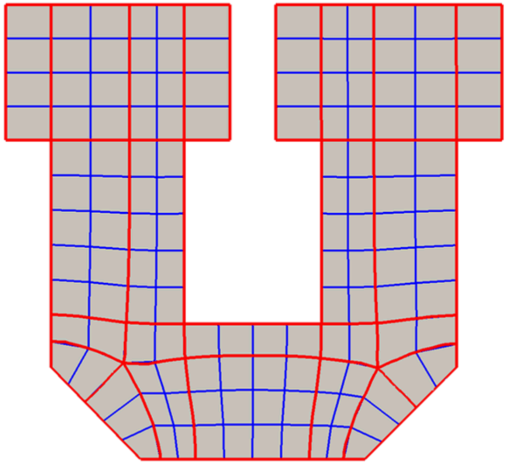
QUAD MESHING, CROSS FIELDS, AND THE GINZBURG-LANDAU THEORY

SAND2017-8794C



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The 3rd Annual Meeting of SIAM Central States Section

Colorado State University

Fort Collins, CO

a University of Utah
b Sandia National Laboratories

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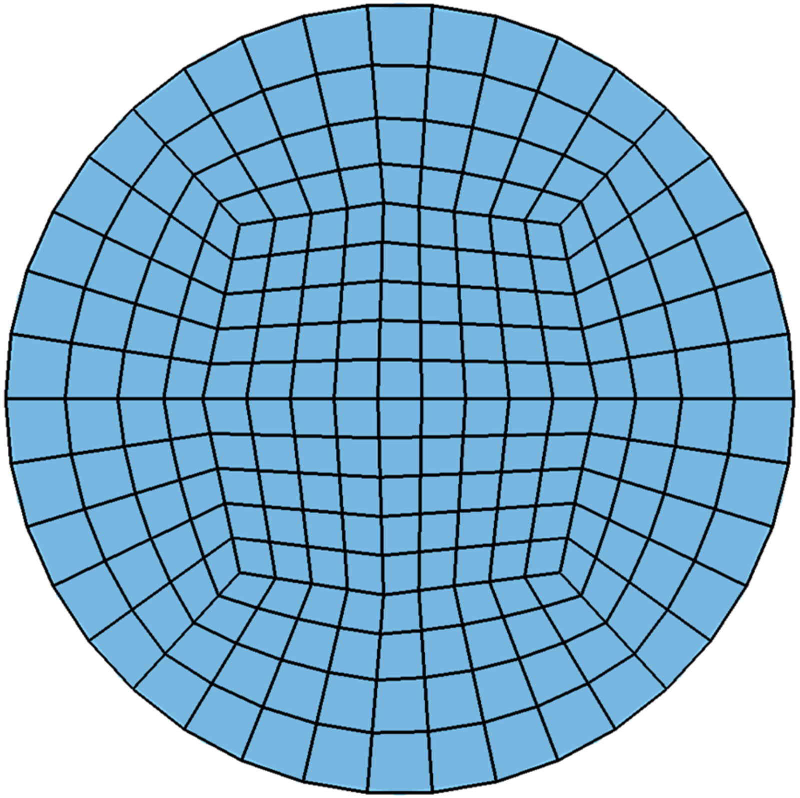
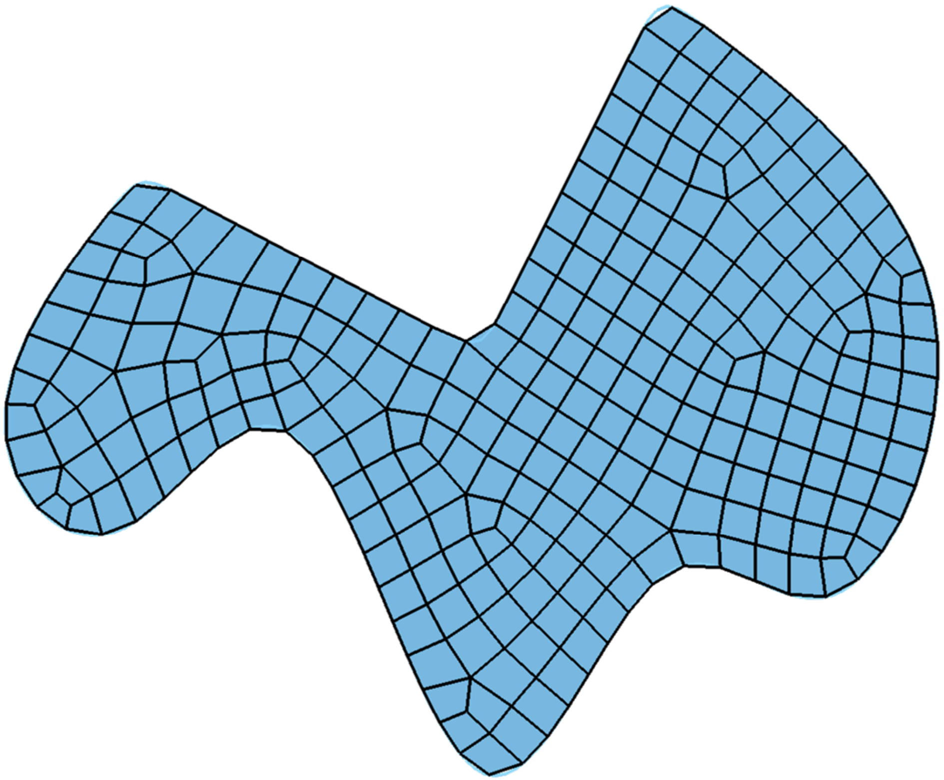
Introduction

Classical Quad Meshing Methods



Pattern Based

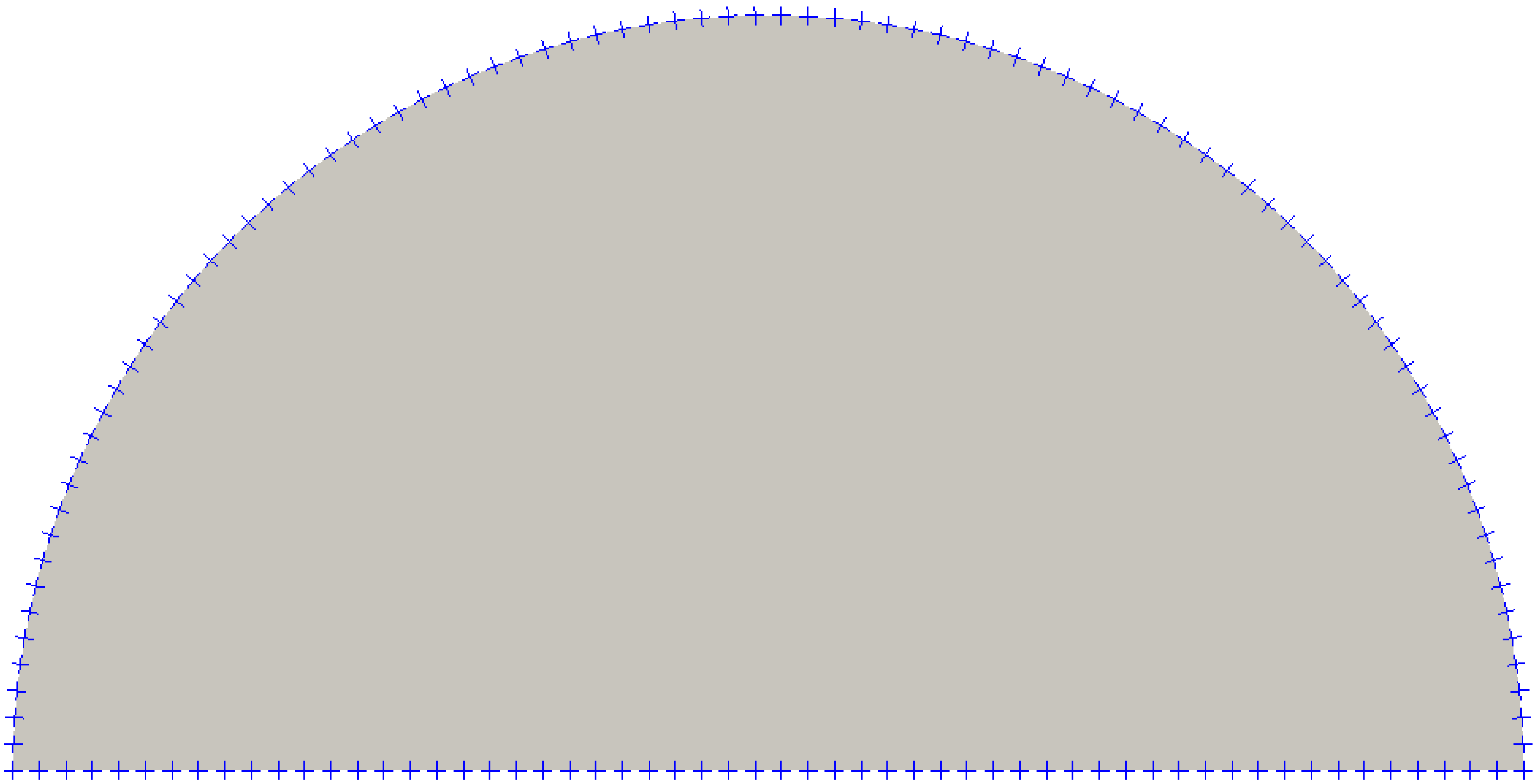


Unstructured - Paving

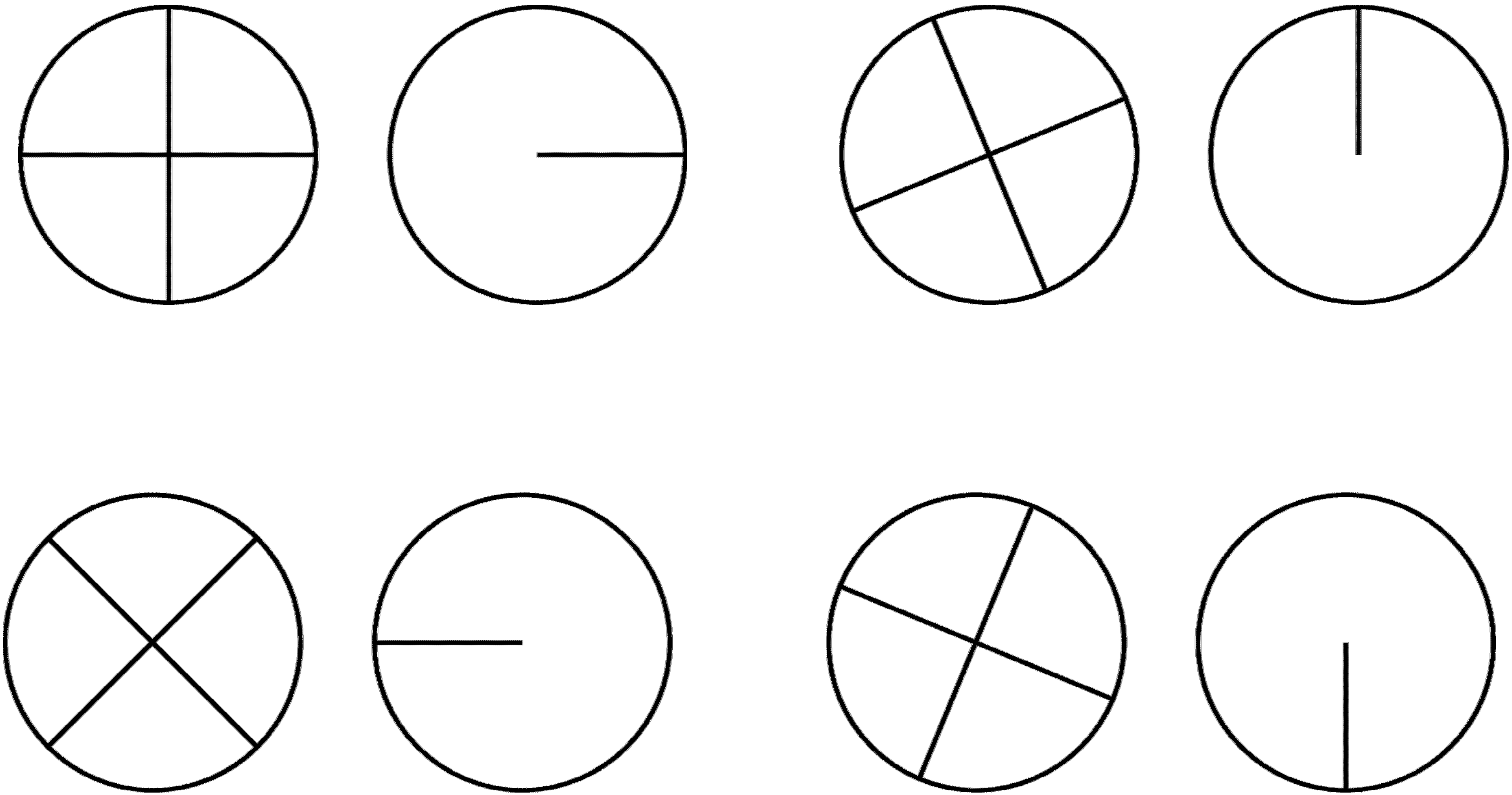
Basic Cross Field Meshing Algorithm (Kowalski et al. 2013)

