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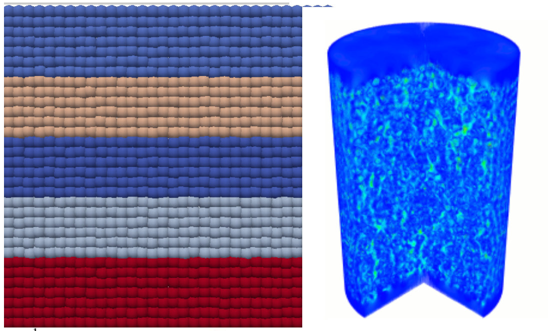

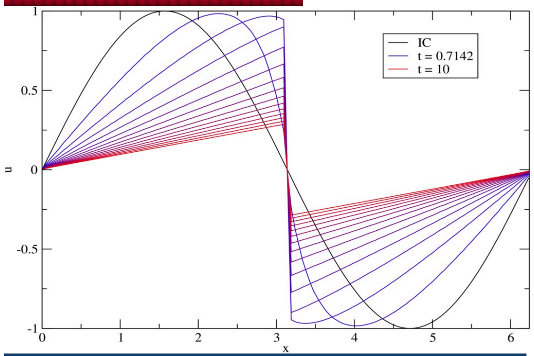

Locally compatible meshless methods for PDEs

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The Compadre project **COM**patible **PA**rticle **D**isc**RE**tizations

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Why are we interested in particle/meshfree discretizations?

1. Next Generation Platforms (NGP):

- NGP will have greater concurrency on node
- Memory, I/O bandwidth and cache lag FP density
- Communication will remain a challenge
- Meshing will be part of the solution & NGP workflow

Particles can:

Increase concurrency

Reduce communication

Enable automatic "meshing"

2. Sandia/DOE Mission

- EMPIRE/EMPRESS: electromagnetic plasma and environment simulator: kinetic particle or hybrid particle fluid, PIC discretization
- Kodiak/CTH/Sierra SM: large deformations, material damage
- Meshfree data transfeR for multiphysics code couplings:
  - Climate: Coupling Approaches for Next-Generation Architectures (CANGA)
  - Energy: DTK: meshless data transfer for code coupling in CASL

ACME Accelerated Climate Modeling for Energy

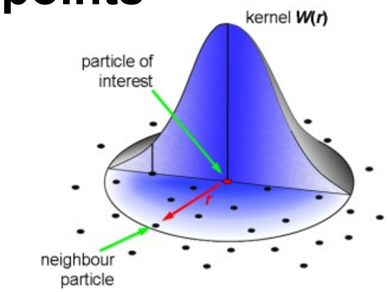
CASL

3. An opportunity for theoretical and practical advances (great science)

- There's no theory for "compatible" particle/meshless methods
- Methods are still a niche player, not ready for the mainstream mission needs

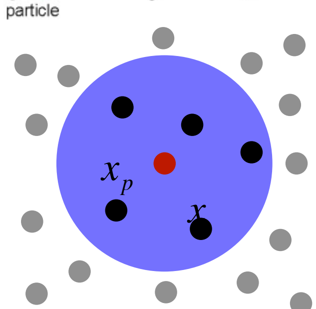
The currently dominant meshfree paradigm

Foundational concept: local kernel estimates of fields from points



$$u_\varepsilon^h(x) = \sum_p u(x_p) W_\varepsilon(x - x_p)$$

Differentiable kernels yield derivative approximations:



$$D^\alpha u_\varepsilon^h(x) = \sum_p u(x_p) D^\alpha W_\varepsilon(x - x_p)$$

Derivative approximations yield the PDE discretization

$$D^\alpha u(x) = f(x) \longrightarrow \sum_p u(x_p) D^\alpha W_\varepsilon(x - x_p) = f(x)$$

Method	Approximation Function												Solution Scheme (Discretization)								
	Local												Weak form					Strong form			
	With Polynomial Reproduction (PR)									without PR			Bubnov-Galerkin			Petrov-Galerkin					
	Local Polynomials	MLS/RK Direct	MLS/RK Diffuse	WLS	Enriched Polynomials	Enriched MLS/RK (PU)	Natural neighbor	Maximum Entropy	Splines/NURBS	Smooth Kernel	Local RBF	RBF (Global)	Gauss integration	Stabilized Integration	Point Collocation	Gauss integration	Variationally Consistent	Point Collocation	Subdomain Collocation	Weighted Collocation	Lagrangian-Eulerian form
FEM[a], SFEM[a]	•												•	•							
SPH[b,c]										•					•						
GFD			•															•			
RBCM											•	•						•	•	•	
DEM			•										•								
EFG, XEFG		•				•							•	•							
MPM, GIMP, CPDI, PFEM-2	•																				•
RKPM, SLRKPM[c]		•											•	•							
GFEM, XFEM					•								•								
MPS[d]				•														•			
PUM	•				•	•							•								
hp clouds						•							•								
FPM				•														•			
FMM[c]	•												•								
C-SPH/MLSPH[b,c]		•													•						
MLPG		•								•						•					
NEM							•						•								
PPU						•							•								
MFS						•							•								
RKCM, GRKCM		•	•															•		•	
Meshfree SCNI		•												•							
LRPIM	•									•	•					•					
RPIM	•										•		•								
MFEM[a], PFEM[c]							•						•								
MaxEnt								•					•								
IGA[a]									•				•				•				
Peridynamics (PD)[e]			•															•			
OTM[c]								•					•								
Meshfree VCI[f]		•											•	•	•		•				

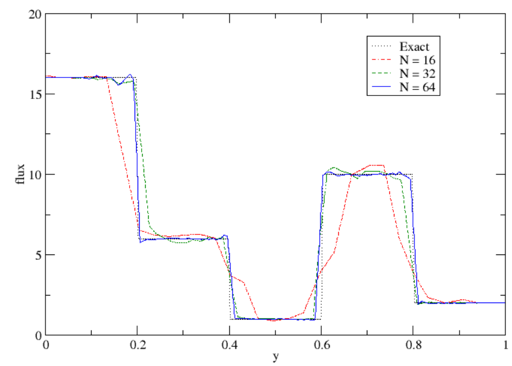
C. Jiun-Shyan, H. Michael, and C. Sheng-Wei. Meshfree methods: Progress made after 20 years. Journal of Engineering Mechanics, 143(4):04017001, 2017/09/08

Is far from perfect…

- These are point collocation type methods!

