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Low-rank functional decompositions with applications to stochastic optimal control

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Takeaways

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1. Low-rank functions $=$ representing multivariate functions with separable expressions
2. Low-rank functions are related to tensors when performing approximation with a linear *basis*
3. Low-rank functions can use *nonlinear* approximation $\rightarrow \neq$ tensors
4. ALS is not always best for finding low-rank decompositions

Stochastic optimal control
Challenge: representing a parametric objective function

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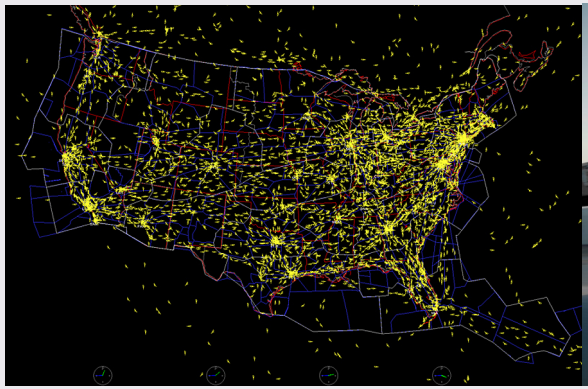
$$J(x_0) = \min_{u(t)} \mathbb{E}\left[\int g(x(t), u(t)dt)\right], \quad x(0) = x_0$$

subject to a stochastic differential equation

$$dx = F(x, u)dt + \sigma(x)dw(t), \quad x \in \mathbb{R}^d,$$



Approximating black-box functions

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Applications and challenges:

- Stochastic control: repeat applications of Bellman Operator $J^{k+1}(x) = T(J^k(x))$
  1. many approximations
  2. high-dimensional states
  3. expensive evaluations
- Data analysis: generate empirical models from data
  1. high-dimensional states
  2. large amounts of data
- Uncertainty quantification: analyze complex physical systems
  1. high-dimensional states
  2. expensive evaluations
  3. expensive analysis

Questions

1. How do we interrogate a function to accurately reconstruct it?
2. Someone else interrogated the function, how do I interpret the results?

Scalable representations of functions

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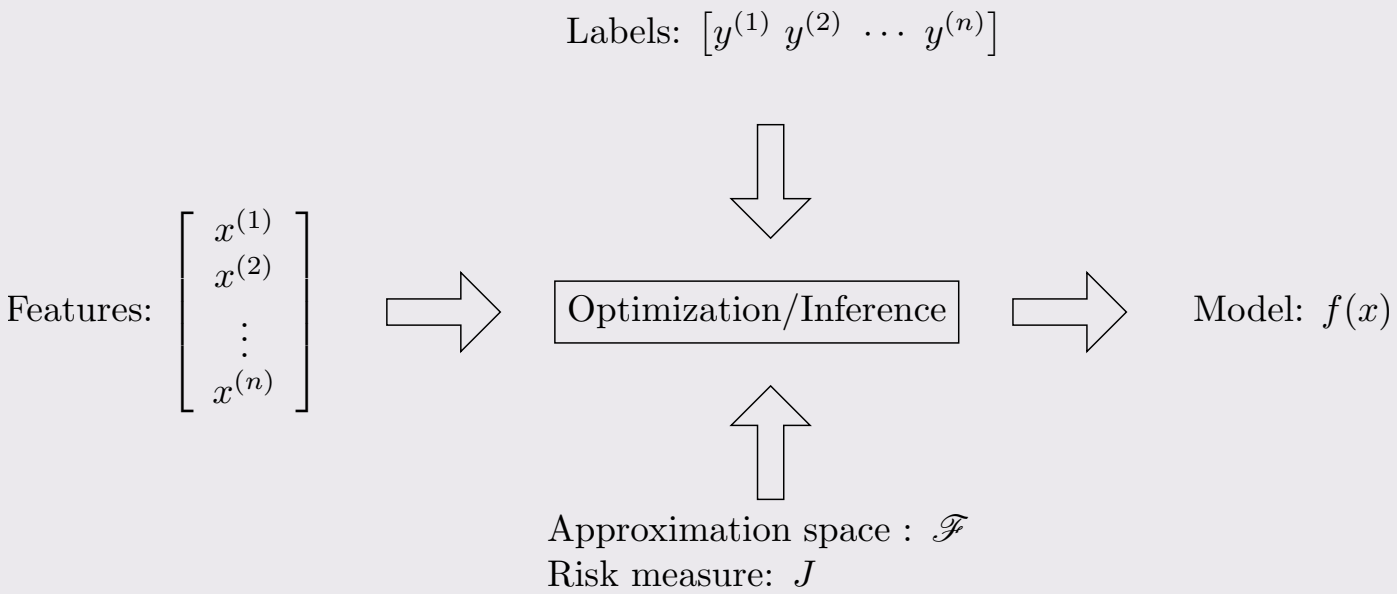

- Parametric approaches for $\mathscr{F}$:
  - Good: statistical properties, stability
  - Bad: curse-of-dimensionality on basis, expressivity
- Nonparametric approaches for $\mathscr{F}$:
  - Good: expressivity
  - Bad: excess risk scales exponentially with dimensionality, expensive

Common parametric approaches

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- Approximation space with linear basis $\left\{\phi^{(i)}\right\}_{i=1}^{P}$

$$\Psi = \left[\phi^{(1)}(\boldsymbol{x}) \ \ldots \ \phi^{(P)}(\boldsymbol{x})\right]$$

- Many algorithmic setups
  - Ridge regression:

$$\min \|\Psi\boldsymbol{\theta} - \boldsymbol{y}\| + \lambda\|\boldsymbol{\theta}\|_2$$

  - $L1$ regression:

$$\min \|\Psi\boldsymbol{\theta} - \boldsymbol{y}\| + \lambda\|\boldsymbol{\theta}\|_1$$

  - $L0$ regression:

$$\min \|\Psi\boldsymbol{\theta} - \boldsymbol{y}\| + \lambda\|\boldsymbol{\theta}\|_0$$

- Constructing Ψ is problematic in high-dimensions
  - Total order expansions: exponential growth of terms
  - Low order expansion: limited expressivity

Typical setup
Linear setup yields curse-of-dimensionality

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Inputs are from a *tensor product space* and inputs are *real*

$$x_i \in \mathcal{X}_i, \quad x = (x_1, x_2, \ldots, x_d) \in \mathcal{X}, \quad \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_d$$

Determine a basis for each dimension

$$\left(\phi_k^{(i)} : \mathcal{X}_k \to \mathbb{R}\right)_{k=1}^{n}$$

1. Global basis, e.g., orthonormal polynomials
2. Multi-resolution basis, e.g., wavelets
3. Localized or kernels

Basis for f obtained by *combinations*

$$\phi^{\boldsymbol{i}} = \phi_1^{(i_1)} \times \phi_2^{(i_2)} \times \cdots \times \phi_d^{(i_d)}, \ \boldsymbol{i} = (i_1, i_2, \ldots, i_d)$$

1. Tensor-product: all combinations $(i_k)_1^{n_k} \to n^d$
2. Some subsets: $\|\boldsymbol{i}\| < n \to \mathcal{O}(n^d)$

Tensors arise!
Linear approximation with tensor-product basis

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The coefficients of a tensor-product basis form a tensor

$$f(x) = \sum_{i_1=1}^{n_1} \sum_{i_2=2}^{n_2} \cdots \sum_{i_d=d}^{n_d} \boldsymbol{\theta}_{i_1 i_2 \ldots i_d} \phi_{1 i_1}(x_1) \phi_{2 i_2}(x_2) \cdots \phi_{d i_d}(x_d)$$

- Exponentially growing number of coefficients reduces feasibility
- Need dimension reduction methods
- Tensor-decompositions are a multilinear compressed representations

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Various decompositions:

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3. Tensor-train: $\boldsymbol{\theta}_{i_1 i_2 \ldots i_d} = \sum_{j_1=1}^{r_1} \sum_{j_2=1}^{r_2} \cdots \sum_{j_d=1}^{r_d} \boldsymbol{a}_{1 j_1 i_1} \boldsymbol{a}_{2 j_1 j_2 i_2} \cdots \boldsymbol{a}_{d j_d i_d}$

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Issues and solutions
Functional tensor-train (FT) as a separable ansatz