2D Cross Field Meshing Algorithm

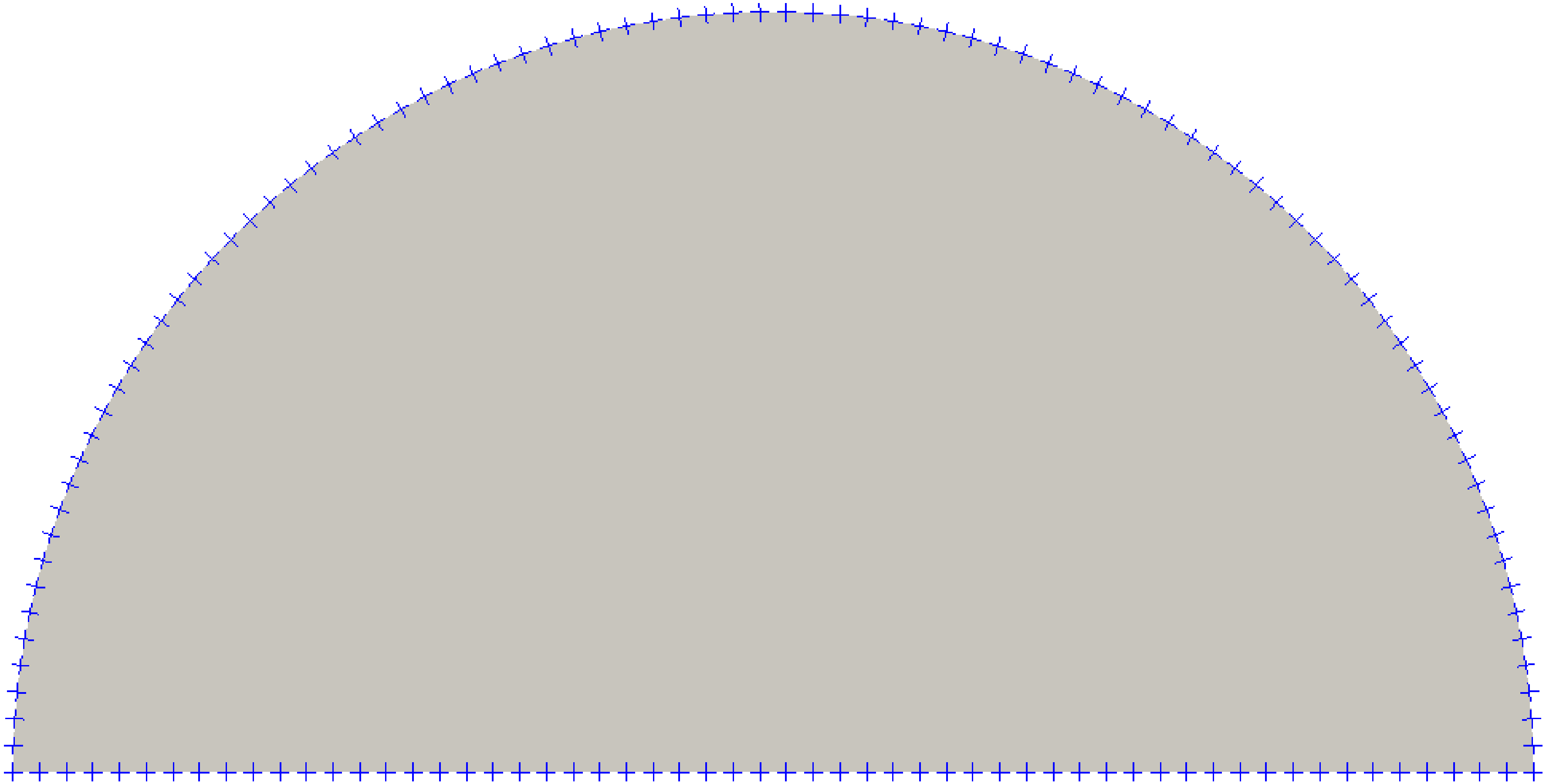
2D Cross Field Meshing Algorithm



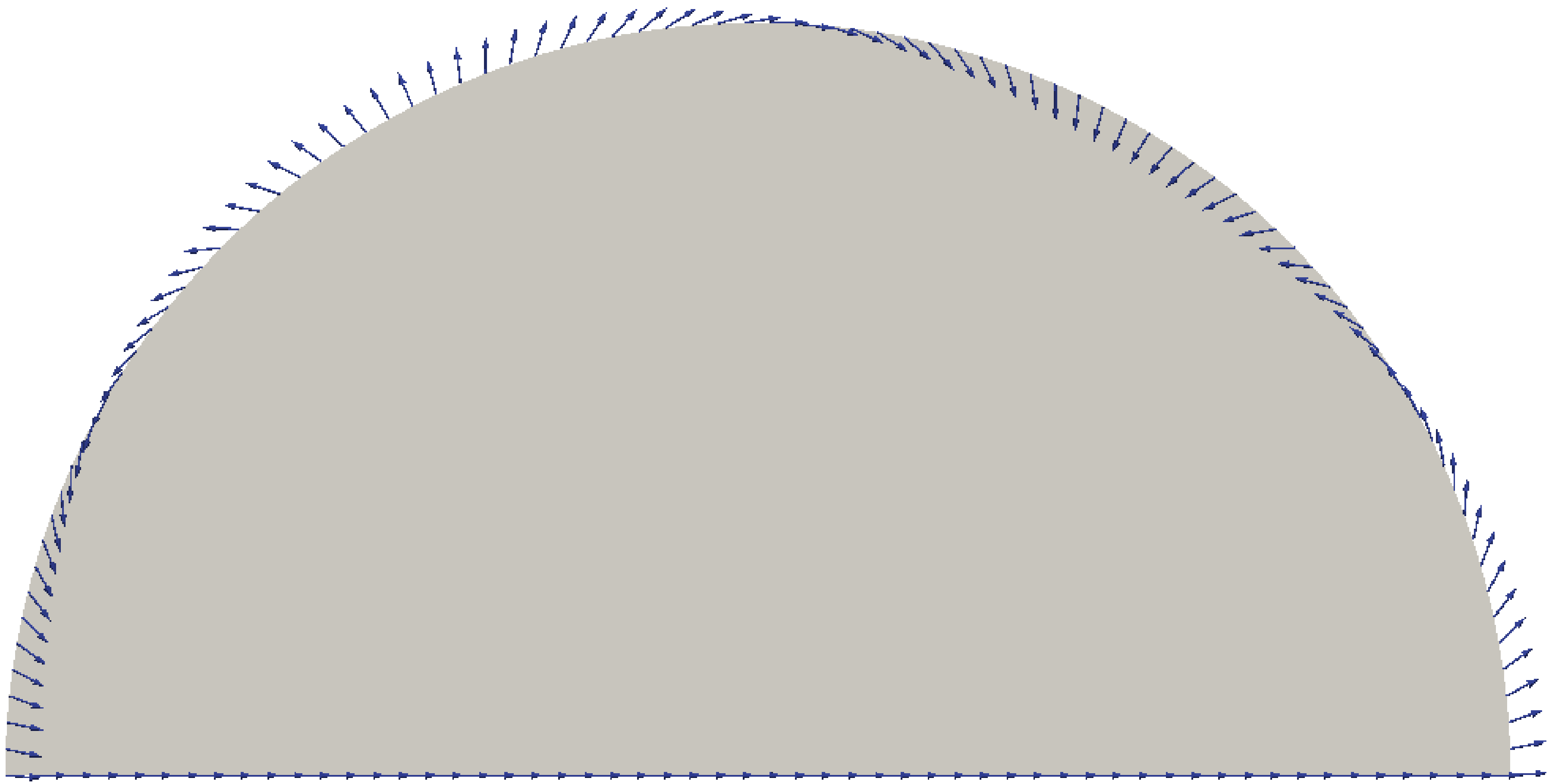
The Representation Map



2D Cross Field Meshing Algorithm



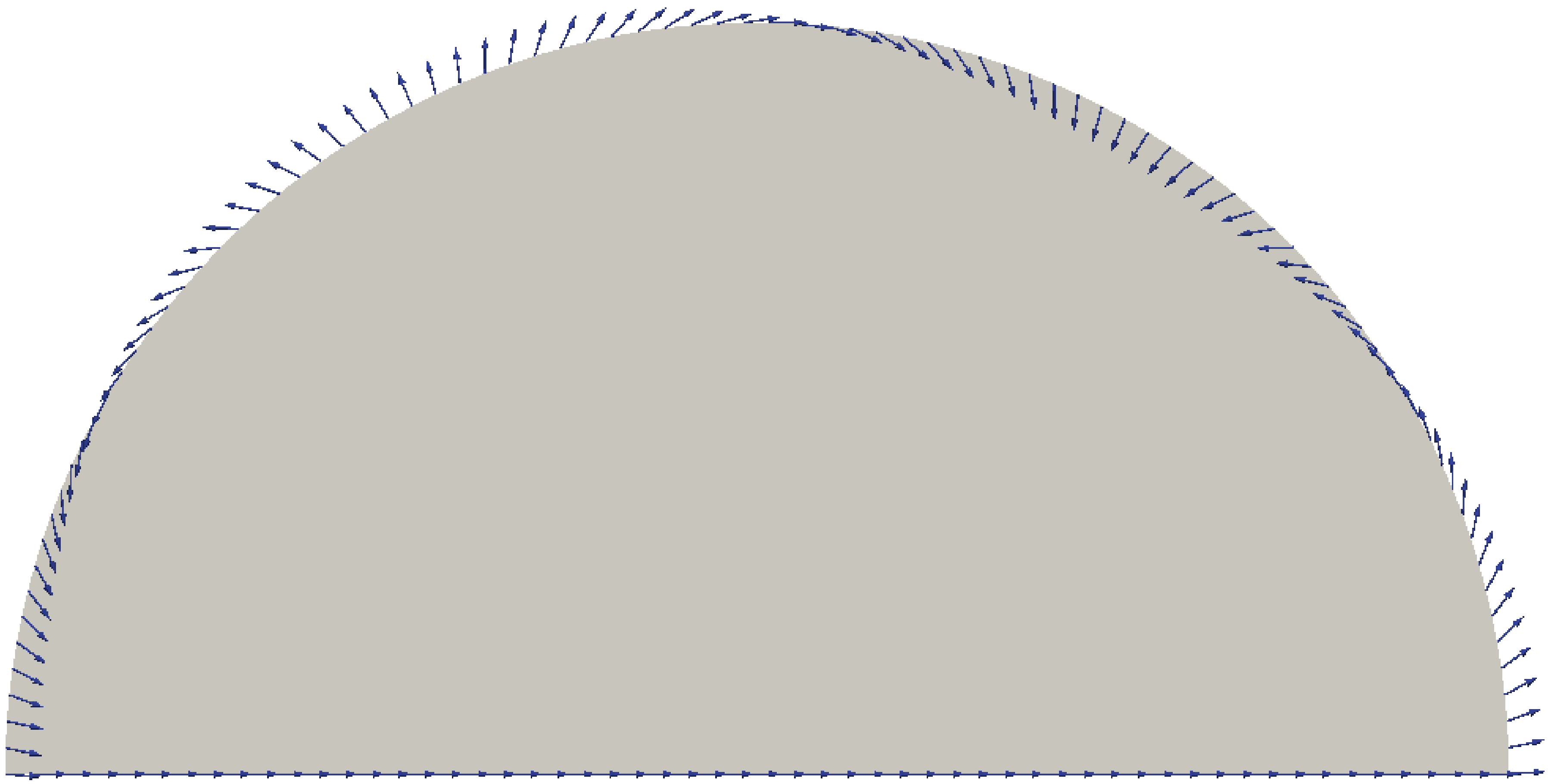
2D Cross Field Meshing Algorithm



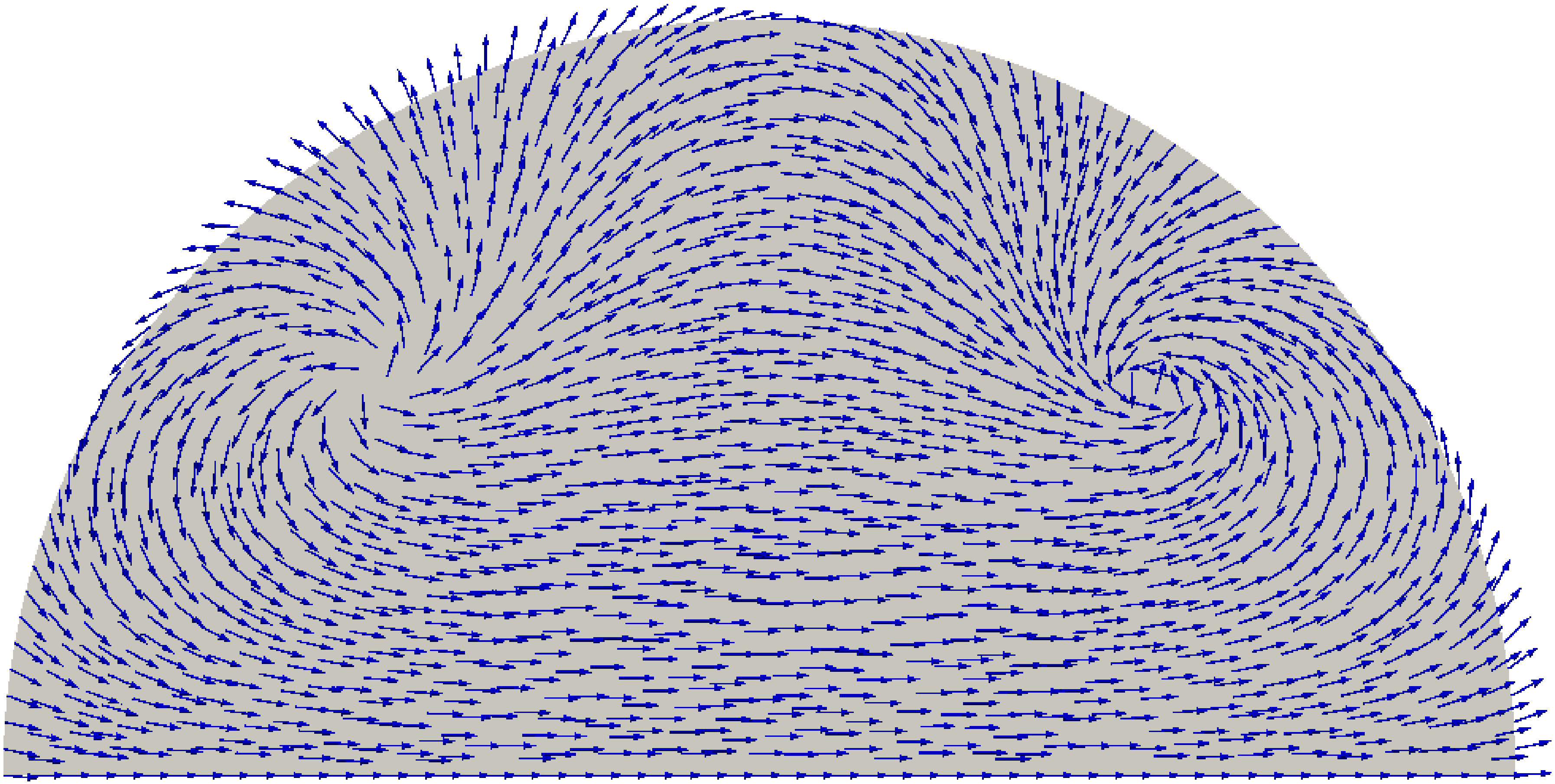
2D Cross Field Meshing Algorithm

$$\begin{cases} \min\limits_{u} E(u) \\ E(u) = \dfrac{1}{2}\displaystyle\int_D |\nabla u|^2 dA \\ u(x) = R(f_0(x)) \quad \forall x \in \partial D \\ |u(x)| = 1 \quad a.e. x \in D \end{cases}$$