- **Require (much) more regularity compared to weak forms!**
- **Conflicts between consistency and conservation**
  - SPH is conservative but does not reproduce constant and linear fields
  - RKPM and other corrections fix the accuracy but can compromise conservation
- **Mathematically equivalent to node-based (or collocated) methods**
  - Unsuitable for mixed discretizations needed in multiple mission problems:
  - Drift-Diffusion, Reactive flow transport, MHD, Incompressible flows
- **“Compatible” meshless methods lag behind their mesh-based cousins.**

Many of these methods fail for the standard 5-strip problem which tests the ability of a method to reproduce fields that are in *H(div)* but not in H^1 – a critical requirement for mixed discretizations

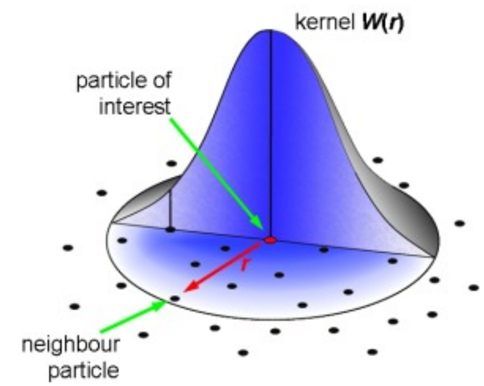


T. J. R. Hughes, A. Masud, and J. Wan. A stabilized mixed Discontinuous Galerkin method for Darcy flow. Comput. Meth. Appl. Mech. Eng., 195:3347–3381, 2006.

Example: SPH

Local kernel representation

$$\langle f \rangle (\mathbf{x}) = \int_{supp(W)} f(\mathbf{y}) W(\mathbf{x} - \mathbf{y}) d\mathbf{y}$$



"Standard": 0^{th}-order accurate ☹

$$\nabla_0 f_i = \sum_{j \in supp(W)} (f_j - f_i) \nabla_{\mathbf{x}_i} W_{ij} V_j$$

$$\nabla_0^2 f_i = 2 \sum_{j \in supp(W)} \frac{f_i - f_j}{|\mathbf{r}_{ij}|} \mathbf{e}_{ij} \cdot \nabla_{\mathbf{x}_i} W_{ij} V_j$$

"Corrected": 1^{st}-order accurate ☺

$$\nabla_1 f_i = \sum_{j \in supp(W)} (f_j - f_i)\, \mathbf{G}_i \nabla_{\mathbf{x}_i} W_{ij} V_j$$

$$\nabla_1^2 f_i = 2 \sum_{j \in supp(W)} \left(\mathbf{L}_i : \mathbf{e}_{ij} \otimes \nabla_{\mathbf{x}_i} W_{ij} \right) \left(\frac{f_i - f_j}{|\mathbf{r}_{ij}|} - \mathbf{e}_{ij} \cdot \nabla_1 f_i \right) V_j$$

Correction tensors (computed locally)

- **Standard: Conservative, low accuracy** (**inconsistent** grad and **Δ** operators)
- **Corrected:** 2^{nd} order accurate, **non-conservative**, fewer neighbors needed.

N. Trask, M. Maxey, K. Kim, M. Perego, M. Parks, K. Yang, J. Xu. A consistent 2^{nd} order projection scheme for simulating transient viscous flow with Smoothed Particle Hydrodynamics. CMAME, vol 289, 2015.

Compadre objectives

It would be great to have a rigorous mathematical basis for particle methods that offers all the nice properties we've come to expect from mesh-based discretizations:

- Ability to use weak variational formulations = less regularity
- Ability to formulate "mixed type" meshless methods
- Ability to have high-order *and* mimetic properties (in some sense)

Compadre aims to build a meshless mathematical foundation that parallels that of, e.g., the Finite Element Exterior Calculus:

- **Meshless Variational Principles (MVP)**
- **Meshless Exterior Calculus (MEC)**

A refresher: why mesh makes compatibility easy

The Stokes theorem $\langle d\omega, \Omega \rangle = \langle \omega, \partial\Omega \rangle$ **is foundational for compatible discretizations:**

- It tells us that the ***derivative*** is dual to ***boundary***.
- This duality is a ***key design principle*** for compatible discretizations:
- **Topology** (*boundary operator*) induces a **derivative** via the *coboundary*

The mesh topology defines the compatible operators for free!

mesh	$C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} C_3$	$\langle a, \partial\sigma \rangle$	$\partial : C_k \rightarrow C_{k-1}$	Boundary	$\partial\partial = 0$
		$=$			$\Downarrow$
field	$C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} C^3$	$\langle \delta a, \sigma \rangle$	$\delta : C^k \rightarrow C^{k+1}$	Coboundary	$\delta\delta = 0$

Example: compatible grad, curl and div for a hex topology

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & & & & & & 1 \\ 0 & \cdots & & & & & & 0 \\ 0 & \cdots & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

grad

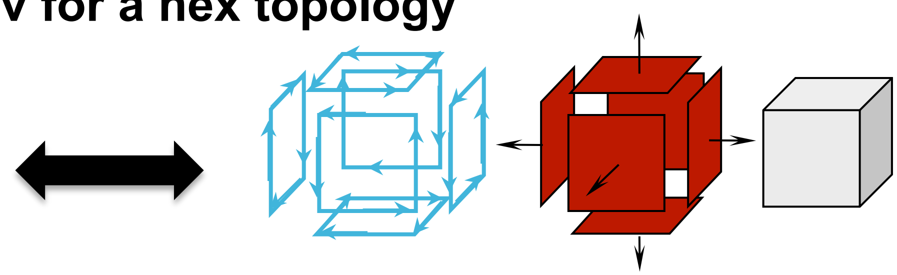
$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

curl

$$\begin{pmatrix} -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

div

curl grad = div curl = 0



$$0 = \partial\partial K^3 \xleftarrow{\partial} \partial K^3 \xleftarrow{\partial} K^3$$

How to be compatible without a mesh?

