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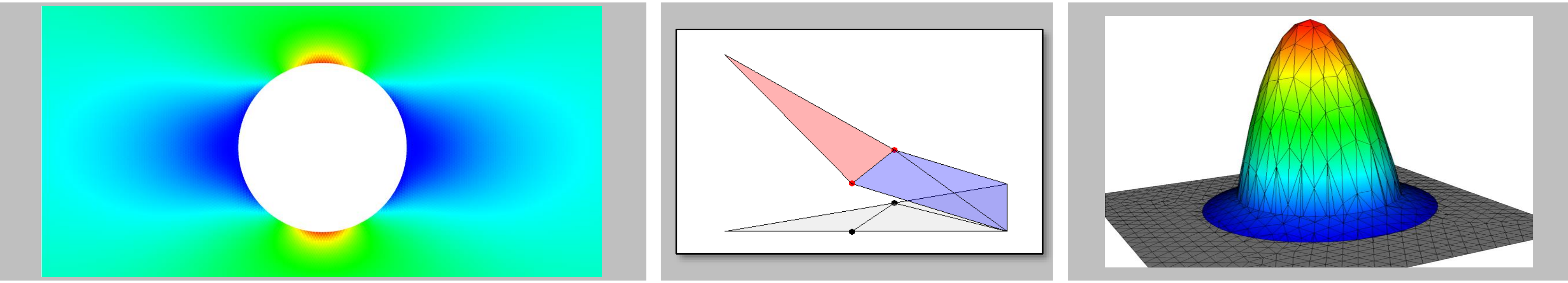

Assessment of Accuracy and Conditioning of Enriched Finite Element Discretizations

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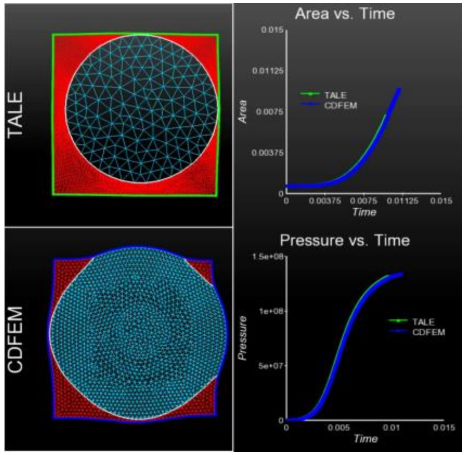
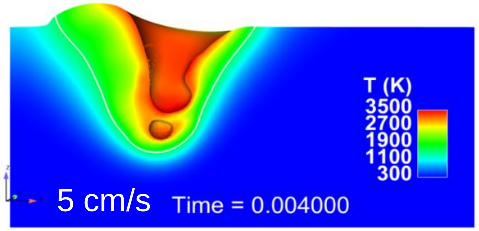
Motivation

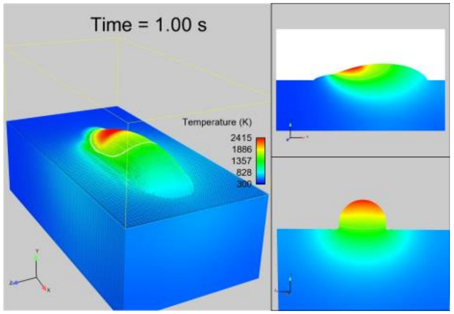
Numerous problems with moving or topologically complex interfaces with discontinuous physics and fields



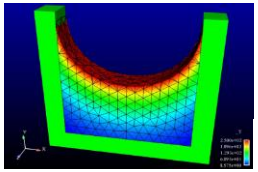
Conductive burn of energetic materials



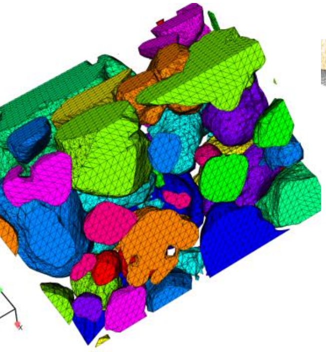
Laser welding

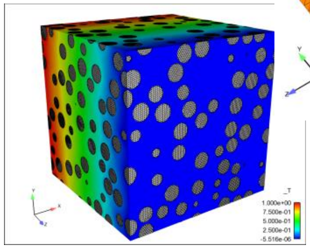


Additive Manufacturing

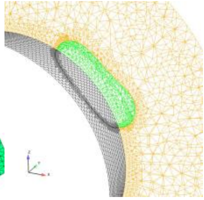


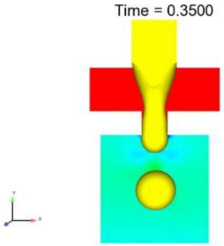
Material death

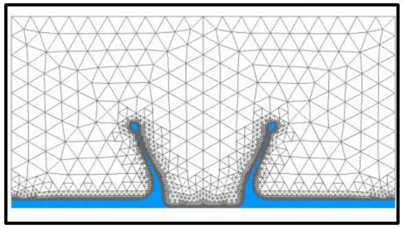




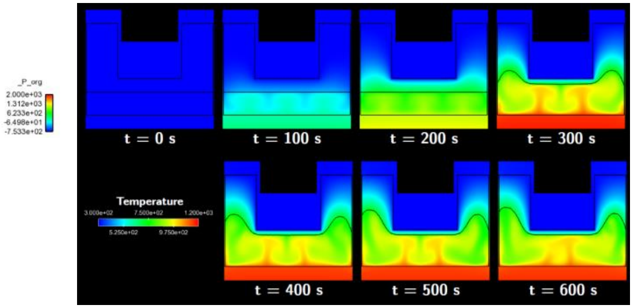
Transport in topologically complex domains including composite energetic materials and batteries





