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Implicit-explicit time integration of multi-fluid plasma models using compatible discretizations

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Multi-fluid plasma modeling

- Multi-fluid plasma models simulate each species independently and couple the species together through collisional and electromagnetic operators.
  - Primarily designed for systems with fast time scales where electron inertial effects and charge separation are resolved.
  - Useful for modeling interactions between ion/neutral species as the closures do not rely on complex equations of state.
- **Main drawback**: model is computational expensive due to the necessity of resolving fast plasma scales associated with electron dynamics and Maxwell's equations.
- **Research objective**: Explore the behavior of strongly-coupled, multi-scale systems, specifically when stepping over fast time scales using **fully implicit** and mixed **implicit-explicit** (IMEX) time integration. Topics include:
  - How to split the multi-fluid plasma model for IMEX integration?
  - What happens to stability/accuracy when stepping over the faster time scales?
  - Is an accurate solution expected for slow dynamics when stepping over fast dynamics?
  - How do we design preconditioning schemes to solve strongly-coupled fully-implicit and IMEX systems?

Multi-fluid plasma scales

- Scaling is strongly dependent on a species α's mass, density, and temperature.
- Can be broken into frequency scales, velocity scales, and diffusion scales:

Plasma frequency

$$\omega_{p\alpha} = \sqrt{\frac{q_\alpha^2 n_\alpha}{m_\alpha \epsilon_0}}$$

Cyclotron frequency

$$\omega_{c\alpha} = \frac{q_\alpha B}{m_\alpha}$$

Collision frequency

$$\nu_{\alpha\beta} \sim \frac{n_\beta}{\sqrt{m_\alpha} T_\alpha^{\frac{3}{2}}} \frac{1 + \frac{m_\alpha}{m_\beta}}{\left(1 + \frac{m_\alpha}{m_\beta}\frac{T_\beta}{T_\alpha}\right)^{\frac{3}{2}}}$$

Flow velocity

$$u_\alpha$$

Speed of sound

$$v_{s\alpha} = \sqrt{\frac{\gamma P_\alpha}{\rho_\alpha}}$$

Speed of light

$$c \gg u_\alpha, v_{s\alpha}$$

Momentum diffusivity

$$\upsilon_\alpha = \frac{\mu_\alpha}{\rho_\alpha}$$

Thermal diffusivity

$$\kappa_\alpha \sim \frac{k_\alpha}{\rho_\alpha}$$

Primitive 5-moment model scaling

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$$\partial_t \rho_\alpha + \boldsymbol{u}_\alpha \cdot \nabla \rho_\alpha = -\rho_\alpha \nabla \cdot \boldsymbol{u}_\alpha$$

$$u_\alpha < \frac{\Delta x}{\Delta t}$$

Each operator is associated with one or more plasma scales. Here are the dominant explicit stability limits:

$$\partial_t \boldsymbol{u}_\alpha + \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha = -\boldsymbol{u}_\alpha \nabla \cdot \boldsymbol{u}_\alpha - \frac{1}{\rho_\alpha} \nabla P_\alpha + \frac{1}{\rho_\alpha} \nabla \cdot \left(\mu_\alpha \left(\nabla \boldsymbol{u}_\alpha + \nabla \boldsymbol{u}_\alpha^T - \frac{2}{3} \boldsymbol{I} \nabla \cdot \boldsymbol{u}_\alpha \right) \right)$$

$$u_\alpha < \frac{\Delta x}{\Delta t} \qquad v_{s\alpha} < \frac{\Delta x}{\Delta t} \qquad \upsilon_\alpha < \frac{\Delta x^2}{\Delta t}$$

$$+ \frac{q_\alpha}{m_\alpha} \boldsymbol{E} + \frac{q_\alpha}{m_\alpha} \boldsymbol{u}_\alpha \times \boldsymbol{B} - \sum_\beta \nu_{\alpha\beta} \left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta \right)$$

$$\omega_{p\alpha} \Delta t < 1 \qquad \omega_{c\alpha} \Delta t < 1 \qquad \nu_{\alpha\beta} \Delta t < 1$$

$$\partial_t P_\alpha + \boldsymbol{u}_\alpha \cdot \nabla P_\alpha = -\gamma P_\alpha \nabla \cdot \boldsymbol{u}_\alpha + \nabla \cdot \left((\gamma - 1) k_\alpha \nabla T_\alpha \right) - \sum_\beta \frac{(\gamma - 1) \nu_{\alpha\beta} \rho_\alpha}{m_\alpha + m_\beta} \left(3 \left(T_\alpha - T_\beta \right) - m_\beta \left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta \right)^2 \right)$$

$$u_\alpha < \frac{\Delta x}{\Delta t} \qquad \kappa_\alpha < \frac{\Delta x^2}{\Delta t} \qquad \nu_{\alpha\beta} \Delta t < 1$$

$$\partial_t \boldsymbol{E} - c^2 \nabla \times \boldsymbol{B} = -\frac{1}{\epsilon_0} \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \boldsymbol{u}_\alpha$$

$$c < \frac{\Delta x}{\Delta t} \qquad \omega_{p\alpha} \Delta t < 1$$

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0$$

The compressible multi-fluid plasma model is a strongly-coupled, non-linear system of equations representing many stiff scales. How do we solve this?

Explicit time integration

- Explicit integration is a classic method for evaluating time derivatives.
- For **Runga-Kutta** methods, explicit time integration can be written in the form:

$$\partial_t u = f(u,t)$$

$$u^{(i)} = u^n + \Delta t \sum_{j=0}^{j<i} \hat{A}_{ij} f\left(u^{(j)}, t_n + \hat{c}_j \Delta t\right)$$

$$u^{(0)} = u^n$$

$$u^{n+1} = u^n + \Delta t \sum_{i=0}^{i<s} \hat{b}_i f\left(u^{(i)}, t_n + \hat{c}_i \Delta t\right)$$

Explicit tableau

$$\begin{array}{c|c} \hat{c} & \hat{A} \\ \hline & \hat{b}^t \end{array}$$

- **Pros:**
  - Computationally efficient to evaluate -> No global solve required
  - Easy to implement and debug
- **Cons:**
  - Stability depends on resolving frequency, velocity, and diffusive scales -> Long runtimes for stiff models.

