the production term is

$$P_k=\left[\nu_t\left(2S_{ij}-\frac{2}{3}\frac{\partial \bar{u}_k}{\partial x_k}\delta_{ij}\right)-\frac{2}{3}k\delta_{ij}\right]\frac{\partial \bar{u}_i}{\partial x_j}$$

with

$$S_{ij}=\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right).$$

The model is subject to a *k* turbulence production limiter that suppresses turbulence in regions where stagnated flow occurs, namely,

$$\tilde{P}_k=\min\left(P_k,\ 10\beta^* k\omega\right).$$

The ω PDE is as follows,

$$\frac{\partial \omega}{\partial t}+\bar{u}_j\frac{\partial \omega}{\partial x_j}=\alpha\frac{\tilde{P}_\omega}{\nu_t}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma_\omega\nu_t\right)\frac{\partial \omega}{\partial x_j}\right]+2\sigma_{\omega 2}\left(1-F_1\right)\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}.$$

This ω PDE version includes a correction noted by NASA [NASA2, 2018; Goldberg and Batten, 2015], whereby the ω production term <u>is not</u>

$$P_\omega=\alpha S^2\ \left(\text{incorrect}\right),$$

but is instead,

$$P_\omega=R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}\ \left(\text{correct}\right),$$

Production in ω is also subject to a turbulence production limiter that suppresses turbulence in regions where stagnated flow occurs, namely

$$\tilde{P}_\omega=\min\left(P_\omega,\ 10\beta^* k\omega\right).$$

The turbulent kinematic viscosity is

$$\nu_t=\frac{k}{\tilde{\omega}}$$

where

$$\tilde{\omega} = \max\left(\omega,\ \frac{F_2 S}{\mathrm{a}_1}\right).$$

Therefore,

$$\nu_t = \frac{\mathrm{a}_1 k}{\max\left(\mathrm{a}_1 \omega,\ F_2 S\right)},$$

whereby

$$S = \sqrt{2 S_{ij} S_{ij}}\ .$$

The blending function F_1 is defined as

$$F_1 = \tanh\left(\left\{\min\left[\max\left(\frac{\sqrt{k}}{\beta^* \omega y}, \frac{500\nu}{\omega y^2}\right), \frac{4\rho\sigma_{\omega 2} k}{CD_{k\omega} y^2}\right]\right\}^4\right),$$

where the scalar $CD_{k\omega}$ is defined as

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j},\ 10^{-10}\right)$$

and y is the distance to the closest surface.

The stress limiter function F_2 is

$$F_2 = \tanh\left\{\left[\max\left(\frac{2\sqrt{k}}{\beta^* \omega y}, \frac{500\nu}{\omega y^2}\right)\right]^2\right\}.$$

Finally, due to blending, the closure coefficients for the SST are computed using the following weighted function,

$$\phi = \phi_1 F_1 + \phi_2\left(1 - F_1\right),$$

where ϕ represents any of the Menter closure coefficients that have subscripts 1 and 2 (e.g., α_1, α_2, β_1, β_2, σ_{k1}, and σ_{k2}). In this context, "1" represents the 1988 k-ω model and "2" represents the SKE model.

The model closure coefficients are as follows,

$$\mathrm{a}_1 = 0.31,$$

$$\alpha_1 = \frac{5}{9},$$

$\alpha_2 = 0.44,$

$\beta^* = 0.09,$

$\beta_1 = \frac{3}{40},$

$\beta_2 = 0.0828,$

$\sigma_{k1} = 0.85,$

$\sigma_{k2} = 1.0,$

$\sigma_{\omega} = 0.5,$ and

$\sigma_{\omega 2} = 0.856.$

Though originally conceived for turbulent aeronautics flows, the SST is commonly employed in many industries, with various modifications that extend the usefulness of the model for use in rough surfaces [Hellsten and Laine, 1997; Knopp, Eisfeld, and Calvo, 2009], as well as rotation [Hellsten, 1997], and a wall-distance-free version [Goldberg and Batten, 2015].

The 2003 SST model is good for

- Adverse pressure gradients,
- Separated flows,
- Turbulent heat transfer,
- Mixed low and high *Re* problems, and
- Aerospace applications.

That said, the 2006 k-ω model DOES NOT have the free stream boundary condition sensitivity of its 1988 predecessor, which was a key reason why the SST was developed. To be blunt, had the 2006 k-ω been developed two decades earlier, the need for the SST's blending of the SKE and the 1988 k-ω would not have existed. Nevertheless, the turbulence community is fortunate to have both the 2003 SST and the 2006 k-ω models. To say the least, both the 2006 k-ω and the 2003 SST are very good, all-around models suitable for both near-wall and free stream turbulence.

But, if a choice exists, the 2006 k-ω is recommended over the 2003 k-ω SST model, especially regarding higher *Ma* flows (supersonic through hypersonic flows) [Wilcox, 2006]. An exception to this recommendation is that the 1992 SST (which is very similar to the 2003 SST), generally outperforms the 2006 k-ω in the transonic regime [Wilcox, 2006]; as noted by Wilcox, the SST was fine-tuned for transonic *Ma*. Furthermore, the SST relies on the k-ε, which has two transport variables are not exactly self-consistent. Finally, there is the issue of the much higher number of closure formulations and constants in the SST vs. the 2006 k-ω.

Note that the improvements in the 2006 k-ω model are attributable to the cross-diffusion term and the incorporation of blending. In this sense, the 2006 k-ω is very much SST-like. As will be shown in Section 4.8, the 2006 k-ω and the 2003 Menter SST are remarkably similar, after all!