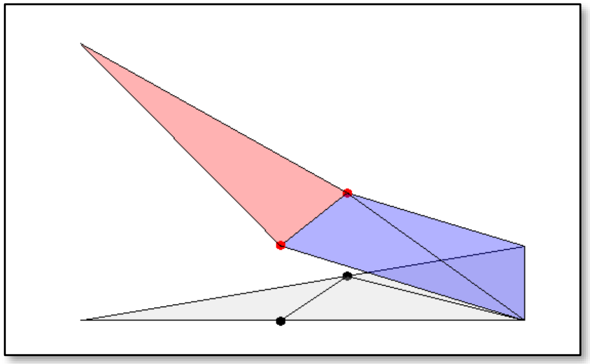


Capillary Hydrodynamics

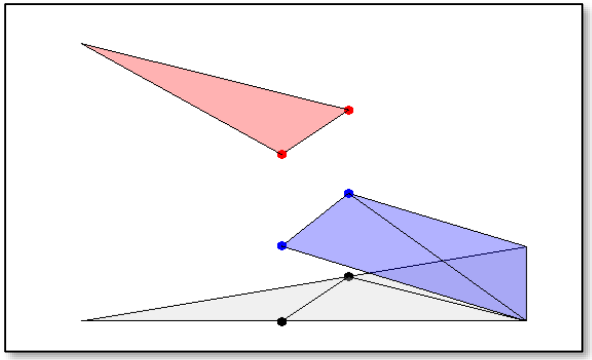


Organic Material Decomposition (OMD) with coupled porous and low Ma flow

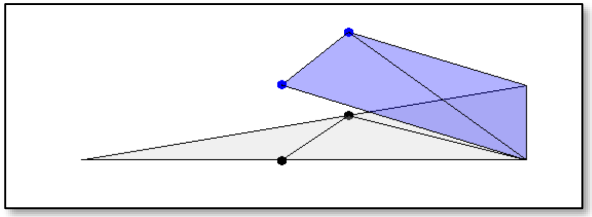
Interfacial Discontinuities in Sandia Applications



- Weak Discontinuities
  - Temperature due to discontinuous conductivity
  - Velocity due to discontinuous viscosity
  - Pressure due to discontinuous body forces
  - Displacement due to discontinuous moduli



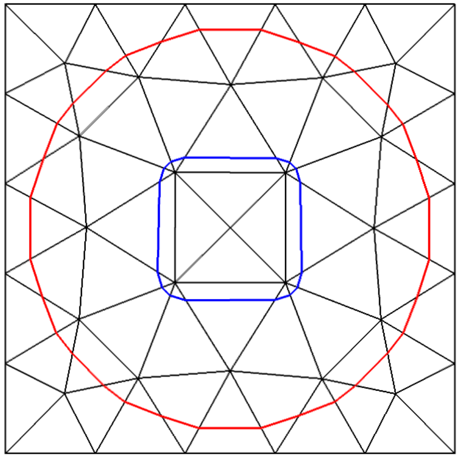
- Strong Discontinuities
  - Pressure due to capillary forces
  - Velocity due to phase change
  - Displacement due to fracture

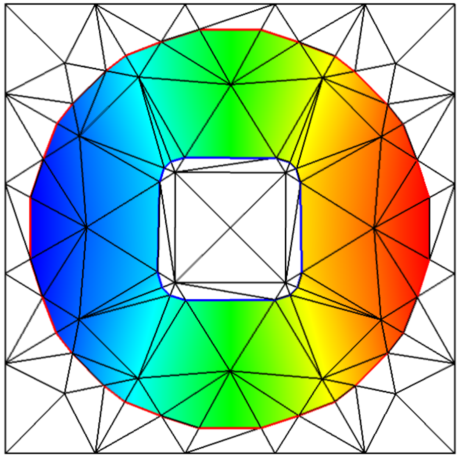


- One-sided Fields
  - Transport in evolving domains
  - Multiphysics transport

Conformal Decomposition Finite Element Method (CDFEM)

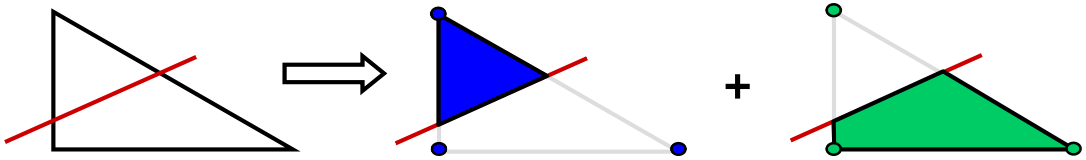
- Simple Concept (Noble, et al. 2010)
  - Use one or more level set fields to define materials or phases
  - Decompose non-conformal elements into conformal ones
  - Obtain solutions on conformal elements
- Related Work
  - Li et al. (2003) FEM on Cartesian Grid with Added Nodes
  - Ilinca and Hetu (2010) Finite Element Immersed Boundary
  - S. Soghrati and P.H. Geubelle (2012) Interface Enriched Finite Element
- Properties
  - Supports wide variety of interfacial conditions (identical to boundary fitted mesh)
  - Avoids manual generation of boundary fitted mesh
  - Supports general topological evolution (subject to mesh resolution)
- Similar to finite element adaptivity
  - Uses standard finite element assembly including data structures, interpolation, quadrature



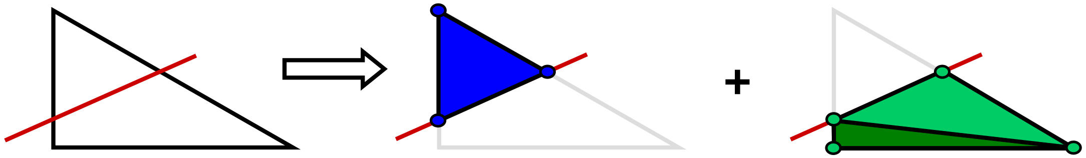


How Does CDFEM Compare to "Traditional" Enriched FEM?

- Enrichment via additional DOFs at added nodes vs. additional DOFs at parent nodes
- Enrichment (hierarchical) vs. replacement of cut element shape functions
- Subelements used to replace (or augment) approximation space vs. subelements used only for integration
  - Does this mean CDFEM is subject to mesh quality requirements that other enriched FEM is immune from?
- How does the accuracy compare?
- How does the conditioning of the resulting matrices compare?