Implicit time integration

- Implicit time integration attempts to solve for a future state as a whole using an optimization process.
- Singly diagonally implicit time integrators (**SDIRK**) can be written in the form:

$$\partial_t u = g(u,t)$$

$$u^{(i)} = u^n + \Delta t \sum_{j=0}^{j\leq i} A_{ij} g\left(u^{(j)}, t_n + c_j \Delta t\right)$$

$$u^{n+1} = u^n + \Delta t \sum_{i=0}^{i\leq s} b_i g\left(u^{(i)}, t_n + c_i \Delta t\right)$$

Implicit tableau

$$\begin{array}{c|c} c & A \\ \hline & b^t \end{array}$$

- **Pros:**
  - Can step over frequency, velocity, and diffusive time scales, but this sacrifices accuracy for the fast under-resolved dynamics
- **Cons:**
  - Requires a global solve across a parallel system
  - Multi-scale systems (e.g. plasmas) require complex preconditioning schemes

IMEX integration

- Splitting the model up based on stiffness allows us to choose what goes into the implicit solve:
  - Explicit for slow, non-stiff terms
  - Implicit for fast, stiff terms

Implicit tableau

$$\begin{array}{c|c} c & A \\ \hline & b^t \end{array}$$

Explicit tableau

$$\begin{array}{c|c} \hat{c} & \hat{A} \\ \hline & \hat{b}^t \end{array}$$

$$\partial_t u = f(u,t) + g(u,t)$$

$$u^{(i)} = u^n + \Delta t \sum_{j=0}^{j<i} \hat{A}_{ij} f\left(u^{(j)}, t_n + \hat{c}_j \Delta t\right) + \Delta t \sum_{j=0}^{j\le i} A_{ij} g\left(u^{(j)}, t_n + c_j \Delta t\right)$$

$$u^{n+1} = u^n + \Delta t \sum_{i=0}^{i<s} \hat{b}_i f\left(u^{(i)}, t_n + \hat{c}_i \Delta t\right) + \Delta t \sum_{i=0}^{i\le s} b_i g\left(u^{(i)}, t_n + c_i \Delta t\right)$$

- **Objective**: Combine the advantages of implicit and explicit solvers
  - Take advantage of slow implicit solver to overstep fast scales, and fast explicit solver to resolve slow scales.

Example 3-stage IMEX algorithm

Implicit Solves

Explicit Solves

$$u^{(0)} = u^n + \Delta t\left(A_{00}\, g^{(0)}\right)$$

$$g^{(0)} = g(u^{(0)}, t_n + c_0 \Delta t)$$
$$f^{(0)} = f\left(u^{(0)}, t_n + \hat{c}_0 \Delta t\right)$$

$$u^{(1)} = u^n + \Delta t\left(\hat{A}_{10} f^{(0)} + A_{10} g^{(0)} + A_{11}\, g^{(1)}\right)$$

$$g^{(1)} = g\left(u^{(1)}, t_n + c_1 \Delta t\right)$$
$$f^{(1)} = f(u^{(1)}, t_n + \hat{c}_1 \Delta t)$$

$$u^{(2)} = u^n + \Delta t\,(\hat{A}_{20} f^n + \hat{A}_{21} f^{(1)} + A_{20} g^{(0)} + A_{21} g^{(1)} + A_{22} g^{(2)})$$

$$g^{(2)} = g\left(u^{(2)}, t_n + c_2 \Delta t\right)$$
$$f^{(2)} = f(u^{(2)}, t_n + \hat{c}_2 \Delta t)$$

$$u^{n+1} = u^n + \Delta t\,(\hat{b}_0 f^n + \hat{b}_1 f^{(1)} + \hat{b}_2 f^{(2)} + b_0 g^{(0)} + b_1 g^{(1)} + b_2 g^{(2)})$$

Splitting multi-fluid plasma model

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- For most applications, multi-fluid plasma model's can be broken into fast and slow components based on the associated time scales:

$$\partial_t \rho_\alpha + \boldsymbol{u}_\alpha \cdot \nabla \rho_\alpha = -\rho_\alpha \nabla \cdot \boldsymbol{u}_\alpha$$

$$\partial_t \boldsymbol{u}_\alpha + \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha = -\boldsymbol{u}_\alpha \nabla \cdot \boldsymbol{u}_\alpha - \frac{1}{\rho_\alpha} \nabla P_\alpha + \frac{1}{\rho_\alpha} \nabla \cdot \left(\mu_\alpha \left(\nabla \boldsymbol{u}_\alpha + \nabla \boldsymbol{u}_\alpha^T - \frac{2}{3} \boldsymbol{I} \nabla \cdot \boldsymbol{u}_\alpha \right) \right)$$

$$+ \frac{q_\alpha}{m_\alpha} (\boldsymbol{E} + \boldsymbol{u}_\alpha \times \boldsymbol{B}) - \sum_\beta \nu_{\alpha\beta} (\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta)$$

$$\partial_t P_\alpha + \boldsymbol{u}_\alpha \cdot \nabla P_\alpha = -\gamma P_\alpha \nabla \cdot \boldsymbol{u}_\alpha + \nabla \cdot \big((\gamma - 1) k_\alpha \nabla T_\alpha \big) - \sum_\beta \frac{(\gamma - 1) \nu_{\alpha\beta} \rho_\alpha}{m_\alpha + m_\beta} \left(3 \big(T_\alpha - T_\beta \big) - m_\beta \left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta \right)^\wedge 2 \right)$$

$$\partial_t \boldsymbol{E} - c^2 \nabla \times \boldsymbol{B} = -\frac{1}{\epsilon_0} \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \boldsymbol{u}_\alpha \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0$$

- The stiffness of these terms are **problem dependent**, but their application in IMEX is easy to modify

Compatible discretization for EM

- For this research, a compatible finite element discretization is used to enforce the divergence constraints for the electric and magnetic fields.
- Fluids are represented by an HGrad (node) basis $\rho \in V_{\nabla}$.
- The electric field is represented by an HCurl (edge) vector basis $\mathbf{E} \in \boldsymbol{V}_{\nabla\times}$.
- The magnetic field is represented by an HDiv (face) vector basis $\mathbf{B} \in \boldsymbol{V}_{\nabla\cdot}$.
- Compatibility is defined by the discrete preservation of the **De Rham Complex**:

$$\nabla\phi_{\nabla} \in \boldsymbol{V}_{\nabla\times} \longrightarrow \nabla\times\boldsymbol{\phi}_{\nabla\times} \in \boldsymbol{V}_{\nabla\cdot} \longrightarrow \nabla\cdot\boldsymbol{\phi}_{\nabla\cdot} \in V_{\mathrm{L}_2}$$