- We need a more general representation tool that goes beyond nodal collocation.
- The Generalized Moving Least-Squares (GMLS) in one such tool:

GMLS approximates **linear functionals** (**dual objects**) instead of **fields** (**primal objects**)

Statement of the GMLS problem: Given

$V,\, V^*$ - a function space and its dual

$P = span\{p_i\}_{i=1}^{Q} \subset V$ - a finite dimensional "consistency" space, e.g., polynomials

$\Lambda = \{\lambda_1, \ldots, \lambda_N\} \subset V^*$ - a finite set of sampling functionals:

$\omega : V^* \times V^* \to \mathbf{R}$ - a correlation measure between functionals

For every $\tau \in V^*$ (target) find an approximation $\bar{\tau} \in V^*$ such that

$$\bar{\tau}(u) = \sum_{i=1}^{N} a_i(\tau)\lambda_i(u) \quad \text{such that} \quad \begin{cases} \bar{\tau}(p) = \tau(p) \quad \forall p \in P & \text{- consistency} \\ \omega(\tau, \lambda_i) = 0 \quad \Rightarrow a_i(\tau) = 0 & \text{- local support} \\ \|\mathbf{a}(\tau)\|_{L_1} \le C \quad \forall \tau \in V^* & \text{- uniform boundedness} \end{cases}$$

H. Wendland. Scattered data approximation, Vol. 17. Cambridge university press, 2004.

GMLS-101

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Theorem. The GMLS coefficients $a_i(\tau) \in \mathbf{R}$ solve a (local) Quadratic Program (QP)

$$\min \frac{1}{2}\sum_{i=1}^{N} \frac{a_i^2(\tau)}{\omega(\tau,\lambda_i)} \quad \text{such that} \quad \sum_{i=1}^{N} a_i(\tau)\lambda_i(u) = \tau(p) \quad \forall p \in P$$

Algebraic form of the QP

$$\min \frac{1}{2}\mathbf{a}(\tau)^T W^{-1}(\tau)\mathbf{a}(\tau) \quad \text{such that} \quad L\mathbf{a}(\tau) = \tau(\mathbf{p})$$

$$\mathbf{a}(\tau) = \left[a_i(\tau)\right] \in \mathbf{R}^N$$

$$W(\tau) = diag\left[\omega(\tau,\lambda_j)\right] \in \mathbf{R}^{N\times N}$$

$$L = \left[\lambda_j(p_i)\right] \in \mathbf{R}^{Q\times N}$$

$$\tau(\mathbf{p}) = \left[\tau(p_j)\right] \in \mathbf{R}^Q$$

QP solution: GMLS “basis” functions

$$\mathbf{a}(\tau) = W(\tau)L^T\left(LW(\tau)L^T\right)^{-1}\tau(\mathbf{p})$$

$a_i(\tau)$ can be used as a “shape function” to define meshless Galerkin methods:

See, e.g., Nayroles (1992), Belytschko (1994), Atluri (1998), Mirzaei (2013).

GMLS-101

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GMLS approximation of the action of $\tau \in V^*$ on $u \in V$:

$$\tau(u) \approx \overline{\tau}(u) = \sum_{i=1}^{N} a_i(\tau)\lambda_i(u) = \mathbf{u}^T \mathbf{a}(\tau)$$

$\mathbf{u} = \left[\lambda_i(u)\right] \in \mathbf{R}^N$ - sample vector

We can group the terms in the GMLS approximation in **two different ways**

$$\overline{\tau}(u) = \mathbf{u}^T \left[W(\tau)L^T \left(LW(\tau)L^T \right)^{-1} \tau(\mathbf{p}) \right] = \mathbf{u}^T \mathbf{a}(\tau)$$

GMLS basis function form:
sum of field samples $\lambda_i(u)$

$$\overline{\tau}(u) = \left[\mathbf{u}^T W(\tau)L^T \left(LW(\tau)L^T \right)^{-1} \right] \tau(\mathbf{p}) = \mathbf{b}(\tau)^T \tau(\mathbf{p})$$

Polynomial basis function form:
sum of transformed *P*-bases $\tau(p_j)$

The coefficients $\mathbf{b}(\tau)$ solve an algebraic WLS problem:

$$\mathbf{b}(\tau) = \underset{\mathbf{c} \in \mathbf{R}^N}{\operatorname{argmin}} \frac{1}{2} \left(L^T \mathbf{c} - \mathbf{u} \right)^T W(\tau) \left(L^T \mathbf{c} - \mathbf{u} \right) = \underset{\mathbf{c} \in \mathbf{R}^N}{\operatorname{argmin}} \frac{1}{2} \left\| L^T \mathbf{c} - \mathbf{u} \right\|_{W(\tau)}^2$$

These coefficients also happen to define a **function** $p_\tau(x) = \mathbf{b}(\tau)^T \mathbf{p} = \sum_{i=1}^{Q} b_i(\tau) p_i(x) \in P$

GMLS-101

The function $p_\tau(x)$ gives the best WLS fit to the data $\mathbf{u}=[\lambda_i(u)]$ out of *P:*

$$p_\tau = \underset{p\in P}{\operatorname{argmin}} \frac{1}{2}\sum_{i=1}^{N}\left(\lambda_i(p)-\lambda_i(u)\right)^2 \omega(\tau,\lambda)$$

$$P=\left\{p(x)=\mathbf{c}^T\mathbf{p} \,\middle|\, \mathbf{c}\in\mathbf{R}^N\right\}; \quad L^T\mathbf{c}=\left[\lambda_i(p)\right]_{i=1}^{N}$$

The "diffuse" derivative misconception

p_τ is the best fit to the data $\mathbf{u}=[\lambda_i(u)]$ out of *P*. However, it is not the approximation $\overline{\tau}(u)$ of $\tau(u)$ that we seek! Often people neglect the dependence of $b_i(\tau)$ on τ and write:

$$\tau(u)\approx\tau(p_\tau)=\sum_{i=1}^{Q} b_i(\tau)\tau\left(p_i(x)\right)=\mathbf{b}(\tau)^T\tau(\mathbf{p})=\overline{\tau}(u)$$

But this is mathematically incorrect since in general $\tau(uv)\neq u\tau(v)$, i.e.,

$$\tau(u)\approx\tau(p_\tau)=\tau\left(\sum_{i=1}^{Q} b_i(\tau)p_i(x)\right)=\sum_{i=1}^{Q}\tau\left(b_i(\tau)p_i(x)\right) \text{ STOP } \neq \sum_{i=1}^{Q} b_i(\tau)\tau\left(p_i(x)\right)=\mathbf{b}(\tau)^T\tau(\mathbf{p})=\overline{\tau}(u)$$

For this reason people call these approximations "**diffuse**" although they are perfectly OK.

GMLS-101: Point collocation example

Specialization: $\Lambda = \{\delta_{x_1}, \ldots, \delta_{x_N}\}; \tau(u) = \tau_x(u) = \left(\delta_x \circ D^\alpha\right)(u)$ and $\omega\left(\tau_x, \delta_{x_i}\right) = \Phi\left(\|x - x_i\|\right)$

$$p_\tau(x) = \operatorname*{argmin}_{p \in P} \frac{1}{2} \sum_{i=1}^{N} \left(p(x_i) - u(x_i)\right)^2 \Phi\left(\|x - x_i\|\right) \quad \longrightarrow \quad p_\tau(x) = \sum_{i=1}^{Q} b_i(x) p_i(x)$$

The coefficients $b_i(x)$ depend on x which makes **nonsensical** the "calculation"

$$D^\alpha u(x) \approx D^\alpha p_\tau(x) = \sum_{i=1}^{Q} b_i(x) D^\alpha p_i(x)$$

This only makes sense as a *definition*!