XFEM Heaviside Approximation



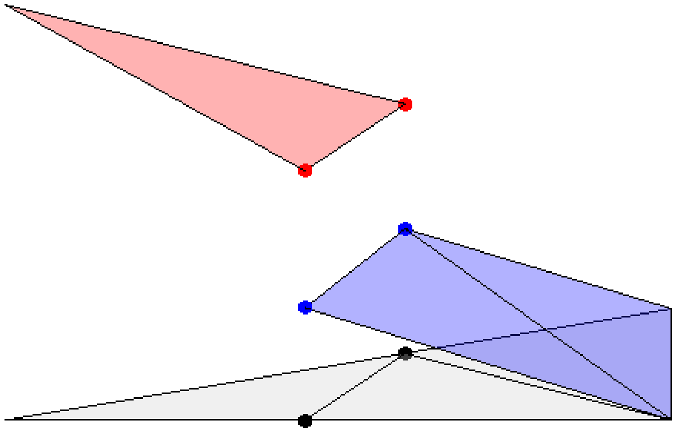
Discontinuous CDFEM Approximation

XFEM - CDFEM Requirements Comparison

	XFEM	CDFEM
Volume Assembly	Conformal subelement integration, specialized element loops to use modified integration rules	Standard Volume Integration
Surface Flux Assembly	Specialized volume element loops with specialized quadrature	Standard Surface Integration
Phase Specific DOFs and Equations	Different variables present at different nodes of the same block	Block has homogenous dofs/equations
Dynamic DOFS and Equations	Require reinitializing linear system	Require reinitializing linear system
Various BC types on Interface	Dirichlet BCs normally cannot be strongly enforced	Standard Techniques available

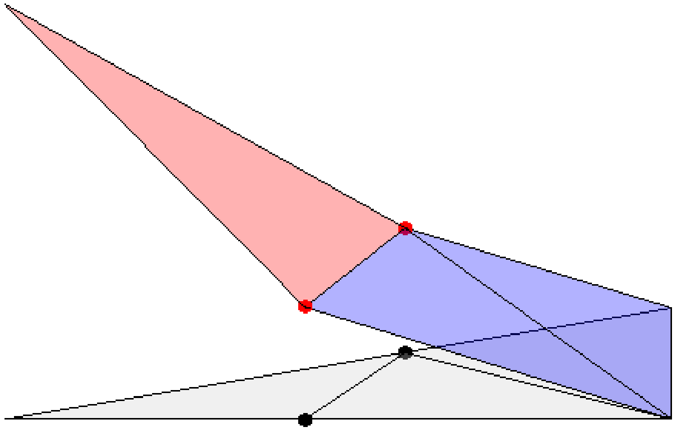
Weak Discontinuity Candidate 1: Discontinuous CDFEM with Weakly Enforced Tie Conditions

- Nodes are added at intersection between interface and background element edges
- Double-valued DOFs are used at added nodes
- Capable of capturing strong discontinuities along interface
- For problems with weak discontinuities, Nitche-type BCs are applied to weakly restore C0 continuity



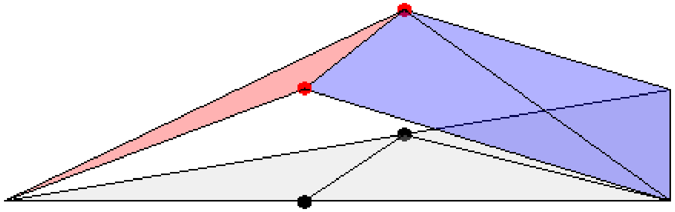
Weak Discontinuity Candidate 2: C0 Continuous CDFEM

- Single-valued DOFs are used at added nodes
- Equivalent to constraining double-valued DOFs to match across interface
- Capable of capturing weak discontinuities along interface



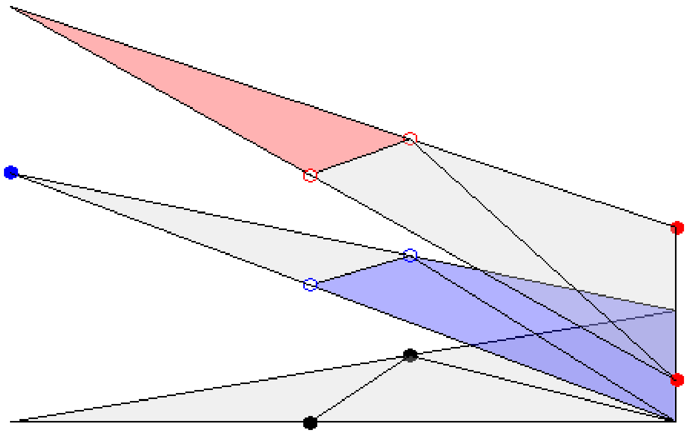
Weak Discontinuity Candidate 2a: Hierarchical Interface Enriched FEM

- Single-valued DOFs are used at added nodes in hierarchical manner, adding to approximation of background element
- Capable of capturing weak discontinuities along interface
- Same approximation space as C0 continuous CDFEM



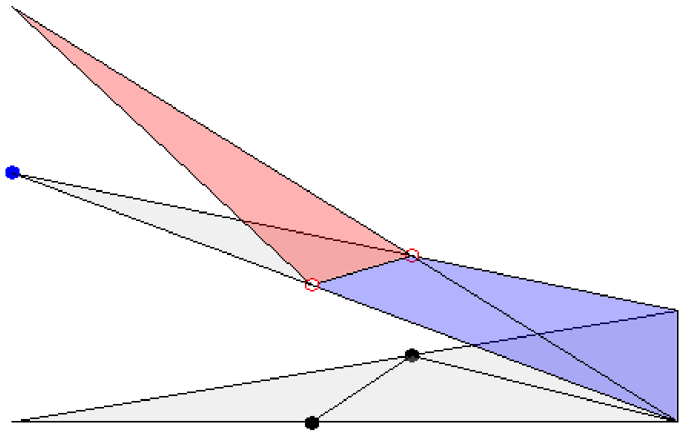
Weak Discontinuity Candidate 3: Heaviside Enriched XFEM with Weakly Enforced Tie Conditions

- Double-valued DOFs at added nodes are constrained to satisfy linear edge constraints effectively replacing the DOFs with Heaviside enriched DOFs at parent nodes
- Capable of capturing strong discontinuities along interface
- For problems with weak discontinuities, Nitche-type BCs are applied to weakly restore C0 continuity