- For Faraday's law, we choose a basis for the electric field such that its curl is fully represented by the basis used by the magnetic field.

$$\partial_t \boldsymbol{B} + \nabla\times\boldsymbol{E} = 0$$

- Since the curl of the electric field is 'globally continuous' w.r.t. a divergence operator, the divergence of that curl is zero over the domain:

$$\nabla\cdot(\partial_t\boldsymbol{B} + \nabla\times\boldsymbol{E}) = \partial_t(\nabla\cdot\boldsymbol{B}) + \nabla\cdot\nabla\times\boldsymbol{E} = \partial_t(\nabla\cdot\boldsymbol{B}) + \sum_i E_i \overset{0}{\nabla\cdot\nabla\times\boldsymbol{\phi}^i_{\nabla\times}} = \partial_t(\nabla\cdot\boldsymbol{B})$$

$= 0$

- **Result**: The curl operator does not add divergence errors to the magnetic field

Satisfying Gauss' laws in plasmas

- **Goal:** Solve plasma-coupled Maxwell's equations and satisfy a divergence constraint:

$$\partial_t \boldsymbol{E} - c^2 \nabla \times \boldsymbol{B} = -\frac{1}{\epsilon_0}\boldsymbol{j} \qquad \partial_t \rho_c + \nabla \cdot \boldsymbol{j} = 0$$

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho_c$$

- In the **strong, non-discretized form:**

$$\nabla \cdot \left(\partial_t \boldsymbol{E} + \frac{1}{\epsilon_0}\boldsymbol{j} - c^2 \nabla \times \boldsymbol{B}\right) = \partial_t \nabla \cdot \boldsymbol{E} + \frac{1}{\epsilon_0}\nabla \cdot \boldsymbol{j} = \partial_t \left(\nabla \cdot \boldsymbol{E} - \frac{1}{\epsilon_0}\rho_c\right) = 0$$

- In the **weak form:** Choose a basis that supports the divergence constraint as HCurl does not support the divergence operation:

$$\int \left(\partial_t \boldsymbol{E} - c^2 \nabla \times \boldsymbol{B} + \frac{1}{\epsilon_0}\boldsymbol{j}\right) \cdot \nabla \phi_\nabla \, dV = \int \left(\partial_t \boldsymbol{E} \cdot \nabla \phi_\nabla + \frac{1}{\epsilon_0}\nabla \cdot \boldsymbol{j}\, \phi_\nabla\right) dV + c^2 \int \boldsymbol{B} \cdot \underbrace{\nabla \times \nabla \phi_\nabla}_{0} dV$$

$$= \int \partial_t \left(\boldsymbol{E} \cdot \nabla \phi_\nabla - \frac{1}{\epsilon_0}\rho_c\, \phi_\nabla\right) dV = 0$$

- Assumes that continuity equation is weakly satisfied:

$$\int (\partial_t \rho_c - \nabla \cdot \boldsymbol{j})\phi_\nabla \, dV = \int (\partial_t \rho_c \phi_\nabla + \boldsymbol{j} \cdot \nabla \phi_\nabla) dV = 0 \rightarrow \int \partial_t \rho_c \phi_\nabla \, dV = -\int \boldsymbol{j} \cdot \nabla \phi_\nabla \, dV$$

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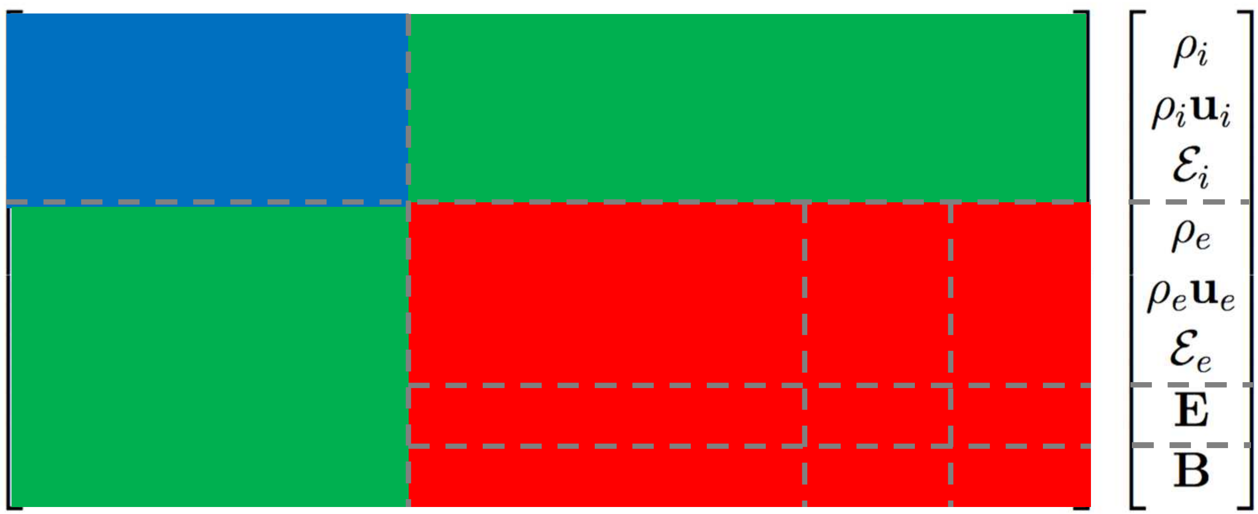
Adaptable block preconditioning

- Solving strongly-coupled, multi-scale systems of equations requires advanced preconditioning techniques.
- **Block preconditioning** is a method of identifying and isolating specific scales in a system by grouping the terms together.
- Many options are available, for instance, when overstepping electron time scales we may block the system into fast, slow, and coupling terms:



- Blocking the ion dynamics (potentially including neutral fluids) into a ‘slow’ block separates the scales from the ‘fast’ block preconditioner*.