To avoid confusion: the legal definitions of GMLS in this talk are:

$\overline{\tau}(u) = \mathbf{u}^T \mathbf{a}(\tau)$ where $\mathbf{a}(\tau)$ solves $\min \frac{1}{2} \mathbf{a}(\tau)^T W^{-1}(\tau) \mathbf{a}(\tau)$ s.t. $L\mathbf{a}(\tau) = \tau(\mathbf{p})$

$\overline{\tau}(u) = \mathbf{b}(\tau)^T \tau(\mathbf{p})$ where $\mathbf{b}(\tau)$ solves $\mathbf{b}(\tau) = \operatorname*{argmin}_{\mathbf{c} \in \mathbf{R}^N} \frac{1}{2} \left\| L^T \mathbf{c} - \mathbf{u} \right\|^2_{W(\tau)}$

D. Mirzaei, R. Schaback, and M. Dehghan. On generalized moving least squares and diffuse derivatives. IMA Journal of Numerical Analysis, 32(3):983–1000, 2012.

Making GMLS compatible

We consider two complementary strategies:

Locally compatible GMLS (this talk)

Mimics ideas from **primal-dual mesh** methods such as MAC, Control Volume or Finite Volume to obtain a "staggered" GMLS discretization.

Globally compatible GMLS (next talk)

Solves an auxiliary graph PDE problem to obtain a global Stokes Theorem mimicking $\langle a, \partial\sigma \rangle = \langle \delta a, \sigma \rangle$

Model problem:

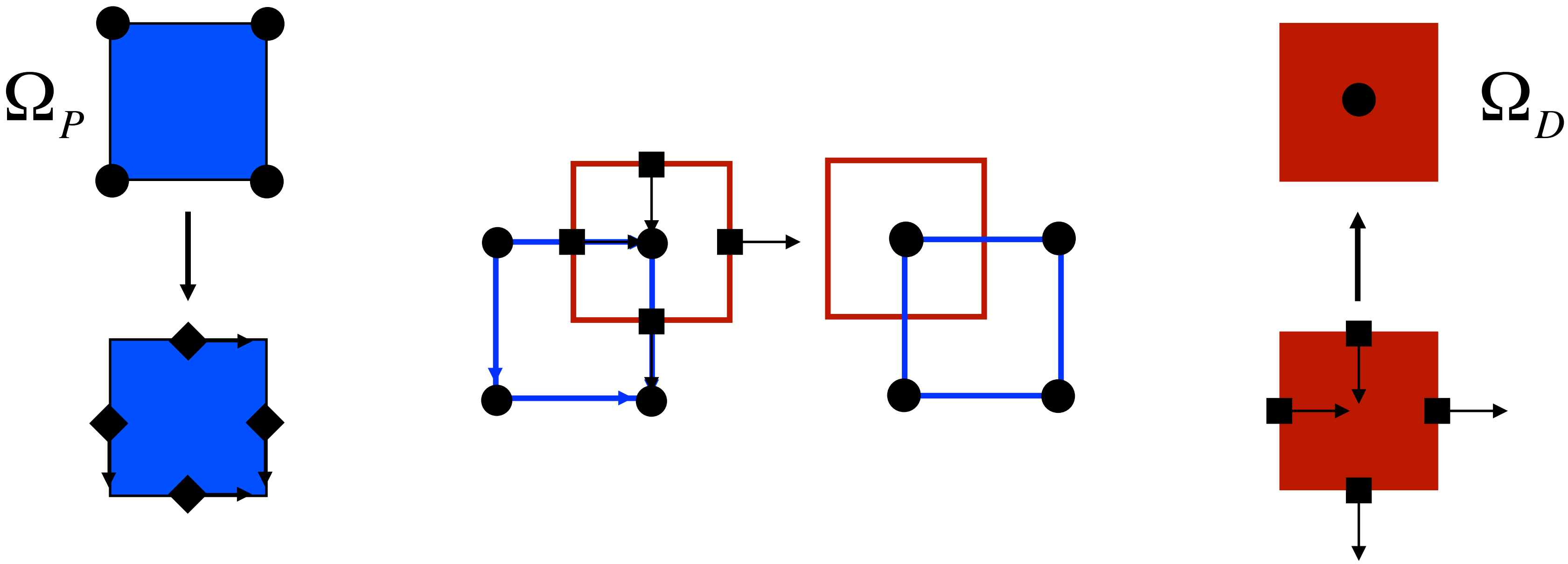
$$\begin{cases} \nabla \cdot \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} + \mathbf{A}^{-1} \nabla \phi = 0 & \text{in } \Omega \end{cases} + \begin{cases} \phi = g & \text{on } \Gamma_D \\ \mathbf{n} \cdot \mathbf{u} = h & \text{on } \Gamma_N \end{cases}$$

We will assume that τ and λ_i can be associated with points x and x_i, resp., so that

$$\omega\left(\tau_x, \delta_{x_i}\right) = \Phi\left(\|x - x_i\|\right)$$

Our template:

"Staggered" grid methods such as Co-volume, Finite Volume, MAC…



$GRAD : V \rightarrow E$ $\qquad$ $DIV \cdot GRAD : V \rightarrow C$ $\qquad$ $DIV : F \rightarrow C$

$$\begin{cases} \nabla \cdot \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} + \kappa \nabla \phi = 0 & \text{in } \Omega \end{cases} \Rightarrow \begin{cases} DIV(\mathbf{u}_h) = f_h & \text{in } \Omega_D \\ \mathbf{u}_h + \kappa GRAD(\phi_h) = 0 & \text{in } \Omega_P \end{cases} \Rightarrow DIV \kappa GRAD(\phi_h) = f_h \;\; \text{in } \Omega_P$$

*Harlow FH, Welch JE. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. Physics of Fluids 1965; **8**(12):2182–2189.*