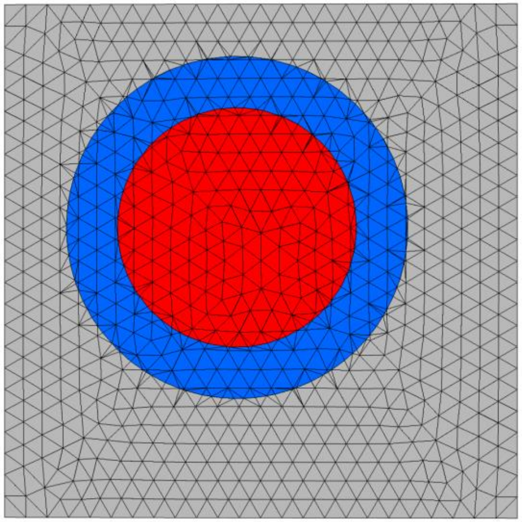


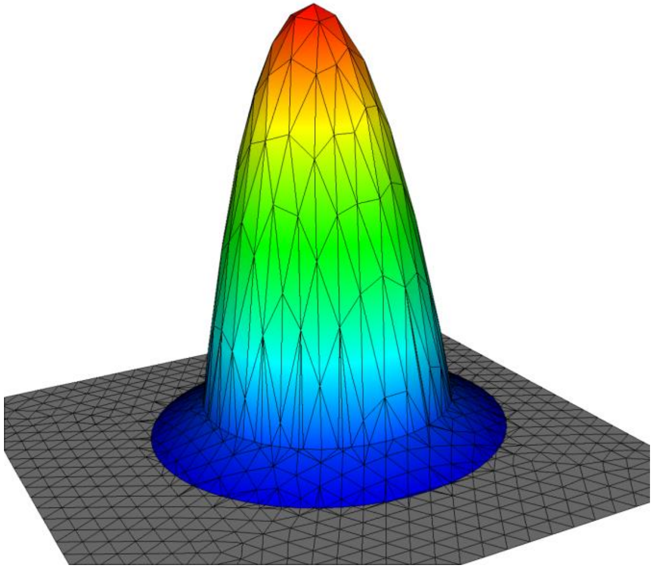
Weak Discontinuity Candidate 4: C0 Enriched, Piecewise Linear XFEM

- Double-valued DOFs at added nodes are constrained to satisfy linear edge constraints effectively replacing the DOFs with Heaviside enriched DOF at parent nodes on one side of the interface
- Capable of capturing weak discontinuities along interface
- Unlike ridge-enrichment, resulting approximation is piecewise linear



Weak Discontinuity Test Problem

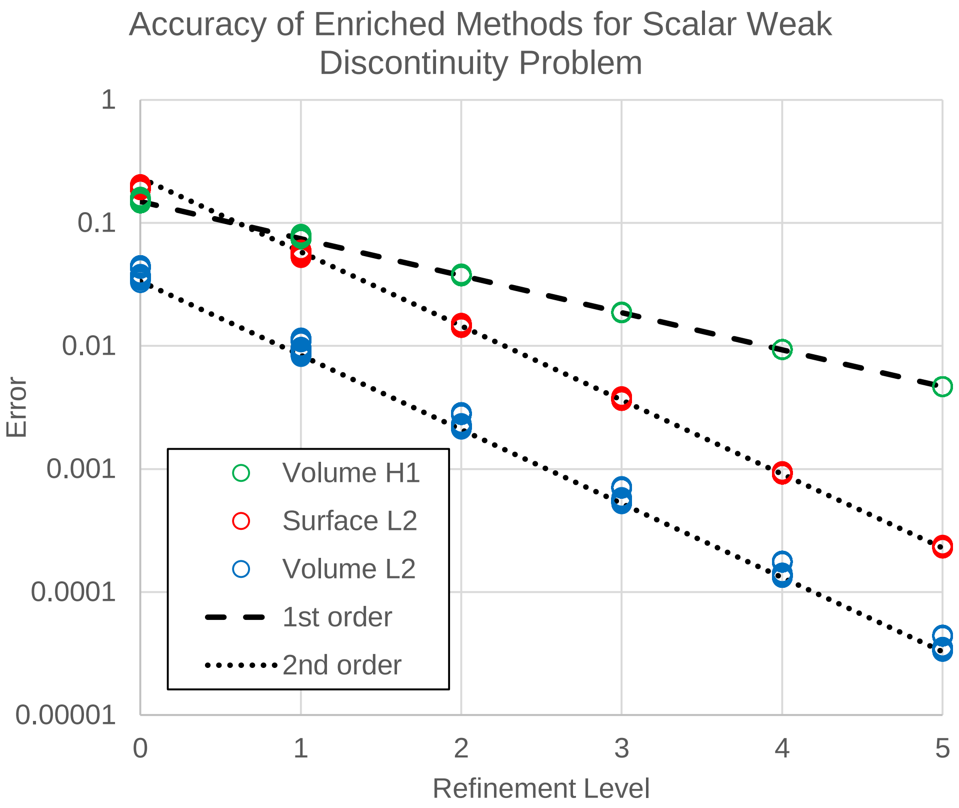




- Scalar Diffusion Problem with Weak Discontinuity
  - Radially symmetric problem solved in x-y
  - 100 realizations of randomly place concentric cylinders to explore impact of element quality
- Measured Quantities
  - L_2 and H_1 error in domain
  - L_2 error on boundary
  - Condition number of Jacobi-preconditioned linear system of equations

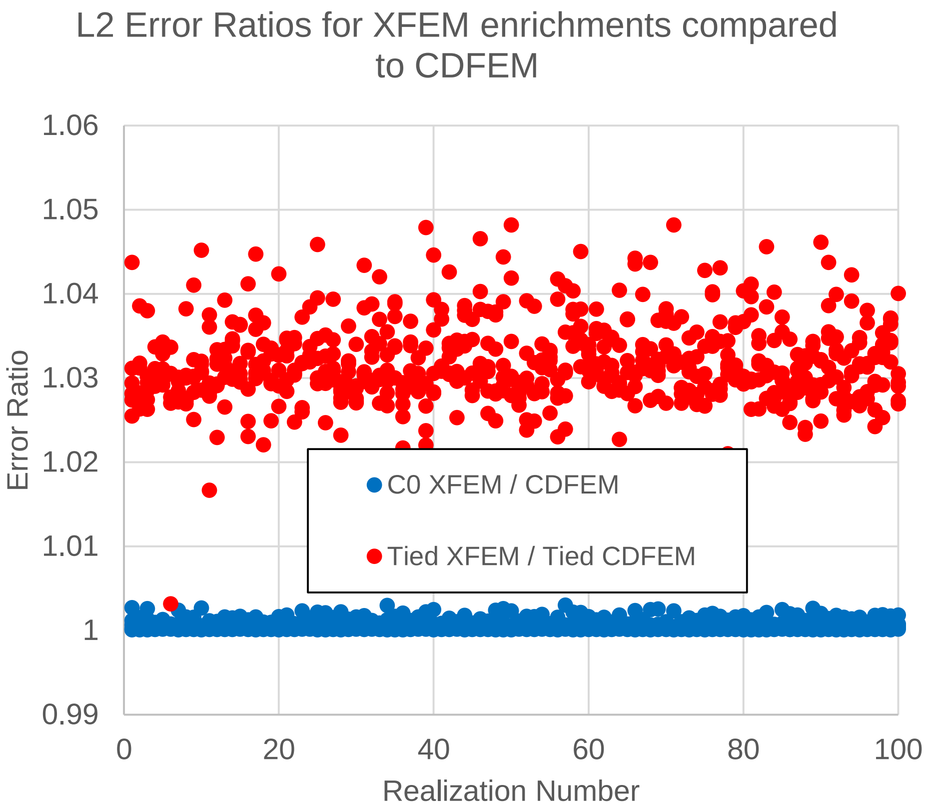
$$T(r) = \begin{cases} A_i r^2 + C_i & r \leq R \\ A_o r^2 + B_o \ln(r) + C_o & r > R \end{cases}$$