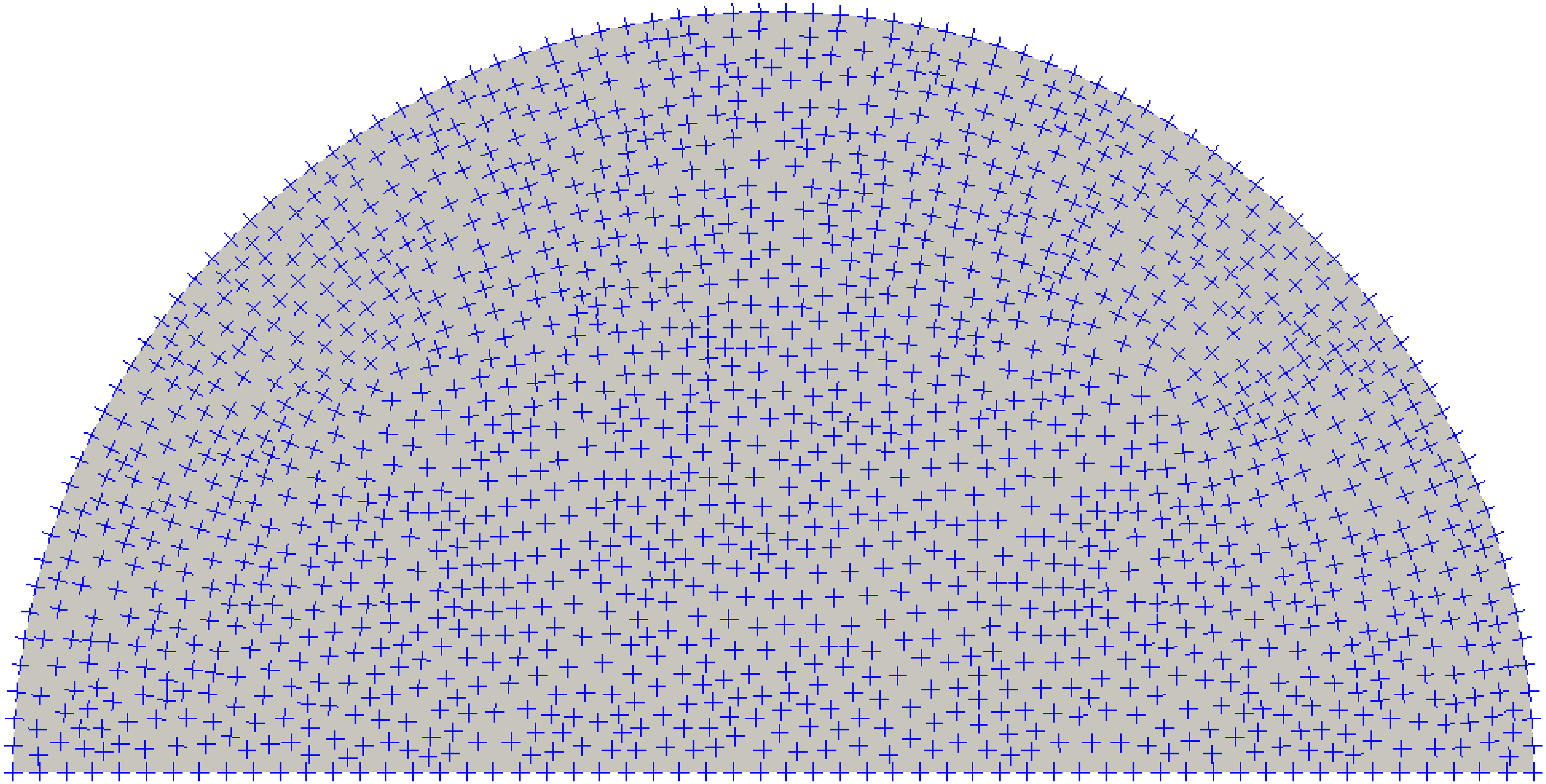
2D Cross Field Meshing Algorithm



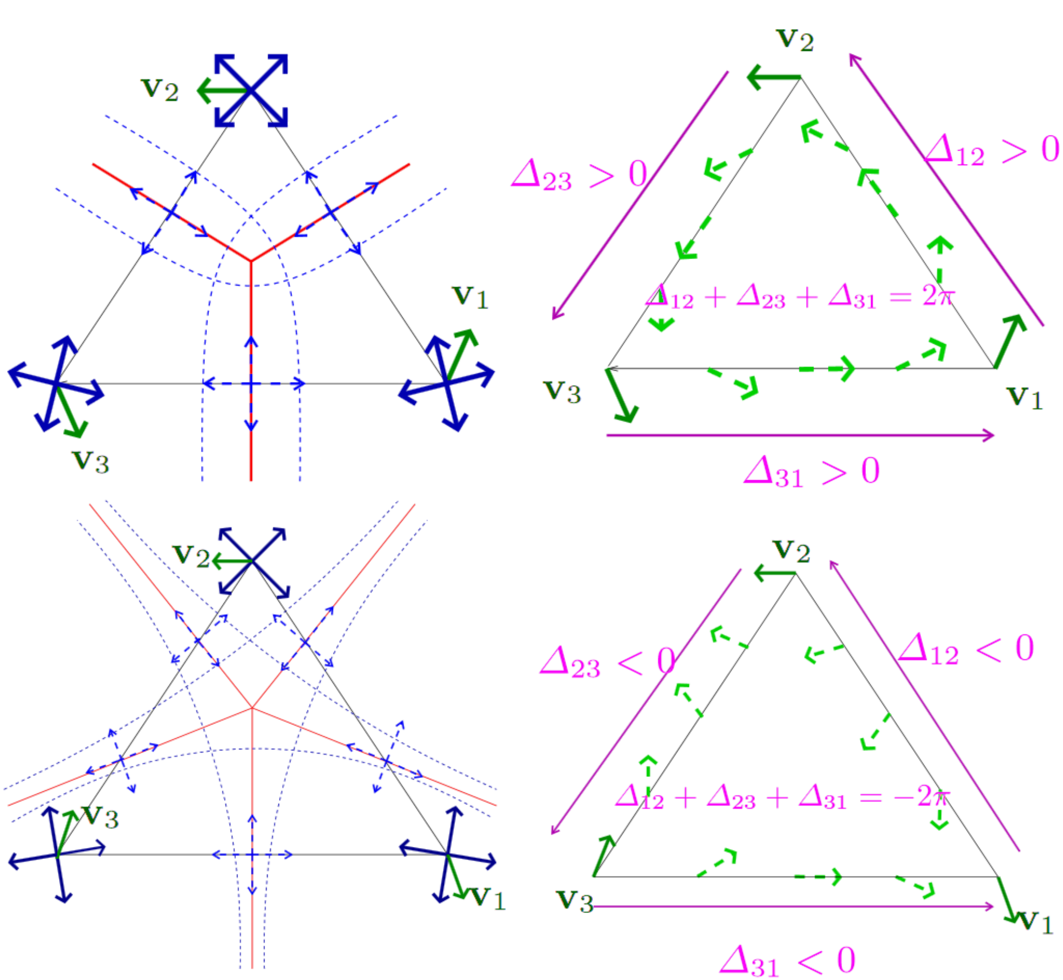
2D Cross Field Meshing Algorithm

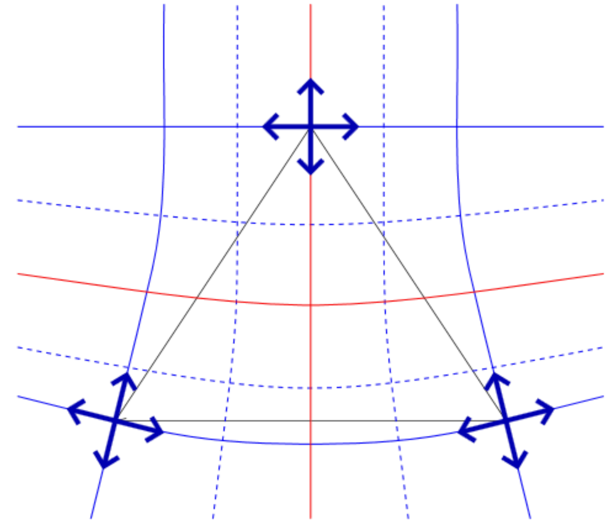


2D Cross Field Meshing Algorithm



Cross Field Singularities





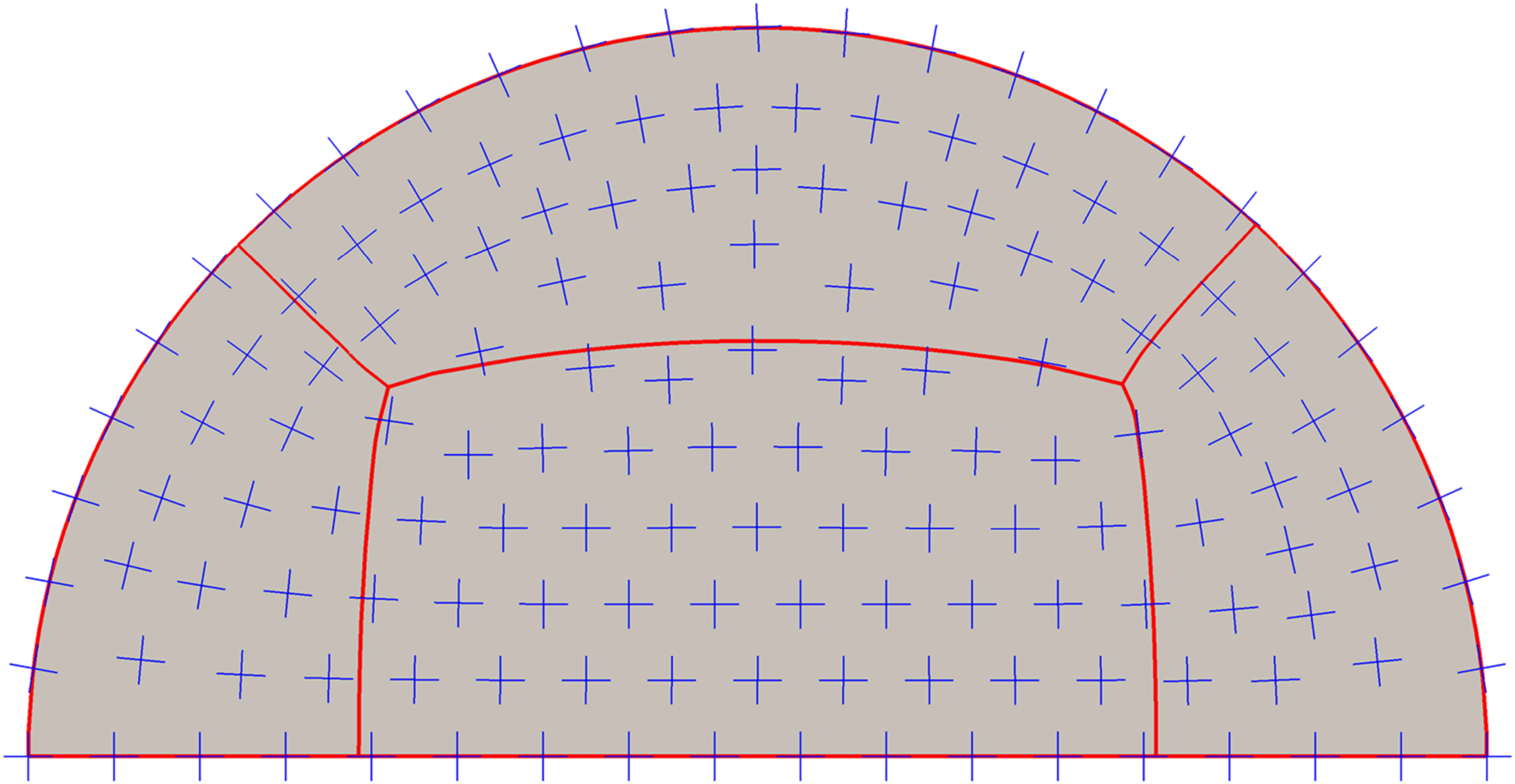
(b)

Nonsingular Triangle

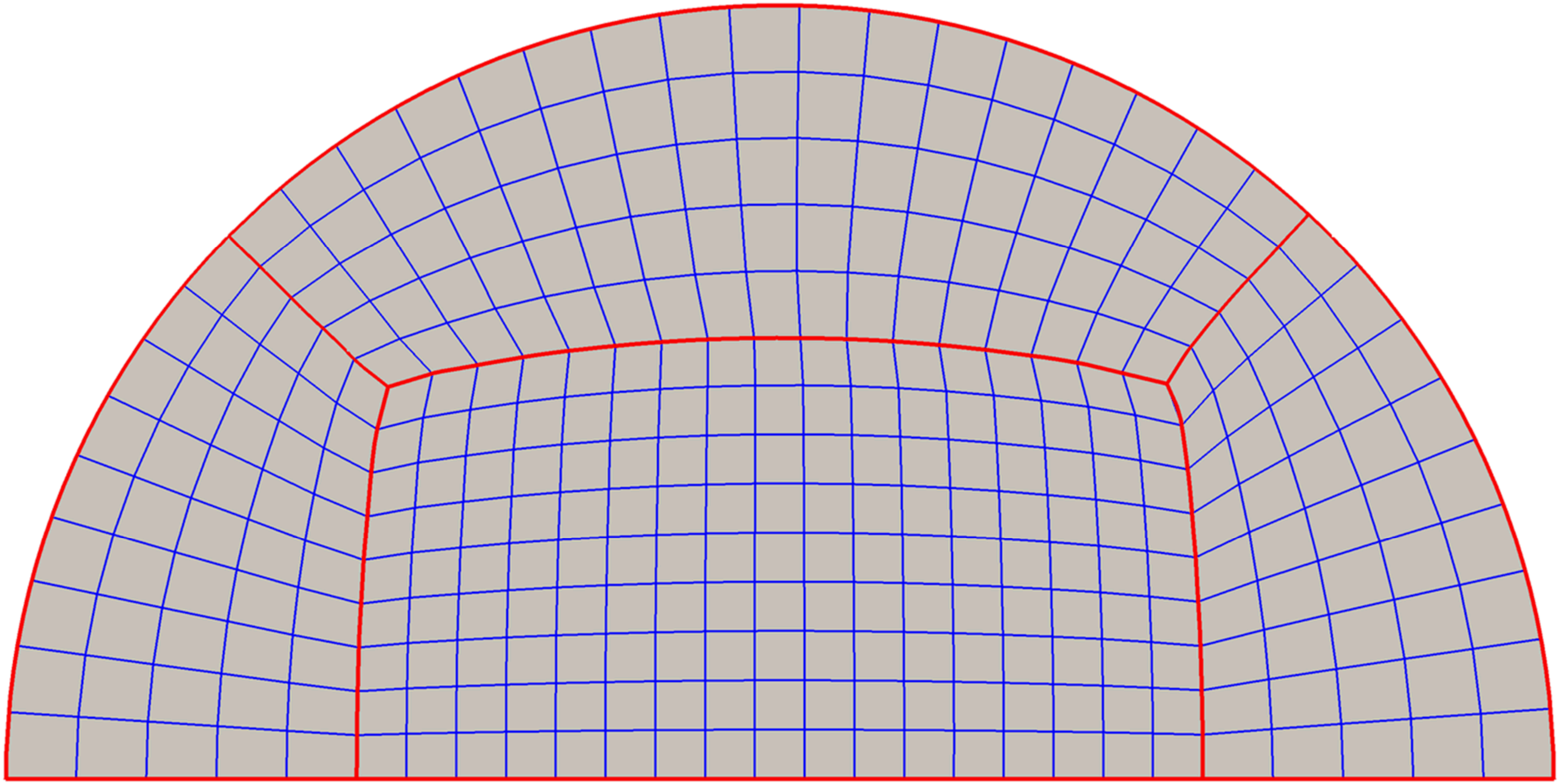
Singular Triangles

Kowalski et al. 2013