We apply this template locally

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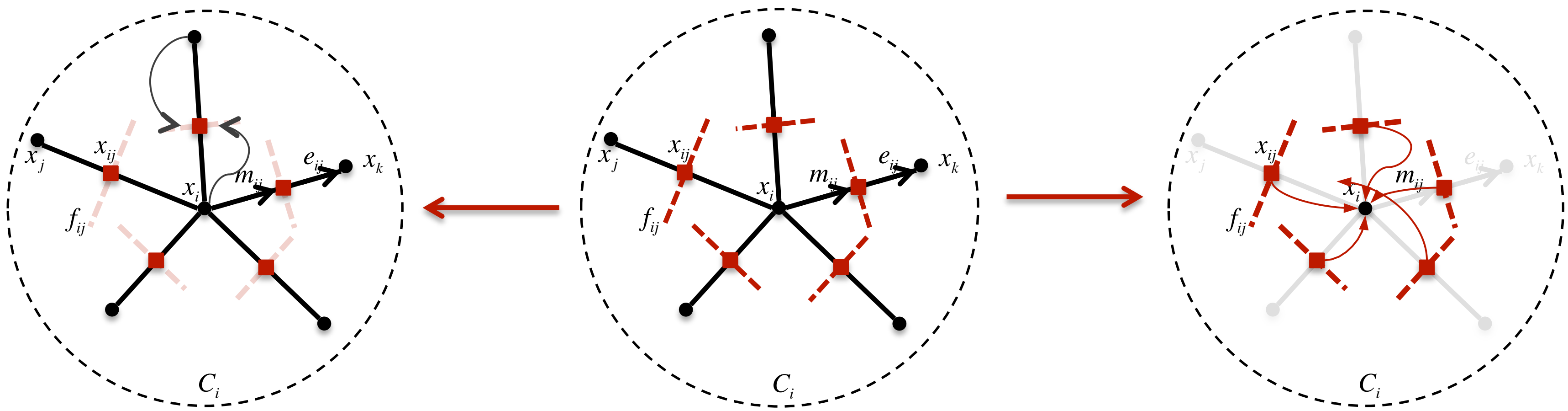
We define a local primal-dual grid complex, with a virtual dual grid, induced by a point x_i and its ε-neighborhood N_i^{ε}:

A local primal grid | **Local primal-dual complex** | **A virtual dual grid**



$$GRAD_i : V^i \rightarrow E^i$$

$$DIV_i \cdot GRAD_i : V^i \rightarrow C^i$$

$$DIV_i : F^i \rightarrow C^i$$

Topological GRAD: defined by local graph connectivity.

Face areas and cell volumes will be implicit in the definition of DIV by GMLS

GMLS DIV: defined by GMLS reconstruction of edge values.

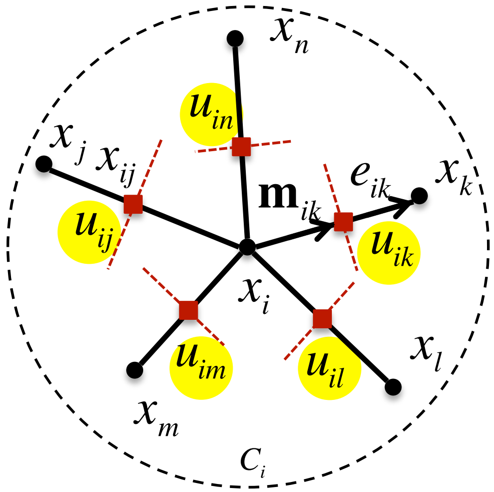
N. Trask, M. Perego and P. Bochev. A high-order staggered meshless method for elliptic problems. SISC, 2017

The details: $DIV_i : F^i \to C^i$

Given a vector field $\mathbf{u}(x) \in \left(C^{m+1}(\Omega)\right)^d$ **define its** ***radial component function***

$$u_{i\to}(x) = \mathbf{u}(x)\cdot(x - x_i) \Rightarrow \begin{array}{l} u_{i\to}(x_{ij}) = \mathbf{u}(x_{ij})\cdot(x_{ij} - x_i) = \mathbf{u}(x_{ij})\cdot\mathbf{m}_{ij} = u_{ij} \\ u_{i\to}(x_i) = \mathbf{u}(x_i)\cdot(x_i - x_i) = 0 \end{array}$$

The value of the radial component $u_{i\to}(x_{ij})$ **at edge midpoint = tangential component of the vector field along edge** $\mathbf{e}_{ij}$**:**



Thus, we can reconstruct the **radial component function** and **any functional acting** on it from the **tangent components** of $\mathbf{u}(x)$

But why would we do this?

Because $\Rightarrow$ $\mathbf{u}(x_i) = \frac{1}{2}\nabla u_{i\to}(x_i)$ $\quad \nabla\cdot\mathbf{u}(x_i) = \frac{1}{4}\nabla\cdot\nabla u_{i\to}(x_i)$

Thus, we can approximate the vector field and its divergence at a point x from the GMLS reconstruction of its radial component function!

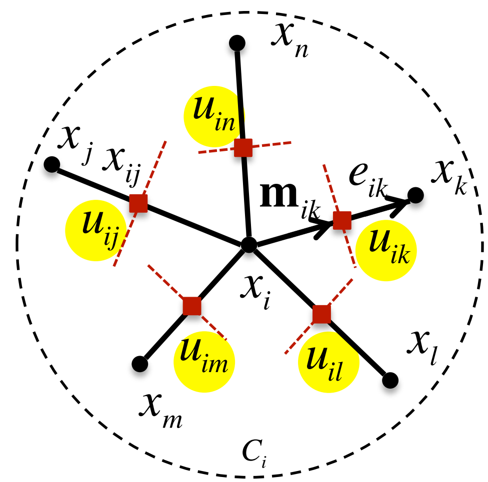
The details: $DIV_i : F^i \to C^i$

Specialization of GMLS to the radial component function

$$V_i = \left\{ v \in C^{m+1}(\Omega) \middle| v(x_i) = 0 \right\} \qquad \mathbf{u} = \left[u_{ij} \right] \qquad \omega(\tau_*, \lambda_{ij}) = \Phi\left(\left\| x_i - x_{ij} \right\| \right)$$

$$P_i = \left\{ p \in P_m \middle| p(x_i) = 0 \right\} \subset V_i \qquad \tau_\nabla(*) = \left(\delta_{x_i} \circ \nabla \right)(*)$$

$$\Lambda = \{ \delta_{ij} | x_{ij} = (x_i + x_j)/2 \} \qquad \tau_\Delta(*) = \left(\delta_{x_i} \circ \Delta \right)(*)$$



1. Solve the algebraic WLS to find the GMLS coefficients:

$$\mathbf{b}(x) = \frac{1}{2} \underset{\mathbf{c} \in \mathbf{R}^N}{\operatorname{argmin}} \left\| L^T \mathbf{c} - \mathbf{u} \right\|^2_{W(\tau)}$$

