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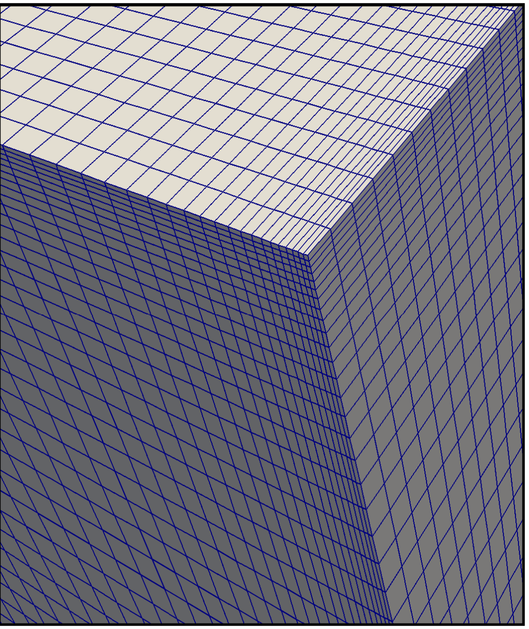

Toward a Paver Replacement

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Sept 18-21, 2017

26th International Meshing Roundtable

Barcelona, Spain

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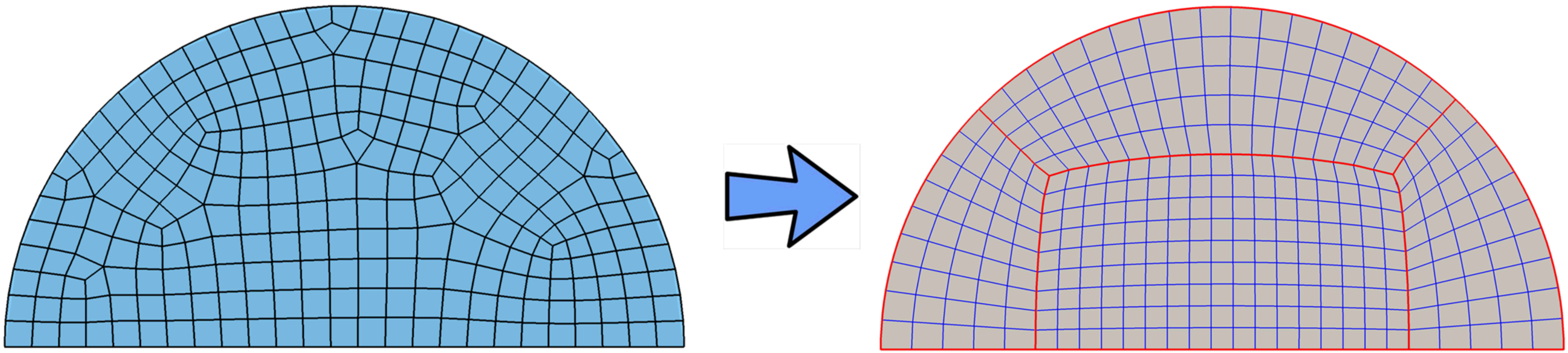
Introduction

Paver Replacement



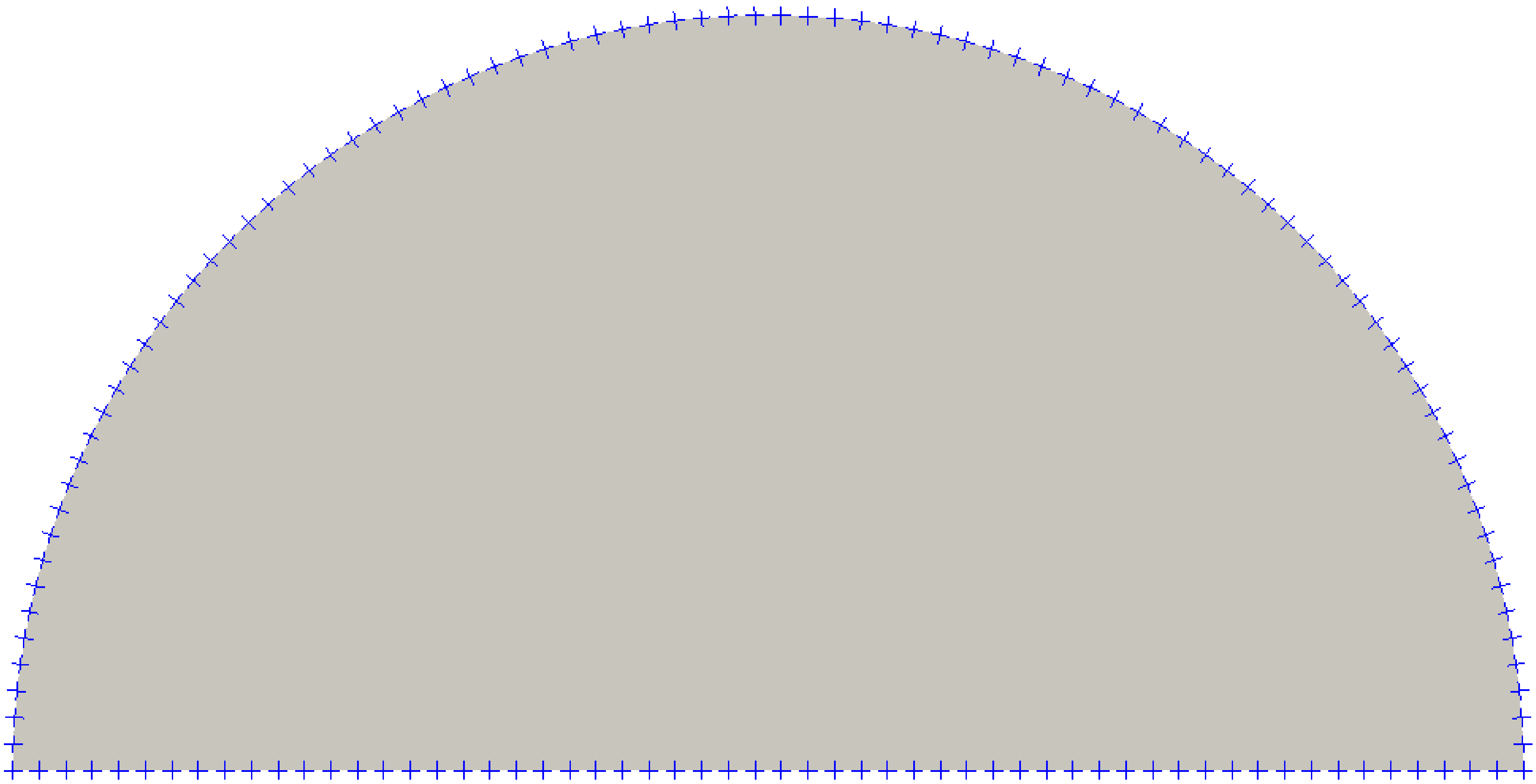
Wish List

1. High element quality
2. Boundary aligned elements
3. Block Structured mesh
4. Prescribed size map
5. Prescribed boundary intervals.
6. Guaranteed results
7. Produces predictable output

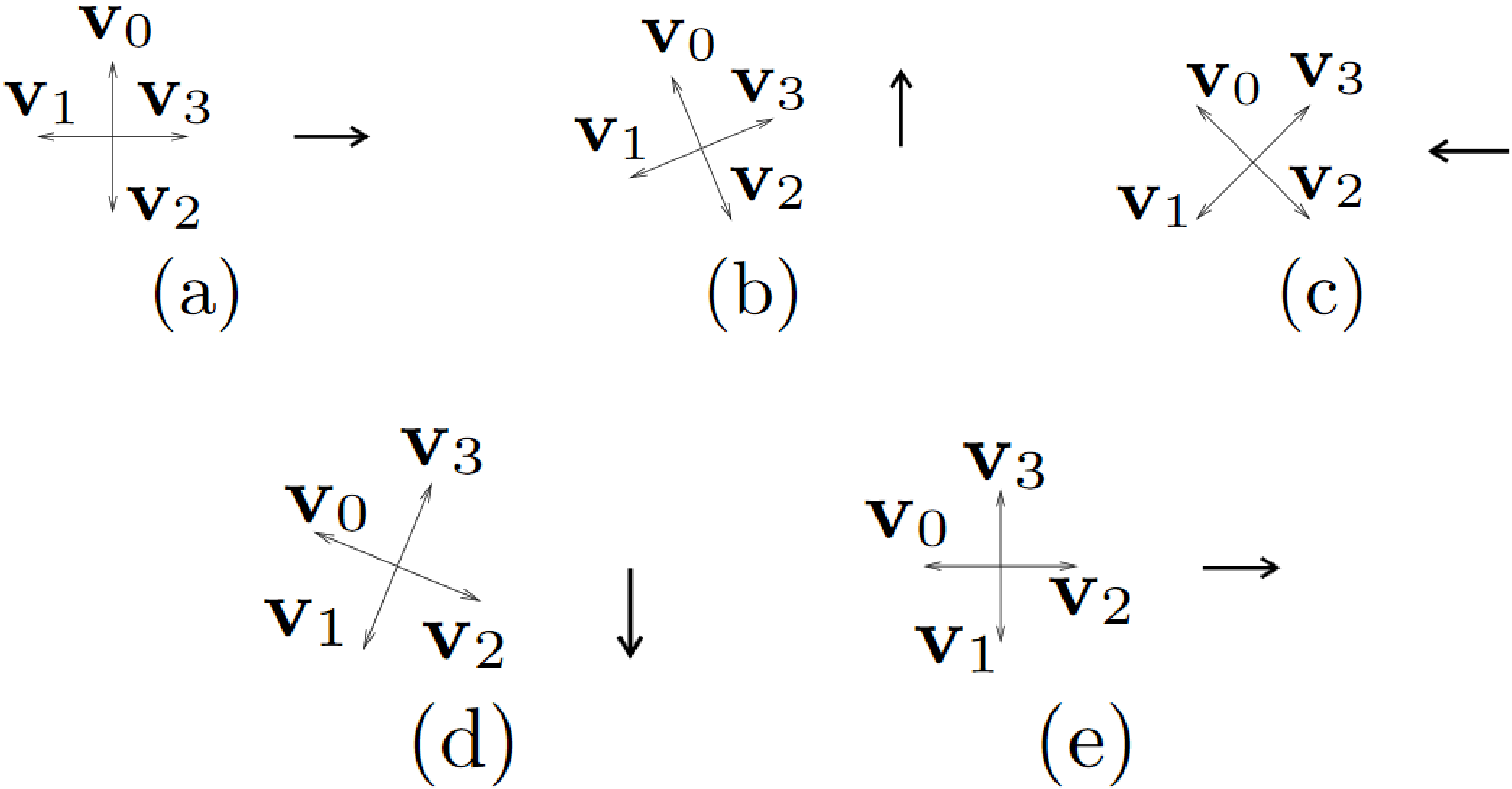
Basic Cross Field Meshing Algorithm (Kowalski et al. 2013)