Weak Discontinuity Test Problem Accuracy



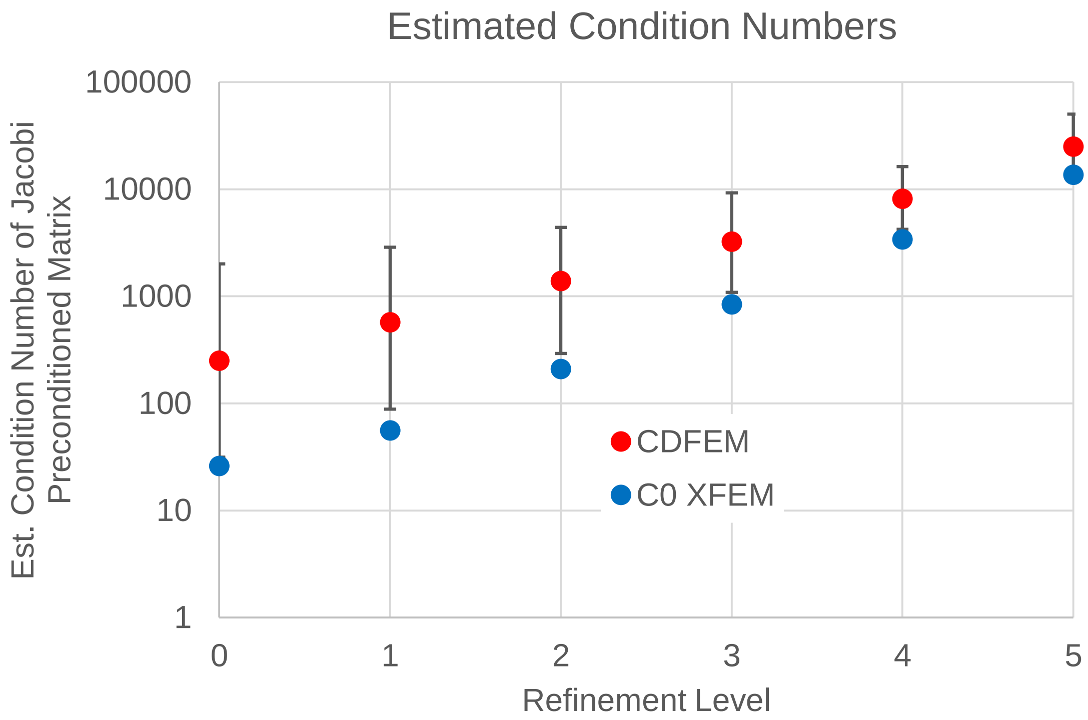
- Comparison
  - All methods converge optimally in L_2 and H_1
  - L_2 errors slight lower for tied, discontinuous formulations compared to C_0 formulations

Weak Discontinuity Test Problem Accuracy



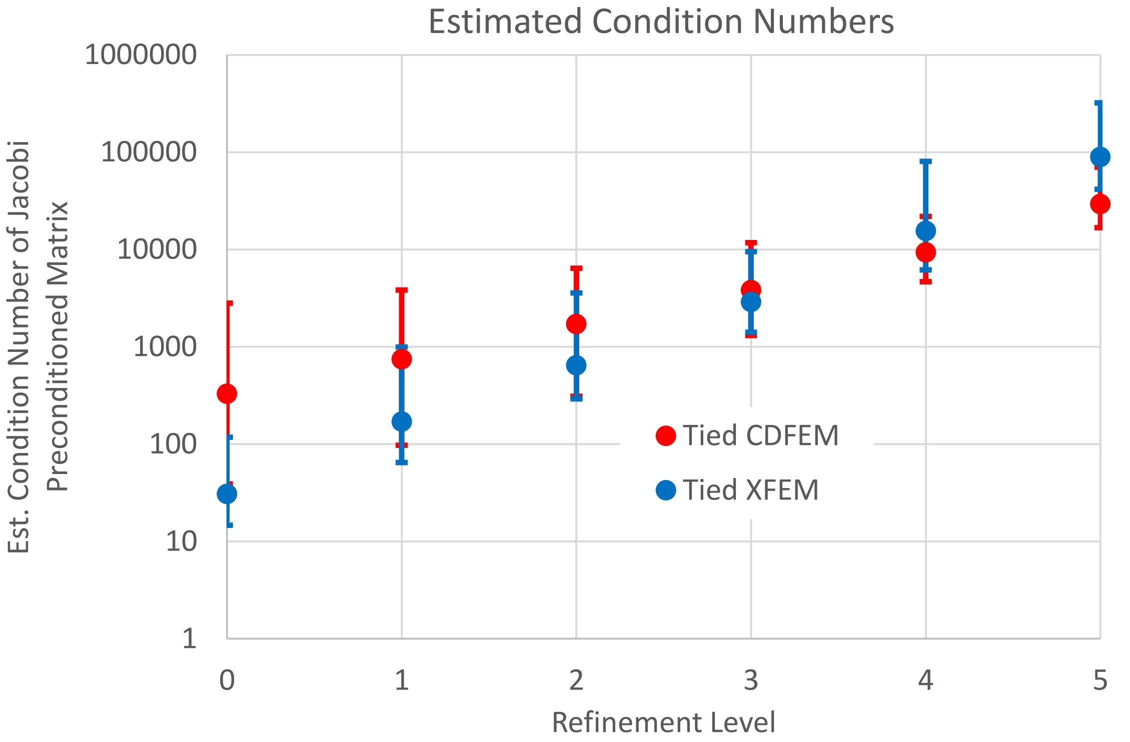
- Comparison
  - L_2 errors for CDFEM always less than or equal to corresponding XFEM
  - H_1 errors within 2% of one another for all formulations
- Conclusions
  - All enriched formulations converge optimally
  - No penalty for poor quality CDFEM elements in terms of accuracy
    - Incorrect to state that subelement quality does not matter because it is not used for the discretization. The fact is that the subelement quality just doesn't matter.