*E.G. Phillips, J.N. Shadid, E.C. Cyr, H.C. Elman, and R.P. Pawlowski, Block Preconditioners for Stable Mixed Nodal and Edge FE Representations of Incompressible Resistive MHD, *SISC*, Vol. 38 (6), 2016.
*E.C. Cyr, J.N. Shadid, R.S. Tuminaro, R.P. Pawlowski, and L. Chacon, A New Approximate Block Factorization Preconditioner for Two Dimensional Incompressible (Reduced) Resistive MHD, *SIAM Journal on Scientific Computing*, 35:B701-B730, 2013.
*E. G. Phillips, J. N. Shadid, and E. C. Cyr, Scalable Preconditioners for Structure Preserving DIscretizations of Maxwell Equations in First Order Form, In preparation, 2017

Linearized two-fluid wave tests

- Linearizing the two-fluid plasma model results in a set of linear waves describing the dynamics of perturbations in a background equilibrium plasma.
- **Electrostatic waves:** Longitudinal waves usually associated with acoustic modes in plasmas
- **Electromagnetic waves:** Transverse electromagnetic wave that drives current in plasma thereby slowing the EM wave's speed

Two-fluid electrostatic

$$\rho_\alpha = m_\alpha n_0 (1 + \delta_\alpha \sin(k_x x - \omega t))$$

$$u_x^\alpha = \delta_\alpha \frac{\omega}{k_x} \sin(k_x x - \omega t)$$

$$P_\alpha = n_0 T_\alpha (1 + \gamma \delta_\alpha \sin(k_x x - \omega t))$$

$$E_x = E_0 \cos(k_x x - \omega t)$$

Two-fluid EM circularly polarized

$$u_y^\alpha = \pm u_0^\alpha \sin(k_x x - \omega t)$$

$$u_z^\alpha = \pm u_0^\alpha \cos(k_x x - \omega t)$$

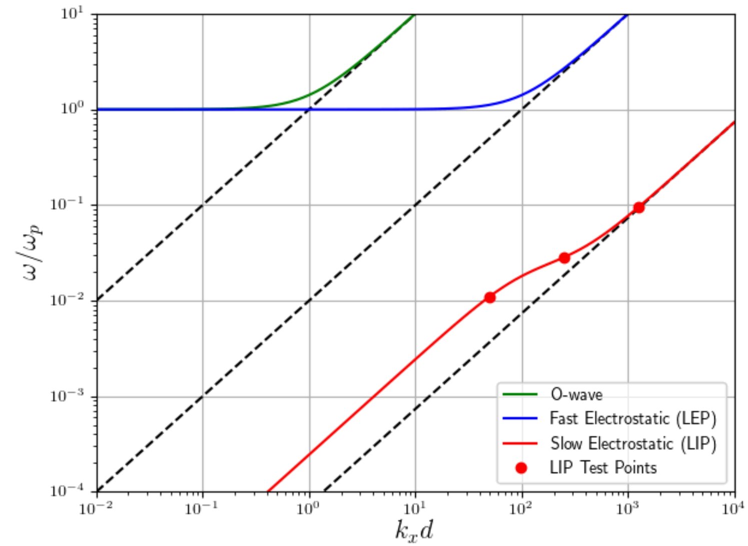
$$E_y = \pm E_0 \cos(k_x x - \omega t)$$

$$E_z = \pm E_0 \sin(k_x x - \omega t)$$

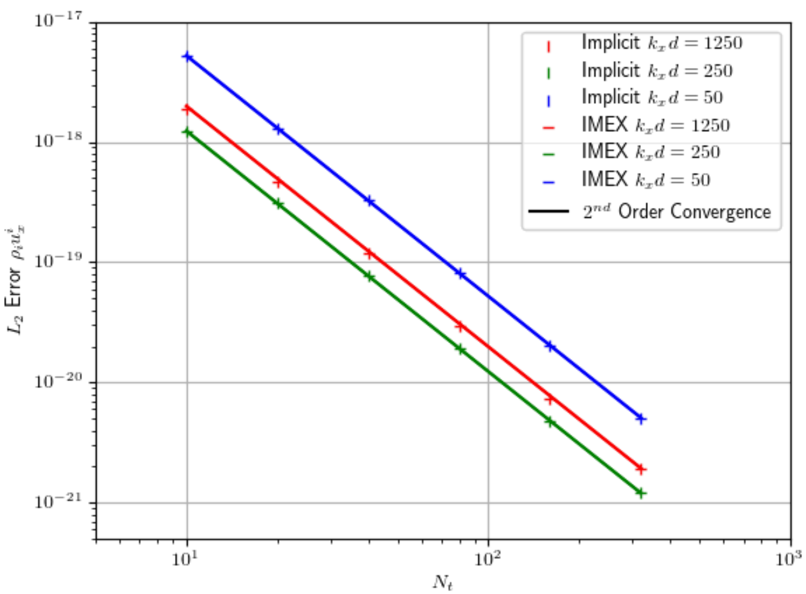
- Notes for test cases:
  - Mass ratio of $m_i = 1836\, m_e$, temperature ratio $T_e = 10\, T_i$
  - Convergence is tested against a constant CFL condition $\Delta t \propto \Delta x$
  - Tests compare SDIRK22 (fully implicit) and SSP3-332 (IMEX) time integrators

Electrostatic wave: LIP

- There are two electrostatic waves:
  - Fast longitudinal electron waves (**LEP**)
  - Slow longitudinal ion waves (**LIP**)
- While LEP can be handled explicitly or implicitly at around the same runtime, LIP is not efficiently simulated by explicit solvers.
- Convergence is equivalent for IMEX and fully implicit when overstepping fast scales.