2D Cross Field Meshing Algorithm

2D Cross Field Meshing Algorithm

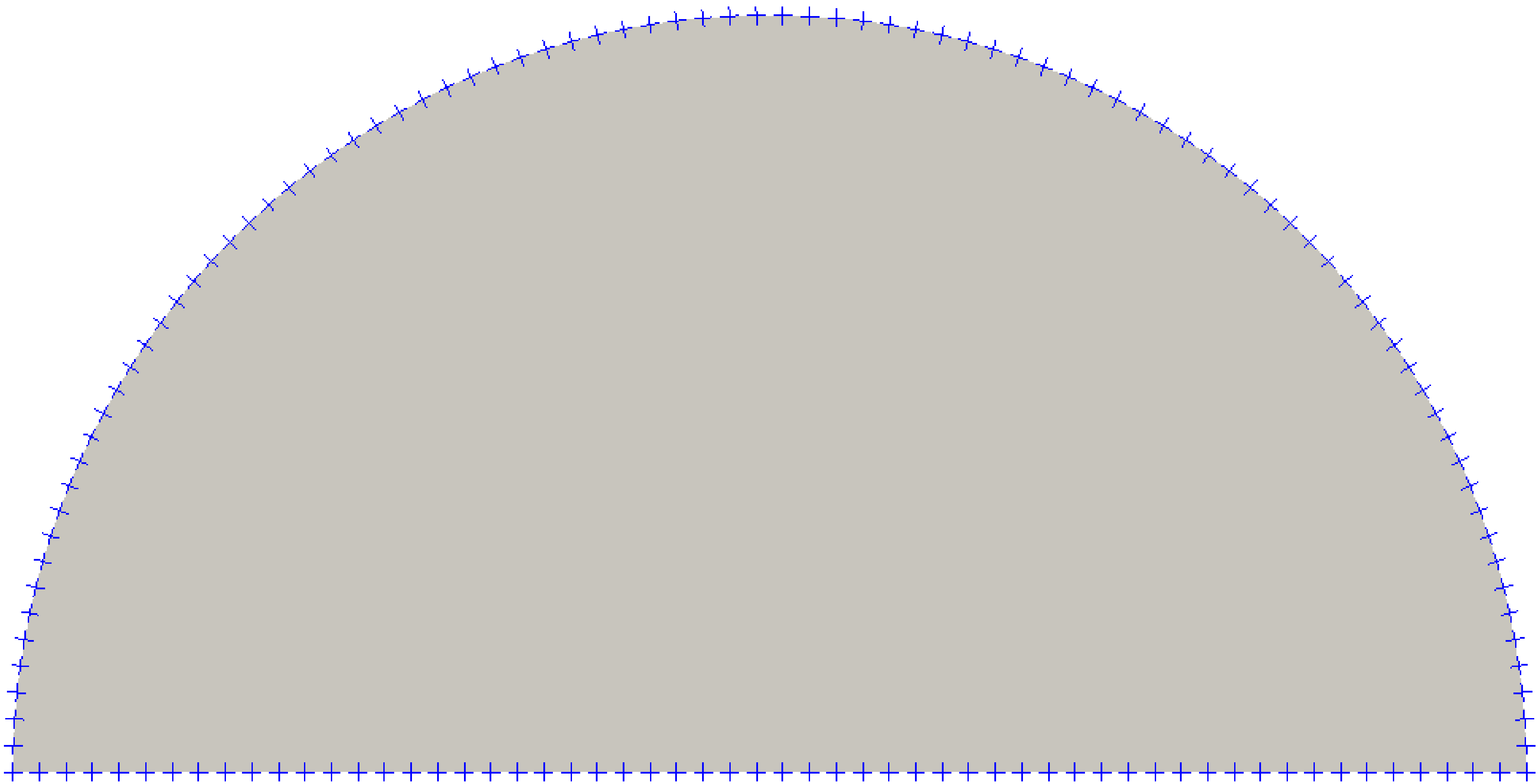


Crosses and Representation Vector

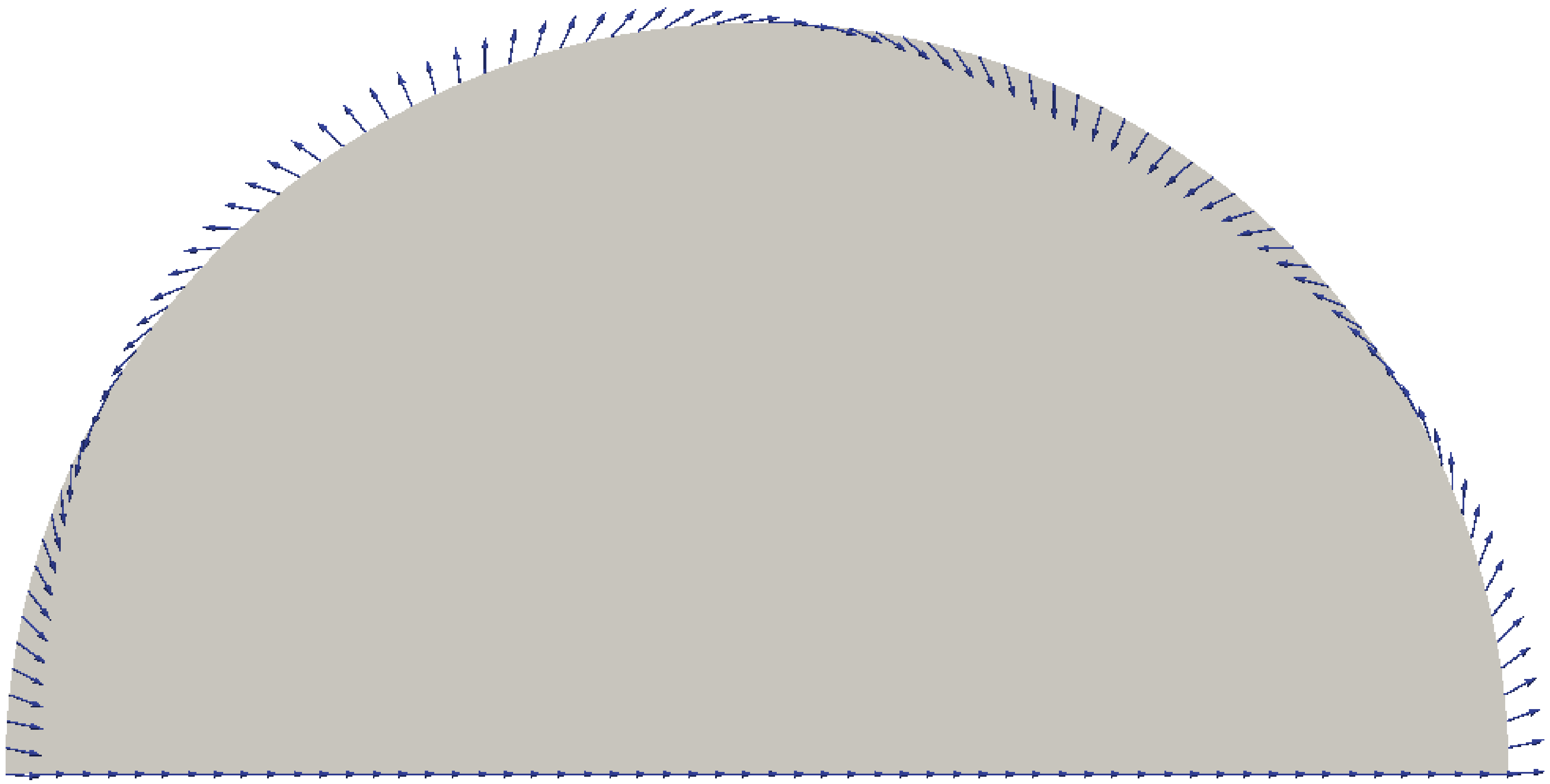


Kowalski et al. 2013

2D Cross Field Meshing Algorithm



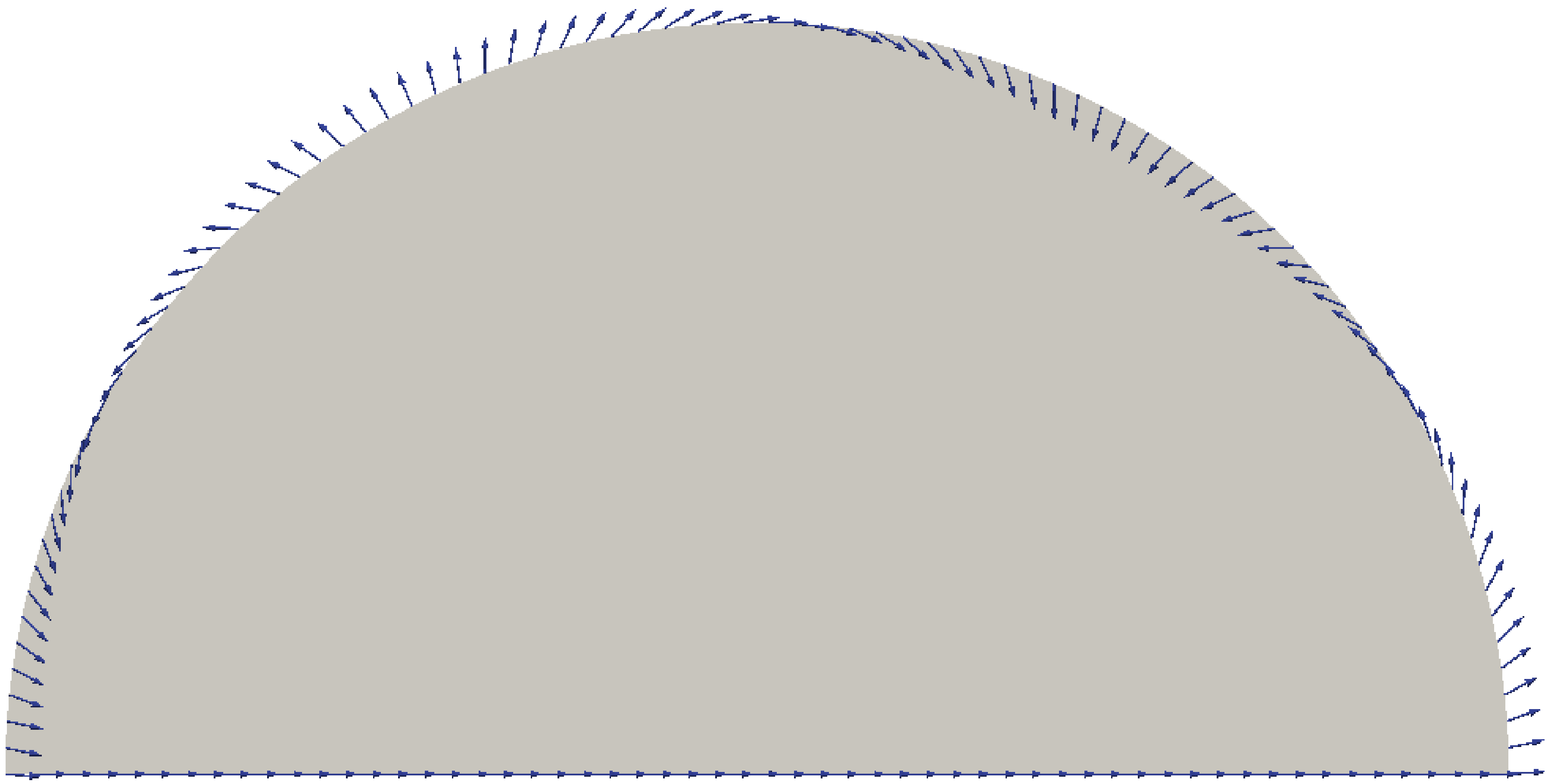