Weak Discontinuity Test Problem Conditioning – Continuous Enrichments



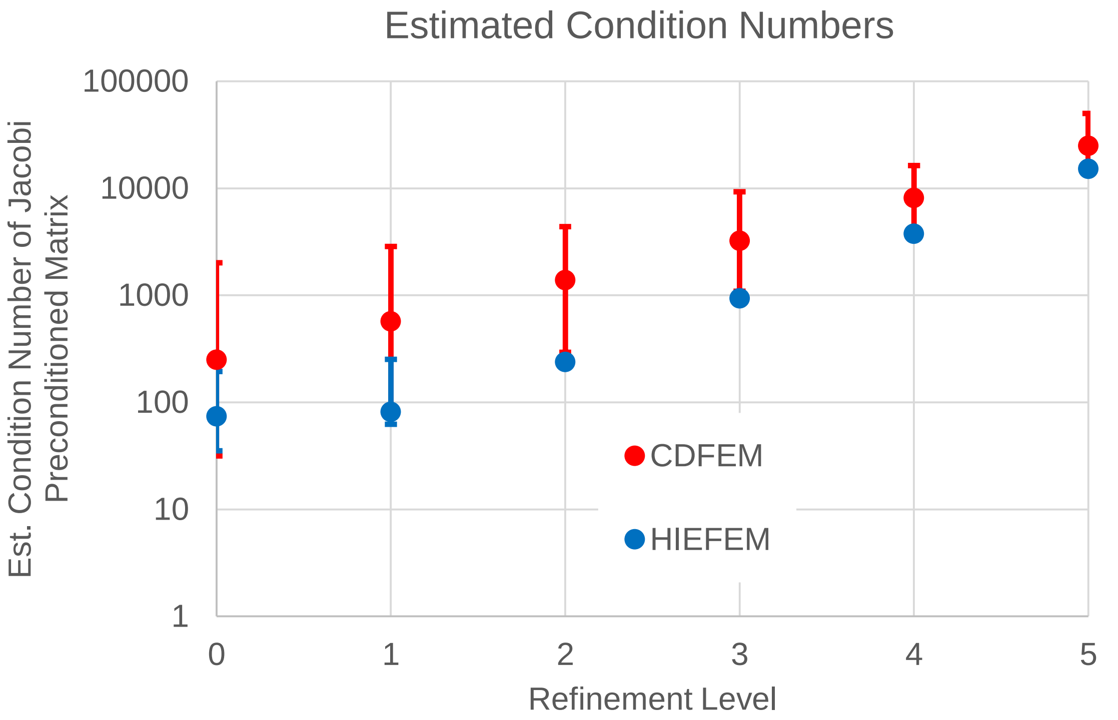
- C_0 XFEM produces much lower condition numbers that are much less dependent on random center location
  - Presumably less dependent on the minimum cut edge size
- Condition Number scales similarly with problem size

Weak Discontinuity Test Problem Conditioning – Discontinuous Enrichments



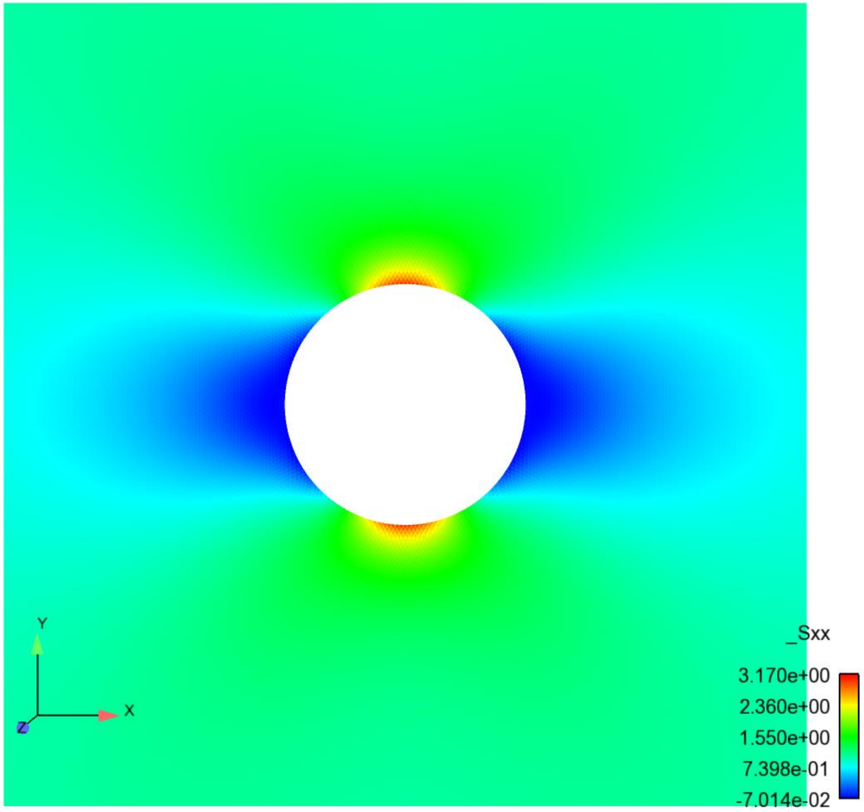
- Discontinuous, Tied versions produce similar (large) sensitivity to interface location
- Condition Number scales somewhat stronger with problem size for XFEM
- Details of XFEM discretization might lead to very different results