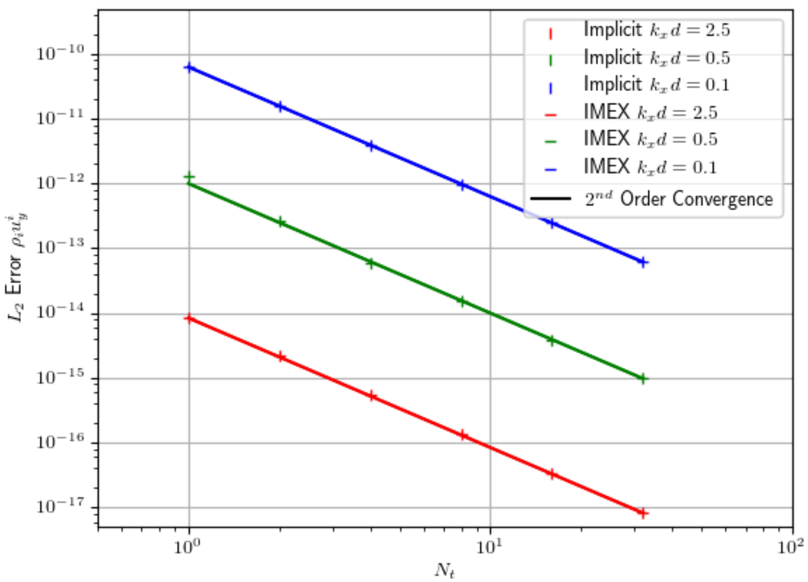
Plasma Scales for $k_x d = 1250$		
	Electrons	Ions
$\omega_p \Delta t$	$0.02 - 0.67$	$4.8 \cdot 10^{-4} - 0.015$
$\omega_c \Delta t$	$2 \cdot 10^{-3} - 0.067$	$1 \cdot 10^{-6} - 3.6 \cdot 10^{-5}$
$v_s \frac{\Delta t}{\Delta x}$	13	0.097
$u \frac{\Delta t}{\Delta x}$	10^{-8}	$1.6 \cdot 10^{-5}$
$c \frac{\Delta t}{\Delta x}$	1314	

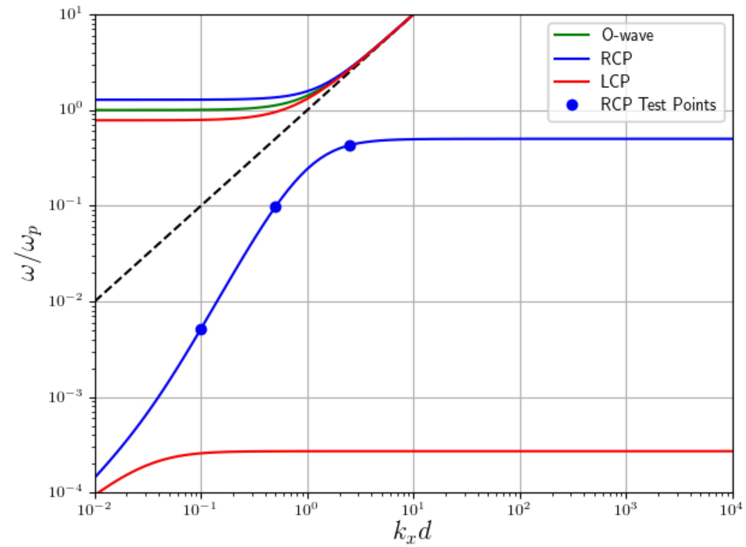


IMEX terms: implicit/explicit

Electromagnetic wave: RCP

- Right handed circularly polarized (RCP) waves have two branches:
  - **Upper branch**: Fast waves, good for explicit schemes
  - **Lower branch**: Slow waves, good for implicit and IMEX schemes
- Both fully implicit and IMEX converge properly when overstepping plasma scales.



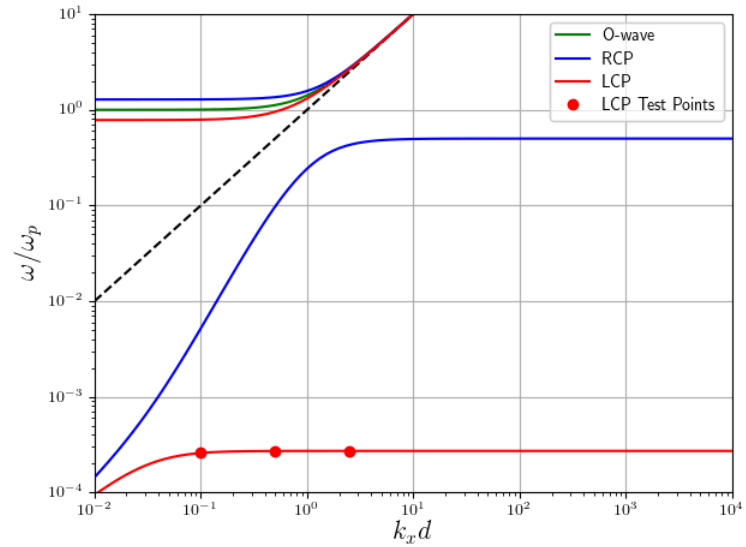


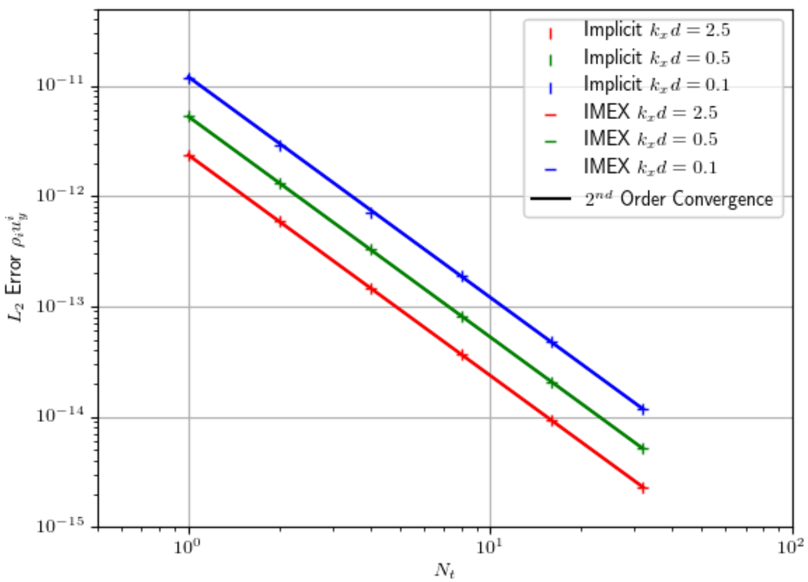
Plasma Scales for $k_x d = 0.1$		
	Electrons	Ions
$\omega_p \Delta t$	$3.8 - 121$	$0.09 - 2.8$
$\omega_c \Delta t$	$1.9 - 60$	$10^{-3} - 0.03$
$v_s \frac{\Delta t}{\Delta x}$	0.19	$1.4 \cdot 10^{-3}$
$u \frac{\Delta t}{\Delta x}$	10^{-5}	$5 \cdot 10^{-7}$
$c \frac{\Delta t}{\Delta x}$	19	

IMEX terms: implicit/explicit

Electromagnetic wave: LCP

- Left handed circularly polarized (LCP) waves are similar to RCP, but the lower branch represents an extremely slow wave making it more difficult to model.
- Results show that the ion dynamics converge as expected even though the electron and EM scales are under-resolved by factors of >1000.