2. Apply the target functional to the polynomial basis:

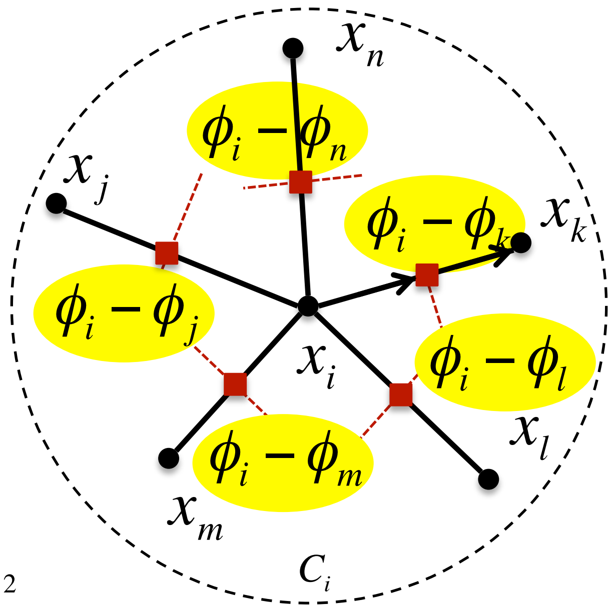
$$\mathbf{u}(x_i) = \frac{1}{2} \nabla u_{i\to}(x_i) \approx \mathbf{b}(x_i)^T \tau_\nabla(\mathbf{p}) = \sum_{k=1}^{Q} b_k(x_i) \nabla p_k(x_i)$$

$$\nabla \cdot \mathbf{u}(x_i) = \frac{1}{2} \Delta u_{i\to}(x_i) \approx \mathbf{b}(x_i)^T \tau_\Delta(\mathbf{p}) = \sum_{k=1}^{Q} b_k(x_i) \Delta p_k(x_i) \longleftarrow DIV_i(\mathbf{u})$$

Putting it all together: $DIV_i \cdot GRAD_i : V^i \rightarrow C^i$

1. Define $GRAD_i : V^i \rightarrow E^i$ as the *topological gradient*

$$GRAD_i(\phi) = \int_{e_{ij}} \nabla\phi \, d\ell = \phi_i - \phi_j$$



2. Define the samples at edge midpoints by :

$$\mathbf{u} = \left[GRAD_i(\phi)\right] = \left[\phi_i - \phi_j\right]$$

3. Compute GMLS coefficients

$$\mathbf{b}(x) = \frac{1}{2}\underset{\mathbf{c}\in\mathbf{R}^N}{\operatorname{argmin}} \left\| L^T\mathbf{c} - \mathbf{u} \right\|^2_{W(\tau)} = \frac{1}{2}\underset{\mathbf{c}\in\mathbf{R}^N}{\operatorname{argmin}} \left\| L^T\mathbf{c} - \left[GRAD_i(\phi)\right] \right\|^2_{W(\tau)}$$

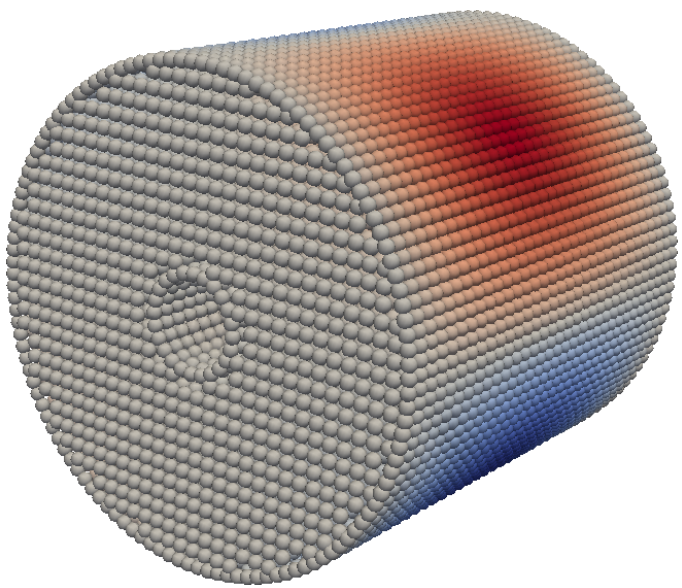
3. Compute the "staggered" DIV GRAD operator

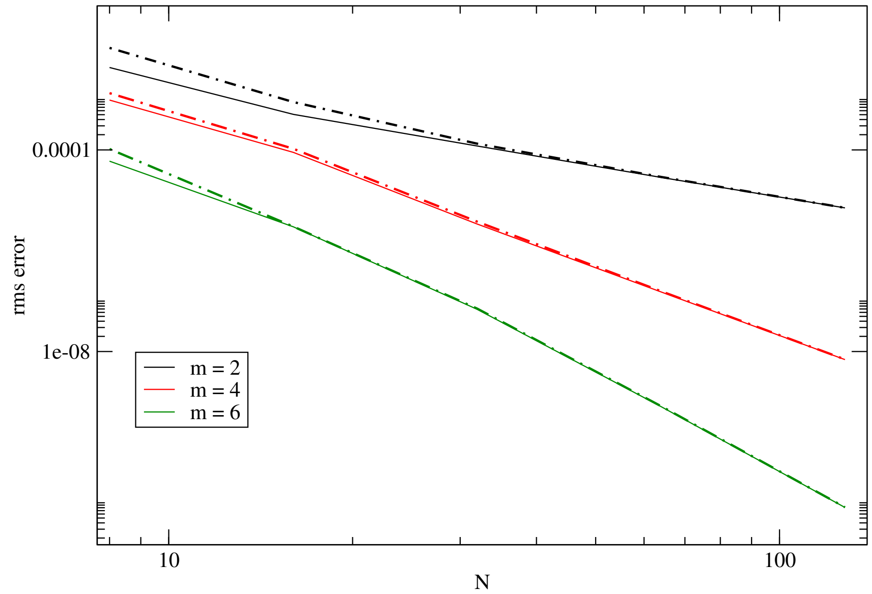
$$DIV_i \cdot GRAD_i\left(\phi\right) = \mathbf{b}(x_i)^T \tau_\Delta(\mathbf{p}) = \sum_{k=1}^{Q} b_k(x_i)\Delta p_k(x_i)$$

Convergence rates

Smooth 3D manufactured solution

dash = collocated
solid = compatible





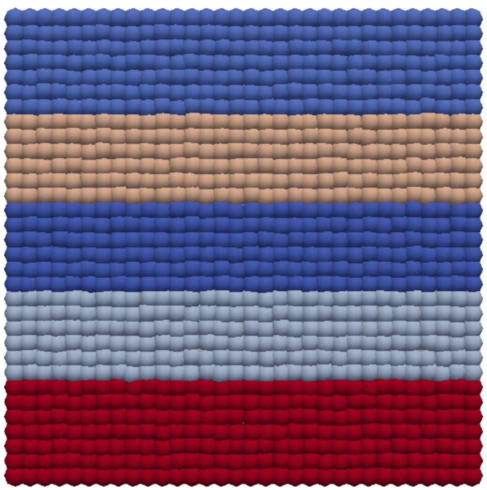
Convergence rates of the compatible meshless method identical with those of a standard nodal (collocated) meshless scheme.
This is to be expected for a smooth solution.