Weak Discontinuity Test Problem Conditioning – CDFEM vs HIEFEM



- HIFEM shows moderate sensitivity to interface location that vanishes with increased problem size
- **Matrix conditioning is highly sensitive to formulation details**
- **Accuracy is relatively insensitive to formulation details**

Linear Elastic Response of Infinite Plate Subject to Unit Traction Test Problem



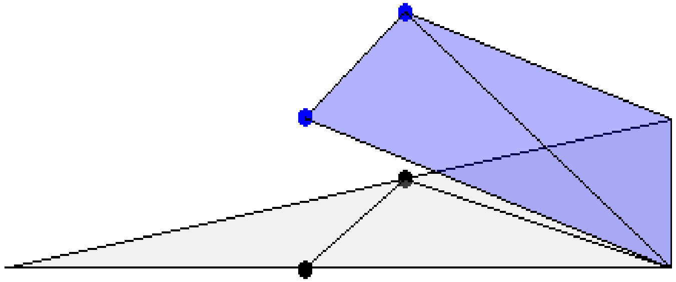
- 2D Vector Problem with Finite Domain (One-sided problem)
  - Plane strain solution
    - Fully 2D, trigonometric solution

$$u_x(r,\theta) = \frac{r(1-\nu^2)}{E}\left[\cos\theta + \frac{2a^2}{r^2}\left(\cos\theta + \frac{r^2-a^2}{4(1-\nu)r^2}\cos 3\theta\right)\right]$$

$$u_y(r,\theta) = \frac{r(1+\nu)}{E}\left[-\nu\sin\theta + \frac{2a^2}{r^2}\left(\frac{(2\nu-1)}{2}\sin\theta + \frac{r^2-a^2}{4r^2}\sin 3\theta\right)\right]$$

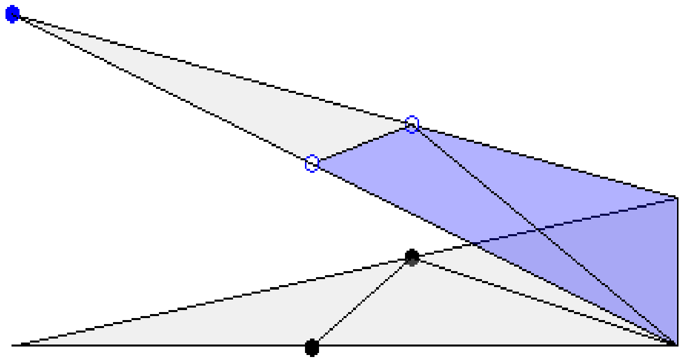
One-sided Field Candidate 1: CDFEM

- Nodes are added at intersection between interface and background element edges
- Assembly performed over conformal elements with domain

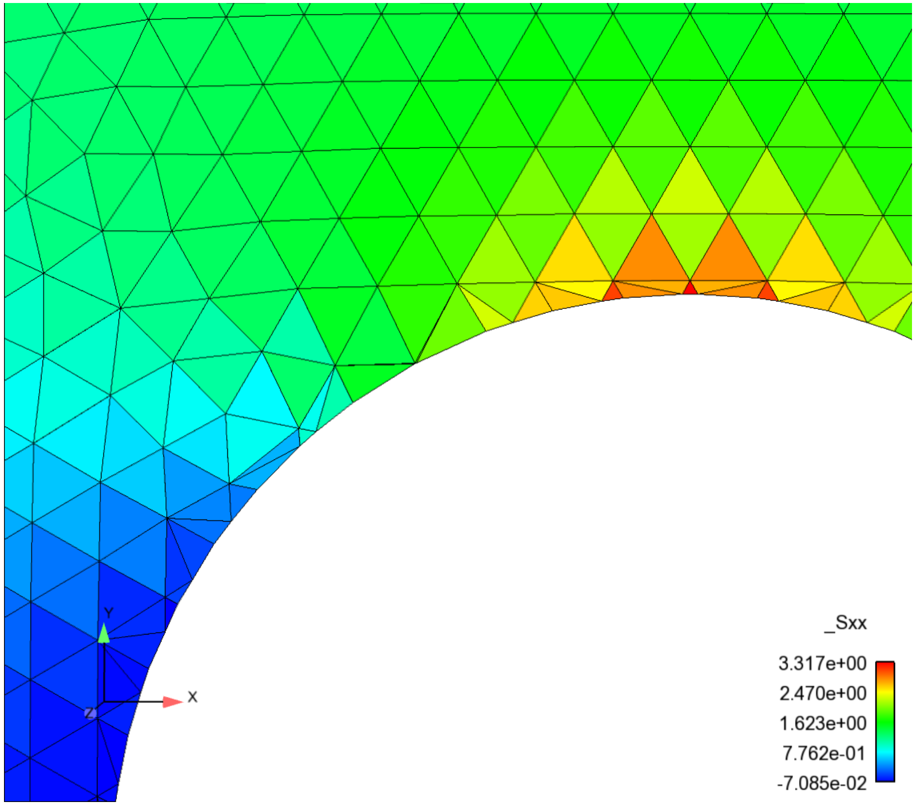


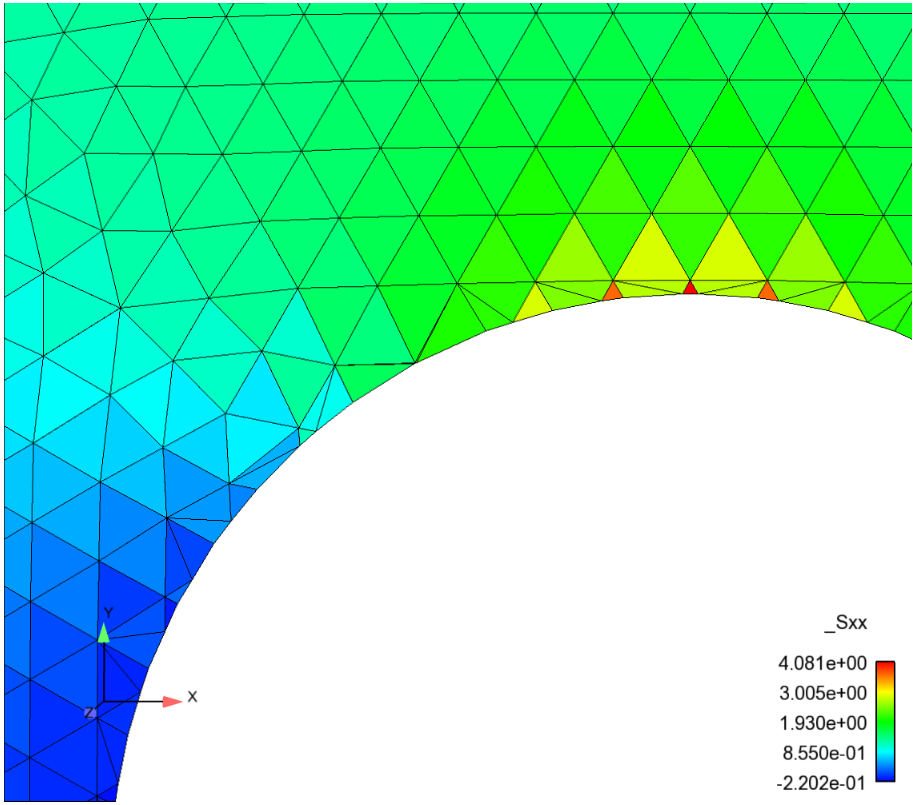
One-sided Field Candidate 2: Heaviside XFEM (CutFEM)