2D Cross Field Meshing Algorithm



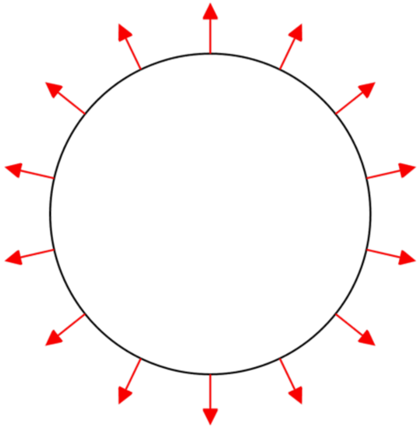
2D Cross Field Meshing Algorithm

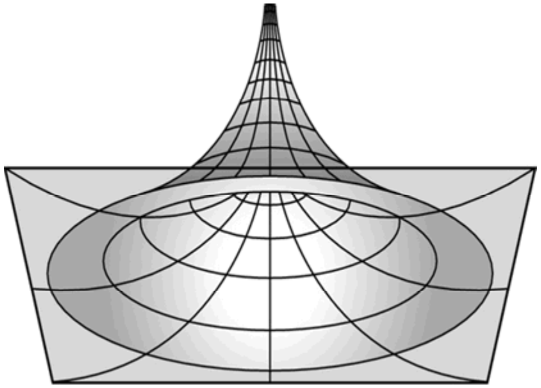


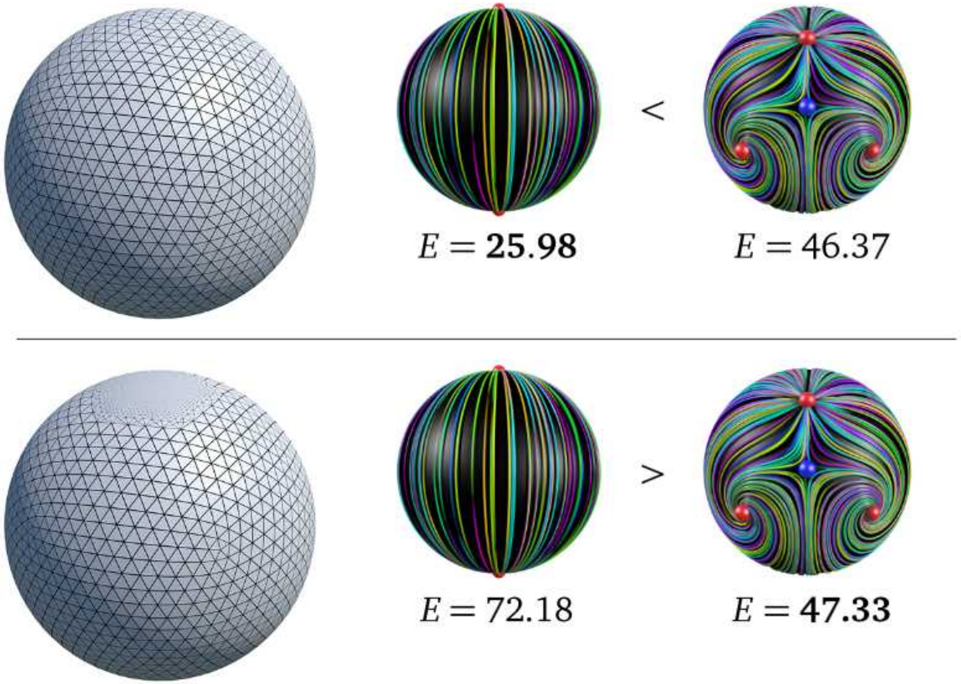
Connection to Ginzburg-Landau Theory

Infinite Energy Problem in Cross Field Design

- Unit vector constraint causes problem to become ill-defined.
- How do you find the minimum between multiple infinite values?