Example 2. Find σ_k for the SST model.

$$\phi_1 = \sigma_{k1} = 0.85$$

$$\phi_2 = \sigma_{k2} = 1.0$$

From $\phi = \phi_1 F_1 + \phi_2 \left(1 - F_1\right)$,

$$\sigma_k = \sigma_{k1} F_1 + \sigma_{k2} \left(1 - F_1\right) = 0.85 F_1 + 1.0\left(1 - F_1\right) = 1.0 - 0.15 F_1$$

If $F_1 = 1 \Rightarrow \sigma_k = 0.85 \; \left(SKE\right)$.

If $F_1 = 0 \Rightarrow \sigma_k = 1.0 \; \left(1988 \text{ k-}\omega\right)$.

4.7 Avoid the Standard k-ε Model?

As a pragmatic matter, the choice of turbulence model is oftentimes analogous to the choice of political party or religion. The authors recognize the psychological human factor in turbulence model choice, and respect the user's right to make choices based on numerous factors and circumstances.

However, as engineers, physicists, mathematicians, and technology enthusiasts, it is always important to reflect why *any* given transport variable is selected in turbulence models, such as *k*, ℓ, ω, ε, etc. Why should there be any "sacred cow" variables, or combinations thereof? Throughout turbulence history, the easiest and most defensible choice is *k* for isotropic flows, because it quantifies the eddy turbulent energy, so its square root provides a great metric for the fluctuating velocity. However, the choice for the second transport variable is not as straightforward.

To proceed, note that the total kinetic energy is held by the total sum of all the eddies within the turbulent flow, which forms a continuous length spectrum for the integral, Taylor, and Kolmogorov eddies,

$$k_{tot} = \sum_{\tilde{i}=1}^{n} k_{\ell,\tilde{i}} + \sum_{\tilde{j}=1}^{m} k_{\lambda,\tilde{j}} + \sum_{\tilde{k}=1}^{o} k_{\eta,\tilde{k}} \; .$$

The Loitsianskii eddies [Hinze, 1987] could be included separately in the turbulent kinetic energy tally, but are instead lumped as part of the large eddy group. Recall that the vast majority of the turbulent kinetic energy is held by the integral eddies, about 80%. The Taylor eddies take the larger fraction of the remaining turbulent kinetic energy, with the Kolmogorov eddies having a progressively smaller fraction, as shown by the curve in Figure 3.1 of Chapter 3. Therefore, as a group ensemble,

$$k_\ell >> k_\lambda >> k_\eta$$

Thus, *k* is most certainly a function of an eddy characteristic length, with that length being essentially a function of ℓ, with practically nothing having to do with η. It is therefore safe to express this mathematically as

$$k = k(\ell)$$

and, conversely, that

$$k \neq k(\eta).$$

So, at the risk of being redundant, but attempting to hone in on a crucial point, the transport variable *k* is based on the *large* eddy scales, not the *small* scales. And of course, this makes physical sense; the integral eddies have the highest velocities and the largest size, and hence the highest kinetic energy.

On the *opposite* extreme of the highly-energetic eddies is decay, whereby the eddies are so small that they dissipate completely out of existence. This is where the eddies approach the tiny Kolmogorov scale, thereby surrendering their meager energy back to the main flow as the viscous force dampens them back to laminarity. As noted in Chapter 2, it is this dissipation, ε, the change of turbulent kinetic energy per unit time, that defines the decaying *small* scales,

$$\varepsilon = -\frac{dk}{dt} = \nu \overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_i'}{\partial x_l}}.$$

Recall that Kolmogorov's definition for the smallest eddies (Section 3.3) shows that the eddy size is purely a function of ν and ε,

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}.$$

Stated differently,

$$\varepsilon = \varepsilon(\eta, \nu),$$

thereby confirming that dissipation is purely a function of the smallest eddy scales and the kinematic viscosity.

As noted in Section 4.6.3.5, the SKE has performed very well for certain flows, but has had spectacular failures in certain, more complex flows. At this point, three fundamental reasons for avoiding the usage of the SKE are summarized and elaborated.

4.7.1 Inconsistent SKE transport scales

The two transported properties for the *k*-ε turbulence model are *k*, which applies strictly to the large eddies, and ε, which applies to the smallest eddies. That is, eddy dissipation occurs at the small scales, where the flow approaches more isotropic conditions. Thus, dissipation usually occurs under high *Re* conditions, in free shear flows, and away from the wall [Sondak, 1992]. By contrast, large eddies tend to form in the buffer layer because of the large stresses and large velocity gradients (despite the lower *Re*). *How, then, can turbulence be calculated consistently when one transport variable calculates the effect of the large eddies, while being <u>mathematically coupled</u> onto a transport variable that calculates the impact of the small eddies?*

4.7.2 Inconsistent SKE closure

Consider the SKE formulation for calculating the turbulent kinematic viscosity, ν_t. Based on dimensional arguments, Taylor postulated in 1935 that,

$$\varepsilon = C\frac{\mathrm{k}^{3/2}}{\ell},$$

where, based on experimental data, Wilcox recommends that $C = C_\mu$ [Wilcox, 2006],

$$\varepsilon = C_\mu\frac{\mathrm{k}^{3/2}}{\ell}$$

Solving for ℓ, yields

$$\ell = C_\mu\frac{\mathrm{k}^{3/2}}{\varepsilon}.$$

To compute ν_t based on two-equation models,

$$\nu_t = \nu_t\left(k,x\right)$$

where x is a transport variable. For the SKE, $x=\ell$, and using dimensional arguments,

$$\nu_t = k^{1/2}\ell.$$

Substituting Taylor's ℓ relationship into the above expression yields

$$\nu_t = k^{1/2}\ell = k^{1/2}\left(C_\mu\frac{k^{3/2}}{\varepsilon}\right) = C_\mu\frac{k^2}{\varepsilon},$$