Darcy flow benchmark

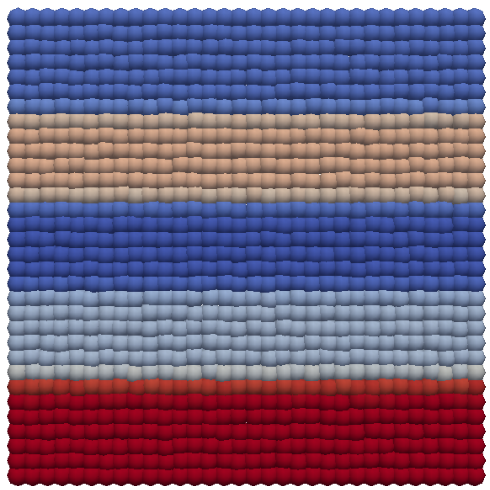
Rough 2D manufactured solution: 5 strip

$$\nabla \cdot \kappa \nabla \phi = f \qquad \phi = 1 - x \qquad \kappa \in \{16, 6, 1, 10, 2\}$$

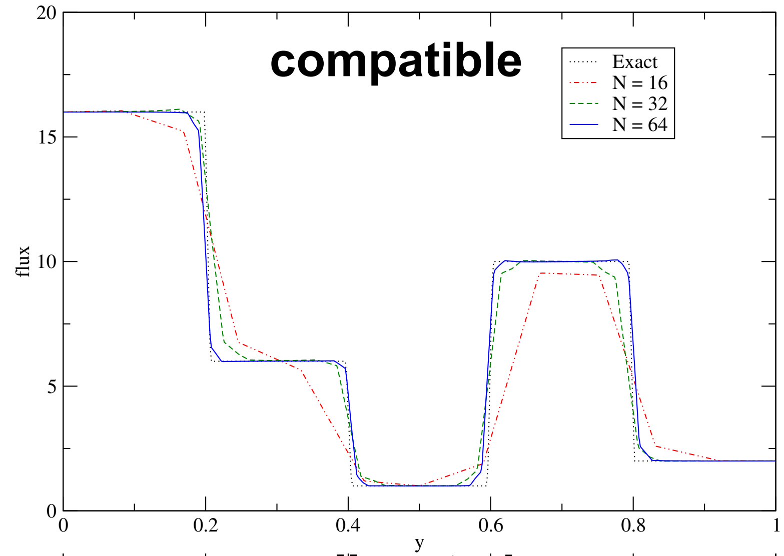
Transverse flux is piecewise constant

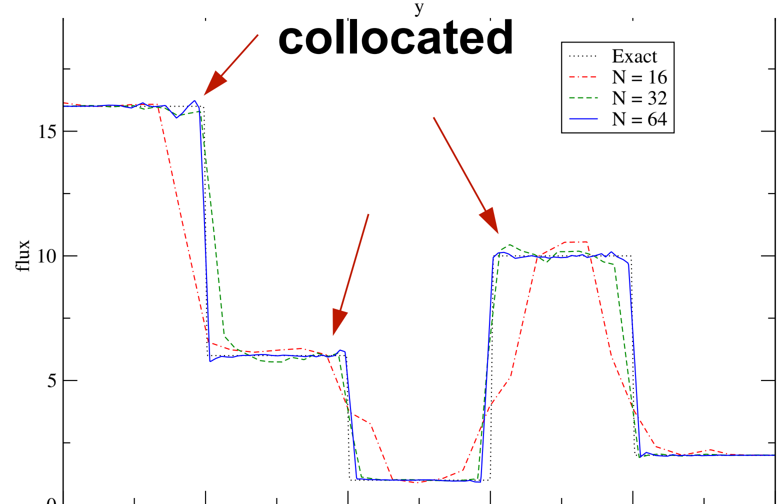


exact



compatible





The locally **compatible meshless** method behaves much like its **mixed finite element** cousin

T. J. R. Hughes, A. Masud, and J. Wan. A stabilized mixed Discontinuous Galerkin method for Darcy flow. CMAME., 195, 2006.