Knöppel et al. 2013

Ginzburg-Landau Functional

Original problem:

$$\begin{cases} \min_u E(u) \\ E(u) = \frac{1}{2} \int_D |\nabla u|^2 dA \\ u(x) = g(x) \qquad \forall x \in \partial D \\ |u(x)| = 1 \quad a.e. x \in D \end{cases}$$

Relaxed problem:

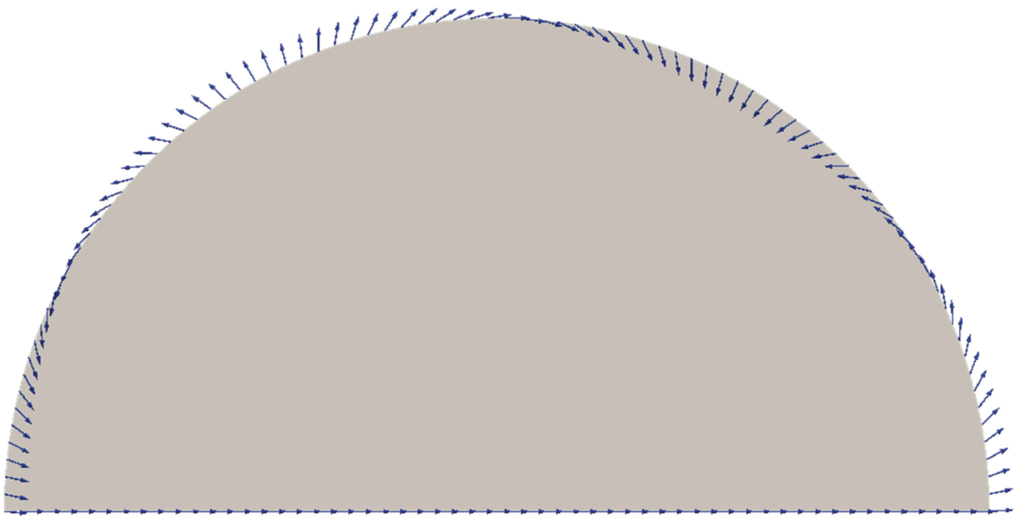
$$\min_{u \in H^1_g(D,\mathbb{C})} E_\varepsilon(u)$$

$$E_\epsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\epsilon^2} \int_G (|u|^2 - 1)^2$$

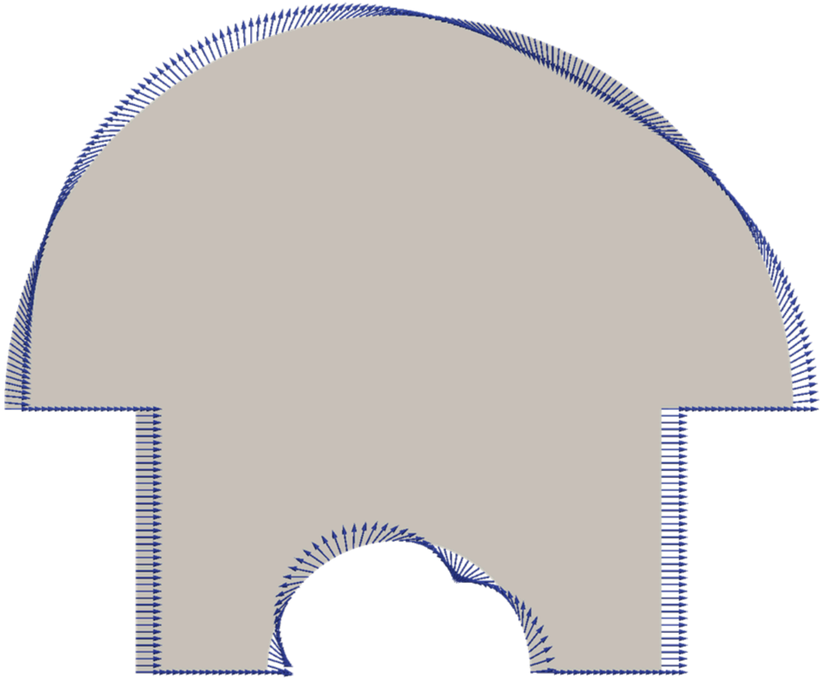
Results of Ginzburg-Landau Theory and Applications to Cross Fields

Brouwer Degree

- Let $g(x)$ be the boundary condition on the domain G.
- Let $d = deg(g, \partial G)$ be the Brouwer degree.



d = 2



d = 0

Result: Well Defined Limit of Relaxed Problem

Theorem 2.2.2 (Bethuel et al. [4]). *Let* $d = deg(g, \partial D)$. *Given a sequence* $\varepsilon_n \to 0$ *there exists a subsequence* ε_{n_i} *and exactly* d *points* $a_1, a_2, ..., a_d$ *in* $D \subset \mathbb{C}$ *and a smooth harmonic map* $u_\star : D \setminus \{a_1, ..., a_d\} \to \mathbb{T}$ *with* $u_\star = g$ *on* ∂D *such that*

$$\boxed{u_{\varepsilon_{n_i}} \to u_\star} \text{ in } C^k_{loc}(D \setminus \bigcup_i (a_i)) \ \forall k \text{ and in } C^{1,\alpha}(\bar{D} \setminus \bigcup_i (a_i)) \ \forall \alpha < 1$$

In addition, if $d \neq 0$ *each singularity of* $u_\star$ *has index* $\operatorname{sgn}(d)$ *and, more precisely, there are complex constants* (α_i) *with* $|\alpha_i| = 1$ *such that*

$$\left| u_\star(z) - \alpha_i \frac{z - a_i}{|z - a_i|} \right| \leq C|z - a_i|^2 \text{ as } z \to a_i, \ \forall i$$

This gives us a generalized sense in which to understand the energy minimization problem

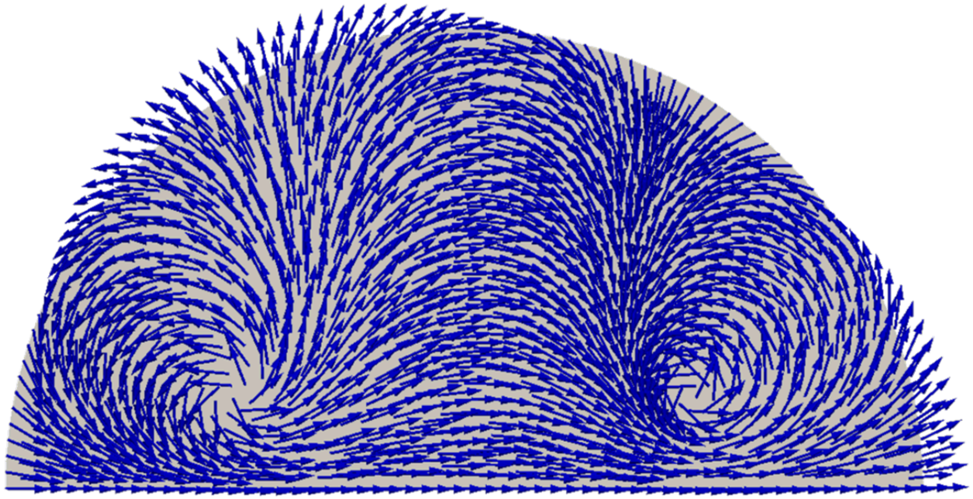
Result: Explicit Formula to Design Field with Fixed Singularities

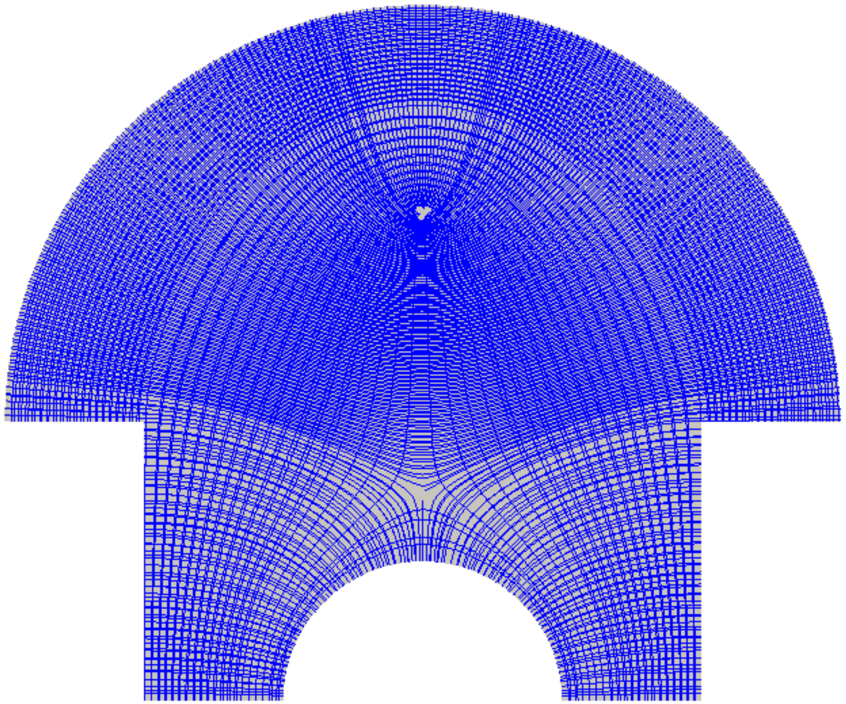
$$e^{i\varphi_0(z)} = g(z)\frac{|z-b_1|^{\alpha_1}}{(z-b_1)^{\alpha_1}}\frac{|z-b_2|^{\alpha_2}}{(z-b_2)^{\alpha_2}}\cdots\frac{|z-b_n|^{\alpha_n}}{(z-b_n)^{\alpha_n}}$$

$$\begin{cases} \Delta\varphi = 0 \text{ in } D \\ \quad \varphi = \varphi_0 \text{ on } \partial D \end{cases}$$

$$u_0 = e^{i\varphi(z)}\frac{(z-b_1)^{\alpha_1}}{|z-b_1|^{\alpha_1}}\frac{(z-b_2)^{\alpha_2}}{|z-b_2|^{\alpha_2}}\cdots\frac{(z-b_n)}{|z-b_n|^{\alpha_n}}$$

Application: New Cross Field Design Method





Merriman-Bence-Osher (MBO) Method

Merriman-Bence-Osher (MBO) Method

Original Method

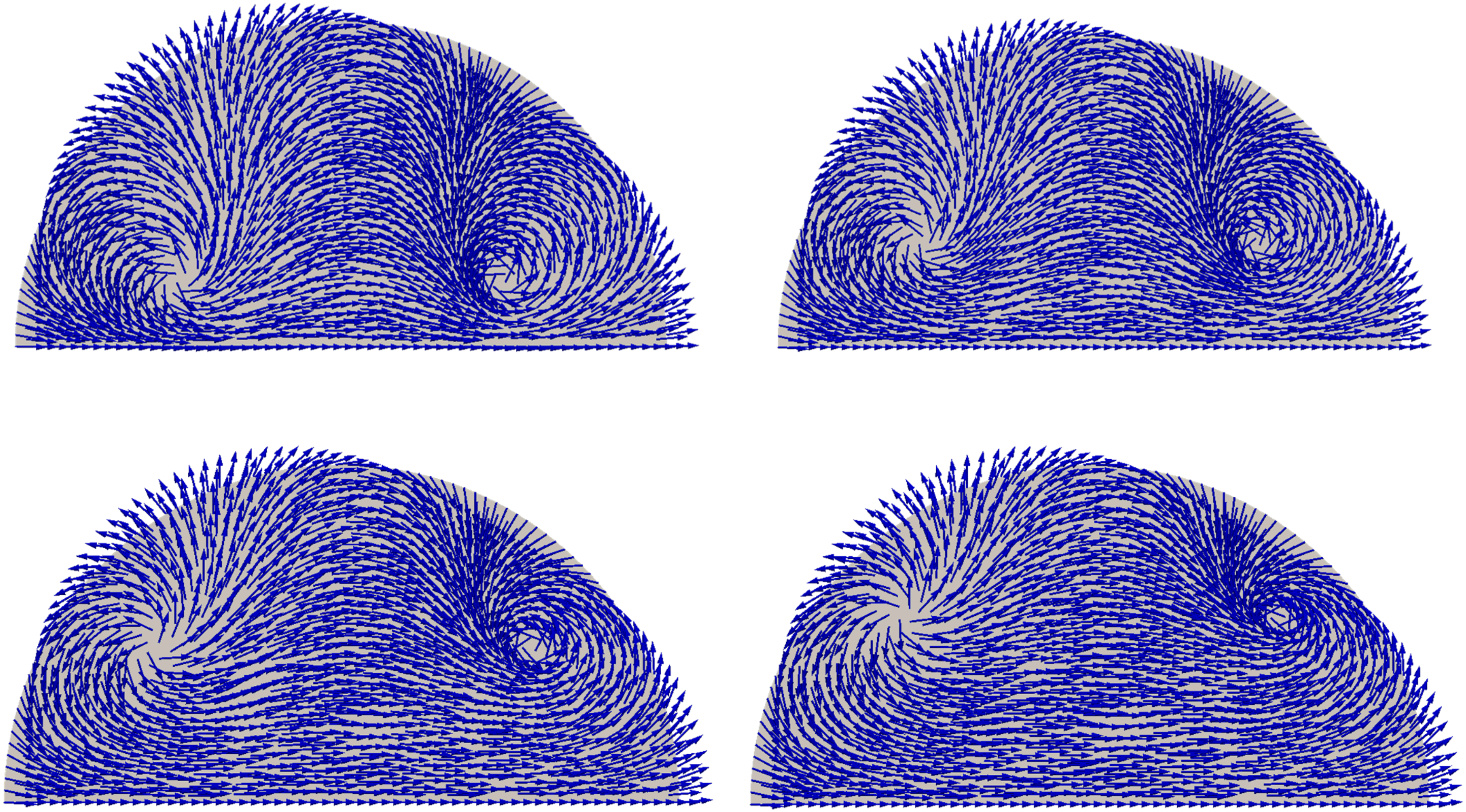
- Introduced as a method for motion by mean curvature
- Minimizes an two-well potential energy analogous to the complex GL energy