which is the well-known Prandtl-Kolmogorov relationship. Therefore, the SKE ν_t is a function of not only k, but also ε. In turn, ε is a function of the larger eddies as shown in this derivation. Namely, the appropriate length scale is ℓ:

$$\nu_t = \nu_t\left(k,\varepsilon\right) = \nu_t\left(k,\ell\right).$$

Thus, the SKE relies on the *larger energetic* eddy scale ℓ to calculate the impact of the *smaller dissipating* eddies to achieve closure, and this puts the SKE in a rather tenuous situation, to say the least! The SKE (as expressed in Launder and Sharma's classic 1974 paper), uses a modified Prandtl-Kolmogorov relationship for closure. This is an attempt to skew the impact of the large eddies with a damping function that reduces in value as Re_T decreases. In particular, the SKE Prandtl-Kolmogorov relationship is based on Re_T, which in turn is based on the *large* eddies, with the explicit goal of dampening turbulence in regions near the wall. The Prandtl-Kolmogorov issue has been noted previously, by noting that the relationship [Myong and Kasagi, 1990],

> "…is approximately valid only at high turbulent Reynolds number flows remote from the wall."

The turbulent Reynolds number as used by the SKE is as follows,

$$\mathrm{Re}_T = \frac{x_{char}\mathrm{v}_{char}\rho}{\mu} = \frac{\ell \mathrm{v}_\ell \rho}{\mu} \approx \frac{\left(\frac{k^{3/2}}{\varepsilon}\right)\left(k^{1/2}\right)\rho}{\mu} = \frac{\rho k^2}{\mu\varepsilon}.$$

Thus, the implied large eddy length in the SKE model is

$$\ell_{SKE} = \frac{k^{3/2}}{\varepsilon}.$$

Note that the SKE uses the Prandtl-Kolmogorov relationship for closure, such that

$$\nu_t = k^{1/2}\ell = C_\mu \frac{\mathrm{k}^2}{\varepsilon},$$

which implies that

$$\ell_{Prandtl\text{-}Kolmogorov} = C_\mu \frac{k^{3/2}}{\varepsilon}.$$

Therefore, the SKE integral scale usage is inconsistent by a factor of C_μ *for the two damping functions that use* Re_T*,* f_μ and C_2.

Furthermore, Launder and Sharma use the following expression,

$$C_{\mu,SKE} = C_\mu f_\mu = 0.09\left\{e^{-\left[3.4/\left(1+\mathrm{Re}_T/50\right)^2\right]}\right\}.$$

Because the Prandtl-Kolmogorov relationship is based on the large eddies, the SKE modifies the relationship in the following manner,

$$\nu_{t,SKE} = C_{\mu,SKE}\frac{\mathrm{k}^2}{\varepsilon} = \left\{0.09e^{-\left[3.4/\left(1+\mathrm{Re}_T/50\right)^2\right]}\right\}\frac{\mathrm{k}^2}{\varepsilon}.$$

To show its impact, the function is plotted in Figure 4.5, where it is noted that C_μ remains constant at 0.09 for Re_T > 2,300 (note, this refers to the turbulent *Reynolds*, not the hydraulic *Reynolds*). As Re_T approaches 0, C_μ approaches 0.003, or 3.3% of the peak value, and hence the large degree of overdamping near the wall [Zhao et al., 2017]. Of course, the direct consequence of a smaller C_μ is that $\nu_{t,SKE}$ will be much smaller near the wall.