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- Would like to *adapt* to local and global structure
- Would like computationally *efficient*, and *flexible* approximation format
- Algorithms for decomposing functions into FT format

$$f(x_1, x_2, \ldots, x_d) = \sum_{i_0=1}^{r_0} \sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} f_1^{(i_0 i_1)}(x_1) f_2^{(i_1 i_2)}(x_2) \ldots f_d^{(i_{d-1} i_d)}(x_d)$$
$$= \mathcal{F}_1(x_1)\mathcal{F}_2(x_2)\ldots\mathcal{F}_d(x_d)$$

$$\underbrace{\begin{bmatrix} f_k^{(11)}(x_k) & \cdot & f_k^{(1 r_k)}(x_k) \\ \vdots & \ddots & \vdots \\ f_k^{(r_{k-1} 1)}(x_k) & \cdot & f_k^{(r_{k-1} r_k)}(x_k) \end{bmatrix}}_{\mathcal{F}_k(x_k)}$$

$$f(x_1, x_2, :, x_4) =$$

→ Oseledets 2013, **G.** et al., A continuous extension of the tensor-train 2016

Linear and nonlinear parameterizations

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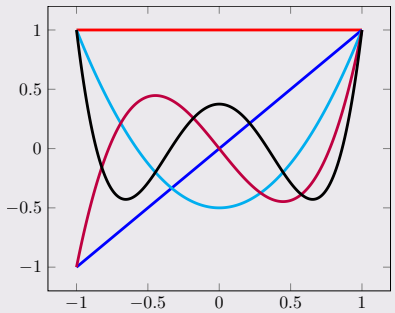
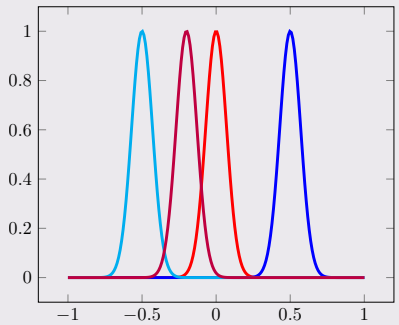
Each $f_k^{(i_{k-1}i_k)}$ can be represented *independently*

	Linear	Nonlinear
		
Evaluation	$\sum_{\ell=1}^{n_{kij}} \boldsymbol{a}_{kij\ell}\phi_{k\ell}^{(ij)}(x_k)$	$\sum_{\ell=1}^{n_{kij}} \exp\left((x_k - \boldsymbol{a}_{kij\ell})^2\right)$
Gradient	$\phi_{k\ell}^{(ij)}(x_k)$	$-2\left(x_k - \boldsymbol{a}_{kij\ell}\right)\exp\left((x_k - \boldsymbol{a}_{kij\ell})^2\right)$

- Linear: gradient independent of parameter
  $\rightarrow$ faster optimization, lower expressivity
- Nonlinear: gradient dependent on parameters
  $\rightarrow$ slower optimization, higher expressivity

Special case: recovering existing models
Or — converting to low-rank tensors

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Modeling choice

- *linear* parameterization univariate functions
- *identitical basis* expansions within a core, i.e., $n_{kij} = n_k$ and $\phi_{k\ell}^{(ij)} = \phi_{k\ell}$

- Define a TT-core $\boldsymbol{\mathcal{F}}_k \in \mathbb{R}^{r_{k-1} \times n_k \times r_k}$ such that

$$\boldsymbol{\mathcal{F}}_k[:, \ell, :] = \begin{bmatrix} \boldsymbol{a}_{k11\ell} & \cdot & \boldsymbol{a}_{k1r_k\ell} \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_{kr_{k-1}1\ell} & \cdot & \boldsymbol{a}_{kr_{k-1}r_k\ell} \end{bmatrix}$$

- TT-core and FT-core related according to

$$\mathcal{F}_k(x_k) = \sum_{\ell=1}^{n_k} \boldsymbol{\mathcal{F}}_k[:, \ell, :] \phi_{k\ell}(x_k).$$

- TT-approximation of coefficients of a tensor-product of basis functions

$$f(x_1, \ldots, x_d) = \sum_{\ell_1=1}^{n_1} \cdots \sum_{\ell_d=1}^{n_d} \boldsymbol{\mathcal{F}}_1[:, \ell_1, :] \cdots \boldsymbol{\mathcal{F}}_d[:, \ell_d, :] \phi_{1\ell_1}(x_1) \cdots \phi_{d\ell_d}(x_d).$$

Two problem setups

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Compression with experimental design

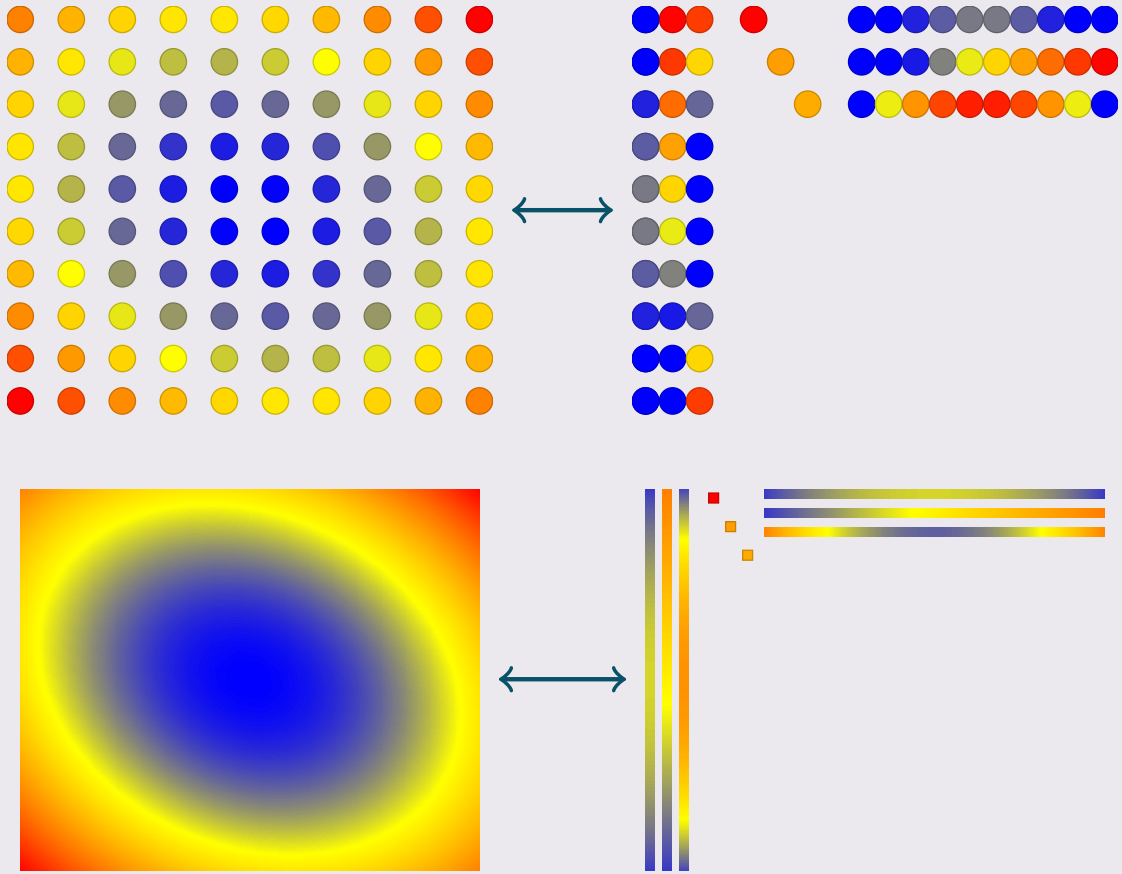
- Have ability to generate samples from f and would like to generate a compressed representation
- **Algorithm:** Cross-approximation with maximum volume
  - Univariate functions chosen from fibers.
  - Algebraically the same as the discrete cross-approximation of Oseledets 2010.
  - However, it is a continuous analogue in the spirt of Chebfun using continuous matrix factorizations (Townsend 2014)

Supervised learning

- We are given data and we parameterized each univariate function $f_k^{(i_{k-1}i_k)}$.
- **Algorithm:** solve an optimization problem for the coefficients
  - Least-squares for real-valued data
  - Cross-entropy for classification
  - Other differentiable objectives...

Compression

Compression of bivariate functions



- Matrix SVD - sum of outer products of vectors
- Functional SVD - sum of products of *univariate functions*