New Application to Frame Fields

- Iterative method to minimize cross field energy:

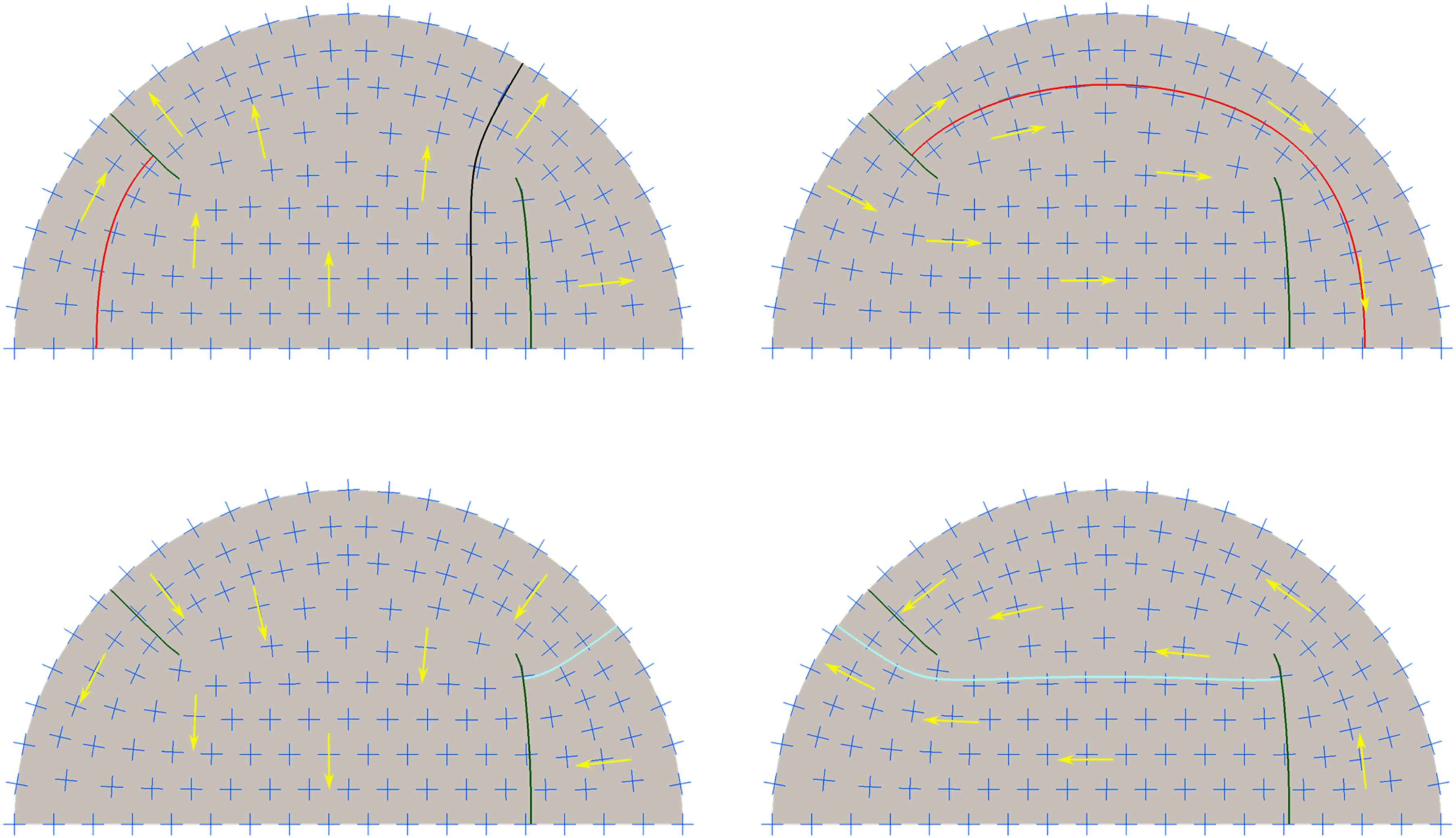
$$u_0 = \frac{\tilde{u}}{|\tilde{u}|} \qquad \text{and} \qquad u_k = \frac{e^{\tau\Delta} u_{k-1}}{|e^{\tau\Delta} u_{k-1}|} \quad k \geq 1.$$

MBO Method



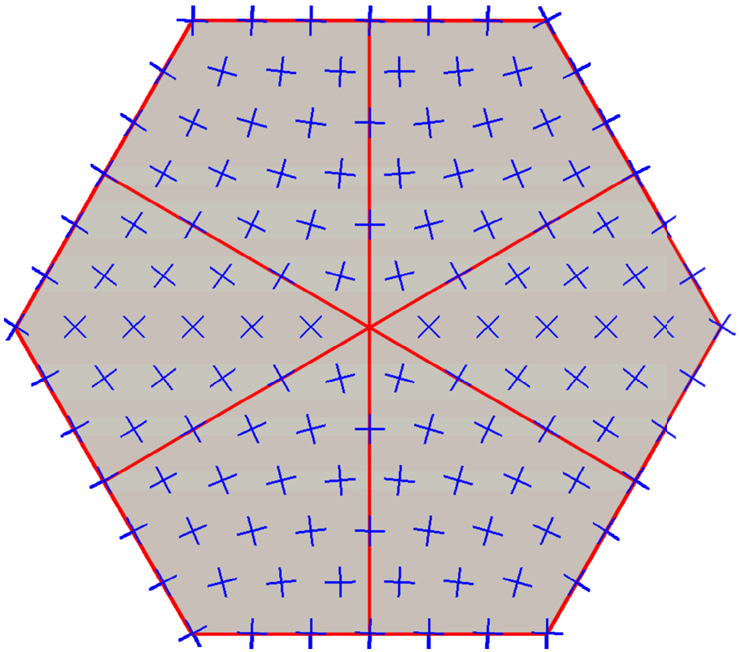
Asymptotic Behavior of Cross Fields Near Singularities

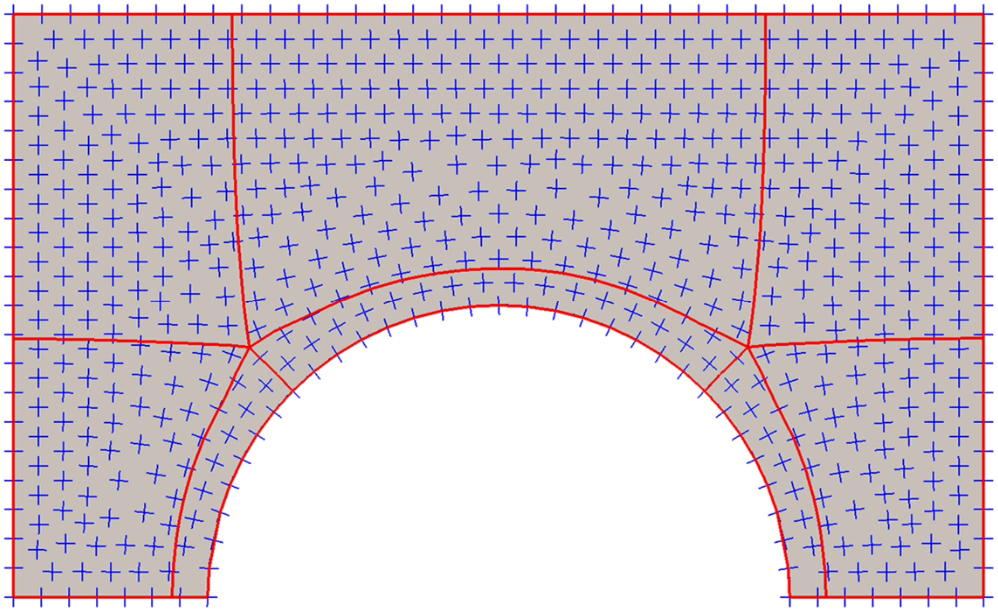
Riemann Surface and Streamlines



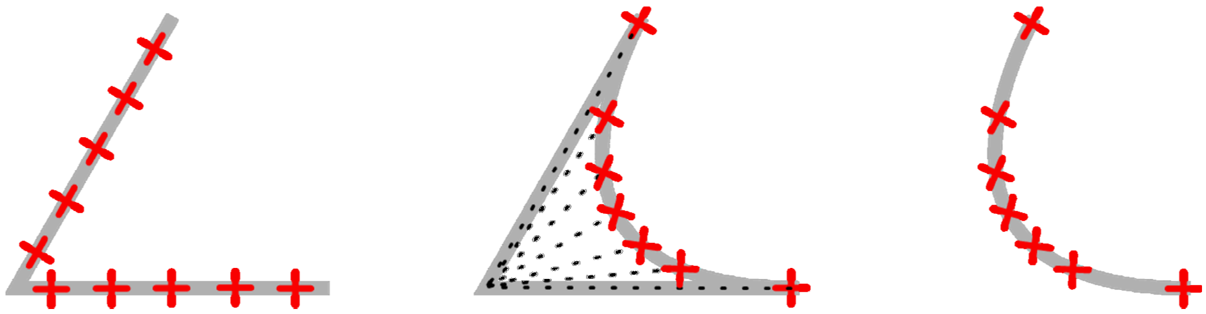
Separatrices of a Singularity

Lemma 5.1. *Let f be a boundary-aligned canonical harmonic cross field on D. Let a be an interior singularity of f of index $d/4$ with $d < 4$. There are exactly $4-d$ separatrices meeting at a. These separatrices partition a neighborhood of a into $4-d$ even-angled sectors.*