2D Cross Field Meshing Algorithm



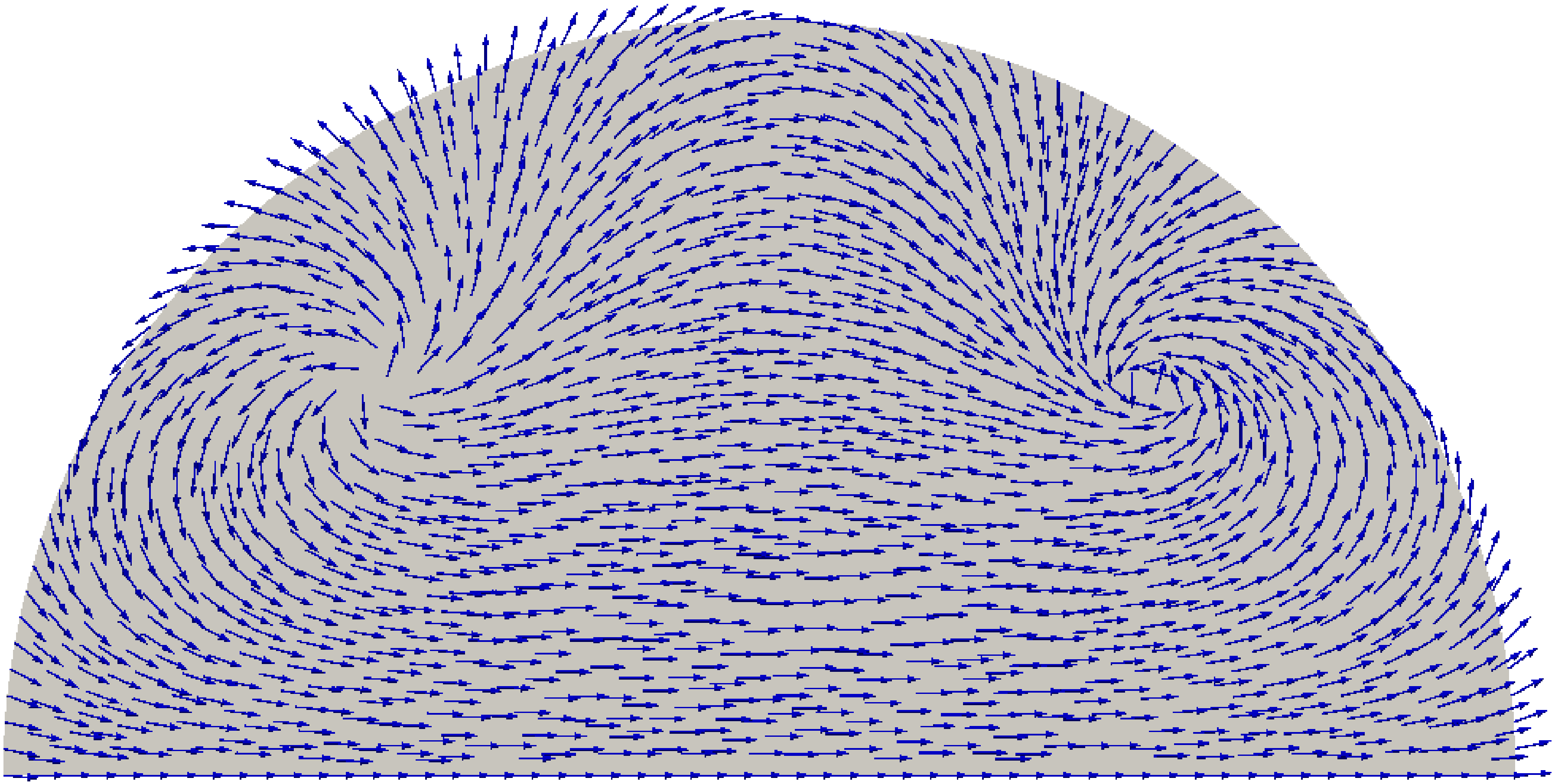
2D Cross Field Meshing Algorithm

$$\begin{cases} \min_u E(u) \\ E(u) = \frac{1}{2}\int_D |\nabla u|^2 dA \\ u(x) = R(f_0(x)) \quad \forall x \in \partial D \\ |u(x)| = 1 \quad a.e. x \in D \end{cases}$$