Replace SVD with a CUR/Skeleton decompositior

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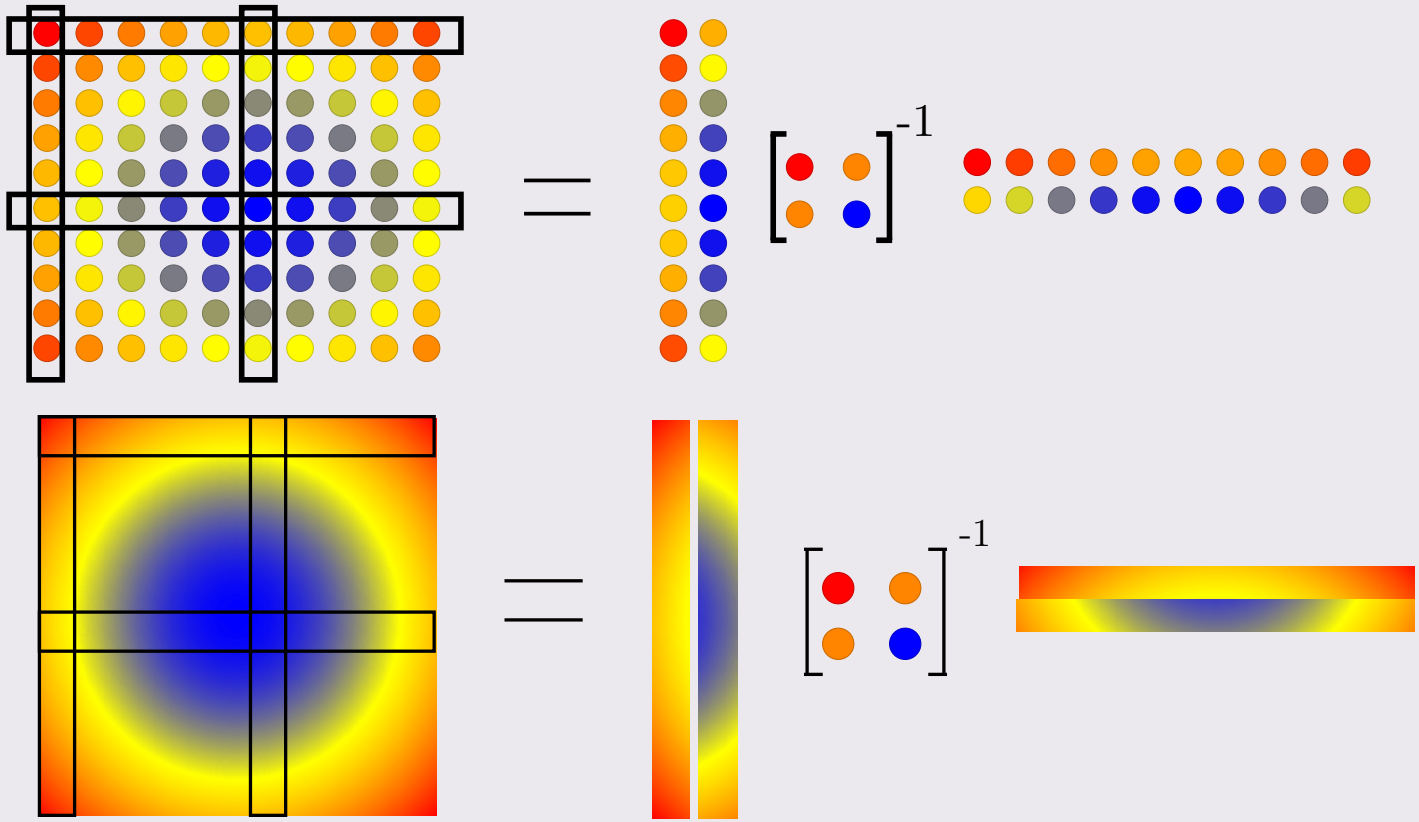

- Compression with SVD expensive, evaluates all elements $\mathcal{O}(N^2)$
- Replace SVD with *another* (suboptimal) factorization: CUR
- Theorem: if $\exists$ exact rank r (f)SVD then $\exists$ exact rank r (f)CUR

Representation of CUR

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1. Fix "rows" $[x_1\ x_2]$ and "columns" $[y_1\ y_2]$

$$f(x,y) = \begin{bmatrix} f(x,y_1) & f(x,y_2) \end{bmatrix} \mathbf{G} \begin{bmatrix} f(x_1,y) \\ f(x_2,y) \end{bmatrix}$$

2. Approximate *univariate* row and column – e.g., in some basis

$$f(x,y) = \begin{bmatrix} \sum_{i=1}^{P} a_i\phi_i(x) & \sum_{i=1}^{P} b_i\phi_i(x) \end{bmatrix} \mathbf{G} \begin{bmatrix} \sum_{i=1}^{P} c_i\psi_i(y) \\ \sum_{i=1}^{P} d_i\psi_i(y) \end{bmatrix}$$

3. Store parameters of *each* row and column and **G** matrix $\mathcal{O}(Pr + r^2)$

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$$f(x,y) = \begin{bmatrix} \sum_{i=1}^{P_1} a_i \phi_i(x;\alpha) & \sum_{i=1}^{P_2} b_i \phi_i(x;\beta) \end{bmatrix} \mathbf{G} \begin{bmatrix} \sum_{i=1}^{P} c_i \psi_i(y) \\ \sum_{i=1}^{P} d_i \psi_i(y) \end{bmatrix}$$

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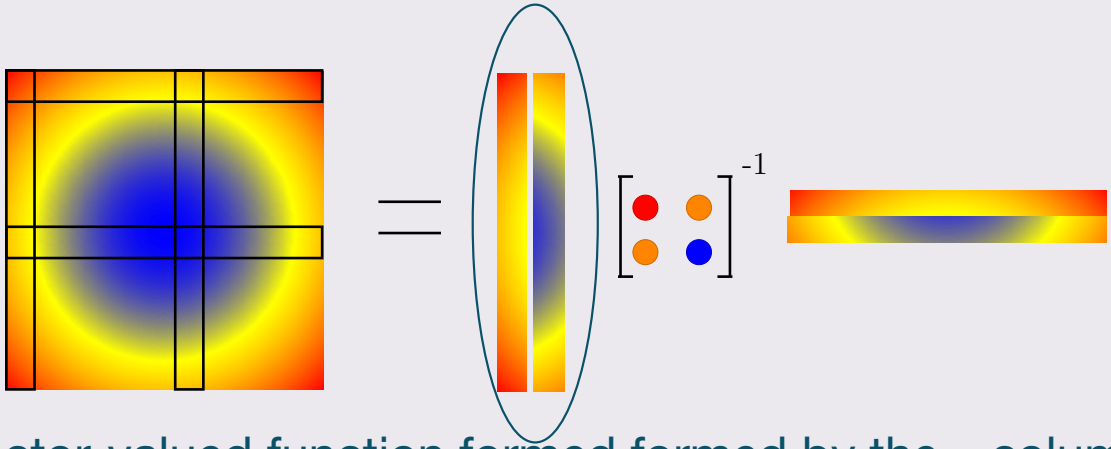
MAXVOL - cross approximation algorithm



- Consider a vector-valued function formed formed by the r columns
- Let $r = 2$

$$\underbrace{\begin{bmatrix} f(x, y_1) & f(x, y_2) \end{bmatrix}}_{\text{vector-valued function}}$$

MAXVOL - cross approximation algorithm

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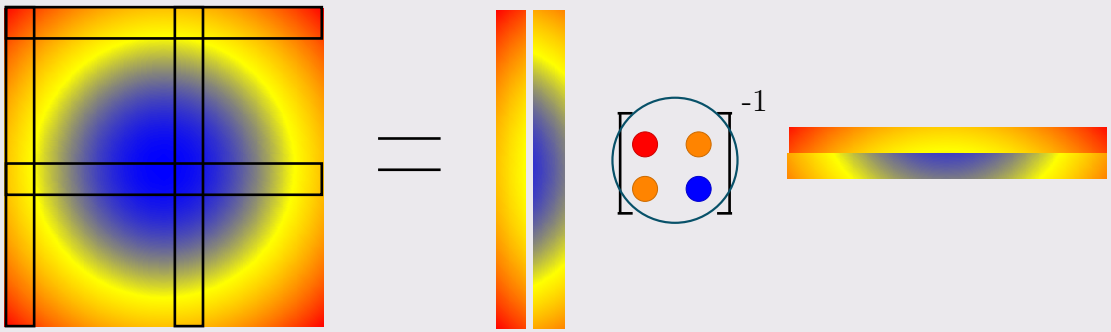

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- Seek values $[x_1, x_2]$ such that

$$\det\left(\begin{bmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_2) & f(x_2, y_2) \end{bmatrix}\right) \to \max$$

MAXVOL - cross approximation algorithm

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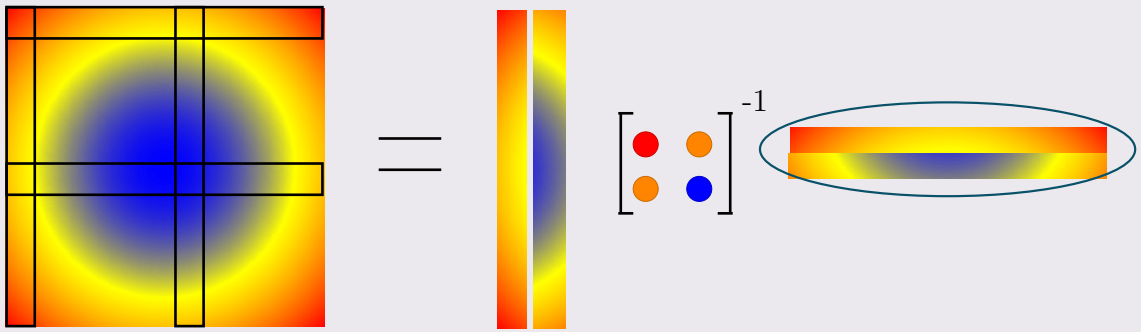

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- Fix $[x_1, x_2]$ and find new values $[y_1, y_2]$.