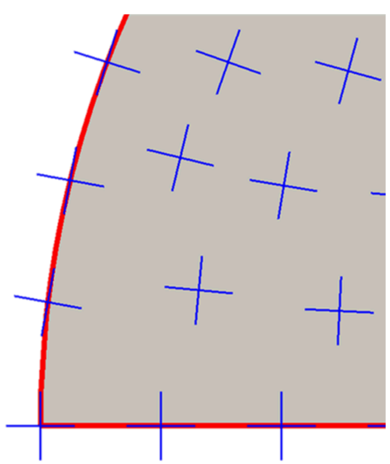


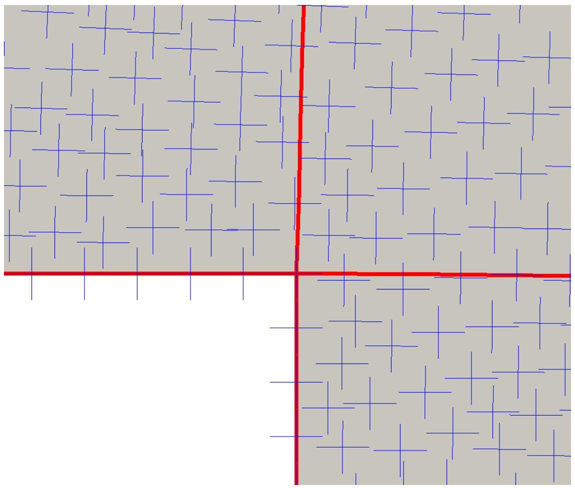


Boundary Singularities

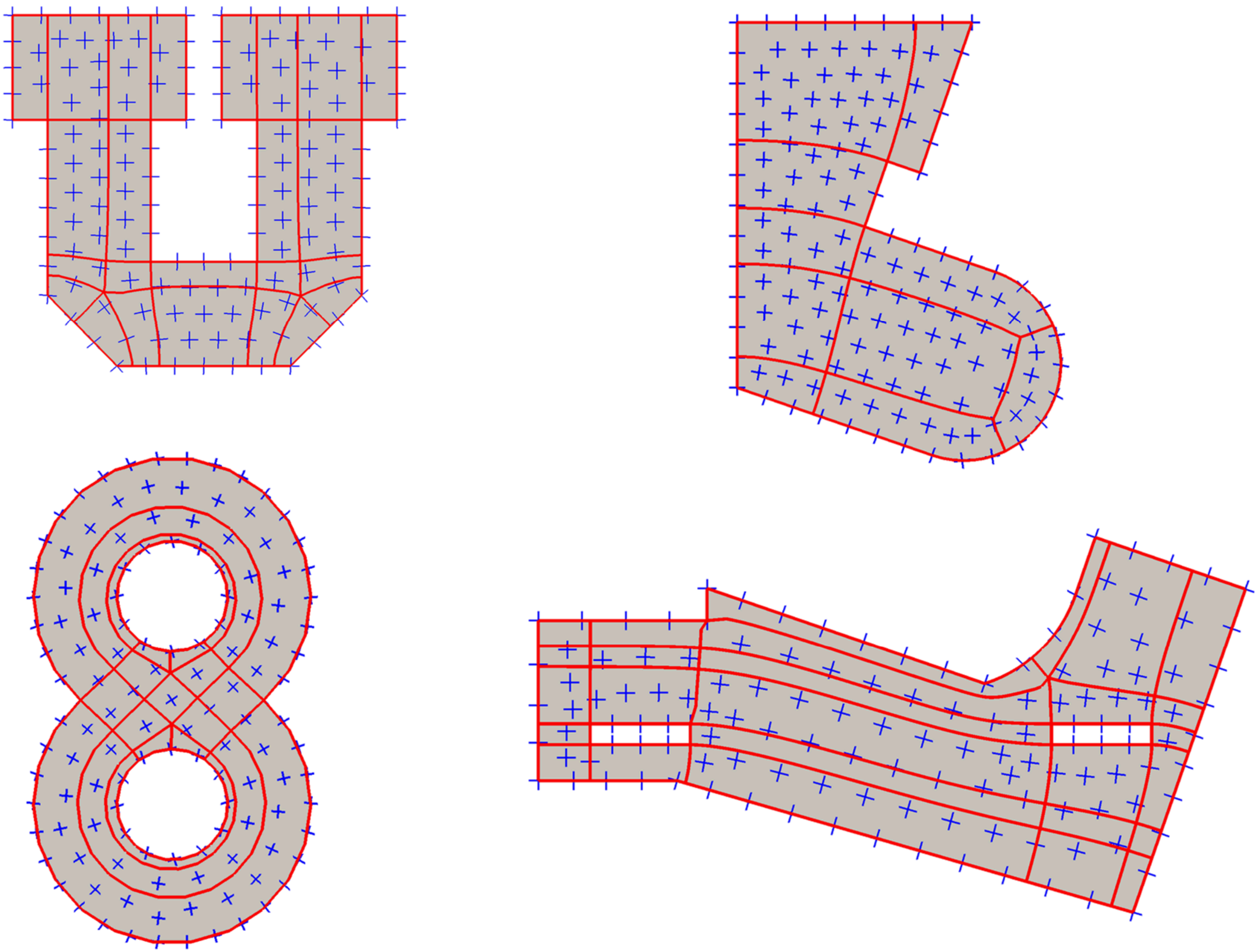


Lemma 5.4. *Let c be a boundary singularity of f of index $d/4$ with $d < 2$. There are exactly $3 - d$ separatrices meeting at c (including the boundaries themselves). These separatrices partition a neighborhood of c into $2 - d$ even-angled sectors.*

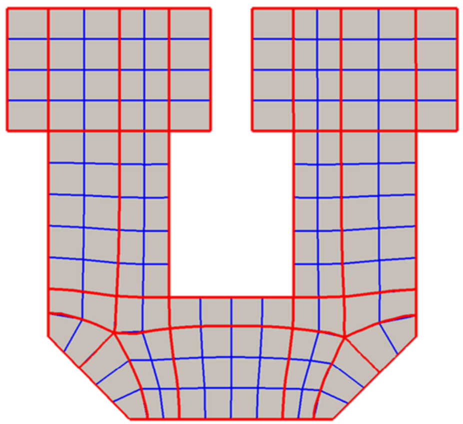


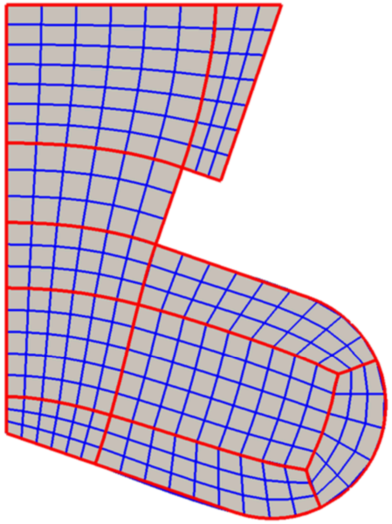


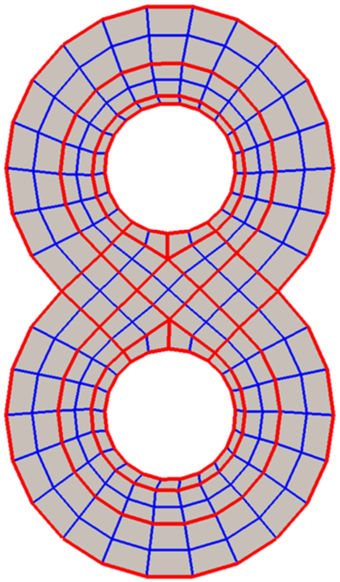
Partition into four-sided regions

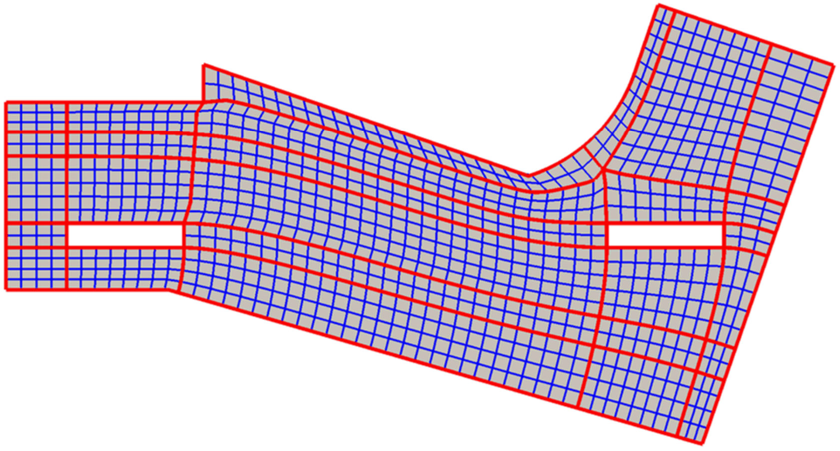


Meshing

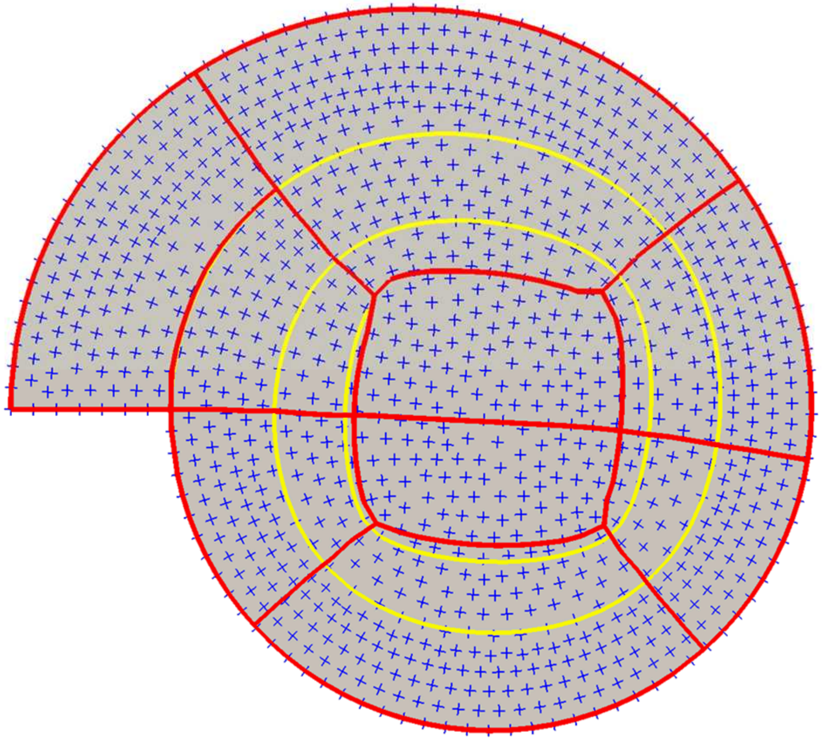


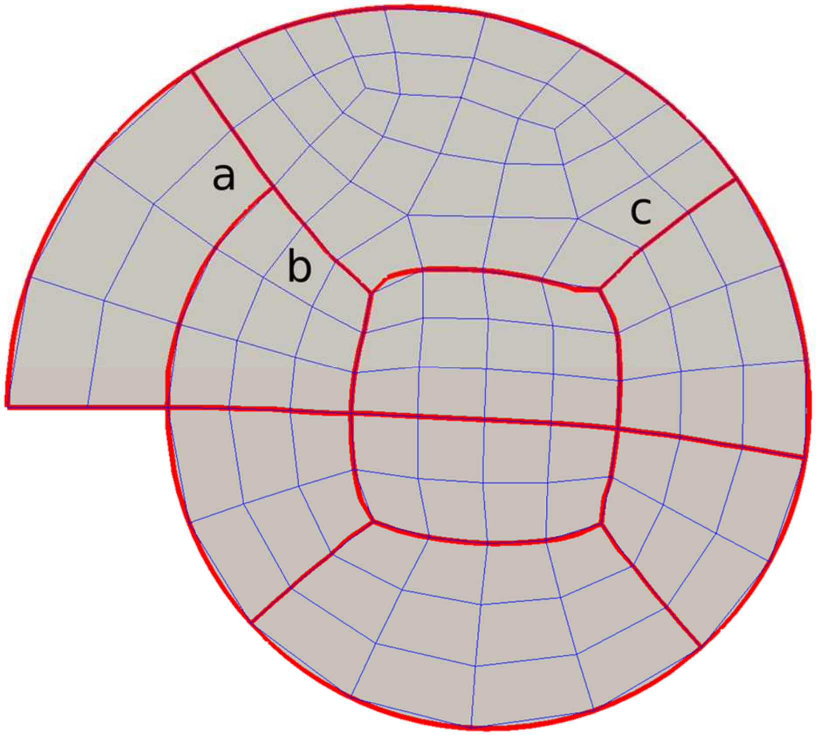






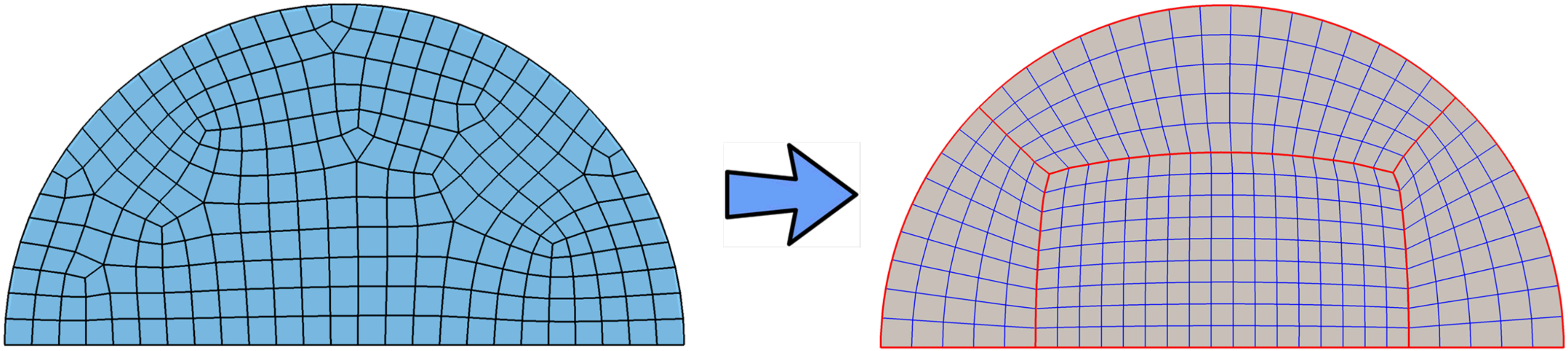
Limit Cycles





Future Research

Future Research: Paver Replacement

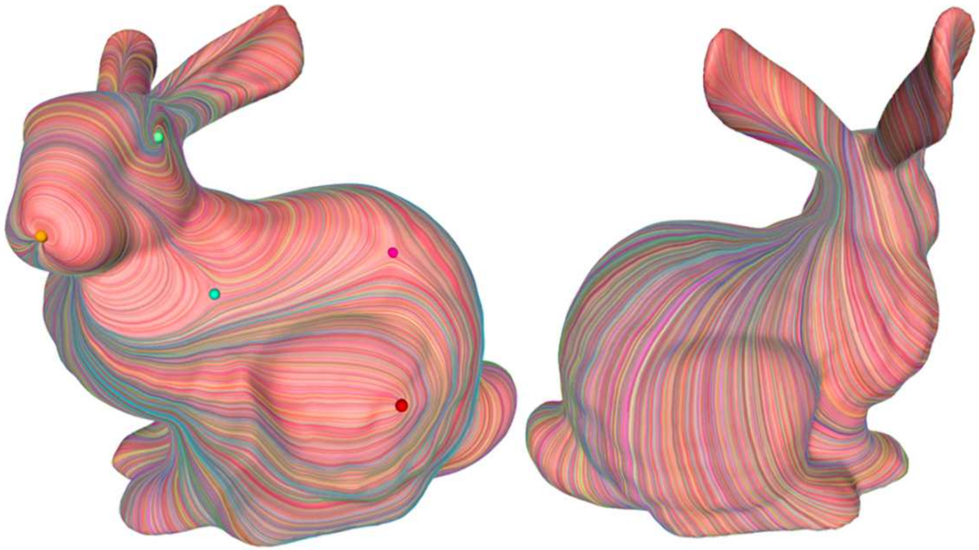


Wish List

1. High element quality – isotropic, close to a perfect squares
2. Boundary aligned elements
3. Block Structured mesh – Minimal number of singularities
4. Prescribed size map
5. Prescribed boundary intervals.
6. Guaranteed results
7. Produces predictable output