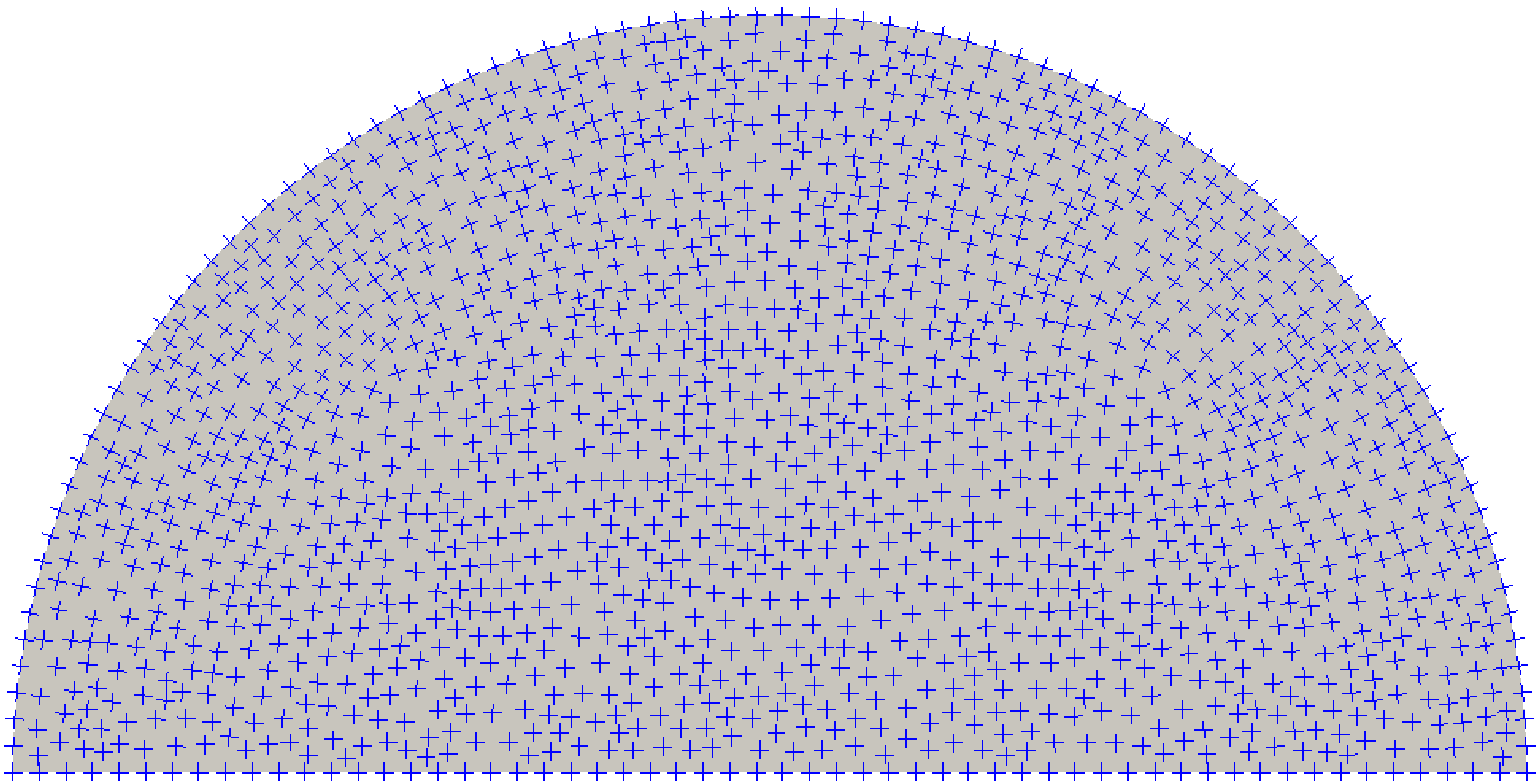
2D Cross Field Meshing Algorithm



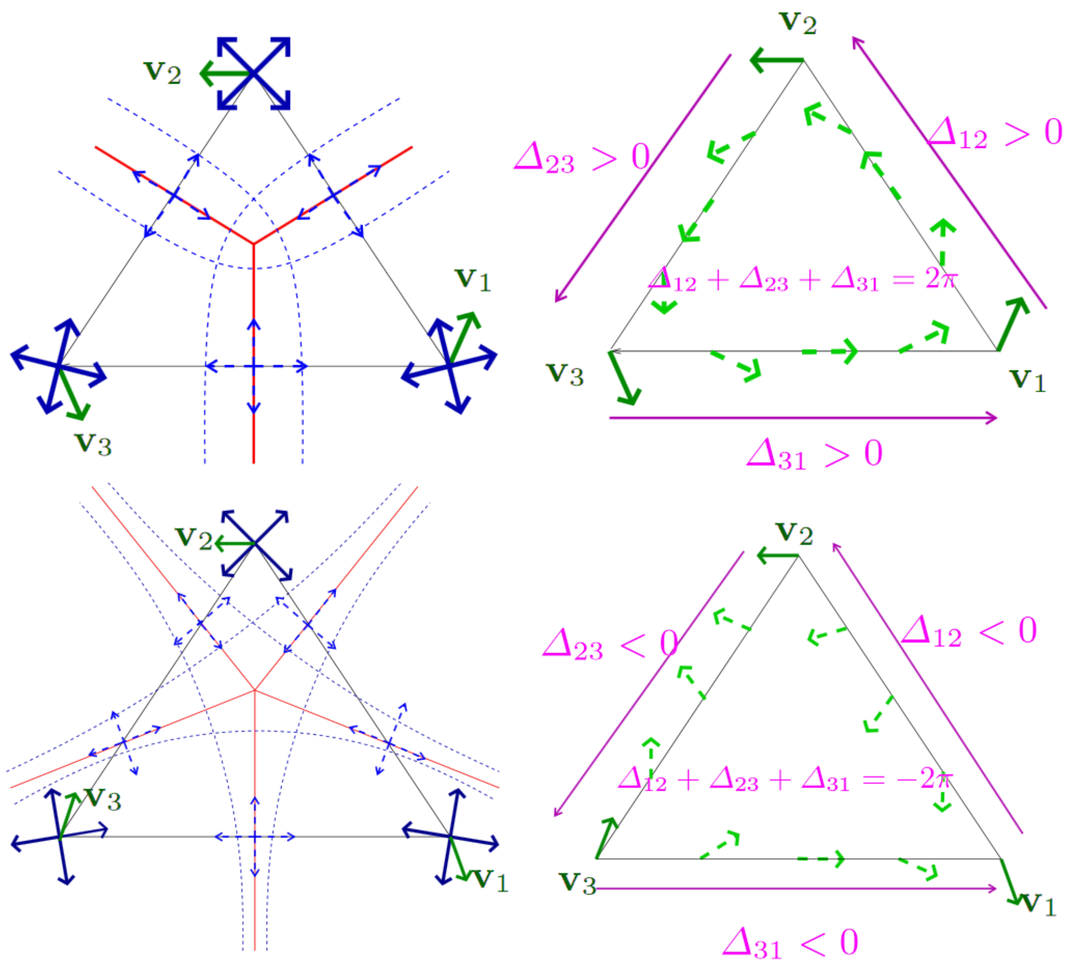
2D Cross Field Meshing Algorithm

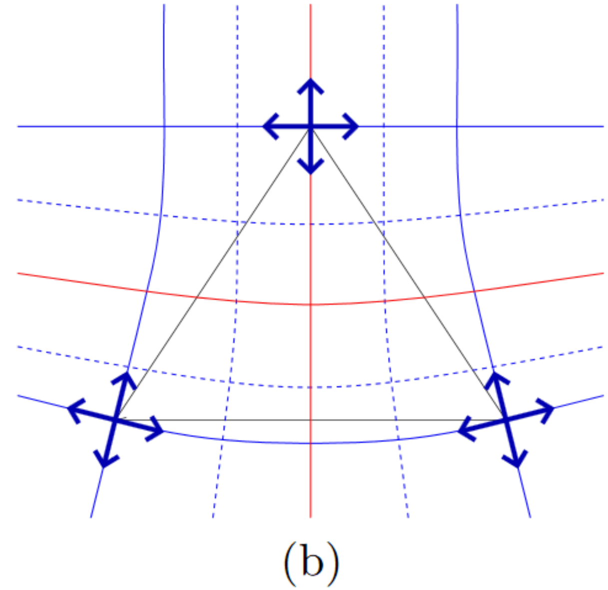


2D Cross Field Meshing Algorithm



Cross Field Singularities



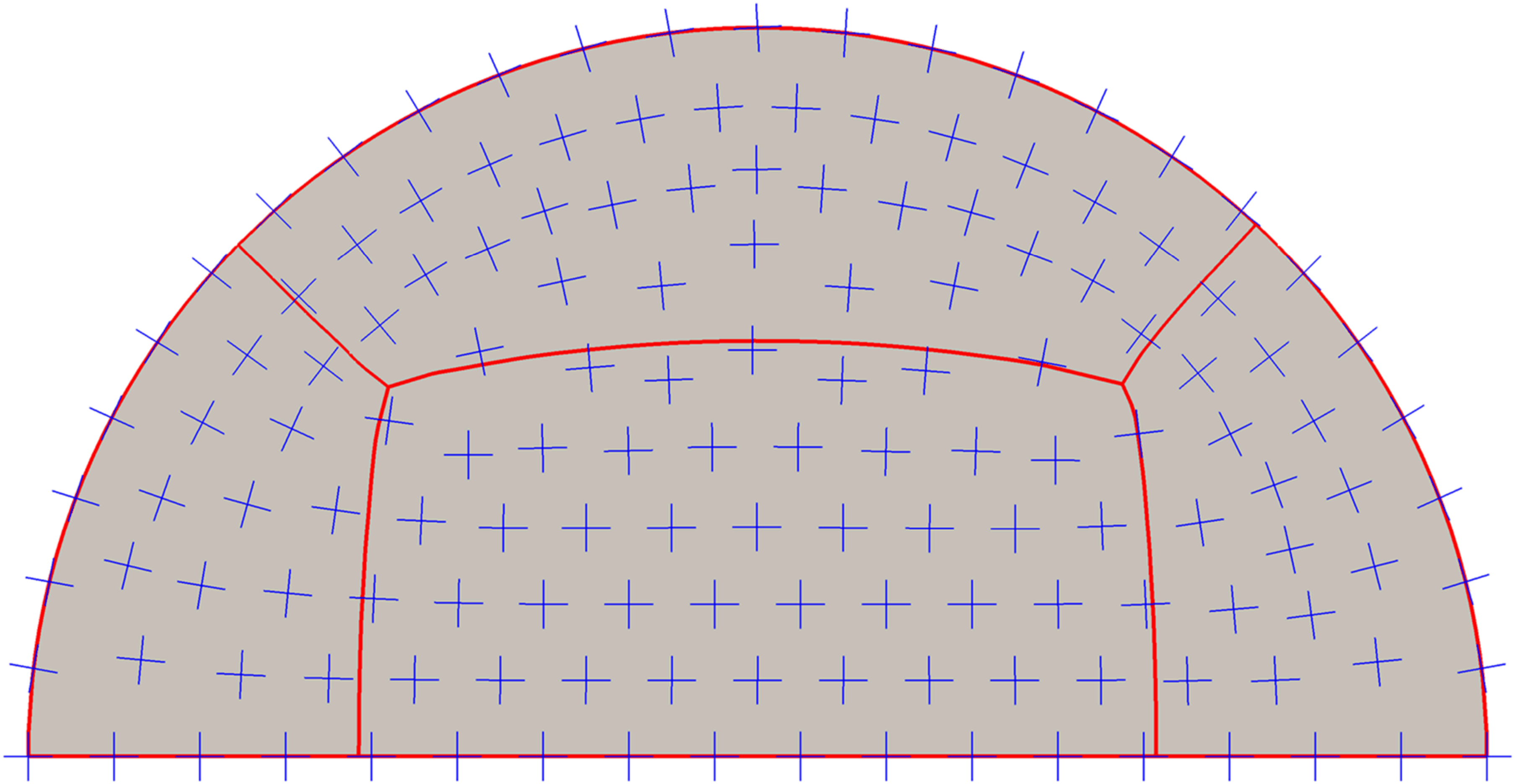


Nonsingular Triangle

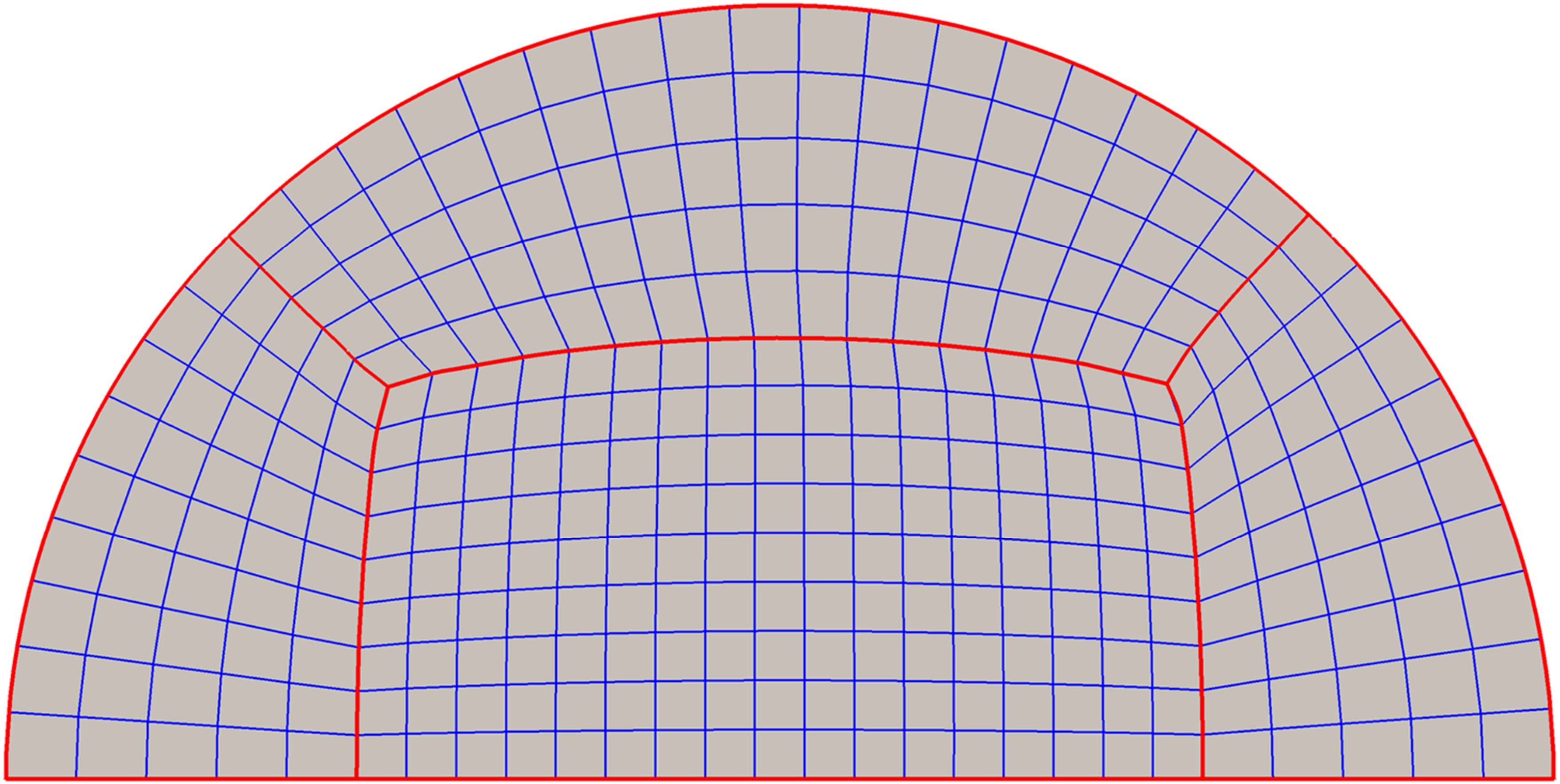
Singular Triangles

Kowalski et al. 2013

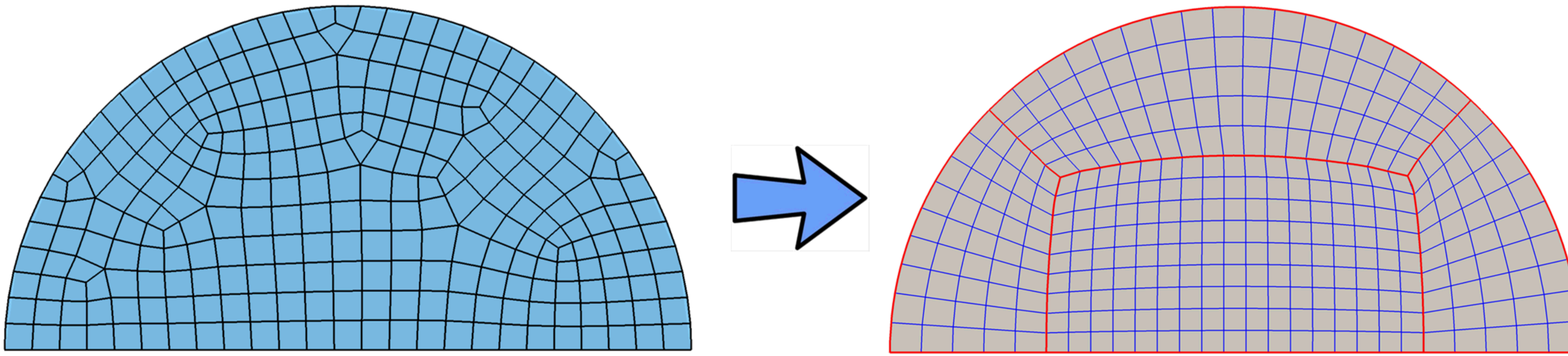
2D Cross Field Meshing Algorithm



2D Cross Field Meshing Algorithm



Candidate: Cross field guided methods



Wish List

1. High element quality ✓
2. Boundary aligned elements ✓
3. Block Structured mesh ?
4. Prescribed size map
5. Prescribed boundary intervals.
6. Guaranteed results
7. Produces predictable output

Connection to Ginzburg-Landau Theory

Ginzburg-Landau Functional

Original problem is not well defined:

$$\begin{cases} \min_u E(u) \\ E(u) = \frac{1}{2}\int_D |\nabla u|^2 dA \\ u(x) = g(x) \qquad \forall x \in \partial D \\ |u(x)| = 1 \quad a.e. x \in D \end{cases}$$

Relaxed problem:

$$\min_{u \in H_g^1(D,\mathbb{C})} E_\varepsilon(u)$$

$$E_\epsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\epsilon^2}\int_G (|u|^2 - 1)^2$$