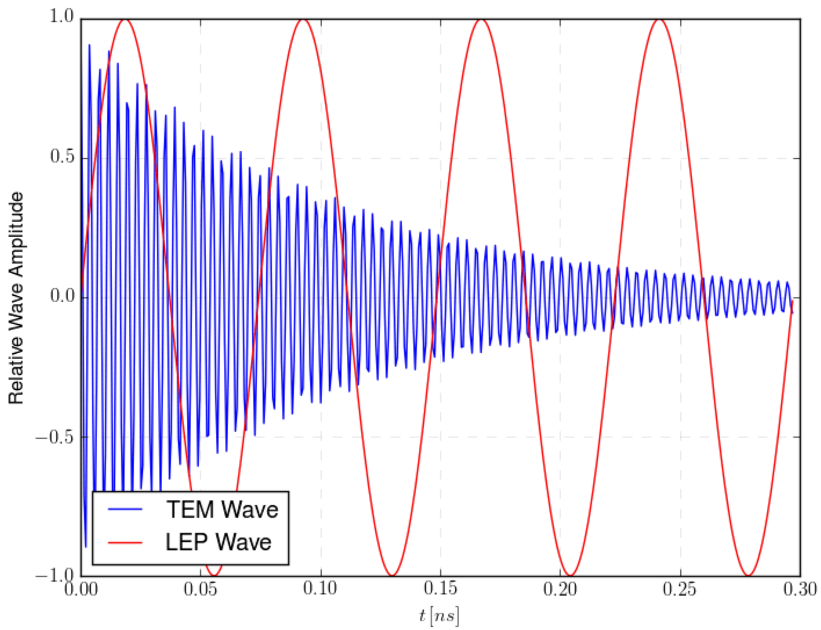


Plasma Scales for $k_x d = 2.5$		
	Electrons	Ions
$\omega_p \Delta t$	72 – 2300	1.7 – 54
$\omega_c \Delta t$	36 – 1154	0.02 – 0.6
$v_s \frac{\Delta t}{\Delta x}$	92	0.7
$u \frac{\Delta t}{\Delta x}$	10^{-7}	10^{-3}
$c \frac{\Delta t}{\Delta x}$	9181	

IMEX terms: implicit/explicit

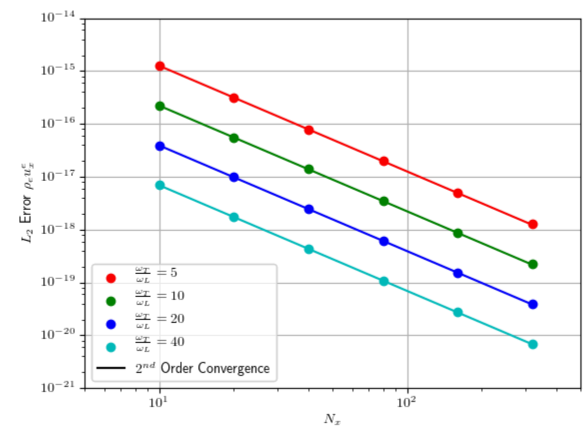
Mixing fast and slow waves

- Both the fully implicit and IMEX results use an **L-stable** time integrator to ensure that fast, under-resolved waves are damped to ensure stability.
- Example: Mixing **O-wave** and **LEP wave**

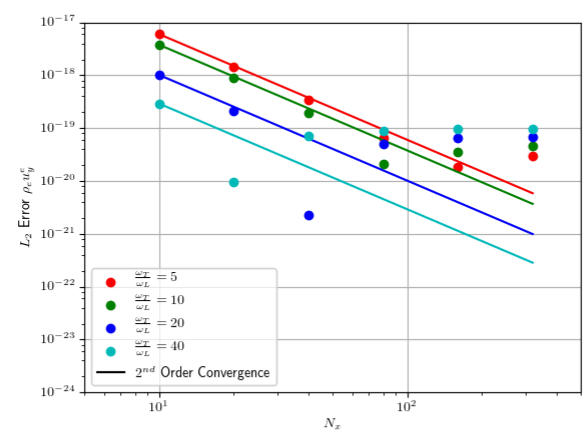


Under-resolved behavior will accrue large phase and amplitude distortion.

Convergence for slow (LEP) modes



Convergence for fast (TEM) modes



MHD asymptote tests

- **Objective**: Run a full multi-fluid solver over MHD plasma scales.
- MHD approximates the multi-fluid plasma model by **analytically removing** electron and electromagnetic time scales.
- Two main assumptions used in deriving MHD:

Slow time scales	Insignificant electron inertia
$c \to \infty, \epsilon_0 \to 0, \omega_p \to \infty$	$\frac{m_e}{m_i} \to 0$

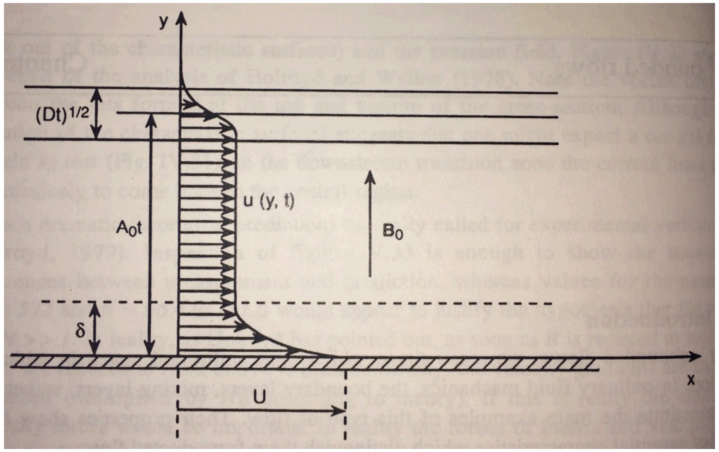
- Process converts frequency scales into velocity scales and diffusion scales:

Flow velocity	Speed of sound	Alfven velocity
$u = \frac{\rho_e u_e + \rho_i u_i}{\rho_e + \rho_i}$	$v_s = \sqrt{\frac{\gamma(P_e + P_i)}{\rho_e + \rho_i}}$	$v_A = \frac{B}{\sqrt{\mu_0(\rho_e + \rho_i)}}$