Maximizing the determinant

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- We extend the MAXVOL algorithm from [Oseledets 2010] to the continuous case
- Idea: find a dominant submatrix; its volume is not much smaller than that of the maximum-volume submatrix
- Dominant submatrix has property that

$$\underbrace{\begin{bmatrix} b_1(x) & b_2(x) \end{bmatrix}}_{B(x)} = \underbrace{\begin{bmatrix} f(x,y_1) & f(x,y_2) \end{bmatrix}}_{F(x)} \underbrace{\begin{bmatrix} f(x_1,y_1) & f(x_1,y_2) \\ f(x_2,y_2) & f(x_2,y_2) \end{bmatrix}^{-1}}_{\mathbf{G}}$$

$$\max_{i\in[1,\ldots,r],x} |b_i(x)| \leq 1$$

MAXVOL sub-optimality

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- SVD provides best rank r approximation

$$\|f(x,y) - f_{SVD}^{(r)}\|_{L^2}^2 = \sum_{i=r+1}^{\infty} \sigma_i^2$$

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- Matrices: error bound relative to SVD singular values [Goreinov et. al. 1997,2001]

$$\|\boldsymbol{f} - \boldsymbol{f}_{CUR}^{(r)}\|_{\text{max}} = (r+1)\sigma_{r+1}$$

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- Functions: error bound relative to functional SVD singular values [**G.**, Karaman, Marzouk 2015]

$$\|f(x,y) - f_{CUR}^{(r)}(x,y)\|_{\text{max}} \leq C(r+1)^{1+2\epsilon}\sigma_{r+1}$$

for $\epsilon > \frac{1}{2}$

High-dimensional cross approximation

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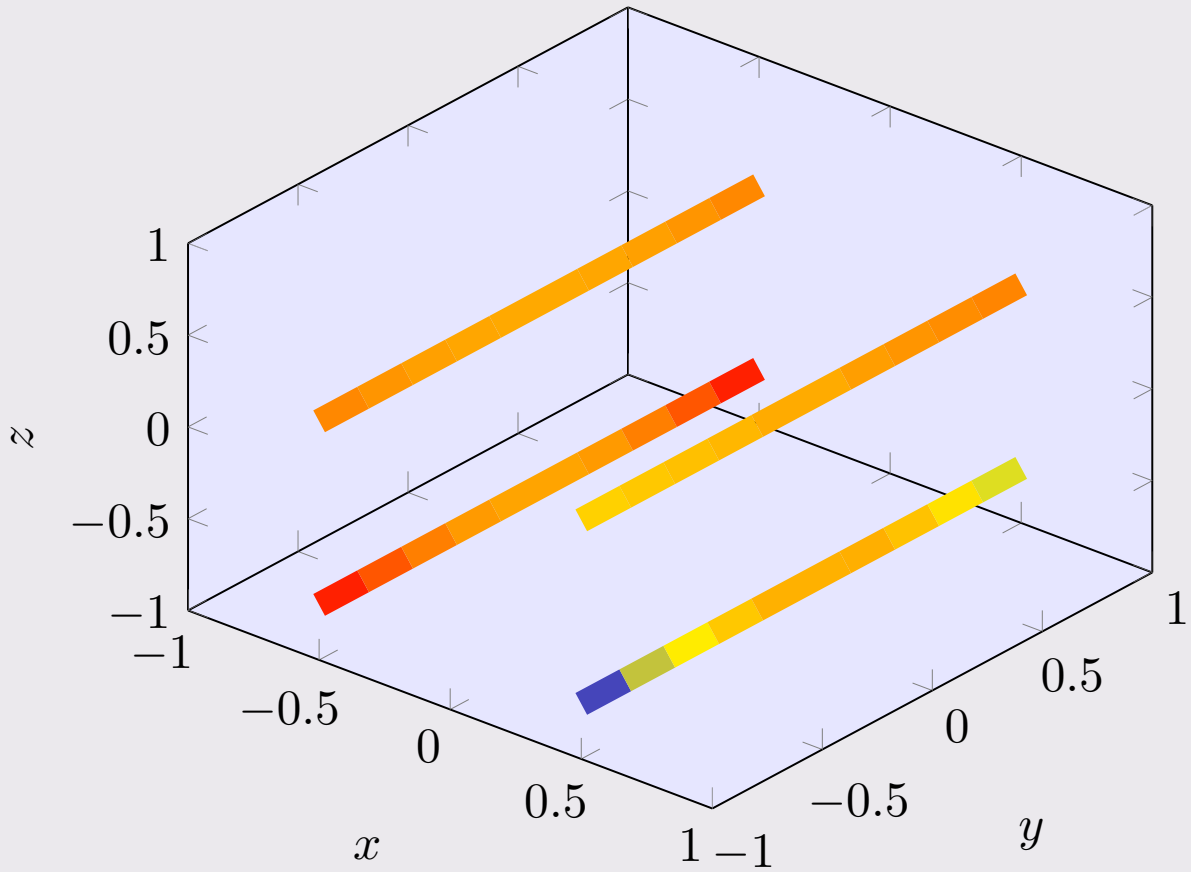
- Dimension sweeping algorithm
- Example – seeking a rank 2 $f(x, y, z)$
- Determine optimal y values from fixed x and z values
  - $[x_1, x_2] = [-0.5, 0.5]$ $[z_1, z_2] = [-0.7, 0.3]$



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1. Generate approximation of

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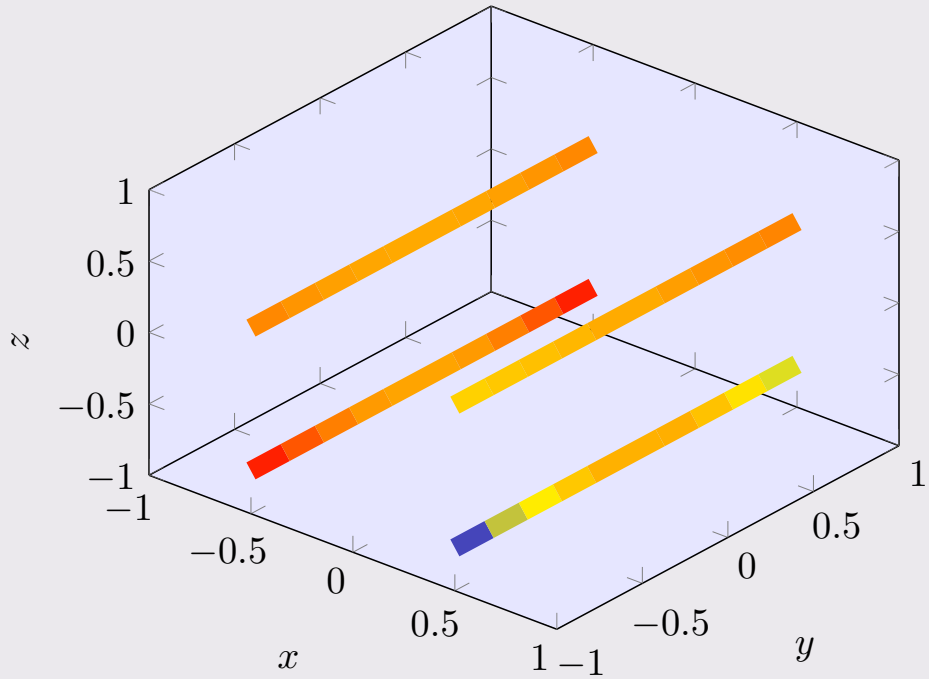

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2. Seek values $[y_1, y_2]$

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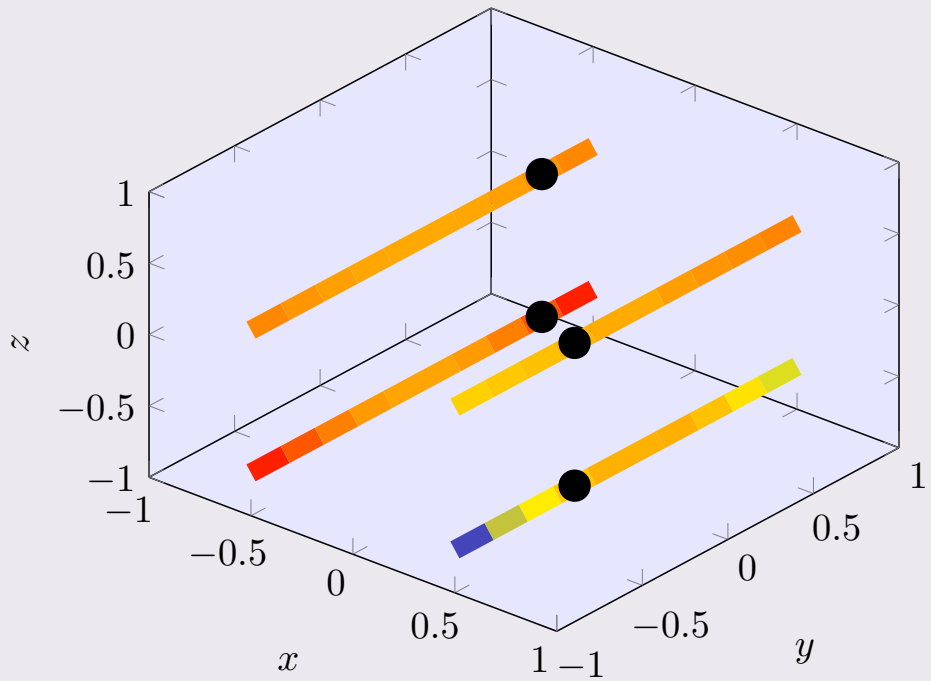