Result: Well Defined Limit of Relaxed Problem

Theorem 2.2.2 (Bethuel et al. [4]). *Let* $d = deg(g, \partial D)$. *Given a sequence* $\varepsilon_n \to 0$ *there exists a subsequence* ε_{n_i} *and exactly* d *points* $a_1, a_2, ..., a_d$ *in* $D \subset \mathbb{C}$ *and a smooth harmonic map* $u_\star : D \setminus \{a_1, ..., a_d\} \to \mathbb{T}$ *with* $u_\star = g$ *on* ∂D *such that*

$$\boxed{u_{\varepsilon_{n_i}} \to u_\star} \text{ in } C^k_{loc}(D \setminus \bigcup_i (a_i)) \ \forall k \text{ and in } C^{1,\alpha}(\bar{D} \setminus \bigcup_i (a_i)) \ \forall \alpha < 1$$

In addition, if $d \neq 0$ *each singularity of* $u_\star$ *has index* $\operatorname{sgn}(d)$ *and, more precisely, there are complex constants* (α_i) *with* $|\alpha_i| = 1$ *such that*

$$\left| u_\star(z) - \alpha_i \frac{z - a_i}{|z - a_i|} \right| \leq C|z - a_i|^2 \text{ as } z \to a_i, \ \forall i$$

This gives us a generalized sense in which to understand the energy minimization problem

Merriman-Bence-Osher (MBO) Method

Merriman-Bence-Osher (MBO) Method

Original Method

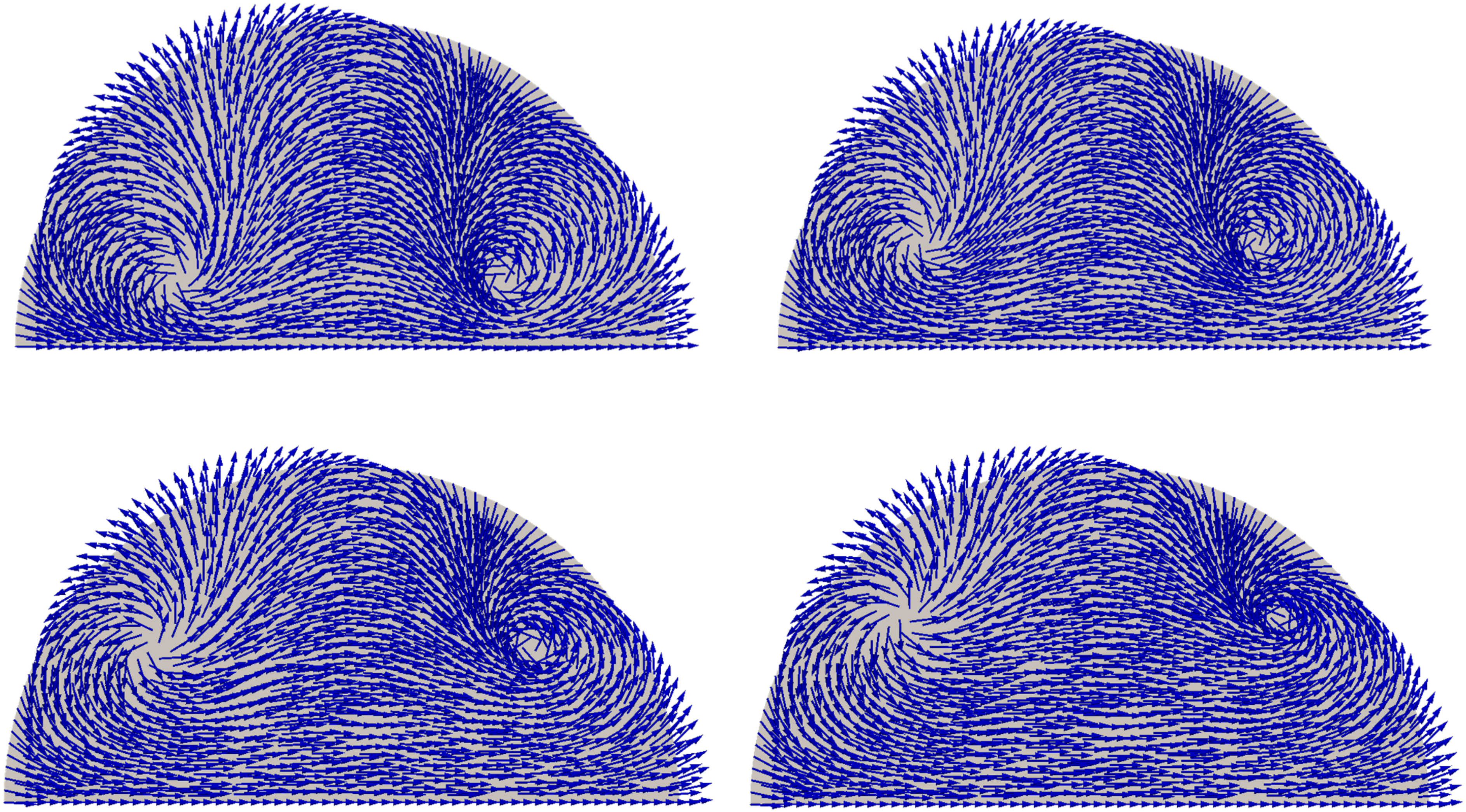
- Introduced as a method for motion by mean curvature
- Minimizes an two-well potential energy analogous to the complex GL energy