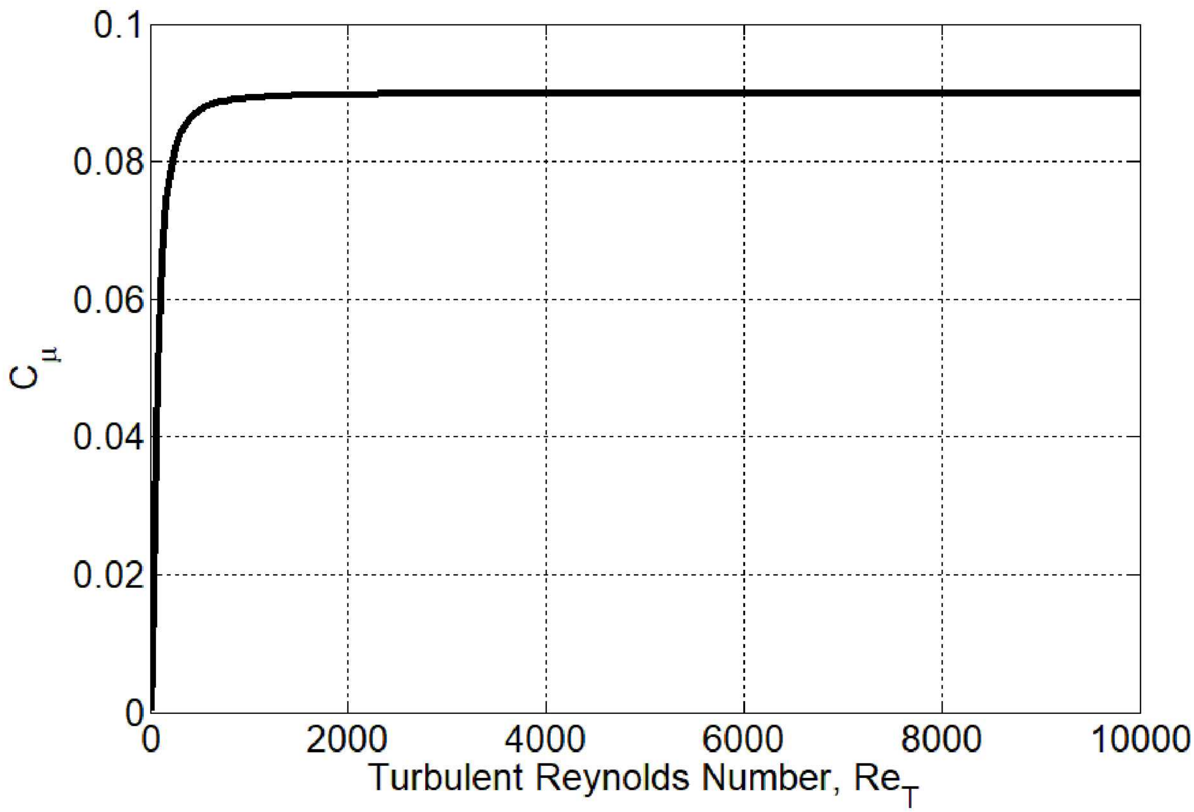

Figure 4.5. C_μ as a function of Re_T.

The same issue applies to $C_2 = C_{\varepsilon 2} = 1.92\left(1.0 - 0.3e^{-\mathrm{Re}_T^2}\right)$, though at a grander scale.

The C_2 coefficient applies to the ε PDE term that represents the "destruction" rate for dissipation associated with eddy velocity fluctuation gradients and velocity fluctuation diffusion. The coefficient function is shown in Figure 4.6. Note that the peak value is 1.92 for $Re_T > 2.5$, and rapidly decays to a minimum of 1.34 as Re_T approaches 0. Thus, its damping effect only occurs at very small Re_T!

Further worsening the situation, the damping function regime can apply to regions that may have significant turbulent eddy motion. Because part of the function domain can be outside the viscous sublayer, any attempt to dampen the eddies will adversely impact the velocity field. By the same token, to mimic eddy behavior in regions that have no eddies will also generate issues. Not surprisingly, recent comparisons with data attribute overdamping of turbulence near the wall, thereby generating "unrealistic results" both in velocity distribution and heat transfer [Zhao et al., 2017].

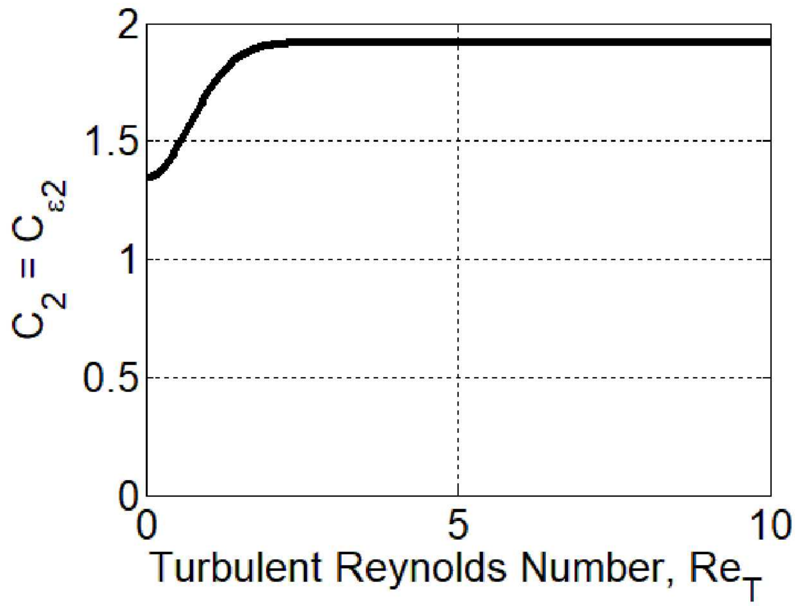

Figure 4.6. $C_2 = C_{\varepsilon 2}$ vs. *Re_T*.

Example 3. Consider internal flow in a pipe with *D*=0.0254 m, wall roughness = 2x10^{-7} m, and $\bar{u}$ =12.5 m/s. The fluid, water in this case, is at 500 K and 7.5x10^{6} Pa. Are there any issues?

For this situation, ρ=835.8 kg/m^3 and μ=1.19x10^{-4} Pa-s. From the LIKE algorithm, *Re* = 2.23x10^{6}, so the flow is highly turbulent and rather isotropic, which is where SKE should perform its best. For this case, the Kolmogorov and Taylor eddies are easily obtained from the LIKE algorithm as η = 5.53x10^{-6} m and λ = 2.67x10^{-4} m, respectively, while the viscous sublayer thickness is $\delta(y^+=7)$ = 2.24x10^{-6} m and $\delta(y^+=1)$ = 3.20x10^{-7} m. The LIKE algorithm also calculates ε = 3.10 m^2/s^3 and *k* = 0.155 m^2/s^2, so the SKE Re_T is quickly calculated as

$$Re_{T,SKE} = \frac{\rho k^2}{\mu \varepsilon} = \frac{835.8*(0.155)^2}{1.19x10^{-4}*3.10} = 54,432 .$$