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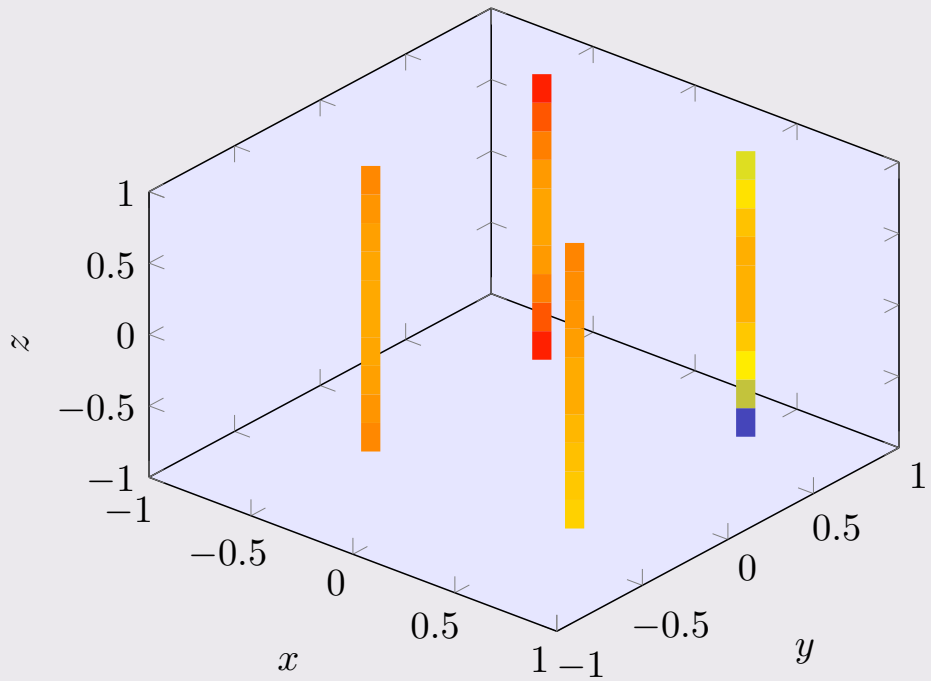
3. Fix $[x_1, x_2]$ and $[y_1, y_2]$ find new values $[z_1, z_2]$.



Comparison with QTT and DMRG - I
Benefits of adaptivity

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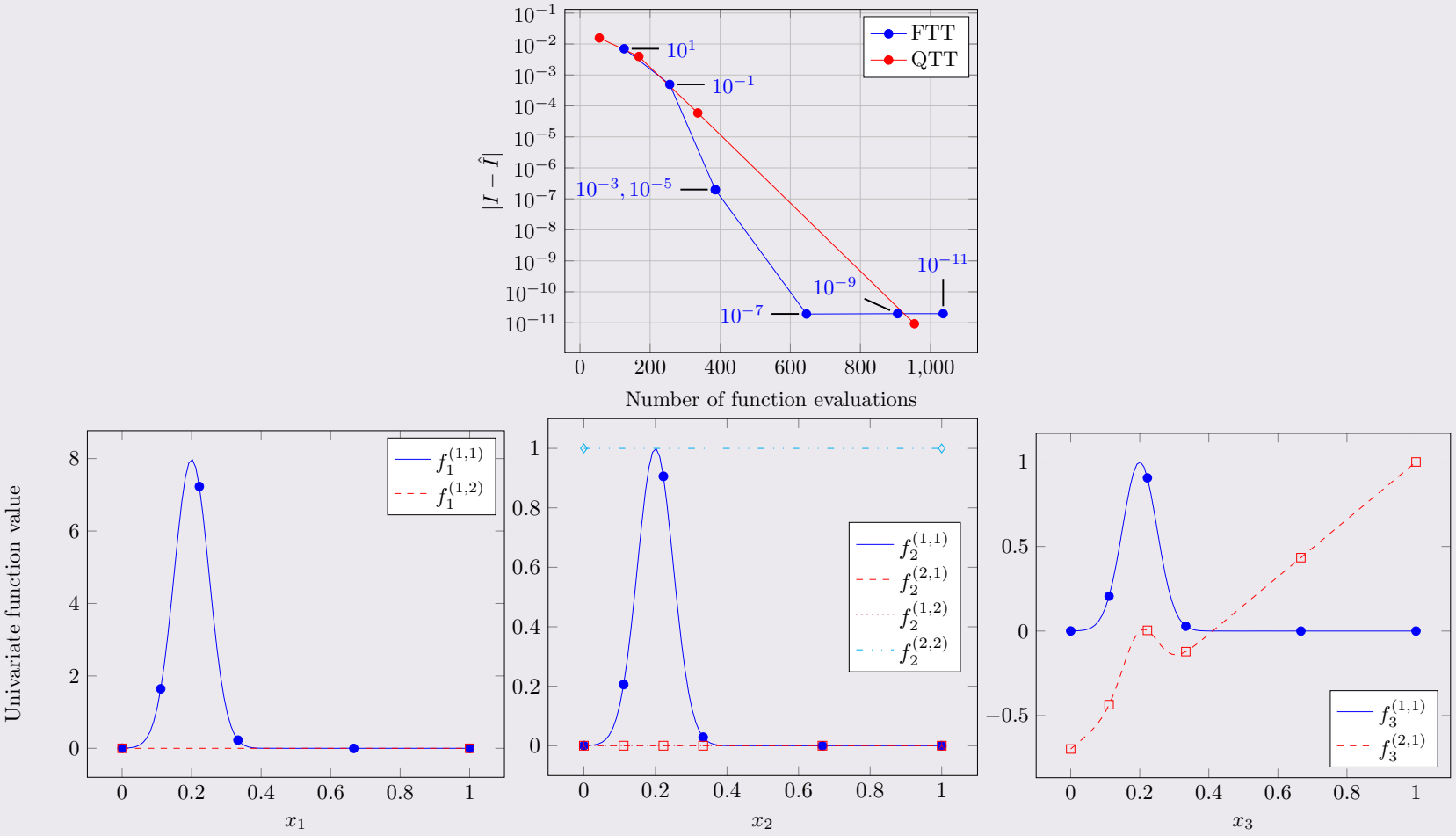

Comparison with QTT and DMRG - II
Benefits of flexibility in representation

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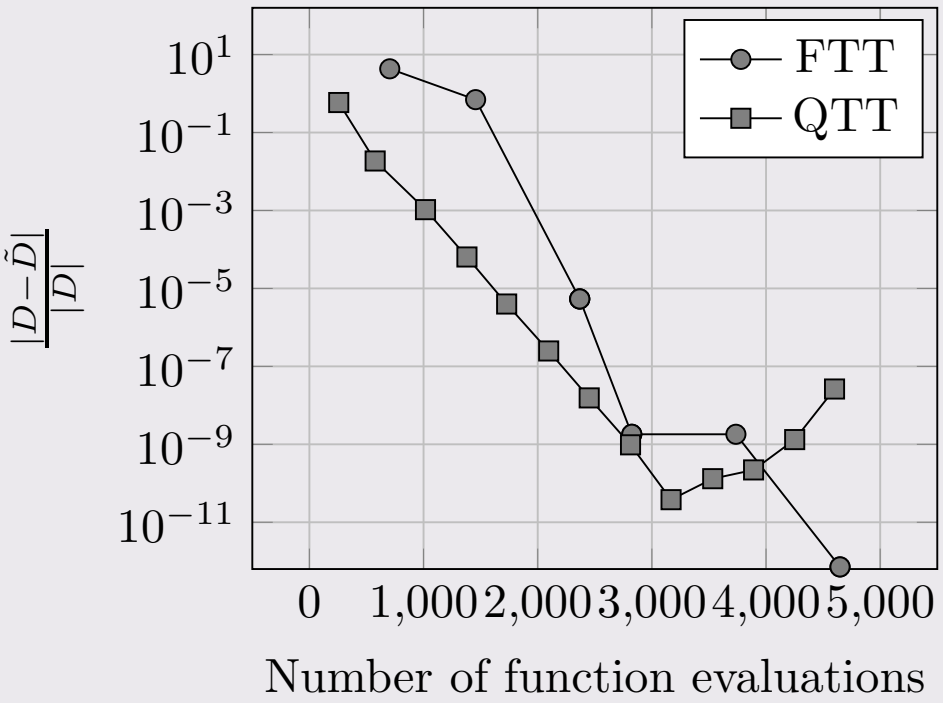
Analytic derivative vs. finite difference approximation