New Application to Frame Fields

- Iterative method to minimize cross field energy:

$$u_0 = \frac{\tilde{u}}{|\tilde{u}|} \quad \text{and} \quad u_k = \frac{e^{\tau\Delta} u_{k-1}}{|e^{\tau\Delta} u_{k-1}|} \quad k \geq 1.$$

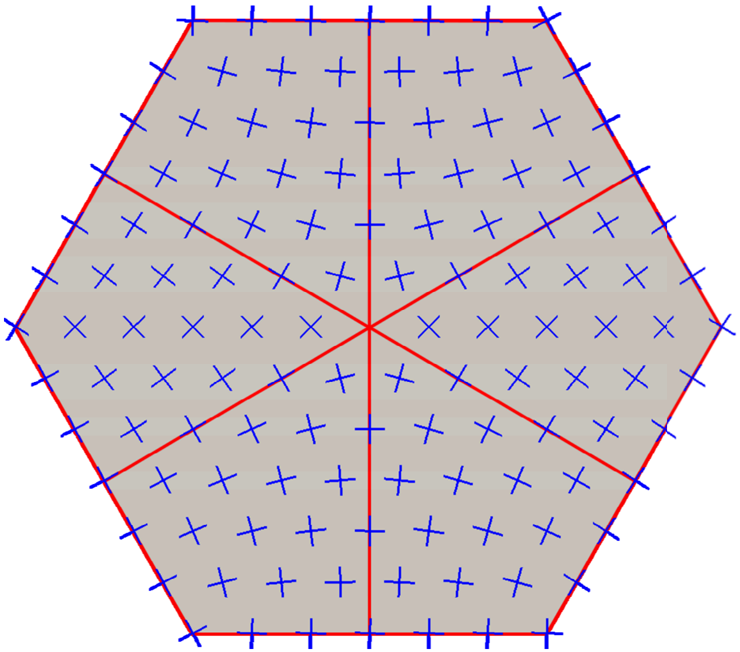
MBO Method

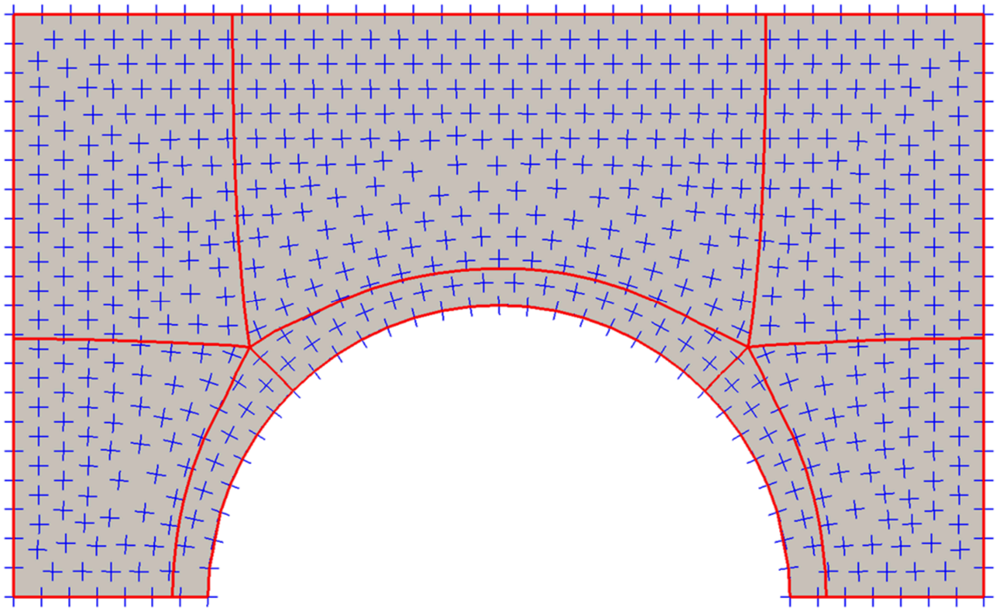


Asymptotic Behavior of Cross Fields Near Singularities

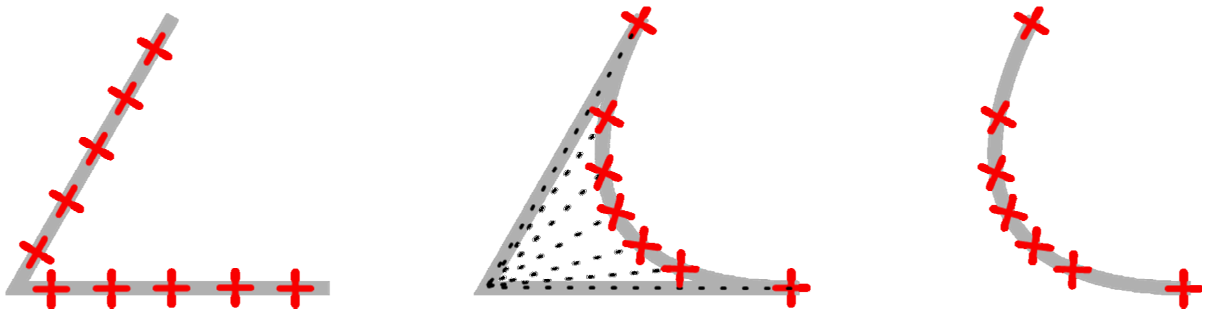
Separatrices of a Singularity

LEMMA 5.1. *Let f be a boundary-aligned canonical harmonic cross field on D. Let a be an interior singularity of f of index $d/4$ with $d < 4$. There are exactly $4 - d$ separatrices meeting at a. These separatrices partition a neighborhood of a into $4 - d$ even-angled sectors.*

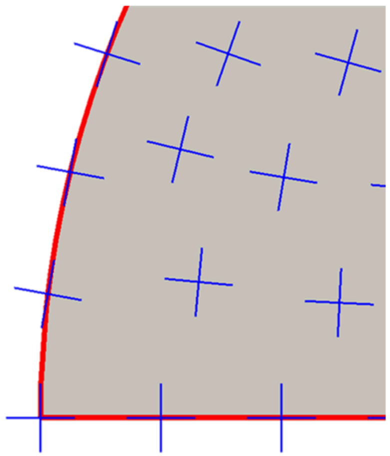


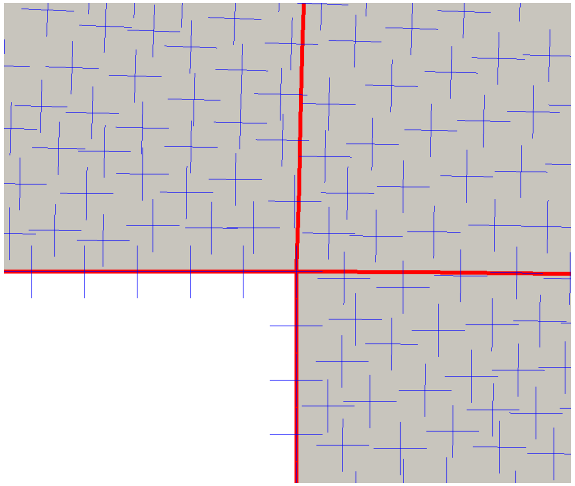


Boundary Singularities

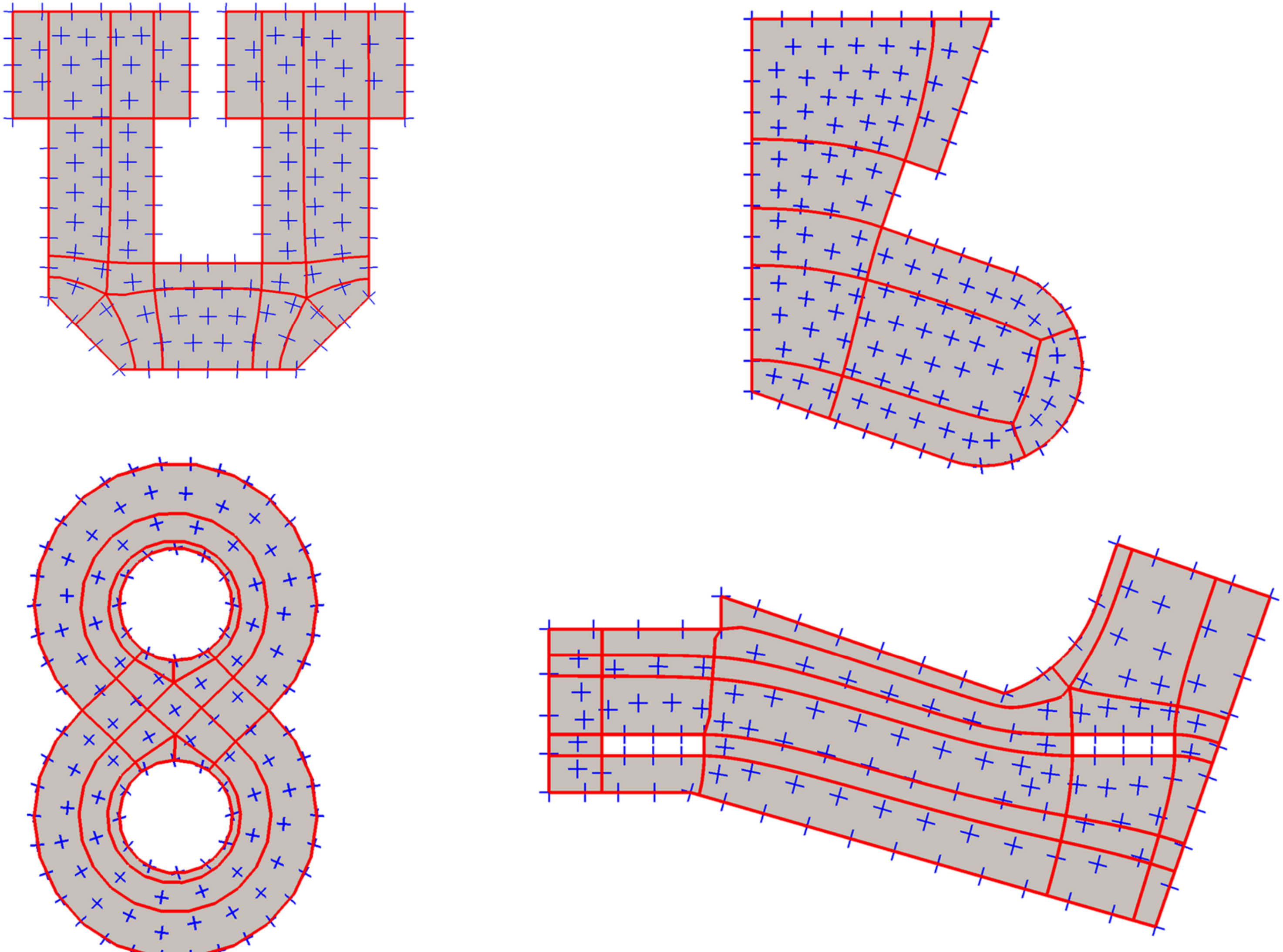


LEMMA 5.4. *Let c be a boundary singularity of f of index $d/4$ with $d < 2$. There are exactly $3 - d$ separatrices meeting at c (including the boundaries themselves). These separatrices partition a neighborhood of c into $2 - d$ even-angled sectors.*