From Figures 4.5 and 4.6, it is clear that $C_2 = C_{\varepsilon 2}$ and C_μ are not damped because $Re_{T,SKE}$ is too large; this comes as no surprise, as the SKE was built for high *Re*. However, the viscous sublayer is 2.5 and 118 times smaller than the Kolmogorov and Taylor eddies, respectively. Safely assuming that there are no Kolmogorov eddies within the viscous sublayer (refer to Section 3.7), then the *first* layer Kolmogorov eddies would be at a distance $y_\eta = \delta(y^+=7) + \eta$ = 2.24x10^{-6} m + 5.53x10^{-6} m = 7.77x10^{-6} m. That is, this assumes that the viscous sublayer thickness and the typical Kolmogorov eddy length are combined. This implies a dimensionless distance $y^+ = y_\eta/\delta(y^+=1)$ = 7.77x10^{-6} m/3.20x10^{-7} m = 24 for the *first* layer of Kolmogorov eddies. By the same token, the *first* layer of Taylor eddies would be at a distance $y_\lambda = \delta(y^+=7) + \lambda$= 2.24x10^{-6} m + 2.67x10^{-4} m = 2.69x10^{-4} m. This implies a dimensionless distance $y^+ = y_\lambda/\delta(y^+=1)$ = 2.69x10^{-4} m/3.20x10^{-7} m = 831 for the *first* layer of Taylor eddies. As a check, note that at $y = D/2$, y^+ = 39,666 (u_* = 0.45 m/s from the LIKE algorithm). This indicates that the damping functions are well outside of the region where eddies would first be expected near the wall—a disconnect to say the least. This is shown conceptually in Figure 4.7, which compares the relative location (y and y^+) and size of the viscous sublayer vs. the Kolmogorov and Taylor eddies; note that the ordinate axis is based on the log of length, because of the large number of decades typically involved in turbulence.

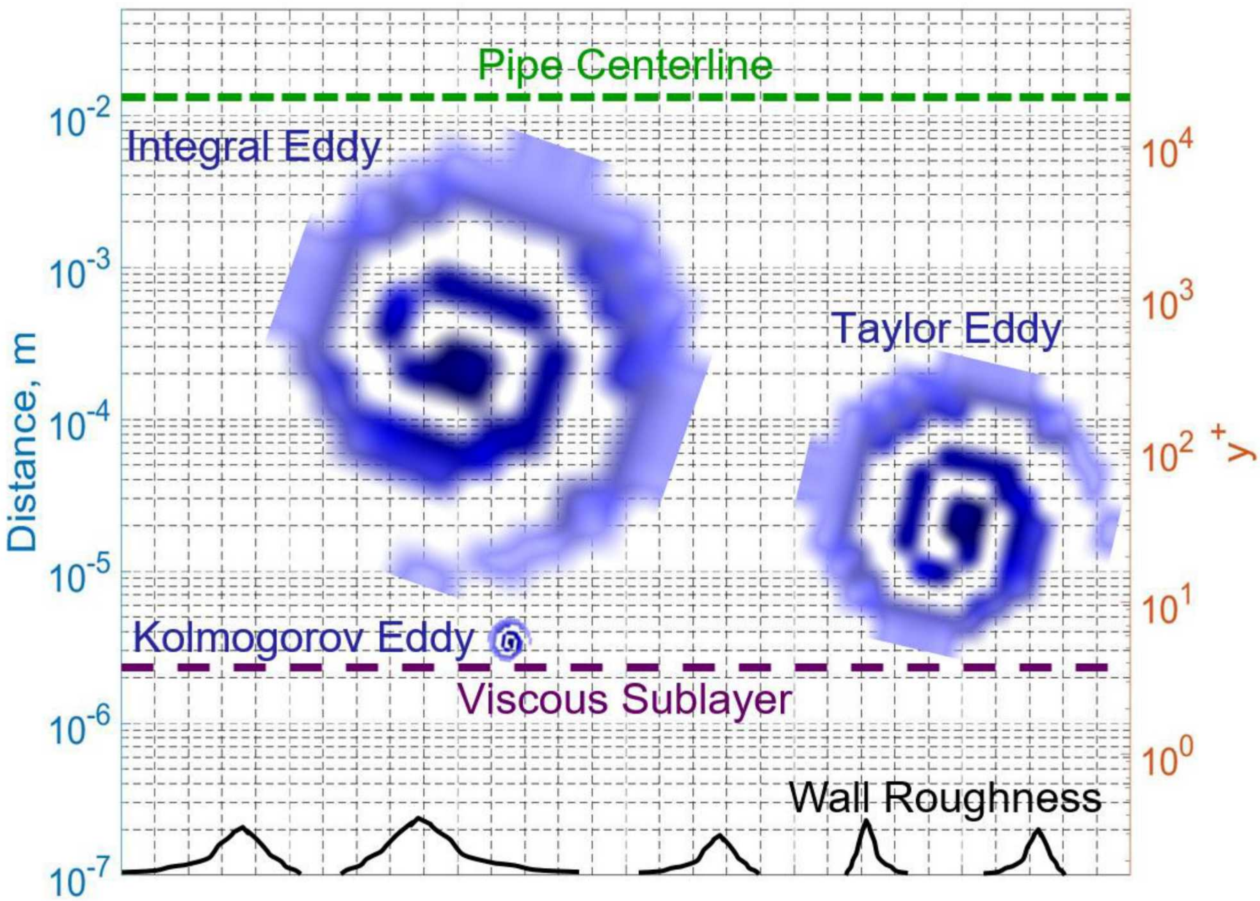

Figure 4.7. Relative size of the viscous sublayer vs. the Kolmogorov and Taylor eddies.

Example 4. Consider Example 3, but modify $\overline{u}$ such that Re=3,000. Are there any issues?

Backtracking $\overline{u}$ from Re, $\overline{u}$ = 0.0168 m/s. From the LIKE algorithm,

u_* = 1.24 x10^{-3} m/s,

y at y^+=1 = 1.15x10^{-4} m,

y at y^+=7 = 8.02x10^{-4} m,

η = 4.23x10^{-4} m,

λ = 4.82x10^{-3} m,

k = 1.47x10^{-6} m^2/s^2, and

ε = 9.04x10^{-8} m^2/s^3.