- DOFs at added nodes are constrained to satisfy linear edge constraints effectively replacing the DOFs with Heaviside enriched DOF at parent nodes outside domain
- Captures natural BC on cut surface



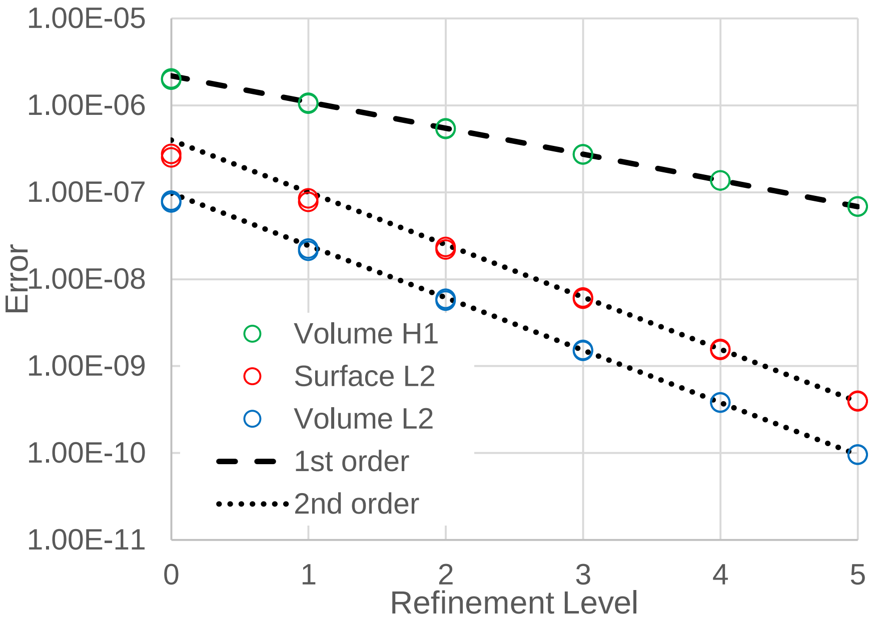
Qualitative Gradient Comparison

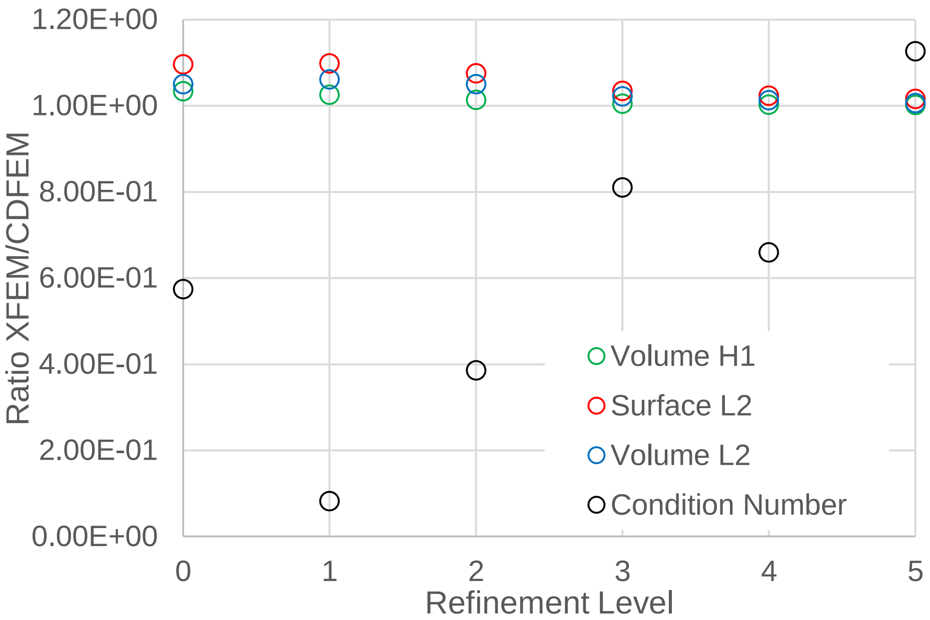




- CDFEM solution has gradient jumps between subelements
- XFEM solution on simplex elements has constant gradient in subelements of common parent element

Infinite Plate Test Problem Accuracy





- Comparison
  - Results extremely similar to scalar diffusion results
  - All methods converge optimally in L_2 and H_1
  - L_2 and H_1 errors always higher for XFEM than for CDFEM
  - Condition numbers typically higher for CDFEM

Conclusions

- CDFEM useful as discretization method or preprocessor for assembling enriched approximations
  - CDFEM, HIFEM, Heaviside XFEM, new C_0 XFEM all assembled from CDFEM by adding algebraic constraints
- Comparison of Enriched Methods
  - CDFEM, HIFEM, XFEM all converge optimally for weak discontinuity problem and elastic plate problem
  - Errors across all methods quite similar
  - L_2 errors for CDFEM always lower than the corresponding XFEM methods obtained by constraining the CDFEM solution to have additional smoothness
- Matrix Conditioning
  - Equations produced directly by CDFEM may be poorly conditioned
  - Equations produced by XFEM may or may not be better conditioned
  - Strategies like change of variables or specialized preconditioners can be effective at removing poor conditioning