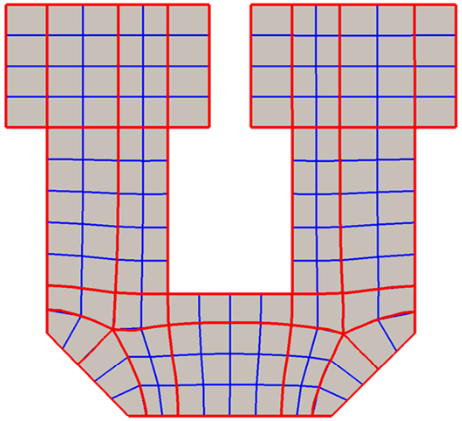


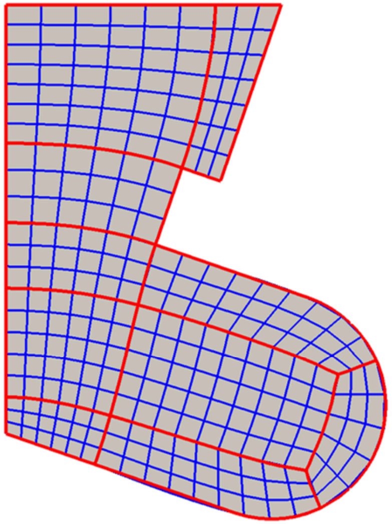


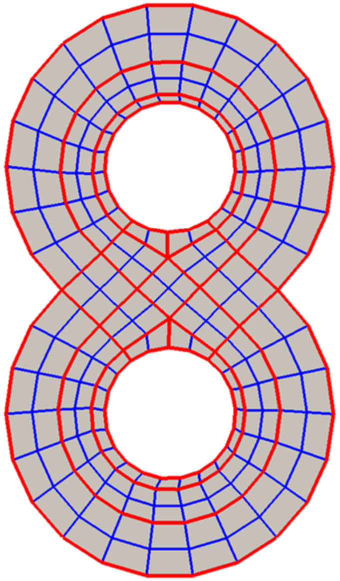
Partition into four-sided regions

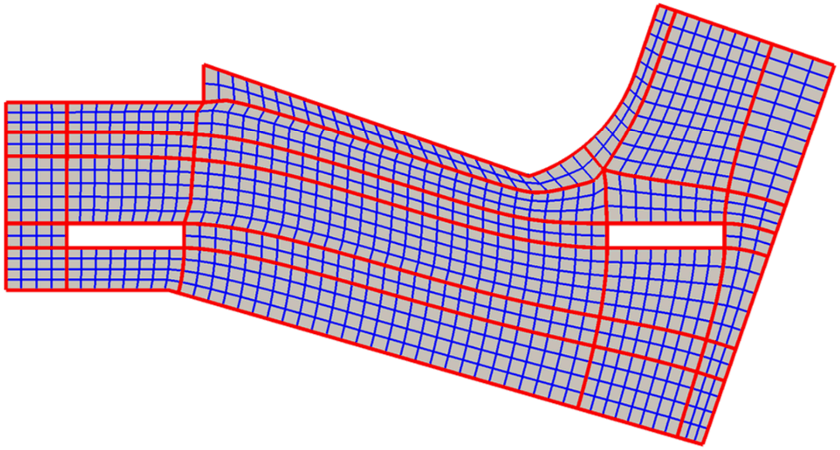


Meshing

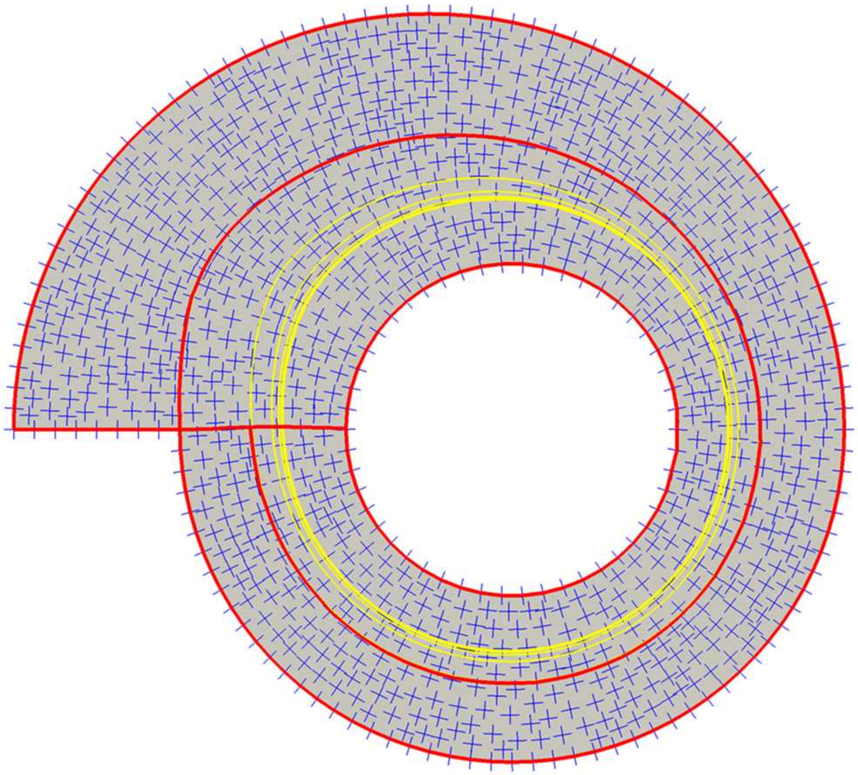


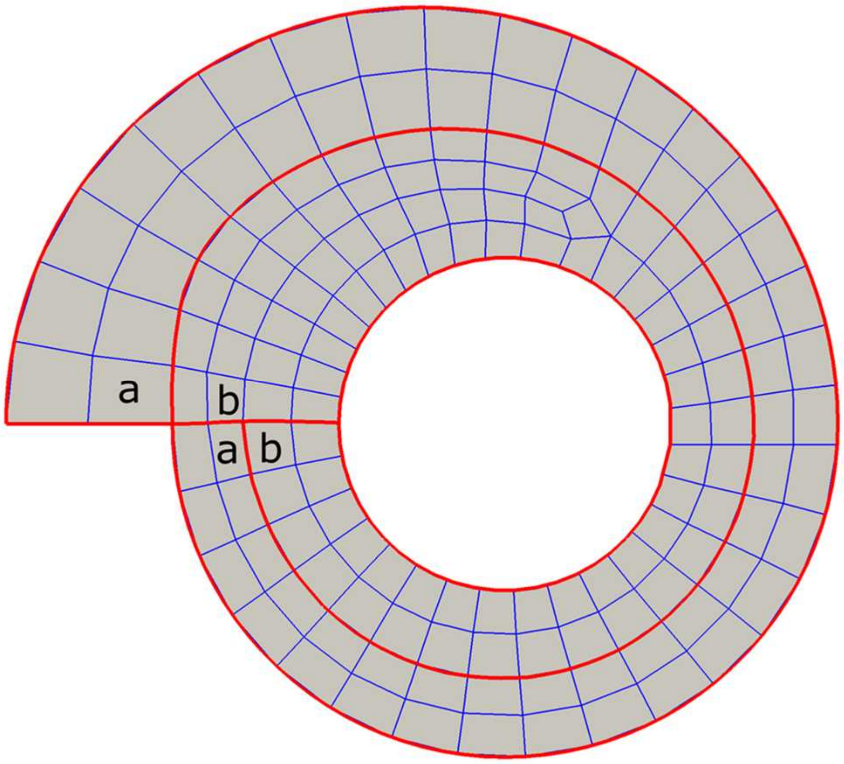




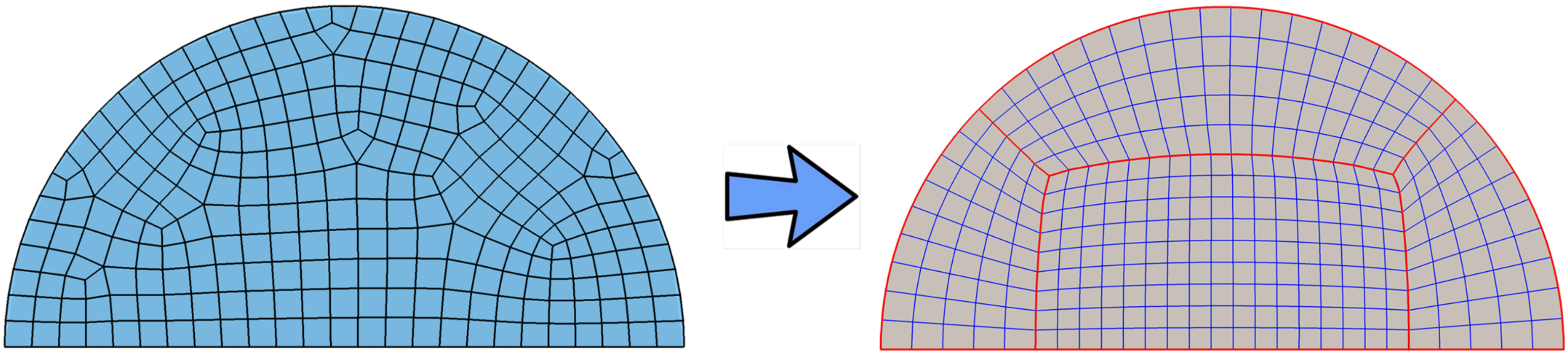


Limit Cycles





Candidate: Cross field guided methods



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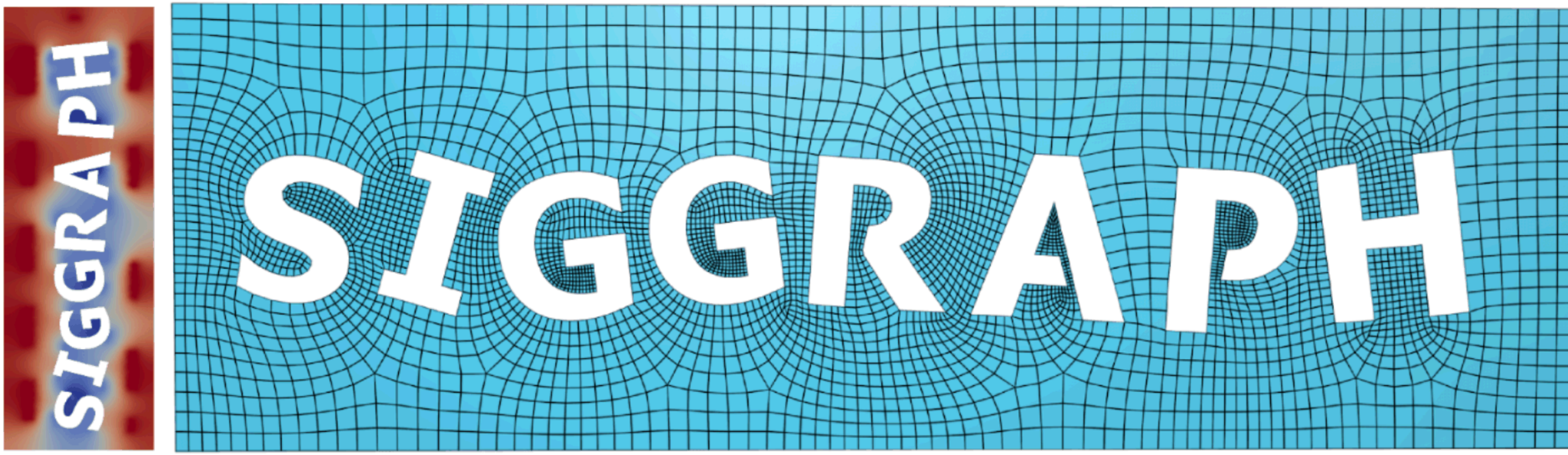
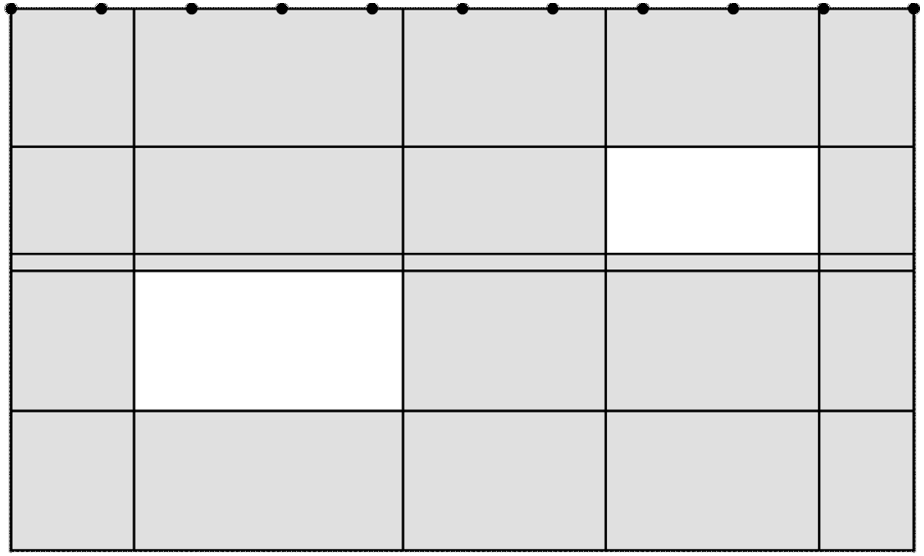
Size Map

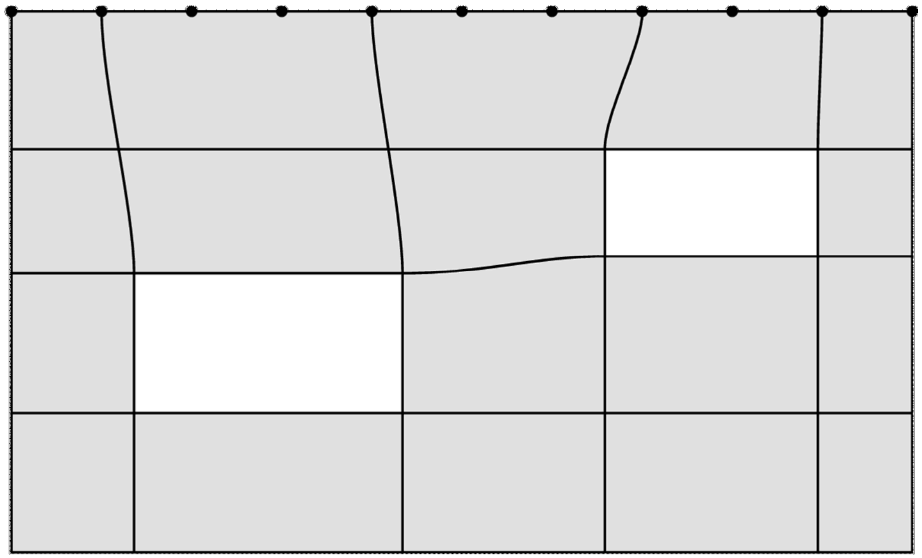


- T. Jiang, X. Fang, J. Huang, H. Bao, Y. Tong, M. Desbrun, Frame field generation through metric customization, ACM Transactions on Graphics 34 (2015).