Supervised Learning

Constrained least squares

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$$f^* = \arg\min_{f \in \mathscr{F}_{\boldsymbol{r}}} \quad \frac{1}{n} \sum_{i=1}^{n} \left(y^{(i)} - f\left(x^{(i)} \right) \right)^2$$

$$\mathscr{F}_{\boldsymbol{r}} = \left\{ f : \mathcal{X} \to \mathbb{R} \;\middle|\; \begin{array}{l} f = \mathcal{F}_1(x_1)\mathcal{F}_2(x_2)\ldots\mathcal{F}_d(x_d) \\ \mathcal{F}_1 : \mathcal{X}_1 \to \mathbb{R}^{1 \times r_1} \\ \mathcal{F}_k : \mathcal{X}_k \to \mathbb{R}^{r_{k-1} \times r_k},\ 2 \leq k < d \\ \mathcal{F}_d : \mathcal{X}_d \to \mathbb{R}^{r_{d-1} \times 1} \end{array} \right\}$$

- Function space $\mathscr{F}_{\boldsymbol{r}}$ includes functions with ranks $< \boldsymbol{r}$.
  - Obtained by constraining certain blocks of $\mathcal{F}_k$ to be zero.
- Gradient based procedure for optimization
  - Alternating least squares
  - Quasi-Newton methods
  - Riemannian manifold

Gradient computation for single core

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Partial Derivative: $\frac{\partial f(x;\boldsymbol{a})}{\partial \boldsymbol{a_i}} = \mathcal{F}_{<k}(x_{<k}) \frac{\partial_i \mathcal{F}_k(x_k)}{\partial \boldsymbol{a_i}} \mathcal{F}_{>k}(x_{>k})$

Evaluation of left cores: $\mathcal{F}_{<k}(x_{<k}) = \mathcal{F}_1(x_1) \cdots \mathcal{F}_{k-1}(x_{k-1})$

Evaluation of right cores: $\mathcal{F}_{>k}(x_{>k}) = \mathcal{F}_{k+1}(x_{k+1}) \cdots \mathcal{F}_d(x_d)$

Gradient computation for single core

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Assumption

Each univariate function in each FT-core is independently parameterized.

Proposition: Partial derivative of core $\mathcal{F}_k$

The partial derivative of core $\mathcal{F}_k$ with respect to $\boldsymbol{a}_{kij\ell}$ is a *sparse* matrix

$$\mathbf{G}[\alpha, \beta] = \begin{cases} \frac{\partial f_k^{(ij)}(x_k)}{\partial \boldsymbol{a}_{kij\ell}} & \text{if } \alpha = i, \beta = j \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha = 1, \ldots, r_{k-1}$, and $\beta = 1, \ldots r_k$.

Gradient computation for single core

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Partial Derivative: $\frac{\partial f(x;\boldsymbol{a})}{\partial \boldsymbol{a_i}} = \mathcal{F}_{<k}(x_{<k}) \frac{\partial_i \mathcal{F}_k(x_k)}{\partial \boldsymbol{a_i}} \mathcal{F}_{>k}(x_{>k})$

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for $\alpha = 1, \ldots, r_{k-1}$, and $\beta = 1, \ldots r_k$.

Given $\mathcal{F}_{<k}(x_{<k})$, $\mathcal{F}_{>k}(x_{>k})$, and $\frac{\partial f_k^{(ij)}(x_k)}{\partial \boldsymbol{a}_{kij\ell}}$, derivative requires $\rightarrow \mathcal{O}(1)$ ops.

Computational complexity

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- Let $\mathbf{\Gamma} = \mathcal{F}_{<k}(x_{<k})$ and $\mathbf{\Theta} = \mathcal{F}_{>k}(x_{>k})$.
- Then $\partial_{\boldsymbol{i}} f(x; \boldsymbol{a}) = \mathbf{\Gamma}^T \partial_{\boldsymbol{i}} \mathcal{F}_k(x_k) \mathbf{\Theta}$
- Let $G(n)$ denote the number of ops. to evaluate the gradient of $f_k^{(ij)}$ with respect to *all* parameters.

Complexity

Computing, $\partial_{\boldsymbol{i}} f(x; \boldsymbol{a})$ for all $\boldsymbol{i}$ within the kth core requires

$$\mathcal{O}\left(\left(G(n) + n\right) r_{k-1} r_k\right)$$

operations.

FT evaluation and gradient algorithm
Forward sweep

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$$f(x_1, x_2, \ldots, x_d) = \mathcal{F}_1(x_1)\mathcal{F}_2(x_2)\cdots \begin{bmatrix} f_k^{(11)}(x_k) & \cdots & f_k^{(1r_k)}(x_k) \\ \vdots & \ddots & \vdots \\ f_k^{(r_{k-1}1)}(x_k) & \cdots & f_k^{(r_{k-1}r_k)}(x_k) \end{bmatrix} \cdots \mathcal{F}_d(x_d)$$

Stage	Task	Operation	Alg. Complexity	Storage Complexity
Stage 1	Eval	$\mathbf{\Gamma} = \mathbf{F}_1 = \mathcal{F}(x_1)$	$E(n)r_1$	r_1
	Eval all	$\partial f_{1ij\ell} = \frac{\partial f_1^{(ij)}}{\partial \boldsymbol{a}_{1ij\ell}}$	$G(n)r_1$	nr_1

FT evaluation and gradient algorithm
Forward sweep

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Stage 2	Eval	$\mathbf{F}_2 = \mathcal{F}(x_2)$	$E(n)r_1r_2$	r_1r_2
	Up. part. deriv	$\partial f_{2ij\ell} = \mathbf{\Gamma}[i]\frac{\partial f_2^{(ij)}}{\partial \boldsymbol{a}_{2ij\ell}}$	$G(n)r_1r_2$	nr_1r_2
	Update	$\mathbf{\Gamma} \leftarrow \mathbf{\Gamma}\mathbf{F}_2$	r_1r_2	-

FT evaluation and gradient algorithm
Forward sweep

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	Update	$\mathbf{\Gamma} \leftarrow \mathbf{\Gamma}\mathbf{F}_2$	r_1r_2	-
Stage k	Eval	$\mathbf{F}_k = \mathcal{F}(x_k)$	$E(n)r_{k-1}r_k$	$r_{k-1}r_k$
	Up. part. deriv	$\partial f_{kij\ell} = \mathbf{\Gamma}[i]\frac{\partial f_k^{(ij)}}{\partial \boldsymbol{a}_{kij\ell}}$	$G(n)r_{k-1}r_k$	$nr_{k-1}r_k$
	Update	$\mathbf{\Gamma} \leftarrow \mathbf{\Gamma}\mathbf{F}_k$	$r_{k-1}r_k$	-

FT evaluation and gradient algorithm
Forward sweep

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$$f(x_1, x_2, \ldots, x_d) = \mathcal{F}_1(x_1)\mathcal{F}_2(x_2)\cdots \begin{bmatrix} f_k^{(11)}(x_k) & \cdots & f_k^{(1r_k)}(x_k) \\ \vdots & \ddots & \vdots \\ f_k^{(r_{k-1}1)}(x_k) & \cdots & f_k^{(r_{k-1}r_k)}(x_k) \end{bmatrix} \cdots \mathcal{F}_d(x_d)$$

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	Update	$\mathbf{\Gamma} \leftarrow \mathbf{\Gamma}\mathbf{F}_2$	r_1r_2	-
Stage k	Eval	$\mathbf{F}_k = \mathcal{F}(x_k)$	$E(n)r_{k-1}r_k$	$r_{k-1}r_k$
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	Update	$\mathbf{\Gamma} \leftarrow \mathbf{\Gamma}\mathbf{F}_k$	$r_{k-1}r_k$	-
Stage d	Eval	$\mathbf{F}_d = \mathcal{F}(x_d)$	$E(n)r_{d-1}$	r_{d-1}
	Part. deriv	$\partial f_{dij\ell} = \mathbf{\Gamma}[i]\frac{\partial f_d^{(ij)}}{\partial \boldsymbol{a}_{dij\ell}}$	$G(n)r_{d-1}$	nr_{d-1}
	Final evaluation	$f(x) = \mathbf{\Gamma}\mathbf{F}_d$	r_{d-1}	-

FT evaluation and gradient algorithm
Backward sweep

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Stage	Task	Operation	Alg. Complexity	Storage Complexity
Stage $d-1$	Initialize	$\boldsymbol{\Theta} = \mathbf{F}_d$	-	-
	Part deriv	$\partial f_{(d-1)ij\ell} = \partial f_{(d-1)ij\ell}\boldsymbol{\Theta}[j]$	r_{d-1}	-

FT evaluation and gradient algorithm
Backward sweep

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Stage	Task	Operation	Alg. Complexity	Storage Complexity
Stage $d-1$	Initialize	$\boldsymbol{\Theta} = \mathbf{F}_d$	-	-
	Part deriv	$\partial f_{(d-1)ij\ell} = \partial f_{(d-1)ij\ell}\boldsymbol{\Theta}[j]$	r_{d-1}	-
Stage k	Up. right product	$\boldsymbol{\Theta} \leftarrow \mathbf{F}_{k+1}\boldsymbol{\Theta}$	$r_k r_{k+1}$	-
	Up. part. deriv	$\partial f_{kij\ell} = \partial f_{kij\ell}\boldsymbol{\Theta}[j]$	$r_{k-1} r_k$	-