Calculation of $Re_{T,SKE} = \frac{\rho k^2}{\mu\varepsilon} = \frac{835.8*\left(1.47x10^{-6}\right)^2}{1.19x10^{-4}*9.04x10^{-8}} = 168$ shows a significantly-lower turbulence Reynolds number, as expected for this situation. The SKE net damping behavior can be estimated from Figure 4.5, or using $C_{\mu,SKE} = 0.09\left\{e^{-\left[3.4/\left(1+\mathrm{Re}_T/50\right)^2\right]}\right\} = 0.09\left\{e^{-\left[3.4/\left(1+168/50\right)^2\right]}\right\} = 0.075$. Hence, the damping factor does little to reduce $C_{\mu,SKE}$, despite the flow occurring at a low *Re*. Furthermore, from Figure 4.6, it is clear that $C_2 = C_{\varepsilon 2}$ does not even come into play because $Re_{T,SKE}$ is still too large! This certainly comes as an unpleasant surprise!

Example 5. Consider Example 3, with all parameters being the same, except that the velocity is unknown. Find $\overline{u}$ such that $Re_{T,SKE}$ = 2.2 (i.e., $C_2 = C_{\varepsilon 2}$ is damped if $Re_{T,SKE}$ < 2.2). What is the maximum *y* distance where $C_2 = C_{\varepsilon 2}$ no longer provides damping?

This problem can be solved in various ways. A not-so-elegant manner is to take the LIKE algorithm, supply it an estimated value for $\overline{u}$, which is then used to output *k* and ε so that $Re_{T,SKE}$ can be evaluated, and the iteration proceeds until $Re_{T,SKE}$ = 2.2. Another approach is to find the number of *n* unknowns and seek the relevant *n* independent equations; this more elegant approach is left as an exercise in Section 4.9 (Problem 4.11). In any case, following the iterative method, it is found that $\overline{u}$ = 0.0012 m/s when $Re_{T,SKE}$ = 2.2. This shockingly-low value ought not to come as a surprise, as $Re_{T,SKE}$ = 2.2 translates to a very, very small *hydraulic Re*. In fact, *Re* = 21.4, which is actually very much in the laminar regime for a pipe! Again, because *Re* is so low, u_* = 1.91x10^{-5} m/s. If the viscous sublayer extends to y^+=7, then *y* is calculated as

$$y = \frac{y^+\nu}{u_*} = \frac{7*\left(\frac{1.19\text{x}10^{-4}\,\text{Pa}\bullet\text{s}}{835.8\ \text{kg}/\text{m}^3}\right)}{1.91\text{x}10^{-5}\text{m}/\text{s}} = 0.052\ \text{m}.$$

That the pipe has *D*=0.0254 m indicates that the entire pipe flow domain is affected.

4.7.3 "Ironic" SKE behavior near the wall

As pointed out by Wilcox [Wilcox, 2006], it is indeed ironic that the SKE does not perform well near walls. The irony stems from the notion that the SKE calculates ε, which is associated with the decay of the *small* eddies. And yet, slightly away from the viscous sublayer, say for y^+ < 30, it would be reasonable to expect that the SKE should perform its best in this region because this is exactly where larger eddies *do not exist* (refer to Figure 3.9, Chapter 3). Therefore, based on the decay of the smallest eddies, shouldn't the SKE perform its calculational best for $y^+ < 30$? Quite the contrary, it is well-known that the SKE requires a wall function to improve its calculational behavior near walls. As experimental data for damping and production indicate, the viscous sublayer decreases as *Re* increases; the same trend applies for the region associated with $y^+ < 30$. This indicates that the near wall region where SKE has computational issues decreases as *Re* increases, and this is the region where poor SKE behavior is consistently found in the literature [Myong and Kasagi, 1990; Speziale, Abid, and Anderson, 1992; Wilcox, 2006; Bae, 2016; Bae, Kim, and Kim, 2016; Bae, Kim, and Kim, 2017]. Finally, it is well-known that the SKE's 1974 damping function f_μ approaches a near-0 value $\left\{ e^{-\left[3.4/(1+0/50)^2\right]} = 0.033 \right\}$ as y^+ approaches 0, which rapidly decreases *k* and ε near the wall [Myong and Kasagi, 1990; Wilcox, 2006]. A more consistent formulation would be [Myong and Kasagi, 1990],

$$f_\mu \rightarrow \frac{1}{y} \text{ as } \mathrm{y}^+ \rightarrow 0 .$$

The Myong-Kasagi model actually enables f_μ to increase near the wall, which is consistent with experimental data. Another approach derives f_μ such that ν_t behaves as y^4 near the wall, with *y* being the distance normal to the wall [Lam and Bremhorst, 1981]. Other researchers have employed similar near wall functions with reasonable success [Shih and Hsu, 1992]. By contrast, the SKE tends to under predict *k* and ε for $y^+ < 10$ to 15 [Bernard, 1986; Myong and Kasagi, 1990]. Curiously, the SKE overpredicts ν_t by > 50% near the pipe centerline at *high Re* [Myong and Kasagi, 1990], despite this being the region where the flow tends to be more isotropic. Again, this is another unexpected behavior.

Finally, the *k* PDE can be used to derive a boundary condition by considering the following at the wall: steady state, no convection, no production, and no turbulent viscous dissipation, respectively,

$$\cancel{\frac{\partial k}{\partial t}} + \cancel{\bar{\mathrm{u}}_\mathrm{j} \frac{\partial k}{\partial x_j}} = \cancel{R_{ij} \frac{\partial \bar{\mathrm{u}}_\mathrm{i}}{\partial x_j}} - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \cancel{\frac{\nu_t}{\sigma_k}}\right)\frac{\partial k}{\partial x_j}\right],$$

where $x_j = y$.