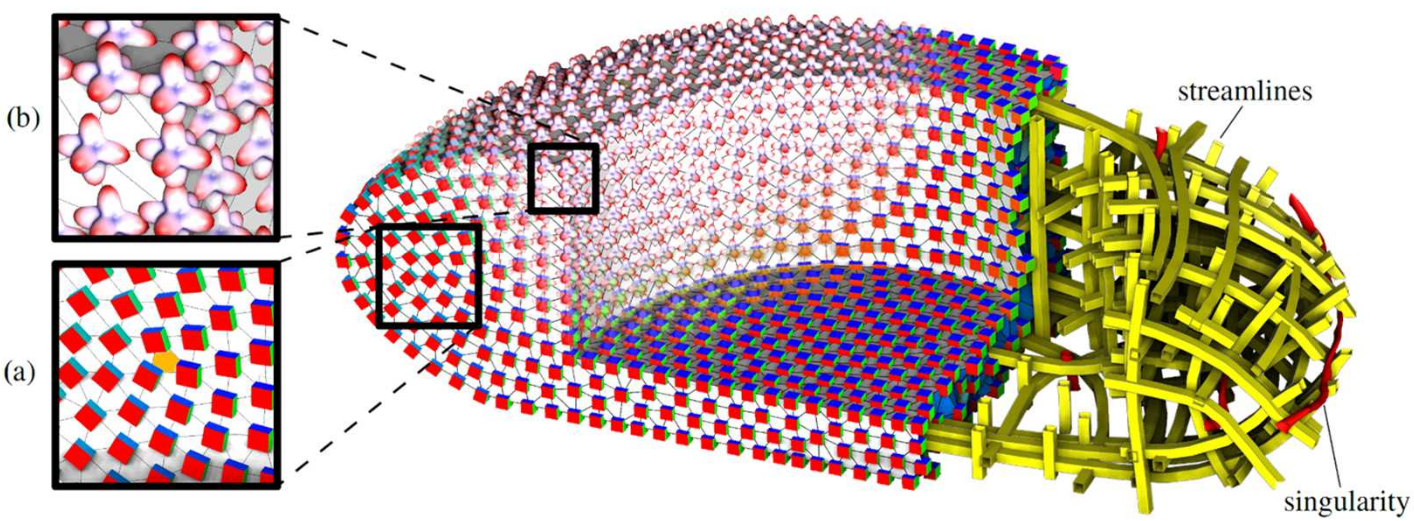
Extend New Algorithms to Higher Dimensions

- Extend fixed frame field design algorithm to 2-manifolds with arbitrary borders



Crane et al. 2010

- Extend MBO method to 3D



Ray et al. 2016

Summary

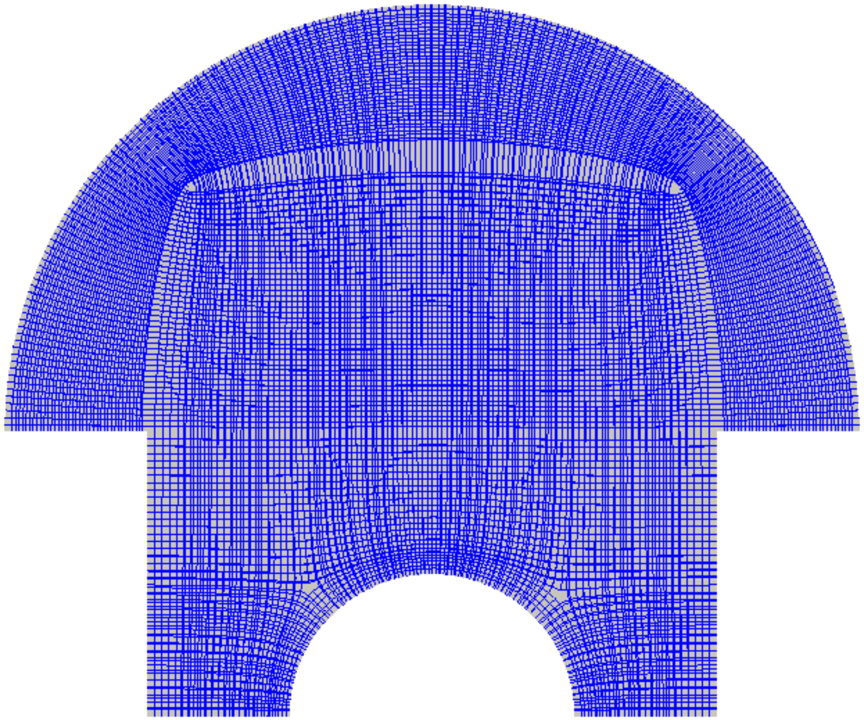
1. Connection with Ginzburg-Landau Theory
2. MBO method for minimizing cross field energy
3. Fixed Frame field design method
4. Asymptotic Behavior of Singularities
5. Cross Field Partitioning Theorem

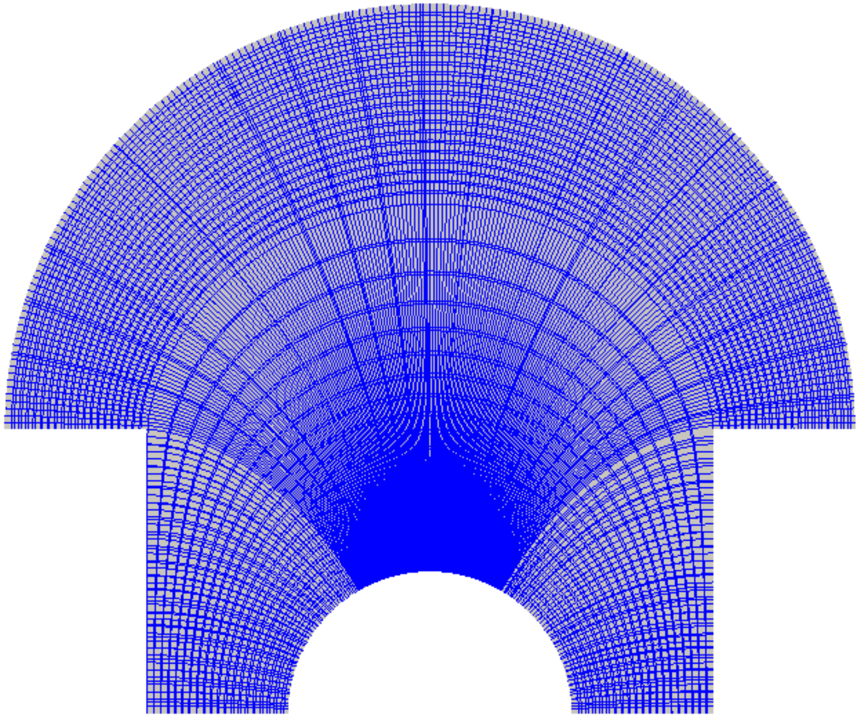
Acknowledgements

- Sandia National Labs
- University of Utah
- NSF DMS 16-19755
- Matt Staten
- Braxton Osting

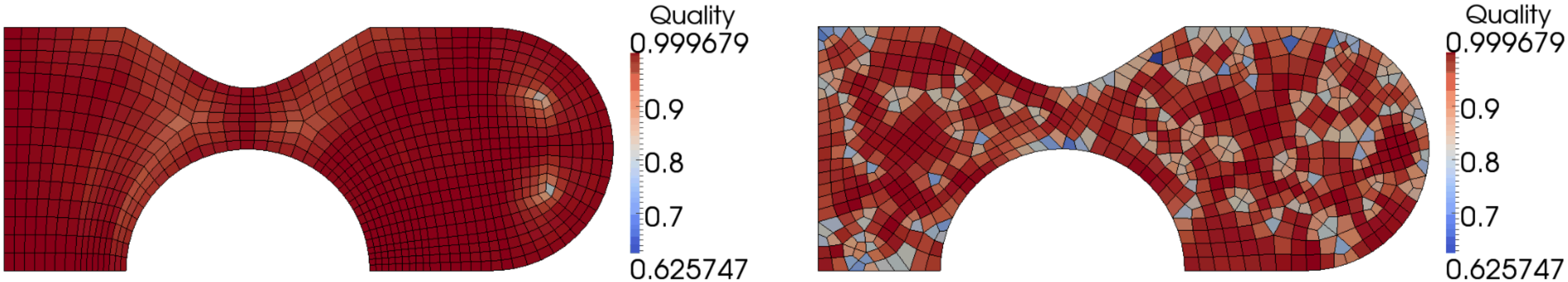
Extra Slides

Implication for Cross Fields: Strange Minimizer





Cross Fields Automatically Generate Good Meshes in 2D



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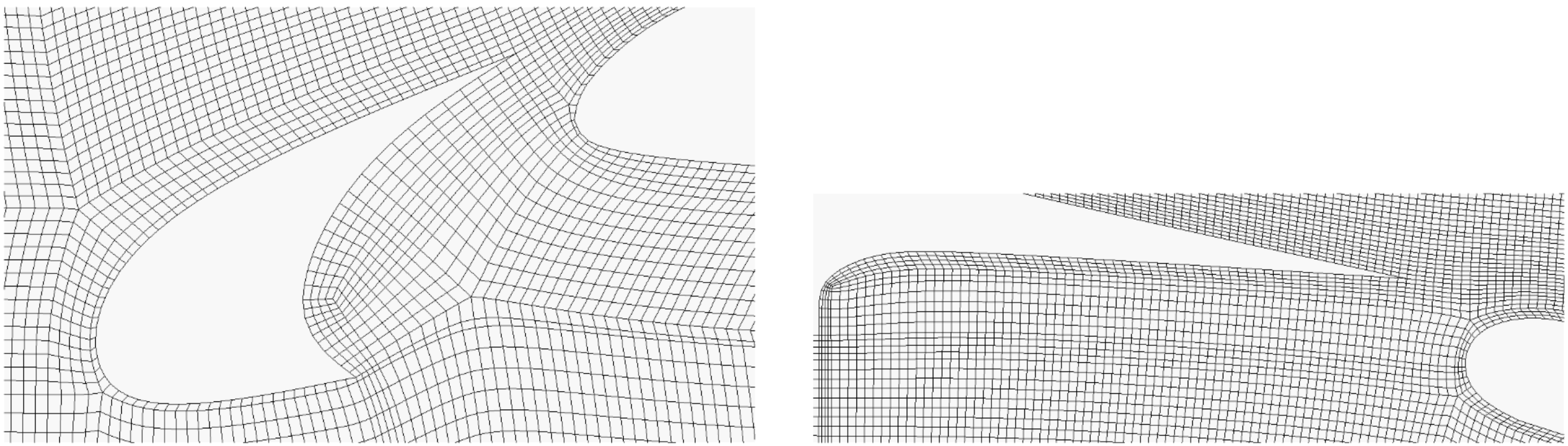

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Kowalski et al. 2013