FT evaluation and gradient algorithm
Backward sweep

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Stage	Task	Operation	Alg. Complexity	Storage Complexity
Stage $d-1$	Initialize	$\boldsymbol{\Theta} = \mathbf{F}_d$	-	-
	Part deriv	$\partial f_{(d-1)ij\ell} = \partial f_{(d-1)ij\ell}\boldsymbol{\Theta}[j]$	r_{d-1}	-
Stage k	Up. right product	$\boldsymbol{\Theta} \leftarrow \mathbf{F}_{k+1}\boldsymbol{\Theta}$	$r_k r_{k+1}$	-
	Up. part. deriv	$\partial f_{kij\ell} = \partial f_{kij\ell}\boldsymbol{\Theta}[j]$	$r_{k-1} r_k$	-
Stage 1	Up. part. deriv	$\partial f_{1ij\ell} = \partial f_{1ij\ell}\boldsymbol{\Theta}[j]$	r_1	-

Alg. complexity: $\mathcal{O}\left(dr^2\left(G(n)+E(n)\right)\right)$

Real data

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- Kernel expansion for each univariate function
- CV over number of basis functions
- Compare with other parametric approaches
- Data and results for non-FT algs from Kandasamy and Yu (2016).

Data Set (d)	FT	RR	LASSO	LAR
Housing (12)	**0.416836**	95.60708	0.44515	0.84410
Galaxy (20)	**0.002675**	0.13902	0.02392	1.02315
Skillcraft (18)	**0.556605**	0.70910	0.66496	1.0048
CCPP (59)	**0.073086**	0.07641	0.07395	1.04527
Airfoil (41)	**0.42956**	0.53187	0.51986	1.00910
Speech (21)	**0.02571**	0.07392	0.07303	0.73916

Use compression within dynamic programming

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- Value iteration – fixed point interation
  - Iterate the Bellman operator:

$$T_h(J) = \min_{u_h(x) \in \mathcal{U}^h} \left[g(x, u_h(x)) + \gamma \sum_{y \in \mathcal{X}^h} p(x, y | u_h(x)) J(y) \right].$$

$$J^{k+1} = T_h(J^{k+1})$$

- Policy iteration – generate a sequence of policies
  1. Update policy

$$\mu_{k+1} = T_h(J_{\mu_k})$$

  2. Update Cost: Solve

$$J_{\mu_{k+1}} = T_h(J_{\mu_{k+1}})$$

- Multigrid
  - Observation: coarse grid solutions converge quickly
  - Algorithm: Solve problem on sequentially finer grids

Convergence result for the approximation

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Theorem

Let J^{k+1} denote a sequence of functions obtained through the iteration

$$J^{(k+1)} = \texttt{FT-cross}\left(T_h(J^{(k+1)}), \epsilon\right).$$

If $T_h(J)$ is a contraction mapping and the sequence $J^{(k+1)}$ has bounded ranks then FT-based VI convergences according to

$$\lim_{k\to\infty} \|J^{(k)} - J_h^*\| \leq \frac{\varepsilon\rho}{1-\gamma},$$

- Convergence is in the norm for which $T_h(J)$ is a contraction mapping
- Bounded ranks ensures that the cross-approximation algorithm can obtain an approximation to a tolerance of ϵ

Underactuated perching glider
Cory 2008

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$$J^*(z) = \min_{u(t)} \mathbb{E}\Big[\int_0^T \bar{x}^T \mathbf{Q}\bar{x} + 0.1u^2 dt + \bar{x}(T)^T \mathbf{Q}_f \bar{x}(T)\Big]$$

subject to nonlinear ODEs

$$\begin{aligned} m\ddot{x} &= -f_w s_\theta - f_e s_{\theta+\phi} + \sigma dw \\ m\ddot{y} &= f_w c_\theta + f_e c_{\theta+\phi} - mg + \sigma dw \\ I\ddot{\theta} &= -f_w l_w - f_e(l c_\phi + l_e) + \sigma dw \end{aligned}$$

$$\mathbf{x}_w = [x - l_w c_\theta, y - l_w s_\theta],$$

$$\mathbf{x}_w = [\dot{x} + l_w \dot{\theta} s_\theta, \dot{y} - l_w \dot{\theta} c_\theta]$$

$$\mathbf{x}_e = [x - l c_\theta - l_e, c_{\theta+\phi}, y - l s_\theta - l_e s_{\theta+\phi}]$$

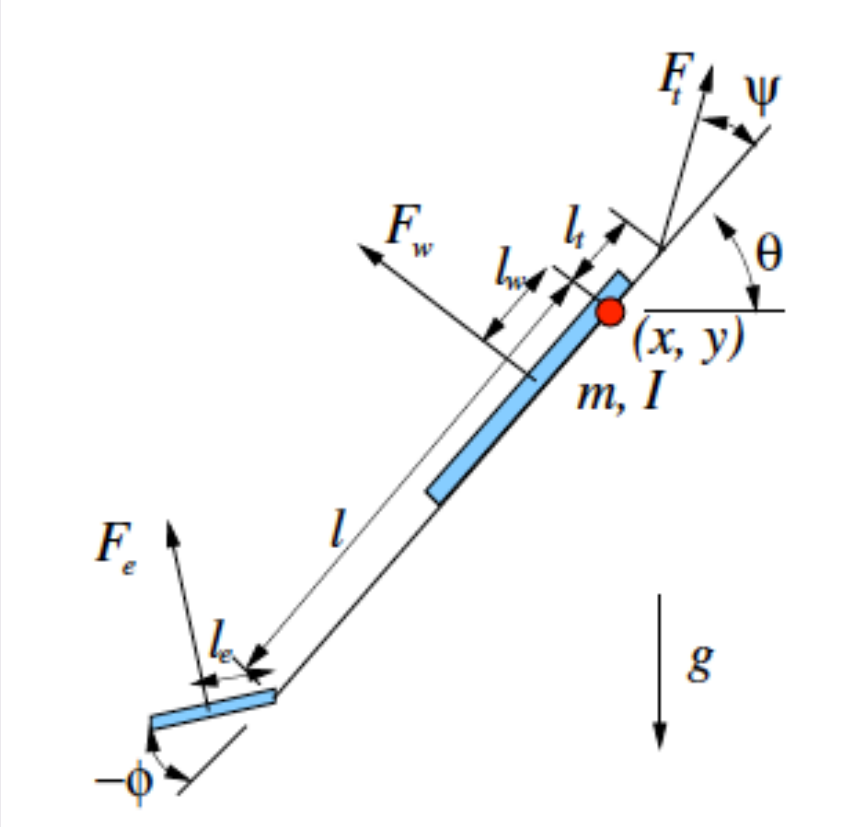
$$\dot{\mathbf{x}}_e = [\dot{x} + l\dot{\theta} s_\theta + l_e(\dot{\theta} + u) s_{\theta+\phi}, \dot{y} - l\dot{\theta} c_\theta - l_e(\dot{\theta} + u) c_{\theta+\phi}]$$