This results in the following second-order derivative boundary condition [Speziale, Abid, and Anderson, 1992; Grunloh, 2016],

$$\nu \frac{\partial k^2}{\partial y^2} = \varepsilon .$$

This boundary condition provides an additional fundamental issue near the wall, because numerical stiffness is introduced. Ways to reduce the stiffness have been proposed, but are generally not satisfactory [Speziale, Abid, and Anderson, 1992].

4.8 The 2003 SST Compared with the 2006 k-ω

The 2003 SST is produced by blending the SKE and 1988 k-ω models. In 2006, David Wilcox developed what would be his final k-ω model (unfortunately, he died on February 24, 2016; he truly led "An Improbable Life", [Wilcox, 2007]). In any case, much to Wilcox's delight, his 2006 model overcame the key weak points of his 1988 model. Had his 2006 model been developed in 1988, the motivating drive for the 2003 SST would not have existed. Nevertheless, a term by term comparison of the 2003 SST and 2006 k-ω models show that both are much more similar than would be expected, as will be shown next.

First, compare the *k* PDE for both models. Beginning with the 2003 Menter SST,

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=\tilde{P}_k-\beta^* k\omega+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma_k\nu_t\right)\frac{\partial k}{\partial x_j}\right]$$

$$P_k=R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}$$

$$\tilde{P}_k=\min\left(P_k,\ 10\beta^* k\omega\right).$$

Now compare the above with the 2006 k-ω,

$$\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}=R_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\beta^* k\omega+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma^*\nu_t\right)\frac{\partial k}{\partial x_j}\right].$$

The two k PDEs are fairly identical, with a couple of differences:

1. The *k* production terms are the nearly the same, except that the 2003 SST uses a production limiter, whereas the 2006 k-ω does not.
2. The multiplier for the SST is $0.85\le\sigma_k\le 1.0$, while the k-ω is fixed at $\sigma^*=\frac{3}{5}$. This indicates that the SST model has a higher degree of turbulent diffusion.

Other than the SST having more *k* diffusion than the 2006 k-ω, the *k* PDEs are the same.

Next, the ω PDEs are compared. Starting again with the 2003 Menter SST,

$$\frac{\partial \omega}{\partial t}+\bar{\mathrm{u}}\frac{\partial \omega}{\partial t}+\bar{\mathrm{u}}_j\frac{\partial \omega}{\partial x_j}=\alpha\frac{\tilde{P}_\omega}{\nu_t}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma_\omega\nu_t\right)\frac{\partial \omega}{\partial x_j}\right]+2\sigma_{\omega 2}\left(1-F_1\right)\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

$$\alpha\frac{\tilde{P}_\omega}{\nu_t}=\frac{\alpha}{\nu_t}R_{ij}\frac{\partial \bar{\mathrm{u}}_i}{\partial x_j}=\alpha\frac{\omega}{k}R_{ij}\frac{\partial \bar{\mathrm{u}}_i}{\partial x_j}$$

$$\tilde{P}_\omega=\min\left(P_\omega,\ 10\beta^* k\omega\right).$$

Now compare the formulation with the 2006 k-ω,

$$\frac{\partial \omega}{\partial t}+\bar{\mathrm{u}}_{\mathrm{j}}\frac{\partial \omega}{\partial x_j}=\alpha\frac{\omega}{k}R_{ij}\frac{\partial \bar{\mathrm{u}}_{\mathrm{i}}}{\partial x_j}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[\left(\nu+\sigma\frac{k}{\omega}\right)\frac{\partial \omega}{\partial x_j}\right]+\frac{\sigma_d}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}.$$

Again, the similarities are striking, though there are two differences:

1. The ω production terms are the nearly the same, except that the 2003 SST uses a production limiter, whereas the 2006 k-ω does not. In addition, the SST α ranges from 0.44 to 5/9, whereas the k-ω has a fixed α of 13/25. Despite this, the α values for both models are within 15% or less, and depending on the value for F_1, can be identical on occasion.
2. The SST uses a blending function in the cross-diffusion term that makes $2\sigma_{\omega 2}\left(1-F_1\right)$ range from 0 to 1.71, vs. the k-ω constant value of $\sigma_d=0.0$ for near the wall, and $\sigma_d=\frac{1}{8}$ for free shear flow. So, except for large F_1 in the boundary layer, the SST employs much more cross-diffusion away from the surface. This is perhaps one of the biggest differences, albeit having essentially the same cross-diffusion derivatives and an inverse ω relationship.
3. Note that the SST uses σ_ω in the viscous term, while the k-ω uses σ. Nevertheless, $\sigma_\omega=\sigma$, so these terms are equal.

Next, it is noted that both models use stress limiters for ω. For the 2003 SST,

$$\tilde{\omega}=\max\left(\omega,\ \frac{\mathrm{F}_2 S}{a_1}\right)=\max\left(\omega,\ \frac{\mathrm{F}_2\sqrt{2S_{ij}S_{ij}}}{a_1}\right)$$

with

$$\nu_t=\frac{k}{\tilde{\omega}}=\frac{k}{\max\left(\omega,\ \frac{\mathrm{F}_2\sqrt{2S_{ij}S_{ij}}}{a_1}\right)}.$$

On the other hand, the 2006 k-ω uses the following similar formulation,

$$\tilde{\omega} = \max\left(\omega,\ C_{\lim}\sqrt{\frac{2S_{ij}S_{ij}}{\beta^*}}\right)$$

with

$$\nu_t = \frac{k}{\tilde{\omega}} = \frac{k}{\max\left(\omega,\ C_{\lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right)}.$$

Note that the stress limiters for the SST and k-ω, respectively, are nearly identical:

$$\frac{1}{a_1} = 3.23$$

and

$$C_{\lim}\sqrt{\frac{1}{\beta^*}} = 2.92.$$

Finally, the 2006 k-ω uses the f_b blending function to distinguish between free shear and near wall flows, while the 2003 SST uses the F_1 "inter-model" blending function to calculate the free shear and near wall flows (refer to Figure 4.4). And as alluded earlier, the 2003 SST uses the P_k and P_ω production stress limiters, while the 2006 k-ω does not.