Application to conservation laws

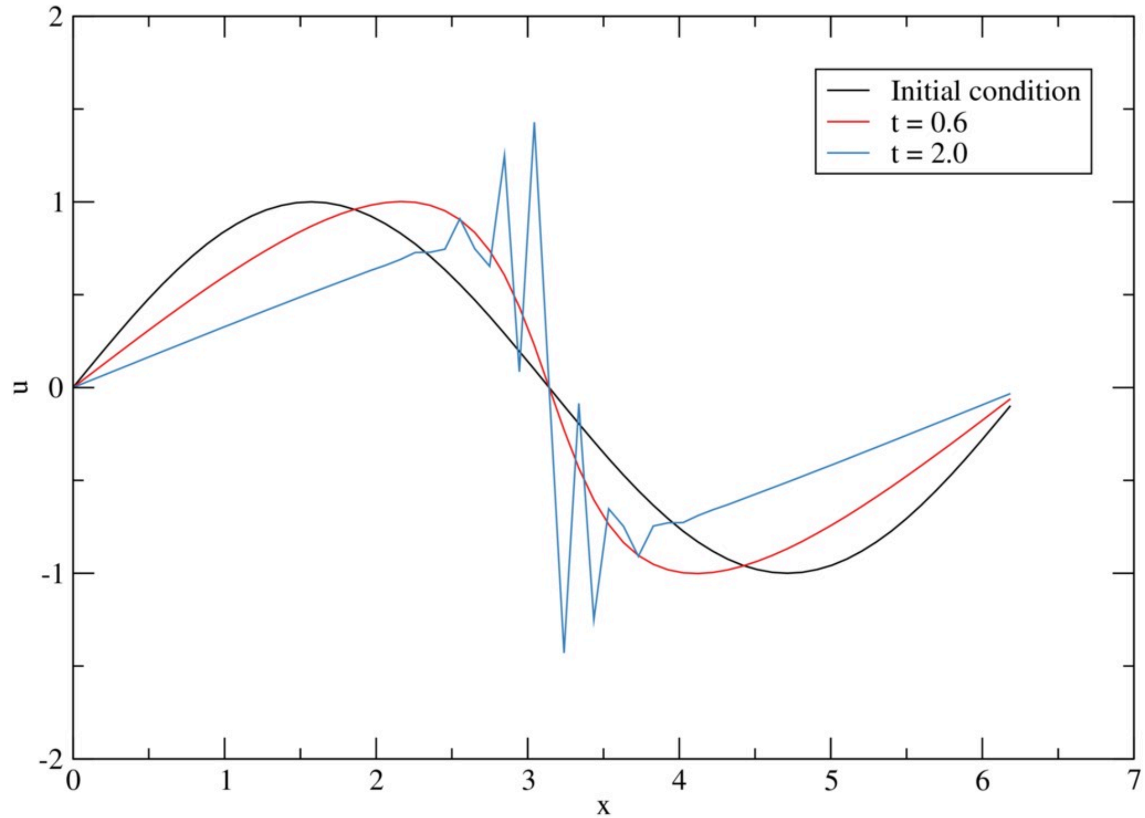
Model problem

$$u_t + F_x = 0$$

$$F = \frac{1}{2}u^2$$

$$u(0,x) = \sin(x)$$

$$u(t,0) = u(t,2\pi)$$

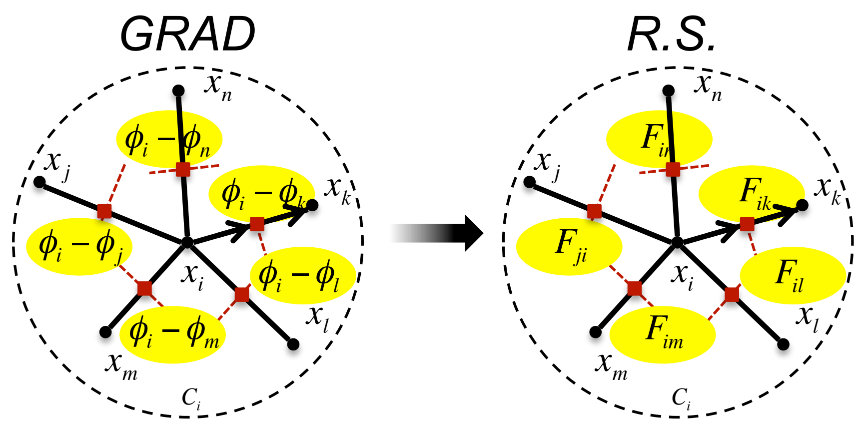


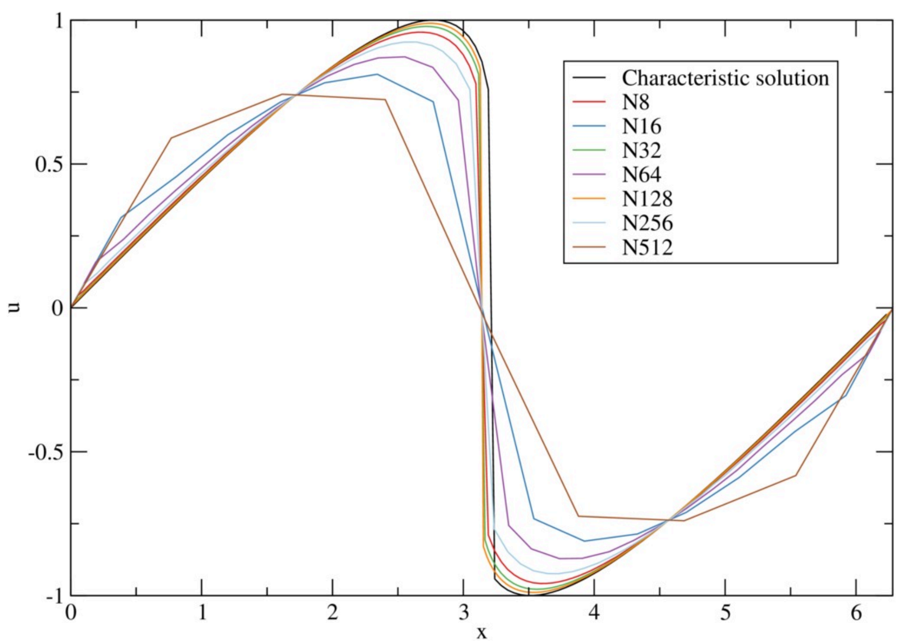
Discretization of Burger’s equation with non-staggered (nodal) RKPM/MLS leads to usual spurious oscillations when shocks form.

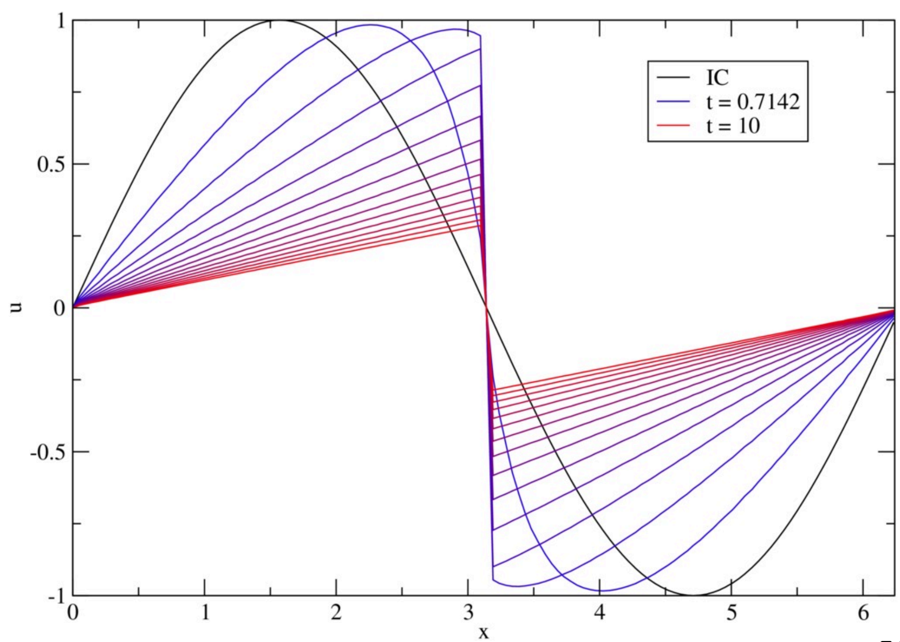
A locally compatible scheme

Approach

1. Replace the topological *GRAD* by *Riemann solvers* to define fluxes F_{ij}
2. Apply the GMLS *DIV* operator.







Conclusions

- GMLS provides a flexible and powerful setting for the development of meshless methods
- It can be used to mimic locally the “staggered” arrangement of classical schemes
- Resulting locally compatible meshles methods behave similarly to their mesh-based analogues.
- The next talk will show how one can use GMLS to develop a globally compatible meshless scheme.

Ongoing work:

- Extension to a complete meshless DeRham sequence
- Meshless variational principles
- Enhancements to MPM, PIC-plasma,…
- ….