$$\alpha_w = \theta - \tan^{-1}(\dot{y}_w, \dot{x}_w),$$

$$\alpha_e = \theta + \phi - \tan^{-1}(\dot{y}_e, \dot{x}_e)$$

$$f_w = \rho S_w |\dot{\mathbf{x}}_w|^2 \sin(\alpha_w),$$

$$f_e = \rho S_e |\dot{\mathbf{x}}_e|^2 \sin(\alpha_e)$$



Simulation and compression

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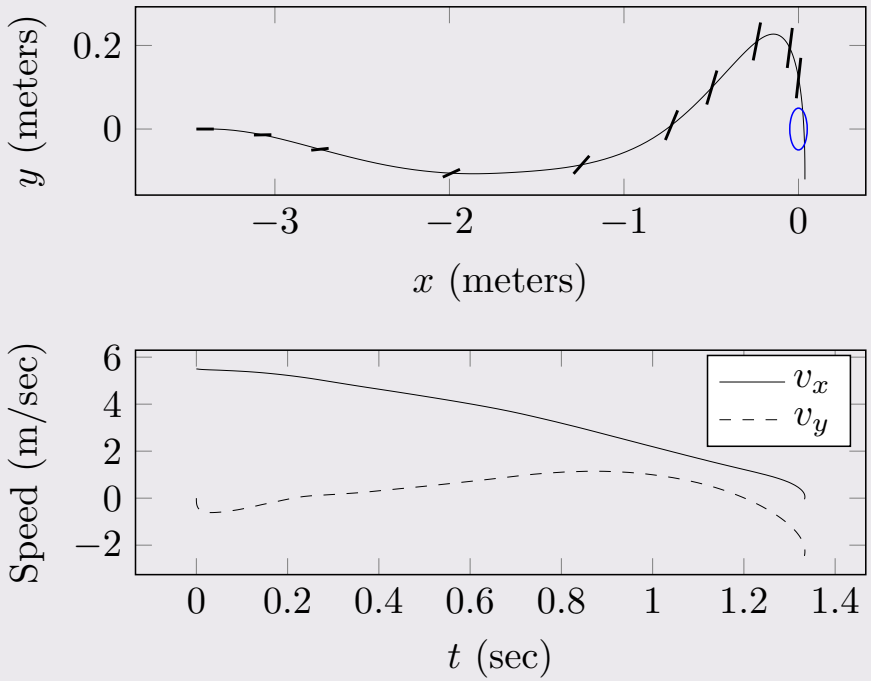
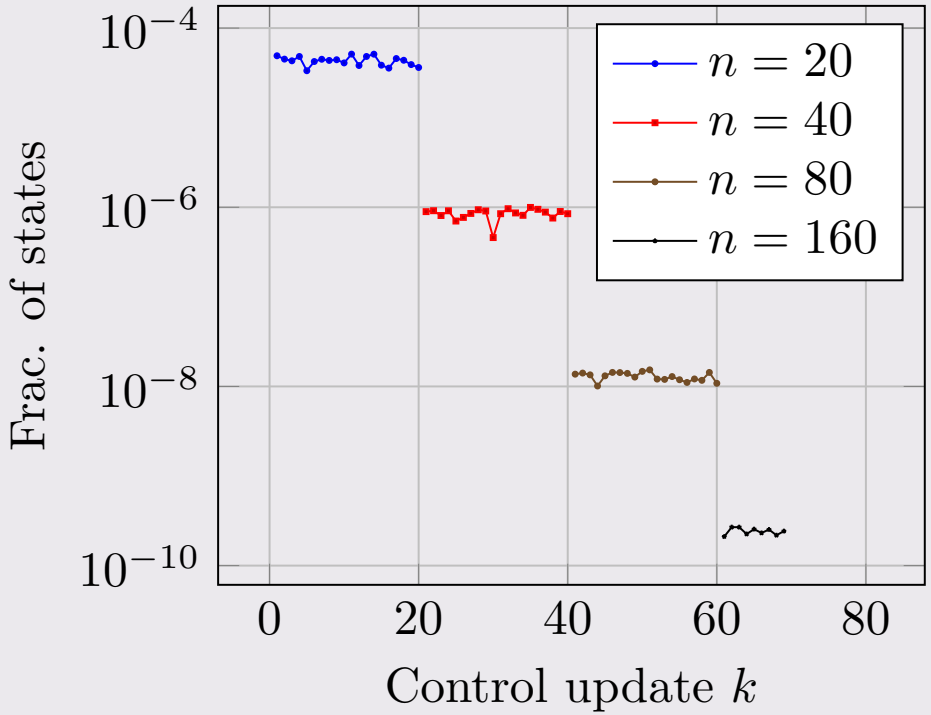
- One-way multigrid [Chow and Tsitsiklis 1991]





Experimental quadcopter

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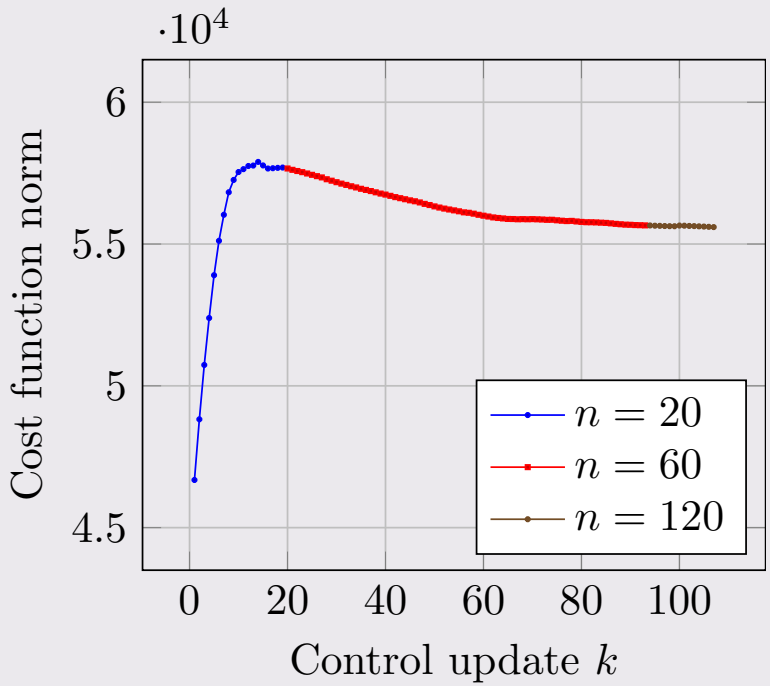
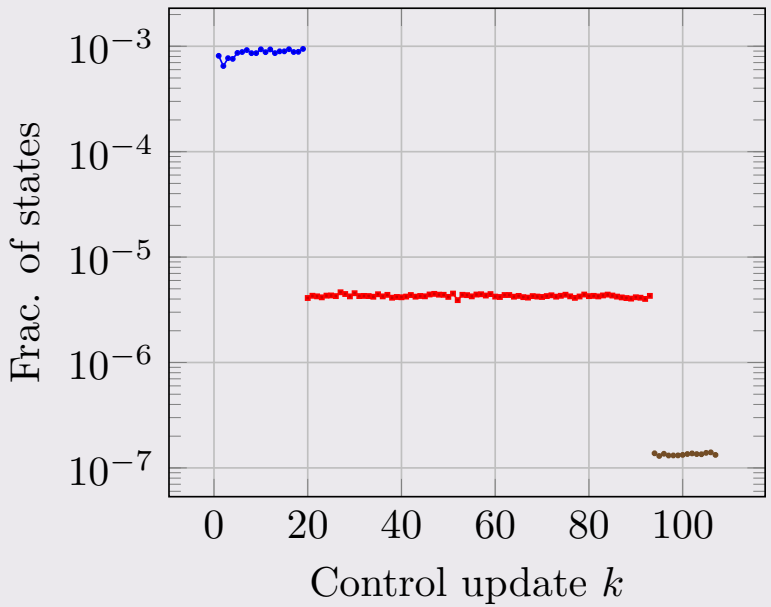
- Goal: maneuver a quadcopter quickly through a window
- Dynamical system has six states and three controls, non-affine control
- Motion capture cameras for state estimation
- Optimal cost function computed offline (1 MB), full controller would require 24 TB





Conclusions

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Summary:

- Tensor-based compression mitigates curse of dimensionality in stochastic optimal control problems
- Works with nonlinear and non-affine-control systems
- Up to four orders of magnitude reduction
- Extensions to real-time systems and state estimation

Related papers: (available on `alexgorodetsky.com` and Arxiv)

- Gorodetsky A., Karaman, S., and Marzouk Y. M. Function-train: a continuous analogue of the tensor-train decomposition. (2016)
- Gorodetsky A., Karaman, S., and Marzouk Y. M. High-dimensional stochastic optimal control using continuous tensor decompositions. (2016)

Code:

- Compressed Continuous Computation (C^3) Library: `github.com/goroda/Compressed-continuous-computation`
- Compressed stochastic optimal control (C^3SC): `github.com/goroda/c3sc`

Acknowledgements

Collaborators:

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- John Jakeman
- Sertac Karaman
- Youssef Marzouk
- Ezra Tal

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Backup Slides

Advantages of the FT format

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- Nonlinear parameterizations: can capture local effects
- Sparse representations:
  - Example: Rank-2 additive function

$$f(x_1, \ldots, x_d) = f_1(x_1) + \cdots + f_d(x_d)$$

  - FT format:

$$f(x_1, x_2, \ldots, x_d) = [f_1(x_1)\ 1] \begin{bmatrix} 1 & 0 \\ f_2(x_2) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 \\ f_d(x_d) \end{bmatrix}$$

  - TT storage requirement: $4p(d-2)$ floats
  - FT storage requirement: $d(p+3)-4$ floats
  - In the limit, FT requires almost 4 times less storage
  - Difference is more striking for higher order interactions

MAXVOL algorithm

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1. Find $\boldsymbol{x} = [x_1, x_2]$ such that cross matrix is nonsingular
2. Form $B(x) = F(x)\mathbf{G}$
3. Find $i^*, x^* = \arg \max_{i \in [1,\ldots,r], x} [b_i(x)]$
   - If basis functions of b_i are orthonormal polynomials then solve eigenvalue problem!
   - If basis functions of b_i are piecewise polynomials then find extremum of each piece
   - If some other representation, can use brute force and discretize to find maximum
4. $b_{\text{max}} = b_{i^*}(x^*)$
5. If $|b_{\text{max}}| \geq 1 \rightarrow \boldsymbol{x}[i^*] = x^*$ (switch "rows")
6. Go to step 2.

Rank adaptation

- Use FT rounding and CV
- Increase ranks until either
  - Rounding lowers all ranks
    $\rightarrow$ data not informative enough
  - CV error increases
    $\rightarrow$ avoid overfitting
- Rounding threshold is a parameter
  - Similar to regularization
  - Relation to other approaches?