Momentum diffusivity	Thermal diffusivity	Magnetic diffusivity
$\upsilon = \frac{\mu}{\rho}$	$\kappa \sim \frac{k}{\rho}$	$\lambda = \frac{\eta}{\mu_0} = \frac{m_e \nu_{ei}}{e^2 n \mu_0}$

Resistive Alfven wave problem

- Solution is derived from resistive/viscous MHD which **ignores Hall effects**:
  - Hall parameter $\mathrm{H} = \frac{\omega_{ce}}{\nu_{ei}} = \frac{\eta\, B}{n_e e} \ll 1$
  - Reducing Hall effects in magnetized multi-fluid model is tricky - requires large collision frequency
- Problem used for verifying resistive, Lorentz force, and viscous operators:
  - Impulse shear due to a moving wall drives a **Hartmann layer**
  - Hartmann layer shear excites **Alfven wave** traveling along magnetic field
  - Alfven wave front diffuses due to momentum and magnetic diffusivity
  - Profile depends on the effective **Lundquist number** $\mathrm{S} = \frac{L\, v_A}{\lambda}$



R. Moreau, Magnetohydrodynamics, 1990

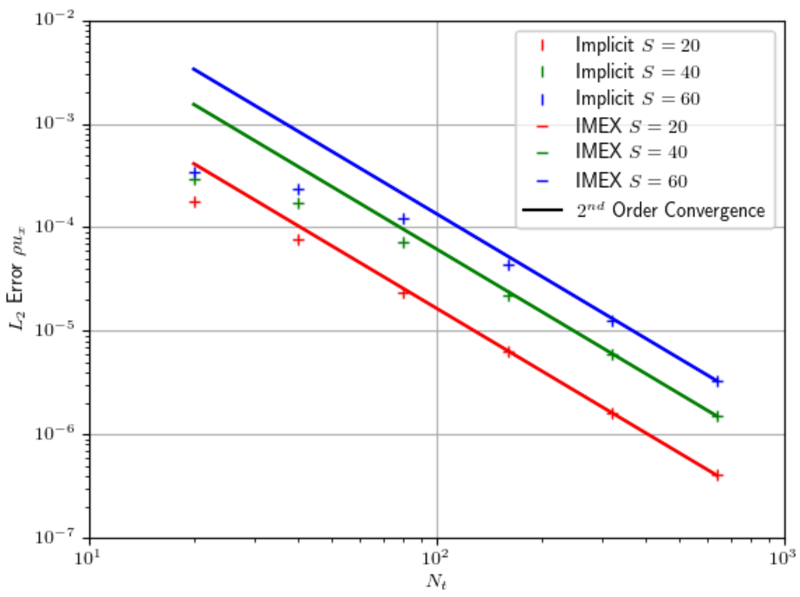
$$u_x = \frac{U}{4}\left(1 + \exp\left(\frac{v_A y}{\lambda}\right)\right)\mathrm{erfc}(\eta_+) + \frac{U}{4}\left(1 + \exp\left(-\frac{v_A y}{\lambda}\right)\right)\mathrm{erfc}(\eta_-)$$

$$B_x = \sqrt{\mu_0 \rho}\,\frac{U}{4}\left(1 - \exp\left(\frac{v_A y}{\lambda}\right)\right)\mathrm{erfc}(\eta_+) - \sqrt{\mu_0 \rho}\,\frac{U}{4}\left(1 - \exp\left(-\frac{v_A y}{\lambda}\right)\right)\mathrm{erfc}(\eta_-)$$

$$\eta_\pm = \frac{y \pm v_A t}{2\sqrt{\lambda t}}$$

Overstepping resistive plasmas

- Convergence tests show expected convergence even when massively overstepping non-MHD plasma scales
- Roll-off at low resolutions due to under-resolving Hartmann layer
  - Large Lundquist number implies thin Hartmann layer



Plasma Scales for $S = 60$		
	Electrons	Ions
$\omega_p \Delta t$	$4 \cdot 10^7 - 1.3 \cdot 10^9$	$9.4 \cdot 10^5 - 3 \cdot 10^7$
$\omega_c \Delta t$	$1.7 \cdot 10^6 - 5.5 \cdot 10^7$	$9.4 \cdot 10^2 - 3 \cdot 10^4$
$\nu_{\alpha\beta} \Delta t$	$1.7 \cdot 10^{10} - 5.5 \cdot 10^{11}$	$9.4 \cdot 10^6 - 3 \cdot 10^8$
$v_s \frac{\Delta t}{\Delta x}$	$7 \cdot 10^{-3}$	$2 \cdot 10^{-4}$
$u \frac{\Delta t}{\Delta x}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-4}$
$\frac{\mu}{\rho} \frac{\Delta t}{\Delta x^2}$	$0.4 - 12$	$0.01 - 0.3$
$c \frac{\Delta t}{\Delta x}$	167	

IMEX terms: implicit/explicit

Overstepping fast time scales is both stable and accurate. The inclusion of a resistive operator adds dissipation to the electron dynamics on top of the L-stable time integrator.

Ideal two-fluid vortex

- Extension of the ideal MHD vortex (*Balsara 2004*) where a current column (Z-pinch) is advected diagonally in a plane.
- Problem balances a pressure gradient with a combination of centrifugal and magnetic forces:

$$u_x^i = u_x^e = 1 - (y - t)\frac{k}{2\pi}\exp\left(\frac{1-r^2}{2}\right)$$

$$u_y^i = u_y^e = 1 + (x - t)\frac{k}{2\pi}\exp\left(\frac{1-r^2}{2}\right)$$

$$u_z^e = -\frac{m_i}{2e\sqrt{\mu_0\pi}}\frac{k}{2\pi}(2 - r^2)\exp\left(\frac{1-r^2}{2}\right)$$