All said, the 2003 SST and 2006 k-ω are much more alike than different, with most terms being the same, a few having slightly different coefficients, and a couple of terms that are not analogous. Not surprisingly, a review of the literature shows that both models tend to generate similar results [Wilcox, 2006; Fraczek and Wroblewski, 2016]. For example, researchers computing drag coefficients in labyrinth seals noted that the results for "the SST and k-omega turbulence model are almost the same" [Fraczek and Wroblewski, 2016].

Finally, Wilcox noted that the 2006 k-ω "predicts reasonably close" when compared with experimental data for incompressible, transonic, supersonic, and hypersonic flows. To investigate this further, he increased C_{lim} to 1.0, so that $C_{\lim}\sqrt{\frac{1}{\beta^*}} = 3.33$ (which is much closer to the SST value of $\frac{1}{a_1} = 3.23$). This change increased the k-ω model's agreement with experimental data in the transonic range, but at reduced accuracy for sonic and hypersonic flows. Wilcox therefore recommended his original value of $C_{\lim} = \frac{7}{8}$ [Wilcox, 2006]. The 1992 SST (which is very similar to the 2003 SST), can outperform the 2006 k-ω in the transonic regime [Wilcox, 2006].

4.9 Problems

4.1. Figure 4.6 indicates that C_2 remains fixed for $Re_T > 2.2$. Assume a cylindrical pipe with D=0.25 m and ν=4.5x10^{-6} m^2/s. Discuss and justify the assumptions used in this conversion. What is the equivalent Re_H?

4.2. Review [Menter, 1992] and derive the 1992 SST model by combining the SKE and the 1988 k-ω models.

4.3. Show that $Re_T = \frac{\rho k^2}{\mu \varepsilon}$ (which is used in the SKE) is not consistent with the implied eddy length ℓ derived from the Prandtl-Kolmogorov relationship, $\nu_t = C_\mu \frac{k^2}{\varepsilon}$. (Hint: recall that the Prandtl-Kolmogorov relationship is valid for $\nu_t = u_{char} x_{char} = k^{1/2} \ell$, whereby it is shown that ℓ is a function of ε. Use that ℓ relationship to derive $Re_T = \frac{x_{char} u_{char} \rho}{\mu}$ and compare with the SKE version.)

4.4. Transform the SKE model into an analogous formulation for the 1988 k-ω model by using a relationship that associates ε with ω (e.g., $\varepsilon = C_\mu \omega k$), where $C_\mu = \beta^*$.

4.5. Take the 2003 SST and modify its coefficients to be equivalent to the 2006 k-ω model. For example, let $\frac{1}{a_1} = C_{\lim} \sqrt{\frac{1}{\beta^*}} = 3.33$, and so forth. Use the newly-derived model to simulate a system that consists of a smooth flat plate under isothermal boundary layer flow, is 0.1 m long and 0.05 m wide, and has air at 300 K and 1 atmosphere (ν=1.58x10^{-5} m^2/s and U_s = 347.3 m/s). The air flows from left to right along the 0.1 m plate at a constant velocity U_∞ based on a specified *Ma*. Conduct simulations with the revised SST model and compare the results vs. the 2006 k-ω simulation using the same geometry, mesh, and initial and boundary conditions. How do the velocity solutions compare for *Ma*=0.25, 0.5, 1.0, 5.0, and 10? Is the flow always turbulent?

4.6. Repeat Exercise 4.5, except that now the 2006 k-ω model is modified so that it more closely approaches the 2003 SST. For example, let $C_{\lim} \sqrt{\frac{1}{\beta^*}} = \frac{1}{a_1} = 2.92$, and so forth. How do the velocity solutions compare for *Ma*=0.25, 0.5, 1.0, 5.0, and 10? Is the flow always turbulent?

4.7. Build a two-equation turbulence model by choosing *k* and *a* (the eddy acceleration). Apply dimensional arguments to develop the transport PDEs; use the Buckingham Pi theorem if necessary. What is the relevant expression for ν_t?

4.8. What is the thickness of the viscous sublayer at the point where C_μ in the SKE drops precipitously at Re_T < 2,300? What is the size of the integral, Taylor, and Kolmogorov eddies at that point? Any potential issues here?

4.9. What is the thickness of the viscous sublayer when C_2 in the SKE drops precipitously at Re_T < 2.5? What is the size of the integral, Taylor, and Kolmogorov eddies? Any potential issues here?

4.10. Consider the MK k-ε model, where its authors added the Taylor length onto the integral length, ℓ. Starting with their length expression, derive a new MK formulation if the Kolmogorov length scale is added as well.

4.11. Consider Example 5 in Chapter 4. Find an analytical expression for the distance *y* from the wall such that the SKE $C_2 = C_{\varepsilon 2}$ is no longer dampening. This point can be assumed as $Re_{T,SKE}$ = 2.2. Hint: the equations for y^+ and $Re_{T,SKE}$ will be needed, and some of the unknowns likely include *k*, ε, u_*, ℓ, and l_T.

4.12. For a cylindrical pipe, how can Re_T be converted to *Re*?

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