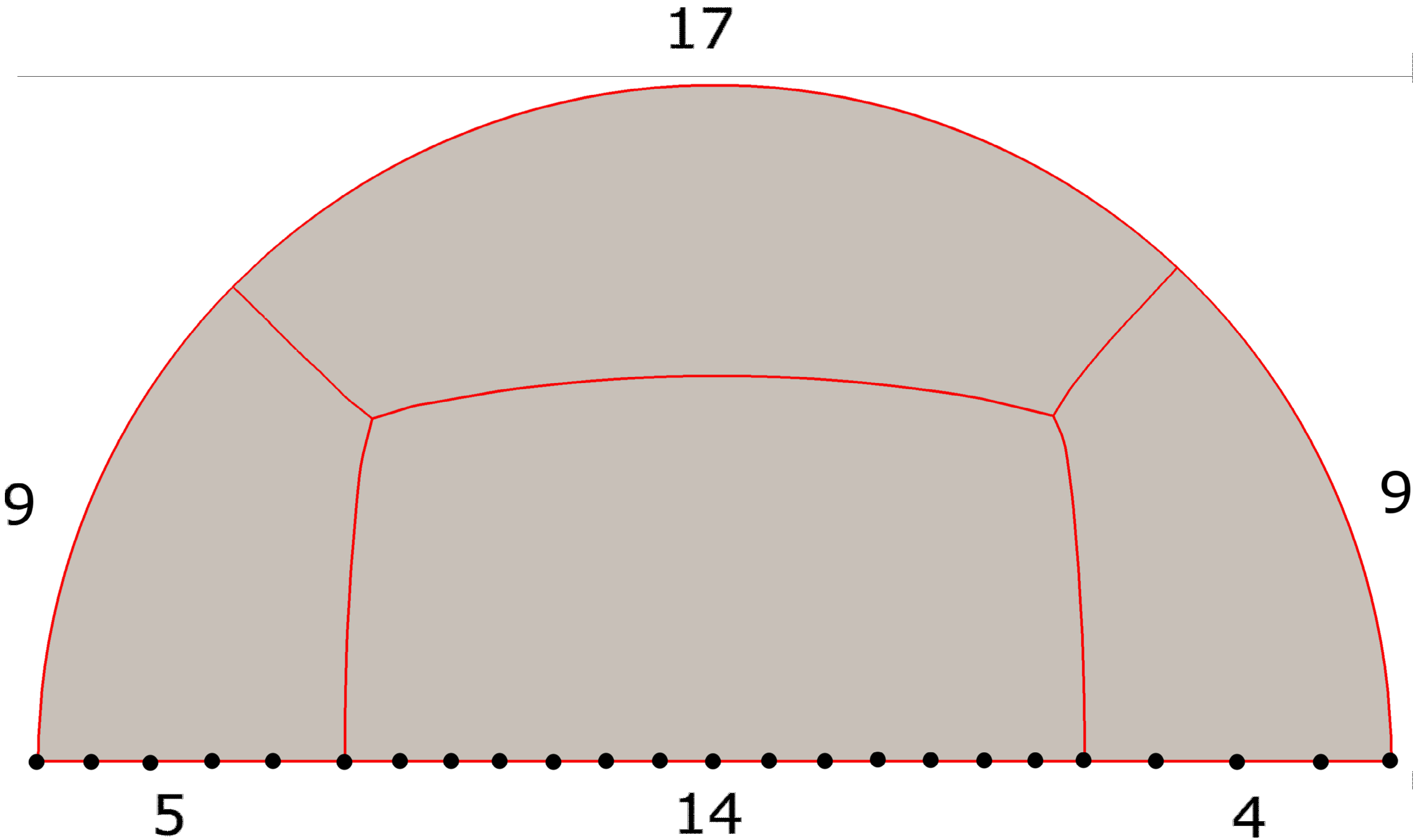
We intend to adapt the work of Jiang et al. to the MBO method for designing cross fields

Partition Modification





Boundary Interval Assignment



- S. A. Mitchell, High fidelity interval assignment, International Journal of Computational Geometry & Applications 10 (2000) 399–415.

Summary

1. Connection with Ginzburg-Landau Theory
2. MBO method for minimizing cross field energy
3. Fixed Frame field design method
4. Asymptotic Behavior of Singularities
5. Cross Field Partitioning Theorem

Acknowledgements

- Sandia National Labs
- University of Utah
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- Matt Staten
- Braxton Osting