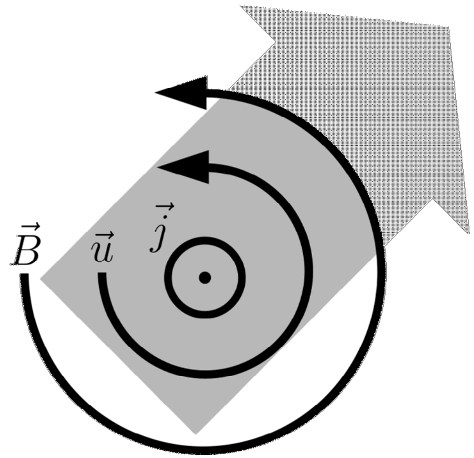
$$u_z^i = \frac{m_e}{2e\sqrt{\mu_0\pi}}\frac{k}{2\pi}(2 - r^2)\exp\left(\frac{1-r^2}{2}\right)$$

$$B_x = -(y - t)\frac{\mu}{2\pi}\sqrt{\frac{\mu_0}{4\pi}}\exp\left(\frac{1-r^2}{2}\right)$$

$$B_y = (x - t)\frac{\mu}{2\pi}\sqrt{\frac{\mu_0}{4\pi}}\exp\left(\frac{1-r^2}{2}\right)$$

$$P_e = \frac{1}{2} + \frac{m_i}{m_i + m_e}P_k + \frac{m_e}{m_i + m_e}P_m$$

$$P_i = \frac{1}{2} + \frac{m_e}{m_i + m_e}P_k + \frac{m_i}{m_i + m_e}P_m$$



$$P_k = -\frac{1}{2}\left(\frac{k}{2\pi}\right)^2 \exp(1 - r^2)$$

$$P_m = \frac{1}{8\pi}\left(\frac{\mu}{2\pi}\right)^2 (1 - r^2)\exp(1 - r^2)$$

$$r^2 = (x - t)^2 + (y - t)^2$$

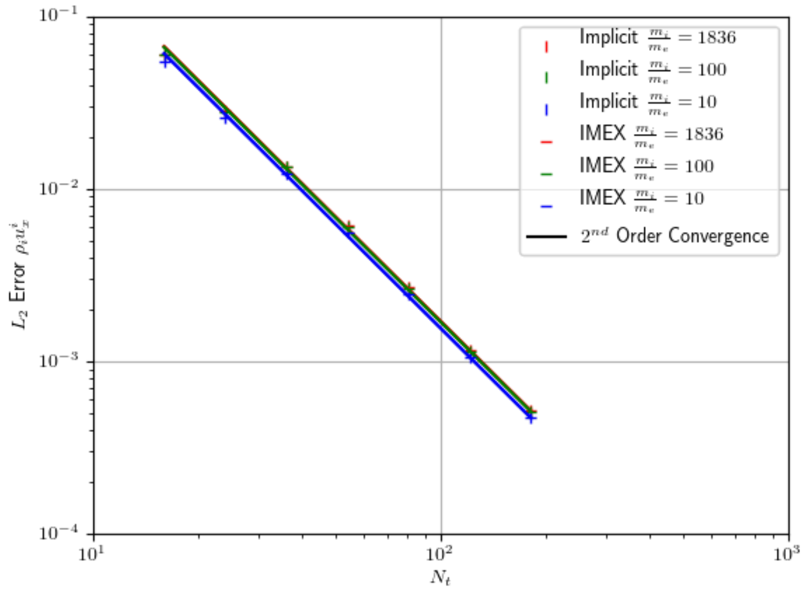
- **Note**: No damping of fast scales through resistive or viscous operators.

Overstepping dispersive plasmas

- Solution is independent of electron mass, but requires a large plasma frequency to avoid a growing displacement current.
- Results show convergence for multiple mass ratios for both IMEX and fully implicit, even when stepping over fast plasma scales.

Plasma Scales for $m_i = 1836\, m_e$		
	Electrons	Ions
$\omega_p \Delta t$	$23 - 270$	$0.55 - 6.3$
$\omega_c \Delta t$	$1 - 12$	$5.5 \cdot 10^{-4} - 6.3 \cdot 10^{-3}$
$v_s \frac{\Delta t}{\Delta x}$	0.25	$5.7 \cdot 10^{-3}$
$u \frac{\Delta t}{\Delta x}$	$6 \cdot 10^{-3}$	$6 \cdot 10^{-3}$
$c \frac{\Delta t}{\Delta x}$	6.3	

IMEX terms: implicit/explicit



L-stable time integrator keeps multi-fluid model stable and accurate, even when there are no dissipation terms in the model.

Summary

- Discussed fully implicit and IMEX time integration in application to the multi-fluid plasma model.
- Discussed the use of compatible bases and block preconditioning as a requirement for stepping over fast plasma scales in strongly-coupled, multi-fluid plasma models.
- Showed results for multi-fluid plasma model when resolving two-fluid behaviors while stepping over faster, less important physics.
  - Showed convergence of slow dynamics to analytic solution for electrostatic and electromagnetic linear dispersion tests when overstepping fast scales.
  - Showed loss of convergence for fast dynamics while retaining convergence for slow dynamics when stepping over fast time scales.
- Showed consistency with MHD asymptotes when stepping over electron scales.
  - Showed large overstepping of frequency time scales when including resistive and viscous effects in the Alfven wave problem.
  - Showed applicability toward multi-dimensional, dispersive plasmas in the ideal two-fluid vortex problem.
- Future work will focus on developing more efficient implementations of IMEX and preconditioners to tackle more advanced plasma dynamics.

Abstract

Multi-fluid plasma models, where an electron fluid is modeled alongside multiple ion and neutral species as well as the full set of Maxwell's equations, can represent physics beyond the scope of classic MHD. The drawback being that these models resolve electron dynamics and electromagnetics characterized by the plasma and cyclotron frequencies as well as the speed of light, which drastically increase runtimes for explicit time integrators. Implicit time integration schemes help alleviate this issue by stepping over these stiff time scales at the cost of accuracy. To do so, implicit schemes must solve a large system of stiff equations which can require complex preconditioning schemes to achieve convergence. For most applications, a fully implicit scheme is overkill since ion and neutral dynamics are much slower than electron and electromagnetic time scales. Mixed implicit-explicit (IMEX) integration provides a mechanism to choose which dynamics to resolve using either a complex and slow implicit solve or a simple and fast explicit solve. Removing slow dynamics from the implicit solve reduces the condition of the solution method thereby reducing runtimes. The use of compatible spatial discretizations, meaning coupling a nodal (HGrad) basis for fluid dynamics with a sets of vector bases (HDiv and HCurl) for Maxwell's equations, allows us to safely evolve electromagnetics without violating Gauss' laws for the electric and magnetic fields. The goal of this research is to develop robust methods for capturing multi-species plasma physics containing electrons and electromagnetics with runtimes more commonly associated with MHD solvers.